Chapter 1
Structure and Structures

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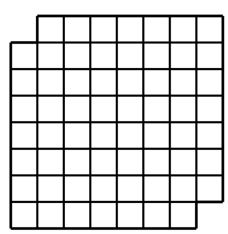
Abstract In this paper we critically evaluate the notion of the structure of the natural numbers with respect to the question how the internal structure of such a structure might be specified.

1 Structure

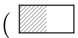
Let us start with two short examples of how structure comes into play in Mathematics (and its applications).

Tiling a mutilated board.

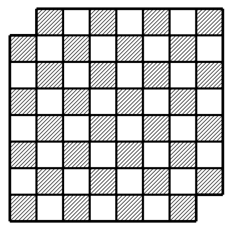
Consider a board with a 8×8 grid on it, dividing it into 64 squares; now remove two opposite squares from the corners so that only 62 squares remain:



Research partially supported by the Portuguese Science Foundations FCT, projects *Hilbert's 24th Problem*, PTDC/MHC-FIL/2583/2014, and *Centro de Matemática e Aplicações*, UID/MAT/00297/2013.

Is it possible to tile 31 dominoes () on this board so that all squares are covered?

The answer is easy to see when we put some extra structure on the board, namely the usual black and white alternation of a chess board; a mutilated chess board looks like this:



As we removed two white squares, the mutilated chess board has 32 black squares but only 30 white squares; but each domino covers exactly one white and one black square, so that we cannot tile a board which doesn't have an equal number of white and black squares.

This example is widely used to illustrate ingenious mathematical reasoning, stressing often the aspect of creativity and/or intuition which is reflected in the idea of replacing the original board by the chess board.[1]

We presented the example here as the addition of the black and white "colouring" is a typical example of how structure can be imposed on an object, structure which is useful in dealing with questions concerning the original object. The crucial point for the following discussion is that this structure, the black-white "colouring", is *not* present in the original board. It is structure we have *imposed* on the original board.

Elliptic orbits

Ellipses as conic sections have been studied in (Euclidean) Geometry since antiquity. They exhibit a lot of "mathematical structure" which can be used to study them geometrically. In particular, an ellipse comes with two focal

[1] Historically, the example of the mutilated chessboard can be traced back to Max Black who posed it in 1946 as a problem in his book *Critical Thinking* [Bla46, exercise 6, p. 142] (but starting off with the chess board, thus, leaving out the creative part of adding this structure as a first step). It is also reported that Emil Artin occasionally used this example in his lectures (see [vRaZ63, Thi06]); it might well be that he took it from Black (or some other later source), but it was stressed in a obituary for Artin that he applied the idea of the solution within his mathematical activity, as he had "the very rare ability to detect, in seemingly highly complex issues, simple and transparent structures". ("[Er hatte d]ie so seltene Gabe, in scheinbar hochkomplexen Sachverhalten einfache, durchsichtige Strukturen aufzuspüren" [Rei06, p. 39].)

points which, together with the (fixed) summed distances of any point of the ellipse to the foci, define a particular ellipse. It was an observation by Kepler that the planets of the solar system move on elliptic orbits around the Sun (within astronomical accuracy). And the center of the Sun is one of the focal points.[2] The natural question is: what is at the other focal points (which are different for the orbits of the different planets)? Apparently, there is no specific *physical* counterpart to the distinguished mathematical (second) focal point of an orbit (while, to repeat, for the first focal point, there is one). Thus, the mathematical structure of ellipses is richer than the physical structure we encounter when we apply the mathematical notion of ellipse to an orbit in the solar system.

The lesson of this example is that (mathematical) structure should not be reduced to physical structure: we legitimately may "add" mathematical objects (here: the second focal point) to those stemming from the empricial data (here: the points of the trajectory of a planet and the Sun). Even without distinct empricial counterparts, their distinction (in a mathematical sense) is unquestionable.

2 Structures

Today it is common to say that mathematicians investigate structures. The term structure is, however, ambigious: it may refer to *concrete structures*, as in the case of the structure of the natural numbers, or to *abstract structures*, which emerged from algebra and which were prominently promoted by Bourbaki. Both notions, as technical terms, as clearly different from the informal notion of structure as sketched in the previous section. Without going into a detailed discussion about the differences between concrete and abstract structures, a discussion which would lead a too far afield, we will concentrate on concrete structures.[3]

[2] More exactly: the center of mass of the solar system, which consists of the Sun together with all objects of the solar system; but, clearly, the Sun dominates this mass in such a way that it is reasonable to identify the center of mass with the center of the Sun.

[3] We will shortly address abstract structures in §4. The distinction of these two froms of structures is ubiquitous for the working mathematician: the different character of natural numbers and groups is conceptionally self-evident. For the syntactic counterpart of structures we find it explicitly discussed in [HM58, §§ 4.2 and 4.3] as *heteronome* and *autonome Axiomensysteme*. Although the authors presuppose an empirical base for the heteronomous axiom system (what we will not do for the concrete structures), they point to the fact that these axioms systems are chosen *a posteriori* ("nachträglich gewählt"). This is in accordance with our understanding of concrete structures relating to mathematical objects which are supposed to preexist.

A concrete structure can be understood as a mathematical realm of discourse of semantic nature.[4] The prime example is the *structure of the natural numbers*, but there are many others, including the structure of the real numbers and the structure set theory is concerned with. The characteristic of such concrete structures is that they start from a certain universe of discourse, which is assumed to be preexistent—somehow in a platonistic manner—and that they provide "tools" (i.e., functions and relations) to investigate such a universe.

The exact shape of such concrete structure is not fully specified. Here we will deal mainly with *first-order structures*, but addressing the question of second-order structures only in §3.5. A first-order structure is a well-defined concept in Mathematics; it presupposes the definition of a *first-order language* which should be interpreted in such a structure.[5] The language dependency is not unproblematic, as it is the convictions that mathematics—or mathematical truth—should be language independent. The following consideration can be taken, however, as an argument that the choice of the language is somehow related to structure (in the informal sense) we impose on the universe of discourse. In the following section, we will look more closely at the seemingly familiar concrete structure of the natural numbers and discuss how its internal structure is given.

3 Structure in a structures

> Mathematics is just the detection and investigation of structures of thinking which lie hidden in the mathematical symbols. The simplest mathematical entity, the chain of integers $1, 2, 3, \ldots$, consists of symbols which are combined according to certain rules, the arithmetical axioms. The most important of these is an internal coordination: to each integer there is one following it. These rules determine a vast number of structures; e.g. the prime numbers with their remarkable properties and complicated distribution, the reciprocity theorems of quadratic residues etc.
>
> Max Born, [Bor66, p. 151f.]

Let us start with the structure of the natural numbers which is often given in the form: $\mathfrak{N} = \langle \mathbb{N}, 0, +, \cdot, \ldots \rangle$. The use of the dots should make one puzzle; they are added in a sloppy presentation of a (concrete) structure, to indicate that one may add some more functions and relations which are supposed to come along naturally. For the natural numbers, for instance, it could make sense to include the less-or-equal relation $\leq$ after addition and multiplication.

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[4] The semantic nature implies, in particular, that it is assumed that any (properly formulated) statement about this realm has a definite truth value.

[5] The distinction of the syntax and semantics of, let say, function symbols and functions themselves is, of course, fundamental in mathematical logic, see [KK15]. In the present context, however, we argue essentially entirely on the semantic side and will neglect the difference as it should not give rise for confusion.

At second glance, however, one should realize that it is far from being obvious what should or could be part of such a structure.

3.1 Constitutive structure

To start from the scratch, let us first consider the raw set $\mathbb{N}$ *without* any further structure on it. Stripped off of any structure, the set of natural numbers should appear as nothing else than a bag with infinitely many elements, and the only condition we have is that any two elements taken out of this bag are different.[6]

By naming the elements, let us say the first one, I pick, by 0, the next one by $S(0)$, then by $S(S(0))$ and so on, we are already *imposing* some structure on the previously completely unstructured set $\mathbb{N}$.[7] In this view, $\mathbb{N}$ "alone" would not have any meaning; in formal terms one would have already more than just a raw set at hand, namely the rudimentary structure $\langle \mathbb{N}, 0, S \rangle$. It is defensible that the very use of $\mathbb{N}$ presupposes this rudimentary structure. One could also consider to put another structure on the raw infinite set, as, for instance, a (binary) word structure; but in such a case, one would probably use the designation $\mathbb{B}$; or, for a tree structure, $\mathbb{T}$.

One can also say the 0 and the successor function S build the *constitutive structure* of the natural numbers; likewise, ϵ and two unary successor functions S_0 and S_1 build the constitutive structure of binary words, and a finite, non-empty set of constants together with a binary successor function provides the consitutive structure of (binary) trees. In this view the constitutive structure of a mathematical structure is nothing other than the set of constructors of a datatype in Computer Science.

The distinction of a constitutive structure for a universe, which is to be used as the basis for a concrete structure, is of importance as it gives a justification for both recursive definitions of functions on the underlying set, as well as the proof scheme of induction for this set.

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[6] The (potential) infinity of $\mathbb{N}$ could also be given by such a condition: by consecutively taking out elements from this bag, one will never completely empty the bag.

[7] Tactically, we use here 0 and S as syntactic entities to give names; if you think of their semantic interpretations, you would have always the corresponding concrete elements of the infinite set at hand; still, S would be supposed to be a semantic function telling you how to go from one element to the next one.

From another, *constructive* perspective one can also proceed the other way around: starting from 0, one *constructs* successively the elements of $\mathbb{N}$ by applying the "successor function" S. In fact, it would be far better to call this "function" in this specific context "constructor". This terminology is known from Computer Science, and the analogy is not accidental: the separation of the definition of a *datatype*, by constants and constructors, from the implementation of functions operating on this datatype corresponds to the distinction we have in mind here. The problem with the constructive perspective is, that it cannot go beyond countable universes.

But starting from a raw infinite set, even the specific constitutive structure is not present in the set, but imposed on it—in the very same way as the black-white colouring is not present on the initial mutilated board.

3.2 First-order structure

With the constitutive structure we get naturally some more structure: for instance, for $\mathbb{N}$ (or, more exactly, $\langle \mathbb{N}, 0, S \rangle$), we obtain immediately an order structure, as the elements of $\mathbb{N}$ are now naturally ordered by the length of their names. But this description is given on the meta level. To get off the ground, in particular in a formal way, we need to "inject" some more structure on $\mathbb{N}$. In (first-order!) Peano-Arithmetic, PA, this is done by presupposing addition and multiplication as primitive functions (see below).

Assuming these functions, the less-or-equal relation can be introduced for the natural numbers by the definition: $t \leq s :\Longleftrightarrow \exists x.t + x = s$. Formally, this can be expressed by *definitorial extensions* (and the possibility to perform such extensions without further explanations give the justification of the dots in a sloppy presentation of a structure), see, e.g., [Rau06§2.6].

It is then an exercise in first-order Arithmetic to show that all first-order definable functions and relations can be given by first-order logic formulas using addition and multiplication.[8] And this leads us to the first 'result':

Structures are supposed to be closed under first-order definability!

That is why they are also called *first-order structures*.

And one can see here a reminder of the distinguished status of *logic*. Whenever one starts off with some structure, one is committed to the structure which can be build up from it by logical (first-order) definitions.

3.3 Primitive structure

First-order definability, however, is not enough to get off the ground from the constitutive structure in the case of the natural numbers: as we said, we need at least addition and multiplication as introduced, for instance, in the now standard Peano Axioms. We may already note, that this extra structure would not be needed, if we had second-order definability at hand; see §3.5 below.

[8] Intentionally, we phrased this exercise as a trivial tautology: first-order definable functions and relations are, by definition, given by first-order logic formulas over addition and multiplication. The interesting question is, of course, which are these functions; it is just an empirical observation that they include essentially all number-theoretic functions used (and defined independently) in the history of mathematics; to show this inclusion is not a trivial exercise!

But it is also notable, that, in PA, we can already dispense with the definition of further primitive-recursive functions, as they are (first-order) definable from addition and multiplication. On the other hand, multiplication is not (first-order) definable from addition alone.

Primitive Recursive Arithmetic, in contrast, requires the inclusion of all primitive-recursive functions as primitive structure, as in the presence of (only) quantifier-free induction, first-order logic is not expressive enough to obtain, let say, exponentiation from multiplication and addition.

Apparently, the role of addition and multiplication in PA is very specific to the first-order framework with full induction. To our knowledge, there is no intrinsic explanation why addition and multiplication are distinguished in PA; they appear just to be the two functions which serve technically for the purpose. Conceptually, the introduction of all primitive-recursive functions seem to be a more natural choice, as they are defined by a general definition scheme over the constitutive structure of the natural numbers (and this general definition scheme can equally be applied to other constitutive structure).

Again, and somehow even more than for the constitutive structure, the primitive structure seems to be imposed by us on the set, even if it is already equipped with constitutive structure, rather than “being there”.

3.4 Some consequences

To take stock, let us briefly reflect on what we have “at hand” in PA. As already noted, addition and multiplication is enough to get all other primitive recursive functions (and even more). Also, the present structure allows one to distinguish numbers in ways which cannot be seen as directly implied by the original structure. As a prominent example, let us mention the number 1729 which, as Ramanujan observed, is the smallest number expressible as the sum of two cubes in two different ways:[9] $1729 = 1^3 + 12^3 = 9^3 + 10^3$.

Such a distinction of a number, induced by the structure, can be seen as a parallel to the physically meaningless second focal point of a planetary orbit, whose distinction is induced only by the elliptic orbit.

But there are much more far-reaching consequences of the introduction of the structure, touching even on the ontological issues. Interestingly, the first-order structure of the natural numbers provides us already with a large part of *ordinal arithmetic*. Conceptually, ordinals seem to transcend the natural numbers by continuing counting into the transfinite, starting with a

[9] The story was recorded by G. H. Hardy [Har21, p. lvii f]: “I remember once going to see him when he was ill at Putney. I had ridden in taxi cab number 1729 and remarked that the number seemed to me rather a dull one, and that I hoped it was not an unfavorable omen. ‘No,’ he replied, ‘it is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways.’” One may note, however, that Bernard Frénicle de Bessy had already published this fact in 1657.

new element ω as the first infinite ordinal. This is supposed, by definition, not to be an element of our “bag of natural numbers”. However, by defining appropriate order relations we can, straightforwardly, encode ordinals in the natural numbers (including, of course, the intended arithmetical operations for them).[10] As consequence, we have—unintentionally—extended our mathematical realm into the transfinite.

3.5 More structure?

Fast growing functions

Even with some primitive structure and first-order definablity there will always be some structure of the natural numbers which cannot be captured by a first-order axiomatization, due to Gödel's First Incompleteness Theorem. In fact, expressing that a function f is *provably total* by the formula $\forall x \exists y. f(x) = y$, it requires only a short reflection on arithmetical first-order theories to see that we can always diagonalize over the provably total functions of a given theory to obtain a new function which is not provably total in this theory. As a matter of fact, PA only proves the totality of ε_0-recursive functions; and the *Hardy Hierarchy* H_α can be used to classify, quite generally, the complexity bounds of arithmetical theories (for these notions and the corresponding results, see [FW98]).

Of course, we expect that our standard structure of the natural numbers will contain all these fast growing functions as total functions; but it remains unclear how far we can actually go in demandind that they make up part of the structure which should “automatically come with” the natural numbers. If they all should “automatically come” we have to admit that we are not able to exhaust the structure *in its very definition.*[11] If we cannot include it all, where to draw the line?

Second-order definability

With respect to the definition of structures, one should recall a historical fact: both, Dedekind (in *Was sind und was sollen die Zahlen?*, [Ded88]) as well as Peano (in *Arithmetices Principia Novo Methodo Exposita*, [Pea89])

[10] For instance, by defining $\prec$ as:

$$1 \prec 2 \prec 3 \prec \cdots \prec 0,$$

where 0 is supposed to be $\prec$-greater than every other natural number, it can be taken to represent ω (and any original n represents $n-1$ of the ordinal world).

[11] Remember the dots in the definition of $\mathfrak{N}$ at the beginning of §3.

define the structure of the natural numbers in a second-order context. In this context, addition and multiplication are logically definable (from 0 and successor). Thus, we would not need any extra primitive structure. But the reason why we no longer follow this path is well-known: second-order logic is not recursively axiomatizable. This fact became known only with Gödel's (First) Incompleteness Theorem;[12] but it descredited second-order logic as a basic framework for mathematics up to today.

From a staunch platonistic standpoint, this is almost an anthropomorphic reason to distinguish first-order structures: the restriction to recursive axiomatizability seems to be a limitation imposed by human capacity to construct formal systems, rather than anything which should be inherent to mathematical ontology. And the non-categoricity of any first-order theory for the natural numbers, expressed in Gödel's First Incompleteness Theorem, gives even more reason to challenge it: after all, we are interested in the standard structure of *the* natural numbers, aren't we?

Kreisel fought forcefully for the consideration of second-order properties when he advocated *informal rigor* [Kre67, pp. 138f and 152]: "Informal rigour wants [...] not to leave undecided questions which can be decided by full use of evident properties of these intuitive notions" and "most people in the field are so accustomed to working with the restricted [first-order] language that they may simply not succeed in taking other properties seriously." He had in mind, first of all, the Continuum Hypothesis. But already for Arithmetic the consequences would be far-reaching: second-order arithmetic already provides us with a concept of real numbers and the very structure already incorporates Analysis. Apparently, today mathematicians are inclined to draw a line here: Analysis doesn't seem to come "automatically" with Arithmetic.[13]

3.6 Less structure?

We have argued above that the first order closure of a constitutive structure should be taken for granted. But there are, at least, two contexts where this presupposition is questioned.

Intuitionisms

From an intuitionistic perspective, the very definition of a first-order structure is problematic: it fixes the meaning of the negation in the classical way and,

[12] In fact, at a time where Skolem had already promoted first-order logic as the "one and only solution" for mathematics; but against the fierce opposition of Zermelo.

[13] This is in contrast to Hilbert who treated Arithmetic and Analysis equally in his second problem in his famous Paris problem list of 1900, as, by that time, he had no reason to restrict himself to first-order theories.

thus, is not constructive. Therefore, intuitionism dispense with the concept of structure but reduces mathematics to the part expressible in constructive terms. This is unproblematic for arithmetic, although it even *justifies* classical arithmetic by use of the double negation interpretation. But for analysis, Brouwer wasn't able to provide the mathematical community with an alternative to the classical conception.[14] In any case, intuitionism is today—quite ironically—more a formal enterprise (but as such, very valuable) and serves the mathematical community more through its conceptual analysis of constructivity than through philosophical reflections.[15] But, as such, it approaches the natural numbers (and other concrete structures) with much less structure than the closure of the primitive structure under (classical) first-order logic provides.

Quantifier-free theories

We already mentioned Primitive Recursive Arithmetic which is a quantifier-free formalization for the natural numbers and, as such, exhibits less structure than Peano Arithematic. Another prominent example for a quantifier-free theory is *Gödel's* $\mathcal{T}$. Of course, both theories should not be considered as closed under first-order logic, as one would incorporate some strength into them which is intended to be controlled. Otherwise, the consistency proof of Peano Arithmetic in terms of Gödel's $\mathcal{T}$ would be pointless. Thus, in foundational studies, one is clearly not bound to the ontological commitments of the intended standard model, as it just the task to give an independent account of it. In this sense, Shoenfield dismissed, quite correctly, any consistency proof by semantic methods [Sho67, p. 214]: "The consistency proof for P by means of the standard model [...] does not even increase our understanding of P, since nothing goes into it which we did not put into P in the first place."

4 Abstract structures

Next to concrete structures, mathematics deals also with *abstract structures*. In this case, one abstracts entirely from any underlying universe but is studying structures entirely on the characteristic properties of operations and re-

[14] "Brouwer and other constructivists were much more successful in their criticisms of classical mathematics than in their efforts to replace it with something better." [Bis67, p. ix].

[15] "Intuitionism was transmuted by Heyting from something which was anti-formal to something which is formal. When one speaks today of intuitionism, one is talking of all sorts of formal systems (studied by the logicians)." Bishop in [Bis75, p. 515]; for the valuable results see, for instance, [Koh08]; for constructivity [Bis67].

lations.[16] In our terminology, one could say that in an abstract structure the closure of the primitive structure under logical definability (i.e., normally first-order definability) is all what is available in the structure—even the constitutive structure of the concrete instances of the abstract structure is intentionally left unspecified. And one does not face the problem concerning the available structure discussed for the concrete structures; all that is there in an abstract structure is exactly that what we have put into it.

The classical example for an abstract structure is the notion of group, but today we have an endless stream of abstract structures at hand. Bourbaki [Bou50, p. 228] even identified some "mother structures", namely algebraic structures, order structures, and topological structures, which received an distinguished status in their "architecture of mathematics".[17] Approaching mathematics from this point of view became an incredible successful programme so that, for instance, the term "abstract algebra" is today essentially a pleonasm.

In principle, one could also treat arithmetic as an abstract study about the consequences in PA; but this would be misleading. We are clearly not just interested in properties which are common to the standard and non-standard models of PA, excluding those which may have different truth values in different models. For the totality statement of a fast growing function we have a clear opinion about its truth value, as well as for the formalized consistency statement as used in Gödel's Incompleteness Theorems.

In contrast to arithmetic, in set theory the situation is not so clear. Using our terminology from above, one could think of a constitutive structure for sets which includes the power set operation among the operation to obtain new sets. It is evident that this operation is far from being clear in the way it works—the Continuum Hypothesis depends on it. Thus, all that we can do is to characterize it insofar as we have intuition about it, and that is what is done in axiomatized set theory. But, in this way, we can study it only in an abstract way—through the axioms we have formulated for it—and the possibility of different interpretations of the formal power set operation in different set-theoretic universes is possible; it is even realized, for ZFC, in Gödel's constructible hierarchy and in other "forced" set-theoretic universes. Adherents of a *multiverse* conception of set theory will probably subscribe to the abstract character of ZFC, but, of course, there are also other views.[18]

[16] Corry [Cor04, p. 259] describes this for Øystein Ore's introduction of the term *structure* in the context of his concept of lattice in the following words: "The leading idea behind this attempt was that the key for understanding the essential properties of any given algebraic system lay in overlooking not only the specific nature, but even the very existence of any elements in it, and in focussing on the properties of the lattice of certain of its distinguished subsystems."

[17] We may note that the way Bourbaki promoted the notion of abstract structure was critically evaluated by Leo Corry [Cor04] with the result that Bourbaki's use lacks a satisfying specification of the notion of structure.

[18] To give just two references: [FFMS00, AFHT15].

5 Conclusion

In this paper we have discussed how internal structure is added to a raw infinite set to obtain what we call the structure of the natural numbers. One may note that this internal structure requires some linguistic tools to express it; these tools are usually given by an underlying first-order language. As for the black-white "colouring" of the mutilated blackboard, this internal structure is not *per se* present in the raw infinite set, but was imposed by us. As a consequence, we are able to distinguish elements in this set, as the number 1729, which does not carry this distinction if they are only considered as the 1730th element in a stream of numbers (somehow in analogy to the second focal point of planatary orbits).

But we may add a word of caution: it is an everlasting controversy whether mathematics invents or discovers its concepts. By saying that one "imposes" structures, we do not advocate the former view. It should be clear, that such an imposement is far from being arbitrary.[19] Thus, there is an element of discovery when a mathematician realizes the *possibility* to put a specific structure on a specific domain. By exploiting this possibility one may speak, indeed, of a discovery.

When it comes to mathematical truth, we take the view that it makes sense only insofar as it refers to structures, see [Kah17]. As we have seen, it is, however, far from being obvious how these structures, together with their internal structure, are given. To introduce (or discover) and investigate this internal structure of mathematical structures is, therefore, the basis of mathematical research—existence[20] and truth are notions which are, then, induced by the structural set-up.

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[19] One will not be able, for instance, to impose the structure of the natural numbers on a finite set; or a field structure on a set with six elements.

[20] In the sense of Bernays's "*bezogene Existenz*", [Ber50].

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