Symmetric Approximate Equilibrium Distributions with Finite Support

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Abstract

We show that a distribution of a game with a continuum of players is an equilibrium distribution if and only if there exists a sequence of symmetric approximate equilibrium distributions of games with finite support that converges to it. Thus, although not all games have symmetric equilibrium distributions, this result shows that all equilibrium distributions can be characterized by symmetric distributions of simpler games (i.e., games with a finite number of characteristics).

1 Introduction

Games with a continuum of players provide a natural framework to describe situations where players are strategically insignificant, which makes them interesting from a conceptual point of view. Furthermore, the large number of players in such games has a convexifying effect that obviates the need to consider mixed strategies. In fact, Mas-Colell [4] has shown the existence of an equilibrium distribution without the use of convexity assumptions on the players' actions and preferences.

In many cases, equilibrium distributions are asymmetric in the sense that players of the same type (i.e., with the same preferences and range of potential actions) choose different actions. In fact, this property is sometimes necessary in order to guarantee the existence of an equilibrium distribution.

Despite the fact that they do not always exist (see Rath et al. [6]), symmetric equilibrium distributions are appealing. The main goal of this paper is to show that every equilibrium distribution can be thought of as the limit of symmetric approximate equilibrium distributions of similar and simpler games. More precisely, we show that a distribution of a game with a continuum of players is an equilibrium distribution if and only if there exists a sequence of symmetric approximate equilibrium distributions of games with a finite number of types that converges to it.

2 The model

The model is the same as in Mas-Colell [4]. Let A be a non-empty, compact metric space of *actions* and $\mathcal{M}$ be the set of Borel probability measures on A endowed with the weak convergence topology. By Parthasarathy [5, Theorem II.6.4], it follows that $\mathcal{M}$ is a compact metric space. We use the following notation: we write $\mu_n \Rightarrow \mu$ whenever $\{\mu_n\}_{n=1}^{\infty} \subseteq \mathcal{M}$ converges to μ and ρ denote a metric on $\mathcal{M}$ that metricizes the weak convergence topology. We let d_A denote the metric on A.

A *player* is described by a continuous utility function $u : A \times \mathcal{M} \to \mathbb{R}$. The interpretation is as follows: if a player with characteristics u takes the action $a \in A$ and ν denotes the distribution of actions across all players, then $u(a, \nu)$ represents the utility enjoyed by that player.

Let $\mathcal{U}$ denote the space of continuous utility functions $u : A \times \mathcal{M} \to \mathbb{R}$ endowed with the supremum norm. The set $\mathcal{U}$ represents the space of *players' characteristics*; it is a complete, separable metric space. A game is characterized by a Borel probability measure on $\mathcal{U}$.

Given a Borel probability measure τ on $\mathcal{U} \times A$, we denote by $\tau_{\mathcal{U}}$ and τ_A the marginal distributions of τ on $\mathcal{U}$ and A respectively. The expression $u(a, \tau) \geq u(A, \tau)$ means $u(a, \tau) \geq u(a', \tau)$ for all $a' \in A$.

Given a game μ and $\varepsilon \geq 0$, a Borel probability measure τ on $\mathcal{U} \times A$ is an $\varepsilon-$ *equilibrium distribution for* μ if

1. $\tau_{\mathcal{U}} = \mu$, and
2. $\tau(\{(u, a) \in \mathcal{U} \times A : u(a, \tau_A) \geq u(A, \tau_A) - \varepsilon\}) \geq 1 - \varepsilon$.

Roughly, in an $\varepsilon-$equilibrium distribution a large fraction of the players are almost optimizing. An *equilibrium distribution* is an $\varepsilon-$equilibrium distribution with $\varepsilon = 0$.

A distribution τ is *symmetric* if there is a measurable function $h : \mathcal{U} \to A$ such that $\tau(\text{graph}(h)) = 1$. The interpretation is that players with the same characteristics play the same action.

We will use the following notation: $B_\tau = \{(u,a) \in \mathcal{U} \times A : u(a, \tau_A) \geq u(A, \tau_A)\}$, and $B_\tau^\varepsilon = \{(u,a) \in \mathcal{U} \times A : u(a,\tau_A) \geq u(A,\tau_A) - \varepsilon\}$. Note that B_τ^ε is closed, and so a Borel set; hence $\tau(B_\tau^\varepsilon)$ is well defined.

3 A Characterization of Equilibrium Distributions

The main result of the paper is:

Theorem 1 *A distribution τ is an equilibrium distribution of a game μ if and only if there exists a sequence $\{\tau_n\}_{n=1}^\infty$ of symmetric distributions with finite support, and a sequence $\{\varepsilon_n\}_{n=1}^\infty$ of positive real numbers such that:*

1. $\tau_n \Rightarrow \tau$,
2. $\varepsilon_n \searrow 0$, *and*
3. τ_n *is an* ε_n-*equilibrium distribution (of* $\tau_{\mathcal{U},n}$*) for all* $n \in \mathbb{N}$.

Proof. (Necessity) Let τ be an equilibrium distribution on $\mathcal{U} \times A$. By Lemma 1 in the Appendix, there exists a sequence $\{\eta_n\}$ of real numbers satisfying $\eta_n \searrow 0$ and a sequence $\{\mu_n\}$ satisfying μ_n has finite support, $\rho(\mu_n, \tau) < 1/n$ and μ_n is an η_n-equilibrium distribution.

Let $n \in \mathbb{N}$ and let $\text{supp}(\mu_n) \subseteq \{u_1, \ldots, u_I\} \times \{a_1, \ldots, a_L\}$. For each $1 \leq i \leq I$, let $L_i = \{1 \leq l \leq L : (u_i, a_l) \in \text{supp}(\mu_n)\}$. Let $\zeta > 0$ be such that $B_\zeta(u_i) \cap B_\zeta(u_j) = \emptyset$ for all $1 \leq i, j \leq I$ such that $i \neq j$. Finally pick, for each $1 \leq i \leq I$ and $l \in L_i$, $u_{i,l,\zeta} \in B_\zeta(u_i)$ in such a way that $u_{i,l',\zeta} \neq u_{i,l'',\zeta}$ whenever $l' \neq l''$ and define $\nu_\zeta(\{(u_{i,l,\zeta}, a_l)\}) = \mu_n(\{(u_i, a_l)\})$. Clearly, ν_ζ is a symmetric distribution: for $1 \leq i \leq I$ and $1 \leq l \leq L$ define $h(u_{i,l,\zeta}) = a_l$ and for $u \neq u_{i,l,\zeta}$ define $h(u) = a$ for some $a \neq a_l$.

Let $\{\zeta_k\}$ be such that $\zeta > \zeta_k \to 0$ and denote $\nu_k = \nu_{\zeta_k}$. We claim that $\nu_k \Rightarrow \mu_n$. Let g be a bounded, uniformly continuous real-valued function on

$\mathcal{U} \times A$. Then

$$\begin{aligned}\left|\int g\mathrm{d}\nu_k - \int g\mathrm{d}\mu_n\right| &\leq \sum_{i=1}^{I}\left|\int_{B_{\zeta_n}(u_i)\times A} g\mathrm{d}\nu_k - \int_{B_{\zeta_n}(u_i)\times A} g\mathrm{d}\mu_n\right| \\ &\leq \sum_{i=1}^{I}\left(\sum_{l\in L_i}|g(u_{i,l,k},a_l) - g(u_i,a_l)|\mu_n(\{(u_i,a_l)\})\right).\end{aligned} \tag{1}$$

Since $(u_{i,l,k}, a_l) \to (u_i, a_l)$ and so $|g(u_{i,l,k}, a_l) - g(u_i, a_l)| \to 0$ as $k \to \infty$, we obtain that $\left|\int g\mathrm{d}\nu_k - \int g\mathrm{d}\mu_n\right| \to 0$ as $k \to \infty$. Since g is arbitrary, it follows that $\nu_k \Rightarrow \mu_n$.

Let $\delta > 0$ be such that $d((a,\mu),(b,\nu)) := \max\{d_A(a,b), \rho(\mu,\nu)\} < \delta$ implies that $|u_i(a,\mu) - u_i(b,\nu)| < \eta_n$ for all $1 \leq i \leq I$. Let $K \in \mathbb{N}$ be such that $\rho(\nu_{A,K}, \mu_{A,n}) < \delta$, $\rho(\nu_K, \mu_n) < 1/n$ and $\zeta_K < \eta_n$. Define $\tau_n = \nu_K$. Clearly, $\rho(\tau_n, \tau) < 2/n \to 0$ and so $\tau_n \Rightarrow \tau$. We claim that τ_n is a $5\eta_n$−equilibrium distribution, from which the theorem will follow by letting $\varepsilon_n = 5\eta_n$, $n \in \mathbb{N}$.

Let $J \subseteq \{u_1, \ldots, u_I\} \times \{a_1, \ldots, a_L\}$ be such that $B^{\eta_n}_{\mu_n} = \{(u_i, a_l)\}_{(i,l)\in J}$. If $(u_i, a_l) \in B^{\eta_n}_{\mu_n}$ then

$$\begin{aligned}u_{i,l,\zeta_K}(a_l, \tau_{A,n}) &> u_i(a_l, \tau_{A,n}) - \eta_n \\ &> u_i(a_l, \mu_{A,n}) - 2\eta_n \\ &\geq u_i(A, \mu_{A,n}) - 3\eta_n \\ &> u_i(A, \tau_{A,n}) - 4\eta_n > u_{i,l,\zeta_K}(A, \tau_{A,n}) - 5\eta_n,\end{aligned} \tag{2}$$

where the first and last inequality follows from $||u_{i,l} - u_i|| < \zeta_K < \eta_n$, and the second and fourth from $\rho(\tau_{A,n}, \mu_{A,n}) < \delta$. Hence $B^{\varepsilon_n}_{\tau_n} = B^{5\eta_n}_{\tau_n} \supseteq \{(u_{i,l,\zeta_K}, a_l)\}_{(i,l)\in J}$. Since $\tau_n(\{(u_{i,l,\zeta_K}, a_l)\}) = \mu_n(\{(u_i, a_l)\})$ for all (i, l) such that $(u_i, a_l) \in \mathrm{supp}(\mu_n)$, it follows that

$$\begin{aligned}\tau_n(B^{\varepsilon_n}_{\tau_n}) &\geq \sum_{(i,l)\in J} \tau_n(\{(u_{i,l,\zeta_K}, a_l)\}) \\ &= \sum_{(i,l)\in J} \mu_n(\{(u_i, a_l)\}) \\ &= \mu_n(B^{\eta_n}_{\mu_n}) \geq 1 - \eta_n \geq 1 - \varepsilon_n\end{aligned} \tag{3}$$

and the claim follows.

(Sufficiency) The sufficiency part follows from Theorem 1 of Carmona [1]. We include its proof for the sake of completeness.

Let $\{\tau_n\}_n$ be a sequence of ε_n-equilibrium distributions, where $\varepsilon_n \searrow 0$ and let τ be such that $\tau_n \Rightarrow \tau$. Then $\tau_{A,n} \Rightarrow \tau_A$; so, taking a subsequence if necessary, we may assume that $\rho(\tau_A, \tau_{A,n}) < 1/n$.

Define, for each $u \in \mathcal{U}$,

$$\beta_n(u) = \sup_{a \in A, \nu \in \mathcal{M}} \{|u(a,\nu) - u(a,\tau_A)| : \rho(\nu,\tau_A) < 1/n\}.$$

Since u is continuous on $A \times \mathcal{M}$, which is compact, it follows that u is uniformly continuous. Thus, $\beta_n(u) \searrow 0$ as $n \to \infty$. We claim that β_n is continuous in $\mathcal{U}$.

Let $\eta > 0$. Define $\delta < \eta/2$. Then if $||u - v|| < \delta$, we have for any $a \in A$, and $\nu \in \mathcal{M}$ such that $\rho(\nu, \tau_A) < 1/n$

$$\begin{aligned} |v(a,\nu) - v(a,\tau_A)| \leq |v(a,\nu) - u(a,\nu)| + |u(a,\nu) + u(a,\tau_A)| + \\ + |v(a,\tau_A) - u(a,\tau_A)| < \delta + \beta_n(u) + \delta, \end{aligned} \tag{4}$$

and so $\beta_n(v) \leq 2\delta + \beta_n(u) < \eta + \beta_n(u)$. By symmetry, $\beta_n(u) < \eta + \beta_n(v)$, and so $|\beta_n(u) - \beta_n(v)| < \eta$. Hence, β_n is continuous, as claimed.

Given the definition of β_n, we have that $B^{\varepsilon_n}_{\tau_n} \subseteq D_n := \{(u,a) : u(a,\tau_A) \geq u(A,\tau_A) - \varepsilon_n - 2\beta_n(u)\}$. Since β_n is continuous, we see that D_n is closed, and so Borel measurable. Thus, $\tau_n(D_n) \geq 1 - \varepsilon_n$. Also, $D_n \searrow B_\tau$. Hence, for fixed $j \in \mathbb{N}, j \geq n$, it follows that $\tau_j(D_n) \geq \tau_j(D_j) \geq 1 - \varepsilon_j \geq 1 - \varepsilon_n$, and so $\tau(D_n) \geq \limsup_j \tau_j(D_n) \geq 1 - \varepsilon_n$. Hence, $\tau(B_\tau) = \lim_n \tau(D_n) = 1$. Therefore, τ is an equilibrium distribution of $\tau_{\mathcal{U}}$. ■

We conclude with some remarks on Theorem 1. The sufficiency part follows from a closedness property of equilibrium correspondences: if, for all $n \in \mathbb{N}$, τ_n is an equilibrium distribution and $\tau_n \Rightarrow \tau$ then τ is an equilibrium distribution (see Green [2] for a related result). Theorem 1's sufficiency part is just a slight extension of this result.

The necessity part is slightly more involved. Part of the argument consists of approximating τ by a sequence of distributions with finite support. Since these approximating distributions may not be symmetric, we symmetrize them as the following example illustrates: if (u, a_1) and (u, a_2) are in its support, then we consider another payoff function $\tilde{u}$ close to u and transfer the mass of (u, a_2) to $(\tilde{u}, a_2)$. In this way, each payoff function corresponds to one and only one action, and the distribution so obtained is indeed symmetric.

A Appendix

In the proof of Theorem 1, we used the following result, which is similar to the necessity part of Theorem 1 of Carmona [1]:

Lemma 1 *Suppose that μ is an equilibrium distribution. Then, there exists a sequence $\{\eta_n\}$ of real numbers and a sequence $\{\mu_n\}$ of Borel measures satisfying:*

1. *μ_n has finite support,*
2. *$\mu_n \Rightarrow \mu$,*
3. *$\eta_n \searrow 0$, and*
4. *μ_n is an η_n-equilibrium distribution.*

Proof. Let $n \in \mathbb{N}$ and define $\varepsilon_n = 1/n$. Since $\mathcal{U}$ is a complete separable metric space, and A is compact, then τ is tight by Parthasarathy [5, Theorem II.3.2], as $\mathcal{U} \times A$ is also a complete separable metric space.

Since B_τ is closed, and so a Borel set, let $K_n \subseteq B_\tau$ be compact, and satisfy $\tau(B_\tau \setminus K_n) < 1/2n$. Since τ is an equilibrium distribution, it follows that $\tau(B_\tau) = 1$, and so $\tau(K_n) > 1 - 1/2n$. If π denotes the projection of $\mathcal{U} \times A$ into $\mathcal{U}$, then $\pi(K_n)$ is compact, and $K_n \subseteq \pi(K_n) \times A$. In particular, $\pi(K_n)$ is equicontinuous by the Ascoli-Arzela Theorem since A, and $\mathcal{M}$ are both compact metric spaces. Furthermore, denoting $C_n = \pi(K_n) \times A$, it follows that $\tau(C_n \cap B_\tau) \geq \tau(K_n \cap B_\tau) = \tau(K_n) > 1 - 1/2n$.

Let $\delta_n > 0$ be such that $d((a,\mu),(b,\nu)) := \max\{d_A(a,b), \rho(\mu,\nu)\} < \delta_n$ implies that $|u(a,\mu) - u(b,\nu)| < 1/2n$ for all $u \in \pi(K_n)$. By Lemma 5 of Carmona [1], which is a generalization of Theorem II.6.3 of Parthasarathy [5], there exists a sequence $\{\psi_j\}$ such that ψ_j has finite support, $\psi_j \Rightarrow \tau$, and $\lim_j \psi_j(C_n \cap B_\tau) = \tau(C_n \cap B_\tau)$ (the last property is the only one that does not follow from Parthasarathy [5, Theorem II.6.3]). Hence, $\psi_{A,j} \Rightarrow \tau_A$, and let $J_n \in \mathbb{N}$ be such that $\rho(\psi_{A,J_n}, \tau_A) < \delta_n$, $|\psi_{J_n}(C_n \cap B_\tau) - \tau(C_n \cap B_\tau)| < 1/2n$ and $\rho(\psi_{J_n}, \tau) < 1/n$. Define $\mu_n = \psi_{J_n}$.

By construction of $\{\mu_n\}_n$ we have $\mu_n \Rightarrow \tau$, and that for every $n \in \mathbb{N}$, $\rho(\mu_n, \tau) < 1/n$, $\rho(\mu_{A,n}, \tau_A) < \delta_n$, and $|\mu_n(C_n \cap B_\tau) - \tau(C_n \cap B_\tau)| < 1/2n$.

We have that $C_n \cap B_\tau \subseteq C_n \cap B_{\mu_n}^{1/n}$, since if $(u,a) \in C_n \cap B_\tau$ then $u(a, \mu_{A,n}) > u(a, \tau_A) - 1/2n \geq u(A, \tau_A) - 1/2n > u(A, \mu_{A,n}) - 1/n$ since $\rho(\mu_{A,n}, \tau_A) < \delta_n$. So $1 - 1/n < \tau(C_n \cap B_\tau) - 1/2n < \mu_n(C_n \cap B_\tau) \leq \mu_n(C_n \cap B_{\mu_n}^{1/n}) \leq \mu_n(B_{\mu_n}^{1/n})$. Hence, μ_n is an ε_n-equilibrium distribution. ■

References

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