Semi-analytical solution for a problem of a uniformly moving oscillator on an infinite beam on a two-parameter visco-elastic foundation

Zuzana Dimitrovová

Departamento de Engenharia Civil, Faculdade de Ciências e Tecnologia, Universidade Nova de Lisboa and
IDMEC, Instituto Superior Técnico, Universidade de Lisboa, Lisboa, Portugal
e-mail: zdim@fct.unl.pt

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Abstract

In this paper a semi-analytical solution for transverse vibrations induced by a moving oscillator is derived and validated. It is assumed that the oscillator is moving uniformly on an infinite beam, which may be subjected to a normal force, and is supported by a two-parameter visco-elastic foundation. Full evolution of deflection shapes is derived with the help of integral transforms and methods of contour integration. Analytical solution of the problem is presented in the Laplace domain. In the time domain, vibrations are given by a sum of truly steady-state part, induced harmonic part and transient vibration that has generally low importance and rapidly decreasing tendency. Except for the transient vibration, solution is expressed as a finite sum of analytical expressions (sum of residues), of which each has at most one parameter that has to be obtained numerically. With the help of iterative techniques proposed in this paper, these parameters can be easily determined with any precision. These parameters are named as the induced frequencies and as such can identify the onset of unstable behaviour. Thus, derivations in this paper allow to predict not only the onset of instability, but also its severity, which is important for mitigation measures.

Transverse vibrations obtained for infinite beams are validated by analyses on long finite beams, exploiting a program written in Matlab software, that was previously validated by finite element software LS-DYNA. The effects of the normal force, Pasternak modulus, harmonic component of the vertical force and foundation damping are discussed.
1. Introduction

Structures subjected to moving loads have several applications in rail, road and bridge engineering. Vibration analyses of beam structures under moving loads undoubtedly contributed to the design of modern railway lines. Recent demands on the railway network capacity and operability renewed the need for better understanding of dynamic phenomena related to the train-track-soil interaction, and therefore, questions regarding moving loads are still important subjects in nowadays investigations. This is supported by the fact that rail transport is the most efficient mean of land transport from an energy and environmental point of view, thus it is essential to ensure that it operates safely and comfortably.

Analytical and semi-analytical solutions have the unquestionable advantages of quickly obtainable high-precision results that can be evaluated without the full-time history, and without the necessity of testing the results convergence. As the associated physical model usually requires substantial simplifications, this means that the results obtained are reduced to essential information that can be simply analysed.

The moving force problem is far simpler than the moving mass or oscillator problem. Analytical and semi-analytical solutions are available for finite as well as infinite beams supported by a visco-elastic foundation, conveniently reviewed for instance in Frýba’s monograph [1]. In the moving force problem, it is important to study the critical velocity. The physical phenomenon behind this behaviour is resonance. When the force is moving at the critical velocity over an infinite beam on massless foundation, then in the absence of damping the beam vibrations are unbounded. Some investigations focused on the critical velocity are presented in [2-7]. If inertial properties of the foundation are considered, then the critical velocity corresponds to the lowest velocity of wave propagation in the supporting structure. Several works have been published on this subject. Recent developments highlighting the importance of dynamic interaction between the beam and the underlying foundation can be found in [8-9], where approximate formula for the critical velocity, having values between the velocity of propagation of bending waves in the beam and Rayleigh waves in the foundation is proposed as a function of mass ratio. Other important developments in this field are presented e.g. in [10-17]. The list cannot be complete, as this has been an active research field for several decades.

In problems with moving mass or oscillator, i.e. when the inertial properties are included in the moving object, it is important to study its instability as another undesirable factor.
Instability of the moving object is not a resonance type behaviour, because instability occurs when the object is moving with a velocity that belongs to a certain interval.

The moving mass problem does not have fully analytical solution, neither in the simplest case when the mass is moving on a finite beam supported by a massless foundation. When unknown deflection shapes are expanded in eigenmodes, then the resulting equations are coupled and must be solved numerically. Other solution methods can utilize Green’s function, or discretization in finite elements.

The moving mass problem was firstly solved on finite beam as early as in 1929 by Jeffcott [18] by a method of successive approximations. Eigenmode expansion is implemented in [19] for moving oscillator and in [20] for moving mass. In the latter, the authors deal with a simply supported Euler-Bernoulli beam, but the term expressing the vertical inertia of the moving mass is not correctly developed from the mathematical point of view, which leads to omission of Coriolis and centrifugal forces. Then a huge amount of other works was published. Green’s function is implemented in [21]. In [22] so-called modal elements are developed. In [23] more general beam structure is considered, in [24] support excitation is introduced and [25] is focused on control issues. Some general aspects are reviewed in [26]. From the most recent works one can mention [27] where the interaction between proximate oscillators is analysed, and [28] where non-linear foundation is implemented, but then the problem is solved numerically.

Much less works has been published on infinite beams. One of the pioneering works is [29] where simple case is solved analytically, but for moving mass problem numerical integration is used. Analytical solution by Fourier transform is presented in [30]. For rail applications it is important to extend the moving mass problem to moving one-mass or two-mass oscillator. Interaction between the moving vehicle and an infinite structure is studied in [31], but, as in many other works, the contact is assumed as rigid. The viability of this assumption is evaluated \textit{a posteriori} by examining the contact force values. The problem is solved in the Fourier domain and the inverse transform is accomplished numerically. In [32-33] Green’s function is implemented and non-linear contact stiffness according to the Hertz theory is assumed. The works are focused on the interaction between the wheel and the rail. Nevertheless, solution methods in [30-33] hide the effect of initial instant and instability. Another approach suitable for infinite structures is based on the moving element method, [34-36].

Significant amount of works was dedicated to instability issues. [37-38] are concerned with the moving mass problem. In [39] deflection field is determined as a function of two unknown
constants. Two-mass oscillator is analysed in [39]. In [40-41] more complicated foundation model is assumed than in this paper, but, full deflection shapes are not derived.

Instability velocity interval is certainly an important result, but for numerical model calibration or unstable region control, it is pertinent to know the exact vibration pattern. Thus, the objective of this paper is to fill the gap in semi-analytical solutions by providing full evolution of beam deflection shapes and oscillator vibrations, accounting for the initial instant and identifying the onset of instability. It is assumed that a two-mass oscillator moves uniformly on an infinite beam supported by a visco-elastic two-parameter foundation. Two vertical forces are acting on the sprung and unsprung masses, respectively. The latter one can have a harmonic component representing the surface irregularity. Similar developments were presented in [42] for moving mass problem. Here, the solution is extended to one- and two-mass oscillators. In addition, transient vibrations are derived to complete the solution and adapt the vibrations to initial conditions. Final solution is thus presented as a sum of a truly steady-state part (fully analytical expression obtainable by Fourier transform), induced harmonic part (analytical expressions except for induced frequencies), and transient vibration (calculated by numerical integration).

Results presented in this paper are validated by analyses on long simply supported finite beams, similarly as in [42]. Vibrations on finite beams are obtained by eigenmode expansion, programmed in Matlab environment and previously validated by software LS-DYNA. Simple supports are assumed because then the mode shapes are given analytically by sine function, thus maintaining numerical stability of higher-order modes, [4]. In order to eliminate supports influence, it is assumed that the oscillator starts to actuate a little further from the left support. In LS-DYNA, the contact stiffness, resisting only to compression, ensures the connection between the unsprung mass and the beam. Therefore, contact loss is possible. For the results obtained by the eigenmode expansion method rigid contact is assumed, as in the derivations in this paper. Therefore, validation by LS-DYNA can only be used in cases where the contact is preserved.

In summary, the new contributions of this paper are:

(i) semi-analytical solution for moving oscillator problem on infinite beams;
(ii) analysis and determination of induced frequencies;
(iii) identification of the source of the transient part of the solution and its analysis;
(iv) investigation of the critical frequency of the harmonic force;
(v) determination of the onset of instability and its severity.
The importance and originality of these developments are based on the fact that the semi-analytical solution is presented as a sum of two parts. The first one forms the essential part of oscillator induced vibrations and is given as a finite sum of analytical expressions, of which each has at most one parameter that has to be obtained numerically as a root of complex equation. These parameters are named as induced frequencies and by the iterative techniques proposed in this paper they can be determined with any precision. Therefore, this part of the solution can be obtained quickly and accurately and is not losing its precision with increasing time, because after having the induced frequencies, vibrations are given by superposition of finite number of harmonic functions whose amplitudes are given by analytical expressions. This part of vibrations will be designated as harmonic solution. The second part of the solution is formed by transient vibrations determined numerically. Nevertheless, as will be seen in the examples presented, the contribution of transient vibrations is rather insignificant and rapidly decreasing in time. Thus, in many cases the transient vibrations can be neglected, especially at the oscillator position. Moreover, due to the decreasing tendency, those numerical calculations are more stable than fully numerical approach. Therefore, there is a clear advantage with respect to the fully numerical solution, that is sensitive to numerical evaluation and is losing its precision with increasing time. Moreover, fully numerical solution mixes all contributions together and does not indicate the unstable case a priori.

The paper is organized in the following way: in Section 2 the problem to be solved is introduced. In Section 3 the semi-analytical formula is derived. In Section 4 variety of numerical examples is presented in order to provide validation for several possible scenarios. Summary of the new developments is given in Section 5, where also main conclusions are drawn. Examples of moving mass with analysis of critical forcing frequency and of moving one-mass oscillator are placed in Appendices A and B, respectively.

2. Definition of the problem

It is assumed that a two-mass oscillator is moving uniformly on a horizontal infinite beam posted on a two-parameter visco-elastic foundation. Assumptions and simplifications for the analysis of vertical vibrations of this system are outlined as follows:
(i) the beam material is homogeneous and isotropic;
(ii) the beam has a uniform cross-section and may be subjected to a normal force acting on its axis (considered positive when inducing compression);
(iii) the beam obeys (linear elastic) Euler-Bernoulli theory;
(iv) the beam vertical displacement is measured from the equilibrium deflection caused by the beam weight;
(v) the unsprung mass is always in contact with the beam (rigid contact) in the way that the unsprung mass displacement and the corresponding beam displacement are the same at all times;
(vi) no friction is acting at the contact point;
(vii) the load and vertical displacements are considered positive when acting downward;
(viii) the horizontal position $x$ of the oscillator is determined by its velocity;
(ix) at zero time the oscillator is located at $x = 0$.

The system under consideration is depicted in Figure 1.

![Figure 1: Infinite beam on a visco-elastic two-parameter foundation subjected to a moving two-mass oscillator and a normal force.](image)

The equation of motion for the unknown vertical displacement field of the beam $\vec{w}(x,t)$ is written as

$$
EI \ddot{w}_{xxx}(x,t) + \left( N - k_p \right) \dot{w}_{xx}(x,t) + m \ddot{w}_x(x,t) + c_b \ddot{w}_x(x,t) + k \ddot{w}(x,t) = p(x,t)
$$

where $EI$, $m$, and $N$ stand for the bending stiffness and mass per unit length of the beam, and a normal force acting on the beam axis. $k$, $k_p$, and $c_b$ are Winkler’s and Pasternak’s moduli of the foundation and the coefficient of viscous damping of the foundation. $x$ is the spatial coordinate and $t$ is the time. Derivatives are designated by the respective variable in subscript position, preceded by a comma. The term referring to viscous damping of the foundation can be equally considered as a viscous damping of the beam. The Pasternak modulus, which effect is usually associated with a shear layer as represented in Figure 1, can also be assumed...
as distributed rotational springs. From Eq. (1) it is seen, that the Pasternak modulus has exactly the opposite effect as the compressive normal force.

The loading term \( p(x,t) \) is given by

\[
p(x,t) = \left( P_u + P_0 e^{i(\omega_f t + \varphi_f + 3\pi/2)} - M_u w_{0u}(t) \right) - k_x \left( w_0(t) - w_s(t) \right) - c_x \left( w_{0s}(t) - w_{s,s}(t) \right) \delta(x-vt)
\]

and the additional equation for the oscillator equilibrium is

\[
M_s w_{s,s}(t) - k_x \left( w_0(t) - w_s(t) \right) - c_x \left( w_{0s}(t) - w_{s,s}(t) \right) = P_s
\]

Here \( M_u \) and \( M_s \) are constant unsprung and sprung masses at which forces \( P_u \) and \( P_s \) are acting. \( P_u \) can have associated harmonic component \( P_0 \sin(\omega_f t + \varphi_f) \) as a result of the beam surface irregularity, where \( \omega_f \) is the forced frequency and \( \varphi_f \) is the phase angle. The harmonic part of the moving force is given more conveniently in the complex domain, thus the phase angle \( \varphi_f + 3\pi/2 \) is used to ensure the correspondence with \( \sin(\omega_f t + \varphi_f) \). \( v \) is the constant velocity of the oscillator and \( k_x, c_x \) are its stiffness and coefficient of viscous damping. \( w_0(t) \) and \( w_s(t) \) designate the vertical displacement of the contact point (unsprung mass) and of the sprung mass, and \( \delta \) is the Dirac delta function. Boundary conditions dictate zero beam deflection and zero slope at positions tending to plus and minus infinity, \( x \to \pm \infty \).

The initial conditions are considered homogeneous

\[
\begin{align*}
\overline{w}(x,t)|_{t=0} &= 0, \quad \overline{w}_x(x,t)|_{t=0} = 0 \quad \forall x \\
\overline{w}_s(t)|_{t=0} &= 0, \quad \overline{w}_{s,s}(t)|_{t=0} = 0
\end{align*}
\]

(4) (5)

To remove the additional unknown \( w_0(t) \) and express it in terms of the unknown beam deflection \( \overline{w}(x,t) \), it is necessary to use the assumption (v): \( w_0(t) = \overline{w}(vt,t) \). The relevant derivatives are obtained by the chain rule

\[
\begin{align*}
w_{0u}(t) &= v \overline{w}_x(x,t) + \overline{w}_x(x,t), \quad w_{0u,s}(t) = v^2 \overline{w}_{xx}(x,t) + 2v \overline{w}_{x,t}(x,t) + \overline{w}_t(x,t) \quad \text{with } x = vt
\end{align*}
\]

(6)

Consequently, the loading term becomes

\[
p(x,t) = \left( P_u + P_0 e^{i(\omega_f t + \varphi_f + 3\pi/2)} - M_u \overline{w}_x(x,t) + 2v \overline{w}_{x,t}(x,t) + v^2 \overline{w}_{xx}(x,t) \right) - k_x \left( \overline{w}(x,t) - w_s(t) \right) - c_x \left( v \overline{w}_x(x,t) + \overline{w}_x(x,t) - w_{s,s}(t) \right) \delta(x-vt)
\]

(7)

Then the oscillator equilibrium reads
\[ M_s w_{s,t} (t) - k_s \left( \bar{w}(vt,t) - w_s(t) \right) - c_s \left( v \bar{w}_{s,t} (vt,t) + \bar{w}_s (vt,t) - w_{s,t} (t) \right) = P_s \]  

(8)

Eq. (8) describes the vertical equilibrium at the actual oscillator position, therefore the beam displacement entering the equation is only related to that particular position.

The objective is to derive displacement fields that fulfil the governing equations given by Eq. (1), (7-8) and (4-5) by employing semi-analytical methods. This will furthermore identify the onset of instability and conditions, under which excessive vibrations occur.

3. Solution of the problem

3.1 Integral transforms

To solve the problem given by Eq. (1), (7-8) and (4-5) it is convenient to introduce the moving coordinate \( r = x - vt, \quad t = t \), \( w(r,t) = w(x-vt,t) = \bar{w}(x,t) \). Then the derivatives are

\[
\begin{align*}
\bar{w}_r(x,t) &= w_r(r,t), \quad \bar{w}_{xx}(x,t) = w_{rr}(r,t), \quad \bar{w}_{xxx}(x,t) = w_{rrr}(r,t) \\
\bar{w}_x(x,t) &= -v w_r(r,t) + w_t(r,t) \quad \text{and} \quad \bar{w}_x(x,t) = v^2 w_{rr}(r,t) - 2vw_{rt}(r,t) + w_{rt}(r,t)
\end{align*}
\]

(9)

(10)

Therefore, the previously introduced derivatives for the loading point can be shorten back to

\[
\begin{align*}
w_0,0 &= v\bar{w}_r + \bar{w}_x = vw_r - vw_r + w_r = w_r \\
w_{0,0} &= v^2 \bar{w}_{xx} + 2v \bar{w}_{xx} + \bar{w}_{rr} = v^2 w_{rr} + 2v( -vw_{rt} + w_{rt} ) + v^2 w_{rr} - 2vw_{rt} + w_{rt} = w_{rt}
\end{align*}
\]

(11)

(12)

In equations above and in what follows, function variables will be generally omitted and included only when necessary. The beam equilibrium in moving coordinates is thus

\[
EI\bar{w}_{rrr} + \left( N - k_p \right) w_{rr} + m \left( w_{rr} - 2vw_{rt} + v^2 w_{rr} \right) + c_p \left( w_t - vw_r \right) + kw = \left( P_s + P_0 e^{i(\omega_p t + \varphi_p t + z \pi / 2)} \right) - M_s w_{rr} - k_s \left( w - w_s \right) - c_s \left( w_r - w_{s,r} \right) \delta (r)
\]

(13)

The oscillator equilibrium reads

\[
M_s w_{s,t} - k_s \left( w - w_s \right) - c_s \left( w_r - w_{s,t} \right) = P_s
\]

(14)

Similarly as in Eq. (8), the beam displacement \( w \) and its derivative \( w_r \) are only considered at position \( r = 0 \). Initial and boundary conditions are unchanged, only \( x \) is replaced by \( r \), thus

\[
w(r,t) = 0, \quad w_r(r,t) = 0 \quad \text{for} \quad r \to \pm \infty
\]

(15)

\[
w(r,t) \big|_{t=0} = 0, \quad w_r(r,t) \big|_{t=0} = 0 \quad \forall r
\]

(16)

\[
w_r(t) \big|_{t=0} = 0, \quad w_{s,t}(t) \big|_{t=0} = 0
\]

(17)
Several dimensionless parameters can be introduced to simplify the resolution and analysis of the results. They include dimensionless spatial coordinate, time, displacement, frequency of the harmonic force and velocity

\[ \xi = \chi r, \quad \tau = \chi v, \quad \omega = \frac{w}{w_{st}}, \quad \alpha = \frac{v}{v_{cr}} \]  

(18)

where \( w_{st} = \frac{P_{u} \chi}{2k} \), with \( \chi = \sqrt[4]{\frac{k}{4EI}} \), is the static displacement caused by the constant force \( P_{u} \) applied on the beam on Winkler’s foundation \( k \) and \( v_{cr} = \sqrt[4]{\frac{4kEI}{m}} = \frac{1}{\chi} \sqrt{\frac{k}{m}} \) is the critical velocity of the constant force \( P_{u} \) moving uniformly on the beam on Winkler’s foundation \( k \), [1]. Further parameters are introduced for convenience. The ones related to the beam parameters are

\[ \eta_{b} = \frac{c_{b}}{2\sqrt{mk}}, \quad \eta_{N} = \frac{N}{N_{cr}} = \frac{N}{2\sqrt{kEI}}, \quad \eta_{S} = \frac{k_{p}}{2\sqrt{kEI}} \]  

(19)

where \( N_{cr} = 2\sqrt{kEI} \) is the critical buckling load of an infinite beam on Winkler’s foundation \( k \) ([43]). This implies that only \( \eta_{N} < 1 \) can be considered, otherwise instability occurs, and assumption of the small displacement theory is violated. There is no such limitation for \( \eta_{S} \). In fact, the analysis could be carried out for single parameter \( \eta_{N} - \eta_{S} \), but it is preferable to keep both parameters with their appropriate physical meaning. Conventionally \( \eta_{b} \) is called the viscous damping ratio, because of the similarity with one degree of freedom oscillator, for which \( 2\sqrt{mk} \) corresponds to the critical damping. It is to be noted that \( \eta_{b} \) is not related to the critical damping of the problem specified here. The parameters associated to the applied load are

\[ \eta_{P_{a}} = \frac{P_{a}}{P_{u}} = 1, \quad \eta_{P_{s}} = \frac{P_{s}}{P_{u}}, \quad \eta_{M_{a}} = \frac{M_{a}}{m}, \quad \eta_{M_{s}} = \frac{M_{s}}{m} \]  

(20)

designated as force and mass ratios.

If \( P_{u} = 0 \) then different force should be used as a reference value, for both, force ratios and \( w_{st} \). Finally, dimensionless parameters expressing the oscillator characteristics are

\[ \tilde{k}_{s} = \frac{k_{s} \chi}{k}, \quad \tilde{c}_{s} = \frac{c_{s} \chi}{2\sqrt{mk}} \]  

(21)

The governing equations in dimensionless form are written as
\begin{equation}
\tilde{w}_{zzzz} + 4(\eta_N - \eta_s + \alpha^2) \tilde{w}_{zz} + 4 \tilde{w}_{z\tau} - 8\alpha \tilde{w}_{z\tau} + 8\eta_h \left( \tilde{w}_{\tau} - \alpha \tilde{w}_{z\tau} \right) + 4 \tilde{w} = 4 \left( 2\eta_p + 2\eta_h e^{i(\omega + \varphi_j + 3\pi/2)} - \eta_{t_{M_s}} \tilde{w}_{z\tau} - \tilde{k}_z (\tilde{w} - \tilde{w}_s) - 2\tilde{c}_j \left( \tilde{w}_{\tau} - \tilde{w}_{z\tau} \right) \right) \delta (\xi)
\end{equation}
\begin{equation}
\eta_{t_{M_s}} \tilde{w}_{z\tau} - \tilde{k}_z (\tilde{w} - \tilde{w}_s) - 2\tilde{c}_j \left( \tilde{w}_{\tau} - \tilde{w}_{z\tau} \right) = 2\eta_p \quad (22)
\end{equation}

Similarly as in Eq. (14), the beam displacement \( \tilde{w} \) and its derivative \( \tilde{w}_{z\tau} \) in Eq. (23) is only considered at position \( \xi = 0 \). Following [6] or [36], the Laplace transform

\begin{equation}
\tilde{F} (\xi, \overline{q}) = \int_0^\infty f (\xi, \tau) e^{-\overline{q} \tau} d\tau \quad \text{with} \quad \overline{q} = iq
\end{equation}

is applied first, in order to catch correctly the initial instant. Laplace transform is also a necessary step for determination of induced frequencies. These important characteristics would be completely hidden, if double Fourier transform would have been used, as exemplified in [36]. For homogeneous initial conditions one obtains

\begin{equation}
\tilde{W}_{zzzz} + 4(\eta_N - \eta_s + \alpha^2) \tilde{W}_{zz} - 4q^2 \tilde{W} - 8iq\alpha \tilde{W}_s + 8\eta_h \left( iq\tilde{W} - \alpha \tilde{W}_s \right) + 4 \tilde{W} = \left( \frac{8\eta_p}{iq} + \frac{8\eta_h e^{i(\omega + \varphi_j + 3\pi/2)}}{iq - i\omega_j} + 4\eta_{M_s} q^2 \tilde{W} - 4\tilde{k}_s (\tilde{W} - \tilde{U}) - 8\tilde{c}_j iq (\tilde{W} - \tilde{U}) \right) \delta (\xi)
\end{equation}

\begin{equation}
-\eta_{M_s} q^2 \tilde{U} - \tilde{k}_s (\tilde{W} - \tilde{U}) - 2\tilde{c}_j iq (\tilde{W} - \tilde{U}) = \frac{2}{iq} \eta_p \quad \text{with} \quad \overline{q} = iq \quad (28)
\end{equation}

where \( \tilde{U} \) is used for transform of \( \tilde{w}_s \) for better clarity. Similarly as in Eq. (14) and (23), the image of the beam displacement \( \tilde{W} \) in Eq. (26) is only considered at position \( \xi = 0 \). Then the Fourier transform

\begin{equation}
F (p, \overline{q}) = \int_{-\infty}^{\infty} \tilde{F} (\xi, \overline{q}) e^{-ip\xi} d\xi
\end{equation}

is applied, yielding

\begin{equation}
W (p, iq) \left[ p^2 - 4p^2 (\eta_N - \eta_s + \alpha^2) - 4q^2 + 8\alpha p + 8i\eta_s q - 8i\eta_h \alpha p + 4 \right] = \frac{8\eta_p}{iq} + \frac{8\eta_h e^{i(\omega + \varphi_j + 3\pi/2)}}{iq - i\omega_j} + 4\eta_{M_s} q^2 \tilde{W} (0, iq) - 4\tilde{k}_s (\tilde{W} (0, iq) - \tilde{U}) - 8\tilde{c}_j iq (\tilde{W} (0, iq) - \tilde{U}) \quad (28)
\end{equation}

and

\begin{equation}
-\eta_{M_s} q^2 \tilde{U} - \tilde{k}_s (\tilde{W} (0, iq) - \tilde{U}) - 2\tilde{c}_j iq (\tilde{W} (0, iq) - \tilde{U}) = \frac{2}{iq} \eta_p \quad (29)
\end{equation}

For convenience the polynomial expression is designated \( D (p, q) \)

\begin{equation}
D (p, q) = p^2 - 4p^2 (\eta_N - \eta_s + \alpha^2) - 4q^2 + 8\alpha p + 8i\eta_s q - 8i\eta_h \alpha p + 4
\end{equation}
Then
\[
W(p,iq) = \frac{8\eta_{p_r}}{iq} + \frac{8\eta_{p_i} e^{i(\phi_0 + 3\pi/2)}}{iq - i\tilde{\omega}_f} + 4\eta_{M_s} q^2 \tilde{W}(0,iq) - 4\tilde{k}_s (\tilde{W}(0,iq) - \tilde{U}) - 8\tilde{c}_s iq (\tilde{W}(0,iq) - \tilde{U}) \right] \frac{1}{D(p,q)}
\]  \hspace{1cm} (31)

At this stage one can accomplish the inverse Fourier transform
\[
\tilde{F}(\xi,iq) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(p,iq) e^{i\xi p} dp
\]  \hspace{1cm} (32)

to get back the Laplace image
\[
\tilde{W}(\xi,iq) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \frac{8\eta_{p_r}}{iq} + \frac{8\eta_{p_i} e^{i(\phi_0 + 3\pi/2)}}{iq - i\tilde{\omega}_f} + 4\eta_{M_s} q^2 \tilde{W}(0,iq) - 4\tilde{k}_s (\tilde{W}(0,iq) - \tilde{U}) - 8\tilde{c}_s iq (\tilde{W}(0,iq) - \tilde{U}) \right] e^{i\xi p} dp \frac{1}{D(p,q)}
\]  \hspace{1cm} (33)

In order to remove \( \tilde{W}(0,iq) \), \( \xi = 0 \) is introduced into Eq. (33).
\[
\tilde{W}(0,iq) = \frac{K(q)}{2\pi} \left[ \frac{8\eta_{p_r}}{iq} + \frac{8\eta_{p_i} e^{i(\phi_0 + 3\pi/2)}}{iq - i\tilde{\omega}_f} + 4\eta_{M_s} q^2 \tilde{W}(0,iq) - 4\tilde{k}_s (\tilde{W}(0,iq) - \tilde{U}) - 8\tilde{c}_s iq (\tilde{W}(0,iq) - \tilde{U}) \right]
\]  \hspace{1cm} (34)

where
\[
K(q) = \int_{-\infty}^{\infty} \frac{dp}{D(p,q)}
\]  \hspace{1cm} (35)

After that, for the two unknown functions \( \tilde{W}(0,iq) \) and \( \tilde{U} \) the following system is obtained
\[
\tilde{W}(0,iq)(\pi + 2K(q)(\tilde{k}_s + 2\tilde{c}_s iq - \eta_{M_s} q^2)) - 2\tilde{U}(\tilde{k}_s + 2\tilde{c}_s iq) K(q)
\]  \hspace{1cm} (36)
\[
= \left( \frac{4\eta_{p_r}}{iq} + \frac{4\eta_{p_i} e^{i(\phi_0 + 3\pi/2)}}{i(q - \tilde{\omega}_f)} \right) K(q)
\]
\[
-(\tilde{k}_s + 2\tilde{c}_s iq) \tilde{W}(0,iq) + (\tilde{k}_s + 2\tilde{c}_s iq - \eta_{M_s} q^2) \tilde{U} = \frac{2}{iq} \eta_{p_r}
\]  \hspace{1cm} (37)

The determinant of the system is
\[
Q = (\tilde{k}_s + 2\tilde{c}_s iq)(\pi - 2K(q)(\eta_{M_s} + \eta_{M_l})q^2) + 2K(q)\eta_{M_s}\eta_{M_l}q^2 - \eta_{M_s}q^2\pi
\]  \hspace{1cm} (38)

and the solution read
\[ \tilde{W}(0,iq) = \frac{-4iK(q)}{Qq(q-\omega_f)} \]

\[ \left( \eta_P(q-\omega_f) + \eta_P q e^{i(\phi+3\pi/2)} \right) \left( \tilde{k}_s + 2\tilde{c}_i q - \eta_M q^2 \right) + \eta_P(q-\omega_f) \left( \tilde{k}_s + 2\tilde{c}_i q \right) \]

\[ \tilde{U} = \frac{-2i}{Qq(q-\omega_f)} \left( \eta_P(q-\omega_f) \left( \pi + 2K(q) \left( \tilde{k}_s + 2\tilde{c}_i q - \eta_M q^2 \right) \right) \right) \]

\[ + 2 \left( \eta_P(q-\omega_f) + \eta_P q e^{i(\phi+3\pi/2)} \right) K(q) \left( \tilde{k}_s + 2\tilde{c}_i q \right) \]

Going back to Eq. (33), Eq. (39) can be substituted and after some manipulations similar relation as in Eq. (39) is obtained

\[ \tilde{W}(\xi,iq) = \frac{-4iK(\xi,q)}{Qq(q-\omega_f)} \]

\[ \left( \eta_P(q-\omega_f) + \eta_P q e^{i(\phi+3\pi/2)} \right) \left( \tilde{k}_s + 2\tilde{c}_i q - \eta_M q^2 \right) + \eta_P(q-\omega_f) \left( \tilde{k}_s + 2\tilde{c}_i q \right) \]

The only difference with respect to Eq. (39) lies in \( K(\xi,q) \), which is now defined as

\[ K(\xi,q) = \int_{-\infty}^{\infty} e^{ip\xi} dp \]

Eqs. (41) and (40) stand for the analytical solution of the problem in the Laplace domain.

The determinant \( Q \), given by Eq. (38), is the crucial term for determining the induced frequencies and consequently the onset of instability. Thus, it is useful to present it also for simplified cases of moving one-mass oscillator \( Q_i = Q(M_i = 0) \) and mass \( Q_2 \), respectively

\[ Q_i = \left( \tilde{k}_s + 2\tilde{c}_i q \right) \left( \pi - 2K(q) \eta_M q^2 \right) - \eta_M q^2 \pi \]

\[ Q_2 = \pi - 2K(q) \eta_M q^2 \]

The form for \( Q_2 \) can only be determined after the complete formula is derived, because besides \( \eta_M = 0 \), it also holds \( \tilde{k}_s = \tilde{c}_s = 0 \) and these two values cannot be directly substituted into Eq. (38). Instead of that \( \eta_P = \eta_M = 0 \) is firstly substituted into Eq. (41), which can then be shortened by \( \left( \tilde{k}_s + 2\tilde{c}_i q \right) \), giving the correct form for \( Q_2 \).

Therefore, solutions for one-mass oscillator acting by \( P_s \) only (\( \eta_P = \eta_P = \eta_M = 0 \) ) are

\[ \tilde{W}(\xi,iq) = \frac{-4iK(\xi,q)\eta_P(\tilde{k}_s + 2\tilde{c}_i q)}{q \left[ \left( \tilde{k}_s + 2\tilde{c}_i q \right) \left( \pi - 2K(q) \eta_M q^2 \right) - \eta_M q^2 \pi \right]} \]
\[
\hat{U} = \frac{-2i\eta_p \left( \pi + 2K(q)(\kappa + 2\zeta, iq) \right)}{q \left[ (\kappa + 2\zeta, iq)(\pi - 2K(q)\eta_M, q^2) - \eta_M, q^2 \pi \right]}.
\] (46)

and for the moving mass one obtains
\[
\hat{W}(\xi, iq) = -\frac{4i\left( (q - \omega_r, q) \eta_p, q + \eta_M, q \xi^{\left( \frac{\phi_r, q + 3\pi}{2} \right)} \right) K(q)}{q \left( (q - \omega_r, q)(\pi - 2\eta_M, q^2 K(q) \right)}.
\] (47)

Finally, the inverse Laplace transform can be performed to obtain the solution of the problem in the time domain. Starting with the definition
\[
f(t) = \frac{1}{2\pi i} \lim_{a+iT \to -\infty} \int_{a-iT}^{a+iT} e^{st} F(s) ds
\] (48)

where \(a\) is positive and real and must be greater than the real part of all singularities, it reads by switching from \(q\) to \(\bar{q}\)
\[
\hat{\omega}(\xi, \tau) = \frac{1}{2\pi i} \lim_{a+iT \to -\infty} \int_{a-iT}^{a+iT} \hat{W}(\xi, \bar{q}) e^{s\bar{q}} d\bar{q} = \frac{1}{2\pi i} \int_{-\infty}^{-ia+\infty} i\hat{\omega}(\xi, q) e^{s\bar{q}} d\bar{q}
\] (49)
\[
\hat{\omega}(\xi, \tau) = \frac{1}{2\pi i} \lim_{a+iT \to -\infty} \int_{a-iT}^{a+iT} \hat{U}(\xi, \bar{q}) e^{s\bar{q}} d\bar{q} = \frac{1}{2\pi i} \int_{-\infty}^{-ia+\infty} i\hat{U}(\xi, q) e^{s\bar{q}} d\bar{q}
\] (50)

Eqs. (49) and (50) can be evaluated by contour integration. In order to clarify what curve should be used to ensure the assumptions of the Cauchy residue theorem, behaviour of \(K(q)\) must be analysed first. This will be accomplished in the next section.

### 3.2 Identification of the domain for the use of Cauchy’s residue theorem

The proper implementation of contour integration methods depends on the behaviour \(K(q)\), as seen from formulas (41) and (40), with (38). Namely, the curve necessary for correct application of Cauchy’s residue theorem must involve a domain where the integrand from Eqs. (49) and (50) is continuous with respect to \(q\). This continuity is dependent on continuity of \(K(q)\).

For some given fixed complex frequency \(q\), \(K(q)\) can also be evaluated by Cauchy’s residue theorem. For fixed \(q\), \(D(p, q)\) is a fourth order polynomial function in \(p\), with zero cubic term, having complex coefficients in linear and constant terms, and real coefficients in fourth order and quadratic terms. Thus \(D(p, q)\) has generally four complex simple roots.
In such a case, $K(q)$ evaluation is straightforward: positive infinite semicircle contour in the upper half-plane of the complex variable $p$ can be used and thus the residues can be evaluated at the roots with positive imaginary parts. This constitutes fully analytical expression, because the roots of $D(p,q)$ can be expressed analytically, [44-45]. By denoting the relevant coefficients

$$c_1 = -4(\eta_N - \eta_S + \alpha^2), \quad c_2 = 8(\alpha q - i\eta_s \alpha), \quad c_3 = -4q^2 + 8i\eta_s q + 4$$

and introducing auxiliary expressions

$$d_1 = c_1^2 + 12c_3, \quad d_2 = 2c_1^3 + 27c_2^2 - 72c_1c_3,$$  

$$d_3 = \left(\frac{1}{2}(d_2 + \sqrt{d_2^2 - 4d_1^3})\right)^\frac{1}{3}, \quad d_4 = \frac{1}{2} - \frac{2c_1}{3} + \frac{1}{3}\left(d_3 + d_1\right) \quad \text{for} \quad d_3 \neq 0$$

the roots are given by

$$p_{1,2} = \mp d_4 + \frac{1}{2} \sqrt{-4d_4^2 - 2c_1 \pm \frac{c_1}{d_4}}, \quad \text{for} \quad d_3, d_4 \neq 0$$  

$$p_{3,4} = \mp d_4 - \frac{1}{2} \sqrt{-4d_4^2 - 2c_1 \pm \frac{c_1}{d_4}}, \quad \text{for} \quad d_3, d_4 \neq 0$$

It is also possible to provide formulas for other cases, when $d_3$ and $d_4$ are zero, but this is not important for the developments presented here. If all the roots are simple and complex, they smoothly vary with the frequency $q$ and so the value of $K(q)$.

Some discontinuity in $K(q)$ can only occur for particular cases of real and/or multiple $p$-roots, which can only happen if the coefficients of $D(p,q)$ are real. It can be easily verified that the polynomial expression $D(p,q)$ can have real coefficients only if $q = q_r + i\eta_s$, where $q_r$ is real. After substitution into $D(p,q)$, one obtains

$$D_r(p,q_r) = p^4 - 4p^2(\eta_N - \eta_S + \alpha^2) + 8\alpha p q_r + 4 - 4q_r^2 - 4\eta_s^2$$

Appropriate theory for fourth order polynomials with real coefficients is well-developed, [38-39]. Possible cases for the nature of the roots are primarily distinguished by the value of the discriminant $\Delta$. For its form it is useful to name the coefficients of $D_r(p,q_r)$ similarly as in Eq. (51)

$$c_{1r} = c_1, \quad c_{2r} = 8\alpha q_r, \quad c_{3r} = -4\left(q_r^2 + \eta_s^2 - 1\right)$$

Then
\[ \Delta = 256c_r^3 - 128c_r^2c_s^2 + 144c_r^2c_s^2c_{3r} - 27c_r^4 + 16c_r^3c_{3r} - 4c_r^3c_{2r} \]  

(58)

Multiple \(p\) -roots can only occur for \(\Delta = 0\). As \(\Delta\) can be written as a cubic polynomial in \(q_r^2\), these roots have also analytical expressions.

The nature of the \(p\) -roots also strongly depends on \(\eta_b\). For instance, for \(\alpha = 0\) the roots of \(\Delta = 0\) are: \(q_{r,1}^2 = 1 - \eta_b^2\) and \(q_{r,2}^2 = q_{r,3}^2 = 1 - \eta_b^2 - (\eta_N - \eta_S)^2\). Therefore, to achieve \(\Delta = 0\) in this limit case, \(\eta_b\) must be below or equal to unity, otherwise there will be no real root for \(q_r\). For \(\alpha = 0\) and \(\eta_b^2 + (\eta_N - \eta_S)^2 < 1\), there are two single real roots and two double real roots for \(q_r\), in pairs with opposite signs. But when \(\alpha > 0\) three real positive roots are kept only for \(\eta_N > \eta_S\) (see Figure 4). Detailed discussion is beyond the objectives of this paper, as this will not be important for further developments.

Only non-negative \(q_r\) will be considered, because for non-positive \(q_r\) the situation is symmetric. The nature of the roots of \(D_r(p,q_r)\) can be shown on some typical cases. \(\eta_N = 0.2\) or \(\eta_S = 0.2\) or both null will be considered, without and with combination with \(\eta_b = 0.2\). For \(q_r \geq 0\), curves defining \(\Delta = 0\) are shown in Figure 2.

![Figure 2: Curves defining \(\Delta = 0\): a) \(\eta_b = 0\) and \(\eta_N = 0.2\), \(\eta_S = 0\) (black), \(\eta_N = 0\), \(\eta_S = 0\) (medium grey), \(\eta_N = 0\), \(\eta_S = 0\) (light grey); b) \(\eta_b = 0.2\) and \(\eta_N = 0.2\), \(\eta_S = 0\) (black), \(\eta_N = 0\), \(\eta_S = 0\) (medium grey), \(\eta_N = 0\), \(\eta_S = 0.2\) (light grey).](image-url)
As already stated, it is seen in Figure 2 that, in some typical cases, there is only one positive real root for $q_r$; three real roots for $q_r$ exist for instance for low $\alpha$ and $\eta_N = 0.2$, as plotted in detail in Figure 3.

![Figure 3: Detail of curves defining $\Delta = 0$.](image)

In summary, the curves defining $\Delta = 0$ can have several branches. Their separation at $q_r = 0$ can only occur for $\eta_b \leq 1$ and is given by

$$\alpha_c = \sqrt{1 - \eta_b^2 - (\eta_N - \eta_S)} \quad \text{and} \quad \tilde{\alpha}_c = \sqrt{-1 - \eta_b^2 - (\eta_N - \eta_S)}$$

(59)

$\tilde{\alpha}_c$ has no physical meaning, it marks a point where $\Delta = 0$, but no associated branches are in the vicinity. $\alpha_c$ marks the critical velocity when $\eta_b = 0$. For this comparison the extended formula for the critical velocity $v_{cr, ex}$ accounting for the normal force and the Pasternak modulus must be considered

$$v_{cr, ex} = v_{cr} \sqrt{1 - (\eta_N - \eta_S)}.$$  

(60)

If $\eta_b > 0$, then $\alpha_c$ has no such meaning. It is not associated with the maximum beam deflection, even if the tendency is the same, i.e. the increase in damping decreases the value of $\alpha_c$ and $\alpha$ at which the maximum beam deflection occurs.

The nature of the roots $p_j, j = 1...4$ of $D_r(p, q_r)$ is clarified in Figures 4 and 5. In Figure 4, the particular situation of $\eta_b = 0$, $\eta_N = 0$, $\eta_S = 0$ is shown.
Figure 4: Nature of the roots \( p_j, j = 1...4 \) of \( D_r(p, q_r) \) as a function of \( \alpha \) and \( q_r \).

Case 7, reported in Figure 4, occurs only on some two-branches curves. For its existence it is necessary that \( \eta_S = \eta_N \). If \( \eta_N < \eta_S \), then Case 4 is extended from the first branch to the initial point with \( \alpha = 0 \). When \( \eta_N > \eta_S \), then four-branches curve is formed, as shown in Figure 5 for \( \eta_b = 0, \eta_N = 0.2 \) and \( \eta_S = 0 \). For better clarity, only the detail that is different from Figure 4 is plotted.

Figure 5: Nature of the roots \( p_j, j = 1...4 \) of \( D_r(p, q_r) \) as a function of \( \alpha \) and \( q_r \).

The cases reported in Figures 4 and 5 have the following meanings:
Case 1: two pairs of complex conjugate roots
Case 2: two distinct real and two complex conjugate roots
Case 3: four distinct real roots
Case 4: real double root and two complex conjugate roots
Case 5: real double root and two distinct real roots
Case 6: two real double roots
Case 7: quadruple real root
Case 8: triple real root and another distinct real root

To conclude this section, it is necessary to analyse, how $K(q)$ behaves on complex $q$-plane, when other parameters of the problem $(\alpha, \eta_N, \eta_S, \eta_b)$ are fixed. Four $q$-values are considered in the form of:

\[ q_{(1)} = q_r + i(\eta_b + q_i), \quad q_{(2)} = q_r + i(\eta_b - q_i), \quad q_{(3)} = -q_r + i(\eta_b + q_i) \]
\[ q_{(4)} = -q_r + i(\eta_b - q_i), \quad \text{where} \quad q_r \text{ and } q_i \text{ are real and positive.} \]

Then the $D(p,q)$-coefficients $c_2$ and $c_3$ are related by:

\[ c_2(q_{(2)}) = (c_2(q_{(1)}))^*, \quad c_2(q_{(4)}) = (c_2(q_{(3)}))^*, \quad c_2(q_{(1)}) = -c_2(q_{(4)}) \] (61)
\[ c_3(q_{(2)}) = (c_3(q_{(1)}))^*, \quad c_3(q_{(4)}) = (c_3(q_{(3)}))^*, \quad c_3(q_{(1)}) = c_3(q_{(4)}) \] (62)

where the star designates the complex conjugate value. In these cases, $D(p,q)$ has complex coefficients and thus four complex distinct roots. They fulfil

\[ p_j(q_{(2)}) = (p_j(q_{(1)}))^*, \quad p_j(q_{(4)}) = (p_j(q_{(3)}))^*, \quad p_j(q_{(1)}) = -p_j(q_{(4)}), \quad j = 1,\ldots, 4 \] (63)

and therefore

\[ K(q_{(2)}) = (K(q_{(1)}))^*, \quad K(q_{(4)}) = (K(q_{(3)}))^*, \quad K(q_{(1)}) = K(q_{(4)}) \] (64)

as follows from the sum of residues. This means that when $q_i$ tends to zero, there is a discontinuity in $\text{Im}(K(q))$ along the horizontal line in complex $q$-plane that cut the imaginary axis at $i\eta_b$. Continuity is assured only in the case when $\text{Im}(K(q)) \overset{q_i \to 0}{\to} 0$, which happens when $\eta_b < 1$, $\alpha < \alpha_c$ and $q_r < q_{\text{cut}}$, where $q_{\text{cut}}$ is the value of $q_r$ lying on the first branch of $\Delta = 0$ curve, because of the nature of the $p$-roots in the region. Existence of $\alpha_c$ also requires that $\sqrt{1-\eta_b^2} - (\eta_N - \eta_S) > 0$. 

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In summary, there is a discontinuity in $K(q)$, which means that branch cuts must be introduced in the contour integration applied to Eq. (49) and (50). When $\eta_s < 1$ and $\sqrt{1-\eta_s^2} - (\eta_N - \eta_N) > 0$, then for $\alpha < \alpha_c$ the branch cuts can be reduced to run along $\langle -\infty + i\eta_s, -q_{cut} + i\eta_s \rangle$ and $\langle q_{cut} + i\eta_s, +\infty + i\eta_s \rangle$. In all other cases full line $\langle -\infty + i\eta_s, +\infty + i\eta_s \rangle$ must be considered.

At this moment, it looks like that it is exaggerating to use Cauchy’s residue theorem, because it seems to be easier just numerically integrate the Laplace image along $\langle -\infty - ia, +\infty - ia \rangle$. But, as already highlighted in the Introduction, sum of residues will separate the oscillations into a finite number of harmonic vibrations, and consequently give clearer physical insight. It will be shown that the necessary input in form of induced frequencies can be determined quickly and accurately by the iteration techniques described in Section 3.4. All harmonic terms can be then expressed by analytical formulas that are not losing their precision with increasing time. Moreover, in the examples shown it will be seen that transient vibrations obtained by the numerical integration along the branch cuts have generally low influence, are rapidly decreasing, are numerically stable and in several cases can be neglected. This presents a clear advantage with respect to the full numerical integration along $\langle -\infty - ia, +\infty - ia \rangle$ is numerically demanding for larger times and would not identify the unstable cases a priori. This also means that without knowing the poles, it is not clear what value for $a$ should be used.

### 3.3 Solution in the time domain

From the analysis given in the previous section, one can conclude that the beam deflection shapes and the oscillator vibrations are given by

$$\tilde{w}(\xi, \tau) = \sum \text{res}(i\tilde{W}(\xi, q)e^{i\tau}, q) + I_{\tilde{W},hc}$$

$$\tilde{w}_s(\tau) = \sum \text{res}(i\tilde{U}(q)e^{i\tau}, q) + I_{\tilde{U},hc}$$

where the sum is performed over all residues and $I_{\tilde{W},hc}$, $I_{\tilde{U},hc}$ stand for the results of the numerical integration along the branch cuts.

To evaluate the residues, it is necessary to determine the poles of $i\tilde{W}(\xi, q)$ and $i\tilde{U}(q)$. Both functions have the same denominator, thus the poles are the roots of $Qq(q - \tilde{\omega}_f)$ with
$Q$ given by Eq. (38). This expression has two obvious roots 0 and $\tilde{\omega}_f$, and others, designated as induced frequencies, $q_{M}$, must be determined from $Q = 0$. By similar analysis as in the previous section, it can be concluded that if $q_2 = (-q_1)^*$ then $K(q_2) = (K(q_1))^*$ and $Q(q_2) = (Q(q_1))^*$. Therefore, if $q_{M_1}$ is a root of $Q = 0$, then also $q_{M_1}^* = (-q_{M_1})^*$ is a root and thus the number of induced frequencies is even. This also indicates that a pair of induced frequencies have always the same imaginary parts and the opposite real parts. From the exponential form $e^{iq\tau}$, it yields that the real parts of induced frequencies will form a harmonic function, while the imaginary parts will indicate whether the amplitude of the induced oscillation will gradually cease in time, will get unstable (gradually increase in time above all limits) or will stay unchanged, which will happen for the positive, negative and zero imaginary values, respectively.

In summary, the complete solution can be written as a sum of several terms: (i) steady-state terms corresponding to zero frequency, (ii) harmonic terms corresponding to excitation frequency, (iii) harmonic terms corresponding to induced frequencies and, (iv) transient terms corresponding to the integration along the branch cuts.

The steady-state terms are obtained as residues at $q = 0$

$$\tilde{w}_i(\xi, \tau) = \frac{4K(\xi, 0)(\eta_{P_s} + \eta_{P_s}^*)}{\pi} \quad (67)$$

$$\tilde{w}_{s,i}(\tau) = \frac{2\eta_{P_s}}{k_s} + \frac{4K(0)(\eta_{P_s} + \eta_{P_s}^*)}{\pi} \quad (68)$$

Both terms include the steady-state solution of the total constant force applied, $P_u + P_s$. The additional term in (68) stays for the static deflection of the oscillator spring caused by $P_s$, because

$$w_{s,i} = \frac{2\eta_{P_s}}{k_s} w_{s} = \frac{2P_s k}{P_s k_s \chi} \frac{P_s}{2k} = \frac{P_s}{k_s} \quad (69)$$

Terms in Eqs. (67) and (68) are stationary and only parameters entering the definition of $D(p, q)$ with $q = 0$ are affecting the deflection value. Neither the harmonic force, nor the moving masses have any influence on this value, defined by the classical formula derived e.g. in [1]. Beam damping effect is also stationary.

Terms induced by the harmonic force are the residues at $q = \tilde{\omega}_f$
\[
\tilde{w}_2(\xi, \tau) = \frac{4K(\xi, \omega_f)}{Q(\omega_f)} \eta_p e^{-i(\phi_f + 3\pi/2)} \left( \tilde{k}_s + 2\tilde{c}_s i\omega_f - \eta_M \omega_f^2 \right) e^{i\omega_f \tau} \tag{70}
\]

\[
\tilde{w}_{s,2}(\tau) = \frac{4K(\omega_f)}{Q(\omega_f)} \eta_p e^{-i(\phi_f + 3\pi/2)} \left( \tilde{k}_s + 2\tilde{c}_s i\omega_f \right) e^{i\omega_f \tau} \tag{71}
\]

with

\[
Q(\omega_f) = \left( \tilde{k}_s + 2\tilde{c}_s i\omega_f \right) \left( \pi - 2K(\omega_f) \eta_m \omega_f^2 \right) + 2K(\omega_f) \eta_m \omega_f^2 \tilde{k}_s - \eta_m \omega_f^2 \pi \tag{72}
\]

Admitting only the moving mass, then

\[
\tilde{w}_2(\xi, \tau) = \frac{4K(\xi, \omega_f)}{Q(\omega_f)} \eta_p e^{-i(\phi_f + 3\pi/2)} \left( \tilde{k}_s + 2\tilde{c}_s i\omega_f \right) e^{i\omega_f \tau} \tag{73}
\]

Eq. (73) reveals that there is generally another kind of resonance related to the excitation frequency, which occur for zero denominator in Eq. (73), or generally for zero value in Eq. (72). As the external frequency is real, this can only happen for \( \tilde{c}_s = 0 \).

Expressions in Eqs. (70) and (71) define harmonic vibrations with the same frequency as the externally applied harmonic force. Besides the terms entering \( D(p, q) \) with \( q = \dot{\omega}_f \), it is seen that all other characteristics of the moving oscillator are affecting the amplitude value. Also here the damping effects of the beam and of the oscillator are stationary, and thus these oscillations theoretically last forever, as long as the external excitation is present.

Other harmonic terms are related to each calculated induced frequency, thus \( Q(q_{M_j}) = 0 \).

The beam deflection shapes can be presented separately, for the constant and harmonic forces

\[
\tilde{w}_3(\xi, \tau) = \frac{4K(\xi, q_{M_j})}{q_{M_j} Q_d(q_{M_j})} \left( \eta_p \left( \tilde{k}_s + 2\tilde{c}_s i q_{M_j} - \eta_M q_{M_j}^2 \right) + \eta_p \left( \tilde{k}_s + 2\tilde{c}_s i q_{M_j} \right) \right) e^{i\omega_f \tau} \tag{74}
\]

\[
\tilde{w}_4(\xi, \tau) = \frac{4\eta_p K(\xi, q_{M_j}) \left( \tilde{k}_s + 2\tilde{c}_s i q_{M_j} - \eta_M q_{M_j}^2 \right) e^{i(\phi_f + 3\pi/2)}}{Q_d(q_{M_j}) \left( q_{M_j} - \dot{\omega}_f \right)} e^{i\omega_f \tau} \tag{75}
\]

The oscillator displacement in the same separation is

\[
\tilde{w}_{s,3}(\tau) = \frac{2 \left( \eta_p \left( \pi + 2K(q_{M_j}) \left( \tilde{k}_s + 2\tilde{c}_s i q_{M_j} - \eta_M q_{M_j}^2 \right) \right) + 2\eta_p K(q_{M_j}) \left( \tilde{k}_s + 2\tilde{c}_s i q_{M_j} \right) \right) e^{i\omega_f \tau}}{q_{M_j} Q_d(q_{M_j})} \tag{76}
\]

\[
\tilde{w}_{s,4}(\tau) = \frac{4\eta_p K(q_{M_j}) \left( \tilde{k}_s + 2\tilde{c}_s i q_{M_j} \right) e^{i(\phi_f + 3\pi/2)}}{Q_d(q_{M_j}) \left( q_{M_j} - \dot{\omega}_f \right)} e^{i\omega_f \tau} \tag{77}
\]
These expressions can be simplified for $\tilde{c}_s = 0$

$$\tilde{w}_3(\xi, \tau) = 2K(\xi, q_M) \left( \tilde{k} \left( \eta_{c} + \eta_{p} \right) - \eta_{c} \eta_{M} q_{M}^2 \right) e^{i\omega_{u} \tau} \tag{78}$$

$$\tilde{w}_{s,3}(\tau) = \frac{\eta_{p} \pi + 2(\eta_{c} + \eta_{p}) K(\xi, q_M) \tilde{k} - 2\eta_{p} K(\xi, q_M) \eta_{M} q_{M}^2 + 2K(\xi, q_M) \eta_{M} \eta_{M} q_{M}^4 - \tilde{k} \pi}{q_{M}^3 K(\xi, q_M) \left( \eta_{M} \eta_{M} q_{M}^2 - \tilde{k} \left( \eta_{M} + \eta_{M} \right) \right) + 2K(\xi, q_M) \eta_{M} \eta_{M} q_{M}^4 - \tilde{k} \pi} \left( q_{M} - \tilde{\omega}_{f} \right) e^{i\omega_{u} \tau} \tag{79}$$

$$\tilde{w}_4(\xi, \tau) = 2q_M K(\xi, q_M) \left( \tilde{k} - \eta_{M} q_{M}^2 \right) \eta_{p} e^{i(\omega_{p} + 3\pi/2)} e^{i\omega_{u} \tau} \tag{80}$$

$$\tilde{w}_{s,4}(\tau) = \frac{2\eta_{p} q_M K(\xi, q_M) \tilde{k} e^{i(\omega_{p} + 3\pi/2)} e^{i\omega_{u} \tau}}{q_{M}^3 K(\xi, q_M) \left( \eta_{M} \eta_{M} q_{M}^2 - \tilde{k} \left( \eta_{M} + \eta_{M} \right) \right) + 2K(\xi, q_M) \eta_{M} \eta_{M} q_{M}^4 - \tilde{k} \pi} \left( q_{M} - \tilde{\omega}_{f} \right) \tag{81}$$

When only moving mass is considered, then

$$\tilde{w}_3(\xi, \tau) = \frac{-2K(\xi, q_M) \eta_{p}}{q_{M}^3 K(\xi, q_M) \eta_{M} + \pi} e^{i\omega_{u} \tau} \tag{82}$$

$$\tilde{w}_4(\xi, \tau) = \frac{-2q_M K(\xi, q_M) \eta_{p} e^{i(\omega_{p} + 3\pi/2)}}{q_{M}^3 K(\xi, q_M) \left( \eta_{M} + \pi \right) + \pi} \left( q_{M} - \tilde{\omega}_{f} \right) e^{i\omega_{u} \tau} \tag{83}$$

which is in agreement with [42]. The contributions of induced vibrations must be summed over all induced frequencies, even in numbers, as already mentioned. Here the damping effect is not stationary.

In the second part of this section, integration along the branch cuts to determine $I_{\tilde{w},bc}$ and $I_{\tilde{U},bc}$ is discussed. For simplicity, $I_{c}$ where $C = \{ -\infty - ia; -ia + \infty \}$ with $a$ specified by the inverse Laplace transform will designate the integral specified in Eqs. (49) or (50). Let $\eta_{b} < 1$ and $\alpha < \alpha_{c}$ be assumed. According to Cauchy’s theorem of residues and by taking into account conclusions from Section 3.2, it holds:

$$I_{C_{j}} = 2\pi i \sum \text{res} - I_{C_{j}} - I_{C_{j}} - I_{C_{j}} - I_{C_{j}} - I_{C_{j}} - I_{C_{j}} \tag{84}$$

where $I_{C_{j}}, j = 1, \ldots, 8$ stand for integrals along the contours identified by the subscript as represented in Figure 6. By analysing the functions $\tilde{W}(\xi, q)/(2\pi)$ and $\tilde{U}(q)/(2\pi)$ it can be concluded that they vanish for $|q| \to \infty$. Thus, when radius of the dashed contour in Figure 6 tends to infinity, $I_{C_{j}} \to 0$ for $j = 6, 7, 8$ and also $I_{c} \to I_{C_{1}}$. 

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Figure 6: Schematic of the contour integration.

Then the term $2\pi i \sum \text{res}$ from Eq. (84) stand for $\sum \text{res}(i\tilde{W}(\xi,q)e^{i\tau q},q)$ and $\sum \text{res}(i\tilde{U}(q)e^{i\tau q},q)$ from Eqs. (65) and (66), respectively, and therefore

$I_{\tilde{W},bc} = -(I_{C_2} + I_{C_3} + I_{C_4} + I_{C_5})$ with integrand equal to $\tilde{W}(\xi,q)e^{i\tau q}/(2\pi)$ and $I_{\tilde{U},bc} = -(I_{C_2} + I_{C_3} + I_{C_4} + I_{C_5})$ with integrand $\tilde{U}(q)e^{i\tau q}/(2\pi)$.

Some simplifications can be introduced. Having in mind the properties of $K(q)$ analysed in Section 3.2, it can be concluded that function values of both, $\tilde{W}(\xi,q)e^{i\tau q}/(2\pi)$ and $\tilde{U}(q)e^{i\tau q}/(2\pi)$ at any point of $C_2$ or $C_3$ and for any $\xi$ and $\tau$ are complex conjugate of function values at symmetrically located points at $C_3$ or $C_4$, respectively, except for cases with external harmonic excitation. This property can significantly facilitate the numerical evaluation.

Nevertheless, although the roots of $D(p,q)$ have analytical expressions and until now everything was presented analytically, an increased effort in providing better approach to integration along branch cuts than numerical, seems hard to justify, because this analytical expression would only be relatively complicated to integrate analytically. For the numerical evaluation of $I_{\tilde{W},bc}$ and $I_{\tilde{U},bc}$, it is convenient to use a software of symbolic calculation with adjustable digits precision, like Maple. It is advantageous to prepare the function values along
the branch cut in fixed points and then perform the sum representing the numerical integration
for each $\xi$ and $\tau$ that is required. It is also helpful to visualize the function for $\tau = 0$,
exploiting the analytical expressions of $p$-roots, to estimate the step for the numerical
integration and the largest $q$-value that is necessary to consider. Then the numerical
integration for calculation of $I_{W_{bc}}$ and $I_{U_{bc}}$ is straightforward and delivers numerically stable
results. Transient vibrations obtained in this way have rapidly decreasing contribution to the
full oscillator induced vibrations. This property is particularly noticeable in the subcritical
region at the oscillator position.

3.4 Induced frequencies and evaluation of other necessary terms

The induced frequencies are the roots of $Q = 0$ given by Eq. (38). It was already proven, that
pair of roots is linked by $q_{M_2} = (-q_{M_1})^*$. Moreover, when $\eta_b = \bar{c}_s = 0$, also
$q_{M_1} = -q_{M_1}$ stand for a valid frequency, so as $q_{M_2} = (q_{M_1})^*$.

For better perception, it is useful to determine a variation of each induced frequency with
respect to the velocity ratio $\alpha$, when other data of the problem are fixed. Such curves will be
named as frequency lines.

Two iterative techniques are proposed. From the numerical point of view, they only require
repetitive calculation of $K(q)$. In the first one, the complex equation to be solved on complex
plane is written as if $K(q)$ and $q$ were independent

$$Q(q, K(q)) = 0$$

(85)

If $q_n$ is an estimate from $n$-th iteration, then the next iteration is calculated as

$$q_{n+1} = \beta q_n + (1 - \beta) q_n, \text{ where } Q(q, K(q_n)) = 0$$

(86)

This means that $K(q)$ is evaluated at the previous estimate $q_n$ and then it is easy to solve
$Q(q, K(q_n)) = 0$ for $q$. Convergence of this technique depends on the initial estimate $q_0$
and some adequate weight $\beta$. The initial estimate can be obtained from the general tendency
of the frequency line. The iteration technique just described is usually convergent in full range
of velocities with $\beta = 0.5$ and relatively high $\eta_b$. If convergence difficulties are experienced,
$\beta$ should be reduced. This can happen close to the discontinuity in $K(q)$, i.e. when tested
frequencies have the imaginary parts near to $\eta_b$. Other difficult region is in the vicinity of the
critical velocity, where \( Q(q) \) values have large variation. Then it is more convenient to use the other proposed iteration method, named as delimiting search. For a reasonable estimate of the induced frequency value, it is possible to delimitate a rectangle domain in the complex \( q \)-plane where the real and imaginary parts of \( Q(q) \) are monotonic and change their sign. Zero values mark two lines on the \( q \)-plane. In further iterations it is necessary to reduce the rectangle domain without losing the intersection of the lines just identified. By step-by-step reduction of the rectangle domain it is possible to determine the root with any required precision. For the purpose of this paper, both iterative techniques were programmed in Maple, in order to take advantage of the symbolic calculation and adjustable digits precision.

It is also possible to search for the roots by implementing the argument principle, but this is computationally more demanding. Nevertheless, the argument principle should always accompany previous calculations to confirm that all roots for given \( \alpha \) have been found.

In the iterative techniques just introduced, repetitive calculation of \( K(q) \) for some fixed \( q \) is required. As already mentioned, \( K(q) \) can be determined by Cauchy’s residue theorem. Due to the discontinuity of \( K(q) \) across the line with \( \text{Im}(q) = \eta_b \), frequency lines are interrupted when the corresponding imaginary part is infinitely close to \( \eta_b \). Therefore, \( K(q) \) is only required at \( q \) for which \( D(p,q) \) have four complex simple \( p \)-roots. Then it is straightforward to select the ones with positive imaginary parts, for instance \( p_1 \) and \( p_2 \) to get

\[
K(q) = \int_{-\infty}^{\infty} \frac{dp}{D(p,q)} = \frac{2\pi i}{2} \sum_{j=1,2} \frac{1}{D(p_j, q)}
\]

as given by the definition of the simple pole residue.

In the formulas of Section 3.3, besides \( K(q) \), also \( K_{\eta}(q) \) and \( K(\xi, q) \) are required for a given frequency \( q \). \( K_{\eta}(q) \) has the same poles as \( K(q) \), but all of them are duplicated, as seen from

\[
K_{\eta}(q) = \int_{-\infty}^{\infty} \frac{-D_{\eta}(p,q)}{D^2(p,q)} dp
\]

Then by implementation of the definition of the double pole residue

\[
\text{res}(f,c) = \lim_{z \to c} \frac{d}{dz} \left( (z-c)^2 \frac{g(z)}{h(z)} \right) = \frac{6g'(c)h''(c) - 2g(c)h'''(c)}{3(h''(c))^2}
\]

one obtains
\[ K_q(q) = 2\pi i \sum_{j=1,2} \frac{-6D_{dp}(p_j,q)D_{pp}^2(p_j,q) + 2D_{qp}(p_j,q)D_{ppp}^2(p_j,q)}{3\left(D_{pp}(p_j,q)\right)^2} \] (90)

\( K(\xi,q) \) is the only term that contains \( \xi \) and as such defines the full beam deflection shape. It can be again expressed analytically as the sum of residues. Poles are already known because they are the roots of \( D(p,q) \). In order to fulfill the boundary conditions from Eq. (15) adapted to the dimensionless form, integration along positive infinite semicircle contour in the upper half-plane of the complex variable \( p \) must be used for the front wave and the negative value of the integration along the negative infinite semicircle contour in the lower half-plane of the complex variable \( p \) must be used for the rare wave.

4. Numerical examples

4.1 Validation and general remarks

All results presented in this section are validated by results obtained by modal expansion method on long finite beams. The resemblance of vibrations induced by moving loads on finite and infinite beams is notorious, when inertia of the moving object and reasonably stiff foundation are used, which is typical for railway applications. For the validation, the load has to be applied further from the support, to eliminate this influence. Then the choice of boundary conditions is immaterial and most convenient ones can be used, which in this case are simple supports that ensure analytically defined and numerically stable mode shapes. As mentioned above, it is also necessary to support the beam by a reasonably stiff foundation, to avoid higher displacements in the central beam sections. Then the vibrations obtained are independent on the beam length and thus can replace the results on infinite beam. This confirmation is presented in the first analysis of Section 4.2. For sufficient accuracy of results obtained by modal expansion method, high number of modes must be used as proven by convergence analysis in [42]. As expected, the necessary number of modes is increasing with the increasing beam length. All results on finite beams presented in this paper are sufficiently accurate, because they were tested for the number of modes, the initial distance of the oscillator from the support and the beam length. Further details about modal expansion method for simplified case of moving mass are given in [42]. Programs that calculate the beam and oscillator response are written in Matlab and were previously validated by commercial finite element software LS-DYNA. Validation by finite element results coming from LS-DYNA software is only possible in situations where the contact between the
oscillator and the beam is preserved, as already mentioned in the Introduction. Such cases can be identified by analysis of the contact force. This subject was already discussed for moving mass problem in [42]. Contact loss is prone to occur in upward displacements, which is typical for unstable cases and cases with harmonic force contribution where the constant force is relatively low. After contact loss, reattachment occurs, and deflections follow the previously determined solution.

In analyses that follow, it is necessary to distinguish the unstable case (the amplitude of the induced vibrations gradually increases above all limits), from the subcritical, critical and supercritical case, which is related to the load critical velocity. It will be shown, in agreement with [37], that the onset of unstable cases is always in the supercritical range. In this context it is worthwhile to summarize cases, where vibrations exceed all limits. There are two cases that can be designated as resonance-type cases, because unbounded amplitudes are developed only under one specific velocity. One of them is related to actuation of constant forces and the other one to harmonic forces. The former indicates the classical critical velocity: for $\eta_b = 0$ and $v = v_{cr,ex}$, $K(0)$ is unbounded. The latter occurs when the external excitation frequency matches one of the induced real frequencies. Further cases of excessive vibrations are related to instability, characterised by exponential increase of amplitudes in time.

In Section 3.3, complete solution of the moving oscillator problem was presented as superposition of steady-state, induced harmonic and transient vibrations. The transient vibrations essentially force the solution to match exactly the initial conditions. Therefore, there are several cases where these vibrations can be omitted. One can recognize them by the fact that the initial conditions are approximately fulfilled by the steady-state and harmonic terms. In what follows and in figure captions the following designations will be used: harmonic (will cover the steady-state and induced harmonics, i.e. the part of solution obtained as sum of residues), full (will cover the previous plus the transient vibrations), finite (will stand for results on finite beams). Agreement between the results on infinite and finite beams is excellent in all presented cases, which means that in graphs with induced vibrations, overlaid curves must be plotted in a distinguishable way. For results on finite beams a grey colour was selected, because then it is possible to plot on top black dotted line representing results on infinite beams. This way, both curves are visible.

Results are presented for two-mass oscillator. Simplified cases of moving mass and one-mass oscillator are placed in Appendices A and B, respectively. Base data for the beam and foundation are related to one rail 60E1 supported by visco-elastic two-parameter foundation
of typical stiffness. In the examples, only dimensionless data are given because formulas were deduced in such a way. Special importance is placed on situations where the number of induced frequencies changes, in order to provide detailed validation of the formulas presented in this paper and of the calculated induced frequencies.

4.2 Moving two-mass oscillator

Numerical data related to the applied load are adapted from [32], thus base values are: 
\[ \bar{k}_s = 0.327, \ \bar{c}_s = 0.269, \ \eta_{M_s} = 6.52, \ \eta_{M_c} = 5.52, \ \eta_{P_c} = 1, \ \eta_{P_p} = 0.108, \ \eta_{P_b} = 0. \]
At first, \( \eta_b = \eta_s = 0 \) and three general cases (listed in Table 1) will be considered for analysis of the induced frequencies; they differ by the values of the beam and oscillator damping.

![Table 1: Selected cases for induced frequencies of two-mass oscillator.](image)

Further: Case 4 will use data from Case 1 and analyse the influence of the harmonic component of the applied force; Case 5 will also use data from Case 1 and analyse the influence of the normal force and Pasternak modulus; finally, Case 6 will deal with the situation when one of the poles is located on the branch cut.

The induced frequencies are plotted in Figures 7-9 for Cases 1-3, respectively. It can be concluded that when \( \eta_b = 0 \), independently on the oscillator damping, the onset of instability is exactly at the critical velocity. For \( \eta_b = 0.1 \) this onset is significantly shifted to higher velocities: \( \alpha = 1.37 \). In all cases it is seen that the number of frequencies is not fixed. If \( \eta_b = \bar{c}_s = 0 \) (Case 1), each point on frequency lines for \( \alpha < 1 \) stands for one pair of frequencies, but, when \( \alpha > 1 \) each point stands for two pairs of frequencies. In Case 2 and 3, each point on frequency lines always stands only for one pair of frequencies. From Figures 7-9 it can thus be concluded that: (i) there are up to four pairs of induced frequencies within the analysed range; (ii) sharp angle in real components has some similarity with curve designating \( \Delta = 0 \), especially in Case 1; (iii) frequency lines are interrupted when their imaginary part reaches \( \eta_b \), as already justified in Section 3.4.
Figure 7: Real and imaginary parts of the induced frequencies in Case 1 (black and grey); $\Delta = 0$ (black dotted).

Figure 8: Real and imaginary parts of the induced frequencies in Case 2 (black, grey and black dashed); $\Delta = 0$ (black dotted).

Figure 9: Real and imaginary parts of the induced frequencies in Case 3 (black, grey and black dashed); $\Delta = 0$ (black dotted).
Displacement at the oscillator location as a function of dimensionless time is analysed for velocities with different number of induced frequencies, to provide more detailed validation. Namely, in Case 1 four values are selected as $\alpha = 0.689$, $\alpha = 0.785$, $\alpha = 1.2$ and $\alpha = 1.4$, having two, one, two and four pairs of induced frequencies. They and the related amplitudes are summarized in Table 2. For completeness, the steady-state deflection (linked to $q = 0$) is also listed. Deflections are plotted in Figures 10-13 for the four velocities, respectively.

![Table 2: Induced frequencies and related amplitudes for Case 1.](image)

Figure 10 shows that for $\alpha = 0.689$ superposition of two harmonics is clearly pronounced in the beam deflection, because the related amplitudes have comparable values, but in the oscillator vibration the lower frequency is dominant, because the associated amplitude is much higher than the one linked to the higher frequency.

The scenario shown in Figure 11 has only one pair of induced frequencies, therefore the beam and the oscillator vibrate with the same frequency and are in phase.
Figure 10: Deflection at the oscillator location in Case 1, $\alpha = 0.689$: a) harmonic (grey), full (dotted); b) full (dotted), finite (grey).

Figure 11: Deflection at the oscillator location in Case 1, $\alpha = 0.785$: a) harmonic (grey), full (dotted); b) full (dotted), finite (grey).

It can be concluded from Figures 10a) and 11a) that only during initial instants there is a slightly noticeable contribution of the transient vibration, to shift the harmonic solution in a way to fulfill the homogeneous initial conditions.

In Figure 12 is it proven that results obtained on finite beams are independent on the beam length and thus can substitute the results on infinite beams. This analysis is done for $\alpha = 0.689$. The number of modes used for varying lengths is 200, 250, 300, 350 and 400, respectively, starting from the number for the shortest beam. In Figure 12a) beam deflections at the oscillator location are shown. The curves are overlaid and thus impossible to distinguish, except when the oscillator reaches the beam end. In Figure 12b) sprung mass
deflections are plotted. Also here the curves are perfectly overlaid. Moreover, there is no indication of the beam end.

![Graph](image1.png)

Figure 12: Solution on four finite beams in Case 1, $\alpha = 0.689$: analysis of results independency on the beam length.

In Figure 13 full deflection shapes of the beam are shown. Naturally, at the initial instant, the contribution of the transient vibration is the largest. Then its influence is rapidly decreasing. It is seen that starting approximately from $\tau = 40$, the transient part of the solution could be completely neglected, and full solution could be replaced by its harmonic part, as already predicted in Figure 10a). It is also seen that the influence of transient vibrations is generally more significant in places further from the oscillator location.
Figure 13: Full deflection shapes of the beam in Case 1 for $\alpha = 0.689$: harmonic (dashed); full (dotted); finite (grey).
The scenario shown in Figure 14 is already unstable. Increase in amplitudes is not very steep and so could be easily controlled. Transient vibrations are insignificant.

The situation shown in Figure 15 is also unstable and has four pairs of induced frequencies. One pair has quite large negative imaginary part (Table 2), dictating steep increase in amplitudes and thus only short time interval is plotted. The increase in amplitudes is better visible in beam vibrations, because larger amplitude relates to the frequency with the larger negative imaginary part. The opposite is verified for oscillator vibrations and therefore the figure looks rather chaotic. In addition, due to low oscillator displacement in the beginning, contribution of transient vibrations is relatively more significant than in beam vibrations. For better clarity beam and oscillator vibrations are plotted in separate parts of the figure.

**Figure 14:** Deflection at the oscillator location in Case 1, $\alpha = 1.2$: a) harmonic (grey), full (dotted); b) full (dotted), finite (grey).

**Figure 15:** Deflection at the oscillator location in Case 1, $\alpha = 1.40$: harmonic (dashed), full (dotted), finite (grey): a) beam; b) oscillator.
Four velocities are equally selected for Case 2, now with $\alpha = 0.5$, $\alpha = 0.9$, $\alpha = 1.2$ and $\alpha = 1.4$. First three situations have two pairs of induced frequencies, only the last one has three. Induced frequencies and the related amplitudes are summarized in Table 3. The deflections are shown in Figures 16-18.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$q = 0$</th>
<th>$q_{M_{1,2}}$</th>
<th>$q_{M_{1,4}}$</th>
<th>$q_{M_{1,6}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha = 0.5$</td>
<td>$\pm0.2190 +0.0300i$</td>
<td>$\pm0.4749 +0.0448i$</td>
<td>$\pm0.2531 +0.0151i$</td>
<td>$\pm0.6602 +0.1210i$</td>
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<tr>
<td>$\tilde{w}_0$</td>
<td>1.2800</td>
<td>0.4629</td>
<td>0.8229</td>
<td></td>
</tr>
<tr>
<td>$\tilde{w}_4$</td>
<td>1.9440</td>
<td>2.2638</td>
<td>0.3498</td>
<td></td>
</tr>
<tr>
<td>$\alpha = 0.9$</td>
<td>$\pm0.1340 +0.0004i$</td>
<td>$\pm0.2531 +0.0151i$</td>
<td>$\pm0.6602 +0.1210i$</td>
<td>$\pm0.6602 +0.1210i$</td>
</tr>
<tr>
<td>$\tilde{w}_0$</td>
<td>2.5431</td>
<td>1.0211</td>
<td>1.4319</td>
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<tr>
<td>$\tilde{w}_4$</td>
<td>3.2071</td>
<td>0.5674</td>
<td>1.9737</td>
<td></td>
</tr>
<tr>
<td>$\alpha = 1.2$</td>
<td>$\pm0.2234 -0.0525i$</td>
<td>$\pm0.2838 -0.0124i$</td>
<td>$\pm0.6602 +0.1210i$</td>
<td>$\pm0.6602 +0.1210i$</td>
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<td>$\tilde{w}_4$</td>
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<td>0.0808</td>
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</table>

Table 3: Induced frequencies and related amplitudes for Case 2.
Figure 16: Deflection at the oscillator location in Case 2: a) $\alpha = 0.5$ harmonic (dotted), finite (grey); b) $\alpha = 1.40$ harmonic (dashed), full (dotted), finite (grey).

In Figure 16a) it is seen that the imaginary parts of the induced frequencies rapidly reduce the vibration amplitudes. Transient vibrations are completely negligible and therefore full solution is replaced by the harmonic one. In Figure 16b) unstable case is plotted. Harmonic solution is added by dashed curve, but it is seen that the difference between the full and harmonic solutions is very small, and therefore the transient vibration contribution could be neglected.

Figure 17: Deflection at the oscillator location in Case 2, $\alpha = 0.9$ : harmonic (dashed), full (dotted), finite (grey): a) beam; b) oscillator.
Figure 18: Deflection at the oscillator location in Case 2, $\alpha = 1.20$: harmonic (dashed), full (dotted), finite (grey): a) beam; b) oscillator.

For $\alpha = 0.9$ and 1.2 reported in Figures 17 and 18, the initial contribution of transient vibrations is more important than in previous cases because the harmonic solution starts further than indicated by homogeneous initial conditions. This is more visible in oscillator vibrations. But also here, this influence is rapidly decreasing. For better clarity, beam and oscillator vibrations are plotted separately in both cases.

To conclude the analysis on the three cases listed in Table 1, two velocities are selected for Case 3: $\alpha = 0.5$ and $\alpha = 1.40$, with two and three pairs of induced frequencies, respectively. Induced frequencies and the related amplitudes are summarized in Table 4.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$q = 0$</th>
<th>$q_{M_{1,2}}$</th>
<th>$q_{M_{3,4}}$</th>
<th>$q_{M_{5,6}}$</th>
</tr>
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<tr>
<td>0.5</td>
<td></td>
<td>±0.2179</td>
<td>±0.4742</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>+0.0315i</td>
<td>+0.0775i</td>
<td></td>
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<td>0.7989</td>
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<tr>
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<td>2.2562</td>
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</tr>
<tr>
<td>1.4</td>
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<td>±0.6577</td>
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</tr>
<tr>
<td></td>
<td></td>
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<td>+0.1945i</td>
</tr>
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<td>0.7273</td>
<td>0.6826</td>
<td>0.0948</td>
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</table>

Table 4: Induced frequencies and related amplitudes for Case 3.
The related displacements are plotted in Figure 19. The contribution of transient solution is negligible for $\alpha = 0.5$ and therefore full solution is replaced by the harmonic one. For $\alpha = 1.40$ very small region close to the initial instants is affected by the transient part of the solution, as seen from Figure 19b).

Figure 19: Deflection at the oscillator location in Case 3: a) $\alpha = 0.5$, harmonic (dotted), finite (grey); b) $\alpha = 1.40$, harmonic (dashed), full (dotted), finite (grey).

As mentioned in the beginning of this subsection, Case 4 will use the data from Case 1 and add the effect of external harmonics. It is assumed that $\eta_{p0} = 0.146$ and the velocity ratio is fixed to $\alpha = 0.5$.

<table>
<thead>
<tr>
<th></th>
<th>$\hat{\omega}_f$</th>
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<th>$q_{M_{3,4}}$, $\eta_{p3}$, $\eta_{p4}$</th>
<th>$q_{M_{5,4}}$, $\eta_{p5}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\omega}<em>0$, $\hat{\omega}</em>{f1}$</td>
<td>0.3491</td>
<td>0.4415</td>
<td>0.1777</td>
<td>0.7711</td>
</tr>
<tr>
<td>$\hat{\omega}<em>1$, $\hat{\omega}</em>{f1}$</td>
<td>2.8791</td>
<td>2.2125</td>
<td>0.8903</td>
<td>0.2617</td>
</tr>
<tr>
<td>$\hat{\omega}<em>0$, $\hat{\omega}</em>{f2}$</td>
<td>5.9251</td>
<td>0.4415</td>
<td>5.7476</td>
<td>0.7711</td>
</tr>
<tr>
<td>$\hat{\omega}<em>1$, $\hat{\omega}</em>{f2}$</td>
<td>28.8560</td>
<td>2.2125</td>
<td>28.8021</td>
<td>0.2617</td>
</tr>
<tr>
<td>$\hat{\omega}<em>0$, $\hat{\omega}</em>{f3}$</td>
<td>0.3196</td>
<td>0.4415</td>
<td>0.0342</td>
<td>0.7711</td>
</tr>
<tr>
<td>$\hat{\omega}<em>1$, $\hat{\omega}</em>{f3}$</td>
<td>0.2083</td>
<td>2.2125</td>
<td>0.1715</td>
<td>0.2617</td>
</tr>
<tr>
<td>$\hat{\omega}<em>0$, $\hat{\omega}</em>{f4}$</td>
<td>9.2683</td>
<td>0.4415</td>
<td>0.0238</td>
<td>0.7711</td>
</tr>
<tr>
<td>$\hat{\omega}<em>1$, $\hat{\omega}</em>{f4}$</td>
<td>3.2001</td>
<td>2.2125</td>
<td>0.1191</td>
<td>0.2617</td>
</tr>
</tbody>
</table>

Table 5: Induced frequencies and related amplitudes for Case 4.

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The induced frequencies are $q_{M_{s_e}} = \pm 0.2177$ and $q_{M_{s_x}} = \pm 0.4833$. Therefore, the external frequencies are selected as $\hat{\omega}_{f_1} = 0.1936$, $\hat{\omega}_{f_2} = 0.2169$; $\hat{\omega}_{f_3} = 0.3873$ and $\hat{\omega}_{f_4} = 0.4803$, i.e. the second and the fourth value is close to one of the induced frequencies. The steady state displacement is the same for all cases, 1.280 and 1.944 for the beam and for the oscillator, respectively. The amplitudes related to the harmonic parts of the solution are summarized in Table 5 and the deflections are plotted in Figure 20. Obviously, the amplitudes related to the application of the constant load are the same in all four cases, but they are repeated in Table 5 for the sake of comparison with the other values.

![Figure 20: Deflection at the oscillator location in Case 4, $\alpha = 0.5$, harmonic (dotted), finite (grey): a) $\hat{\omega}_{f_1} = 0.1936$; b) $\hat{\omega}_{f_2} = 0.2169$; c) $\hat{\omega}_{f_3} = 0.3873$; d) $\hat{\omega}_{f_4} = 0.4803$.](image_url)

In all cases plotted in Figure 20 transient vibrations are completely negligible, therefore full solution can be substituted by the harmonic one. It can be concluded from Figure 20 that especially the oscillator vibration is very similar in all cases, because it is basically governed
by the lower frequency and the amplitude related to the constant forces. The beam deflection at the oscillator location clearly shows superposition of all harmonic vibrations as they have comparable amplitudes. Only in the last case when \( \tilde{\omega}_j \) is close to \( \tilde{\omega}_k \) there is a slight increase in the beam vibrations. The fact that the excessive vibrations are not seen is related to the time interval included in Figure 20. In fact, the external harmonics and the related induced harmonics act in an opposite direction and thus almost cancel each other in the initial time interval. Nevertheless, for instance for \( \tilde{\omega}_j \), the curves encompassing the two dominant beam vibrations is

\[
\pm \sqrt{68.1414 - 68.1099 \cos \left( (q_1 - \tilde{\omega}_j) \tau \right)}
\]

which indicate that at \( \tau = \pi / (q_1 - \tilde{\omega}_j) = 3990 \) the vibrations will reach approximately \( 1.280 \pm 11.6727 \). The case selected is more harmful for the oscillator, where the encompassing curves of the two most dominate parts are

\[
\pm \sqrt{1662.231 - 1662.228 \cos \left( (q_1 - \tilde{\omega}_j) \tau \right)}
\]

which indicate that, again at \( \tau = \pi / (q_1 - \tilde{\omega}_j) = 3990 \), the vibrations will stay approximately within \( 1.944 \pm 57.658 \). These extreme vibrations are shown in Figure 21.

Figure 21: Detail of the time interval of extreme vibrations for \( \tilde{\omega}_j = 0.2169 \), \( \alpha = 0.5 \), harmonic (full), encompassing curves (dashed): a) beam; b) oscillator.

Naturally, if the external frequency matches exactly the induced frequency, then resonance occurs, and in the absence of damping the vibrations are theoretically infinite.
In Case 5, $\alpha = 0.5$ is kept. Harmonic force is removed $\eta_B = 0$ and three combinations are tested for the effect of normal force and Pasternak modulus: $\eta_N = 0.2; \eta_S = 0$, $\eta_N = 0; \eta_S = 0.2$ and $\eta_N = \eta_S = 0$. Results are shown in Figure 22. As expected, the normal force softens the foundation, while the Pasternak modulus stiffens it, [42]. Induced frequencies and the respective amplitudes are summarized in Table 6.

<table>
<thead>
<tr>
<th>$q = 0$</th>
<th>$q_{M1,2}$</th>
<th>$q_{M3,4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\eta_N = 0.2; \eta_S = 0$</td>
<td>$\pm 0.2128$</td>
<td>$\pm 0.4399$</td>
</tr>
<tr>
<td>$\hat{w}_0$</td>
<td>1.4947</td>
<td>0.5917</td>
</tr>
<tr>
<td>$\hat{w}_s$</td>
<td>2.1587</td>
<td>2.5139</td>
</tr>
<tr>
<td>$\eta_N = 0; \eta_S = 0.2$</td>
<td>$\pm 0.2208$</td>
<td>$\pm 0.5137$</td>
</tr>
<tr>
<td>$\hat{w}_0$</td>
<td>1.1373</td>
<td>0.3562</td>
</tr>
<tr>
<td>$\hat{w}_s$</td>
<td>1.8013</td>
<td>2.0184</td>
</tr>
<tr>
<td>$\eta_N = \eta_S = 0$</td>
<td>$\pm 0.2177$</td>
<td>$\pm 0.4833$</td>
</tr>
<tr>
<td>$\hat{w}_0$</td>
<td>1.2800</td>
<td>0.4415</td>
</tr>
<tr>
<td>$\hat{w}_s$</td>
<td>1.9440</td>
<td>2.2125</td>
</tr>
</tbody>
</table>

Table 6: Induced frequencies and related amplitudes for Case 5.
Figure 22: Deflection at the oscillator location in Case 5, $\alpha = 0.5$, harmonic (dotted), finite (grey): $\alpha = 0.5$: a) $\eta_N = 0.2; \eta_S = 0$; b) $\eta_N = 0; \eta_S = 0.2$; c) $\eta_N = \eta_S = 0$.

In all cases plotted in Figure 22 transient vibrations are completely negligible, and therefore full solution can be substituted by the harmonic one. It can be concluded from Table 6 and Figure 22, that the oscillator has one dominant amplitude connected to the lower frequency, and so it looks like that it oscillates with one frequency only. On the other hand, beam vibrations have clear superposition of both harmonics.

By plotting full deflection shapes of the beam, same conclusion about the negligibility of the transient vibration can be taken. In Figure 23 the initial instants distant by $\Delta \tau = 10$ are plotted for the case from Figure 22a). Except for $\tau = 0$, it is seen that there are only small differences between the harmonic and full solutions. They are more pronounced at positions further from the oscillator position. As they are rapidly decreasing, after a short time full solution can be substituted by the harmonic one.
Figure 23: Full deflection shapes of the beam in Case 5 for $\alpha = 0.5$ and $\eta_N = 0.2; \eta_s = 0$: harmonic (dashed), full (dotted), finite (grey).

In all previous examples, integration along the branch cuts could be done along lines located very close to the cuts. This ensured that all residues could be safely used and therefore the transient vibrations affected only the initial short time interval. There is only one exception. When the external excitation frequency lies on the branch cut, then the related residue for the external harmonic steady-state part given by Eqs. (70) and (71) cannot be used. Consequently, the vibrations obtained from the integration along the branch cuts are non-negligible, because they include now not only the transient part, but also the steady-state part of the solution.

In Case 4, $q_{cut} = 0.6169$, therefore all tested $\bar{\omega}_f$ lie in the region of $K(q)$ continuity. For Case 6 it is assumed that $\alpha = 0.95$, $\bar{\omega}_f = 0.1936$ is kept, $\eta_{M_c} = 74.1$ is increased to the value that is the base value in [42], and for better clarity the load is given only by the harmonic force: $\eta_{P_s} = \eta_{P_c} = 0$, $\eta_{P_0} = 1$, $\tilde{k}_s = \tilde{c}_s = 0$ and obviously $\eta_{M_s} = 0$. In such a case $q_{cut} = 0.0698$ and thus $\bar{\omega}_f$ lies on the branch cut and cannot be used. There is only one pair of induced
frequencies $q_{M,2} = \pm 0.0637$, determining the induced harmonics. Results are plotted in Figure 24 and it is seen that the induced harmonic vibrations are quite far from the pretended solution.

![Figure 24: Beam deflection at the mass location in Case 6, $\alpha = 0.95$: induced harmonics (dashed), transient and steady-state (dash-dotted), full (dotted), finite (grey).](image)

5. Conclusions

In this paper a semi-analytical solution for the moving two-mass oscillator problem on infinite beam was derived and validated. The solution was presented for the beam deflection shape and displacement of the sprung mass as a sum of the truly steady-state part, induced harmonic vibrations and transient vibrations. The truly steady-state part is fully analytical. The induced harmonic part is given as a finite sum of analytical expressions, each one having one parameter that has to be determined numerically. Two iterative techniques were proposed for determination of these parameters named as induced frequencies. The transient part is determined numerically. Nevertheless, it was demonstrated by a variety of examples that the contribution of the transient vibrations is generally small and can be neglected in many cases. Thus, in many cases the full solution can be substituted by the harmonic one. In other cases the harmonic solution stand for an acceptable approximation in initial times and can substitute the full solution at larger times.

Using the formulas derived in this paper, less numerical errors are accumulated, since the numerical integration is used only for the transient vibration that do not play the essential role in the final result. The role of transient vibrations is to adapt the harmonic solution in the way to fulfil the initial conditions. Therefore, they can make a difference at initial stages, but, not
at larger times. Nevertheless, there are several situations in which their contribution is completely negligible. One can recognize these cases by the fact that the initial conditions are approximately fulfilled by the steady-state and harmonic terms. Importance of transient vibrations is therefore not affected by other factors like damping or branch cuts length.

One could argue that the complete solution could be obtained directly by the numerical integration. But, the approach presented in this paper has several advantages: (i) it provides the necessary insight into the final solution, because of the separation into several parts; (ii) it allows determination of the unstable cases \textit{a priori}; (iii) it allows evaluation of the severity of an unstable case; (iv) the semi-analytical (harmonic) part of the solution can be quickly evaluated at any time without losing precision and without the necessity to test convergence; (v) the harmonic solution can be used as acceptable approximation of the full solution. On the other hand, full integration would quickly encounter numerical problems at larger times.

In comparison with previous works published by other researchers on this subject, this paper provides full evolution of beam deflection shapes and oscillator vibrations, and as a side result, the onset of instability and its severity which is important for mitigation measures. It was seen in the validation examples that sometimes negative imaginary parts of the induced frequencies are very low, indicating moderate increase of vibration amplitudes, which provides sufficient time to apply mitigation measures before structure reaches harmful vibrations.

Integration along the branch cuts was performed numerically. In most cases such calculations are quick and straightforward. Only when there is a pole located close to the branch cuts, attention must be paid to numerical evaluation and sufficient number of function values must be considered in the integration. There is always a possibility of cutting the complex plane and encompass all poles inside, except when the frequency of the harmonic force lies on the branch cut. Analytical formulation of the function to be integrated allows to easily identify the necessary integration step and length.

Formulas derived are not restricted to railway applications. The requirement of reasonably stiff foundation, discussed in the beginning of Section 4.1, applies only to comparison between finite and infinite beams. Namely, in such comparison results on finite beams must be independent on the beam length and thus the foundation stiffness cannot be unreasonably low. But formulas derived have no restriction on foundation stiffness value. Nevertheless, very low stiffness would imply very high displacements and possibly violation of the assumptions of small displacement theory.
Main limitation of developments presented here is the assumption of the rigid contact between the unsprung mass and the beam, but contact loss can be evaluated \emph{a posteriori}, by analysis of the contact force. Generally, contact loss is prone to happen in upward displacements, which are typical for unstable behaviour.

In all examples excellent agreement between the results on finite and infinite beams was obtained, confirming that the induced frequencies and all other deductions are correct. For the sake of simplicity only homogeneous initial conditions were considered. Extension to other conditions will be the subject of further research.

It was summarized that extensive vibrations occur in two cases that can be designated as resonance-type cases, and in all unstable cases characterised by exponential increase of amplitudes in time. In the resonance cases the unbounded amplitudes are developed only under one specific velocity. One case is related to actuation of constant forces and the other one to harmonic forces. The former indicates the classical critical velocity, the latter occurs when the external excitation frequency matches one of the induced frequencies. Regarding the instability, which happens when there is at least one induced frequency with negative imaginary part, it was concluded that in the absence of beam damping, the onset of instability is always at the critical velocity, independently on the oscillator damping.

In the absence of any damping, induced harmonic vibrations, that cannot be obtained by the double Fourier transform, theoretically last for ever. If damping is present, induced vibrations gradually cease in time and then only the truly steady-state part of the solution remains. This, however, does not mean that the analysis of the initial stages is not important.

**Acknowledgements**

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Appendix A: Moving mass

Many examples related to moving mass problem are presented in [42]. They cover large number of cases with negligible contribution of transient vibrations. Here, therefore, attention is paid to cases where transient vibrations should not be neglected. One typical case is presented in [46], where it was concluded that for very low mass ratio $\eta_{M}$, the harmonic solution at zero time is displaced from the initial conditions, and therefore the transient solution should be added at initial stages. Other cases stand for situations with no induced frequency and unstable vibrations.

The following cases will be analysed: Case A.1: situations with no induced frequency; Case A.2: unstable damped cases with two pairs of induced frequencies; and Case A.3: unstable undamped cases with two pairs of induced frequencies.

Numerical values adopted in Case A.1 are $\eta_N = \eta_S = 0$, $\eta_b = 0.05$, $\eta_{M_1} = 74.1$, $\eta_{P_1} = 1$ and $\eta_{P_2} = 0$. Value $\eta_{M_1} = 74.1$ is chosen in agreement with [42]. This value is related to typical railway application where the beam is modeled by the rail and the moving mass is derived from the moving force corresponding to mass weight. The induced frequencies were derived in [42] and are listed in Table A.1, confirming that the frequency line is interrupted when the imaginary part is close to $\eta_b = 0.05$. How the fact of missing frequencies is affecting displacements at the mass position is shown in Figure A.1.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$q_{M_1}$</th>
<th>$q_{M_2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.96</td>
<td>0.0710 + i0.0485</td>
<td>-0.0710 + i0.0485</td>
</tr>
<tr>
<td>0.97</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>0.98</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>0.99</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>1.00</td>
<td>0.1137 + i0.0484</td>
<td>-0.1137 + i0.0484</td>
</tr>
</tbody>
</table>

Table A.1: Induced frequencies in moving mass problem: Case A.1, [42].
Figure A.1: Deflection at the mass position in Case A.1: harmonic (dashed), transient (dash-dotted), full (dotted), finite (grey): a) $\alpha = 0.96$; b) $\alpha = 0.97$; c) $\alpha = 0.98$; d) $\alpha = 0.99$; e) $\alpha = 1.00$.

It can be concluded that for $\alpha$ with no induced frequencies the initial influence of the transient vibration is significant, nevertheless, the complete time interval under which this effect lasts is basically the same in all tested $\alpha$, i.e. for cases with or without induced
frequencies. In all tested situations, only initial stages are affected by the presence of transient vibrations and after a short time transient vibrations can be neglected.

Numerical values adopted in Case A.2 are $\eta_N = \eta_S = 0$, $\eta_b = 0.2$, $\eta_M = 74.1$, $\eta_p = 1$, $\eta_r = 0$. It was shown in [42] that starting from $\alpha = 1.26$, there is another pair of induced frequencies. Therefore, the influence of these additional values is tested for two velocities, $\alpha = 1.26$ and $\alpha = 1.4$. Induced frequencies are summarized in Table A.2.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$q_{M_1}$</th>
<th>$q_{M_2}$</th>
<th>$q_{M_3}$</th>
<th>$q_{M_4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.26</td>
<td>0.2457 - i0.0706</td>
<td>-0.2457 - i0.0706</td>
<td>0.2053 + i0.2022</td>
<td>-0.2053 + i0.2022</td>
</tr>
<tr>
<td>1.4</td>
<td>0.3175 - i0.1157</td>
<td>-0.3175 - i0.1157</td>
<td>0.2867 + i0.2387</td>
<td>-0.2867 + i0.2387</td>
</tr>
</tbody>
</table>

Table A.2: Induced frequencies in moving mass problem: Case A.2, [42].

How the additional frequencies affect the displacement at the mass position is shown in Figure A.2.

![Figure A.2](image)

Figure A.2: Deflection at the mass position in Case A.2: harmonic (dashed), transient (dash-dotted), full (dotted), finite (grey): a) $\alpha = 1.26$, b) $\alpha = 1.40$.

Existence of induced frequencies with negative imaginary parts indicates unstable cases. In conformity, deflections in Figure A.2 are gradually increasing. The contribution of the additional pair of induced frequencies is small, because it possesses large positive imaginary part. Transient vibrations are also negligible; therefore, it can be concluded, that the importance of transient vibrations is independent on the length of the branch cuts.
Case A.3 stand for unstable undamped vibration. In this case one point on frequency line implies two pairs of frequencies. For $\alpha = 1.20$ they are listed in Table A.3. The corresponding displacements are shown in Figure A.3. It is seen that in the absence of beam damping vibrations are significantly augmented.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$q_{M_1}$</th>
<th>$q_{M_2}$</th>
<th>$q_{M_3}$</th>
<th>$q_{M_4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.2</td>
<td>$0.1992 - 0.0853i$</td>
<td>$-0.1992 - 0.0853i$</td>
<td>$0.1992 + 0.0853i$</td>
<td>$-0.1992 + 0.0853i$</td>
</tr>
</tbody>
</table>

Table A.3: Induced frequencies in moving mass problem: Case A.3, [42].

Figure A.3: Deflection at the mass position in Case A.3 for $\alpha = 1.2$: harmonic (dashed), transient (dash-dotted), full (dotted), finite (grey).

The analyses in this appendix showed that the importance of transient vibrations is independent of the beam damping and the length of the branch cuts. In all presented cases their effect was low and could be neglected. Such analyses are important additions to [42], where only harmonic solution was derived.

**Appendix B: Moving one-mass oscillator**

In this appendix simplified case of one-mass oscillator is dealt with. It will be analysed: (i) how parameters $\tilde{k}_s$ and $\eta_{M_i}$ influence the induced frequencies; and (ii) what differences can be observed by applying the load directly on the beam and on the sprung mass. No damping is considered. Data for selected cases are given in Table B.1.
The induced frequencies are plotted in Figure B.1. It is seen that, as for the moving mass [42], instability starts exactly at the critical velocity. The sharp corner in the real part of these frequencies is approximately fitted to the curve representing $A = 0$ defined in Section 3.2.

It can be concluded that if $\tilde{k}_s$ and $\eta_{Ms}$ change in the same proportion, then the related frequencies have similar values. The number of induced frequencies is not fixed, each point on the frequency lines for $\alpha < 1$ stands for one pair of frequencies, but, when $\alpha > 1$ each point stands for two pairs of frequencies. It is seen in Figure B.1 that some of the negative values of the complex part of the frequencies are very low, indicating that the increase in vibration amplitudes is quite slow and thus, there is sufficient time to apply mitigation measures before structure reaches very harmful vibrations. It is also possible to extract the maximum negative value (in absolute sense) indicating the worst case of very steep amplitude increase.

![Figure B.1: Real and imaginary parts of the induced frequencies: Case B.1 (black); Case B.2 (grey); Case B.3 (black dashed); $A = 0$ (black dotted).](image-url)

Regarding the deflection at the oscillator position, as in the previous section, when the initial conditions are approximately fulfilled by steady-state and harmonic parts, then the
transient vibration is negligible. Two situations will be shown with data of Case A.2. They will differ by the applied force, namely they will use $\eta_P = 0$, $\eta_P = 1$ and $\eta_P = \eta_P = 1$. The velocity ratio is $\alpha = 0.344$ and the induced frequencies are $q_{M,2} = \pm 0.2192$.

The first situation is shown in Figure B.2. This is the typical case, where the effect of the transient solution is completely negligible and therefore the full solution was replaced by the harmonic one. In the second case, when another force is also acting directly on the beam, the equilibrium position of the beam vibrations is lower, which activates initial contribution of transient vibrations of the beam, to shift the vibrations in the way to fulfil the initial conditions. In Figure B.3a) the oscillator vibration is shown, for which the transient part is negligible. In Figure B.3b) the transient vibration of the beam is shown separately. In Figure B.3c) the harmonic solution is compared to the full solution, that is validated by results on finite beams. It is seen that the initial harmonic waves are perturbed, but the transient effect rapidly disappears.
The analysis is important to show that by changing the loading conditions in the way that the harmonic part of the solution is further from the initial conditions, activates the transient vibrations. Nevertheless, even if the initial value is relatively high, the time interval affected by the transient vibrations is quite low and therefore they could be neglected.

References


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