

ESTIMATION OF THE MAXIMUM MULTIPLICITY OF AN EIGENVALUE IN TERMS OF THE VERTEX DEGREES OF THE GRAPH OF A MATRIX *

CHARLES R. JOHNSON [†] AND CARLOS M. SAIAGO [‡]

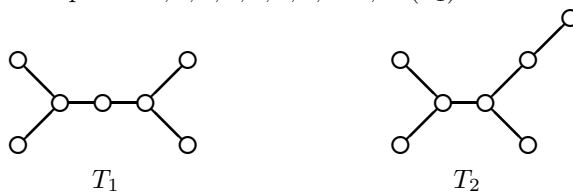
Abstract. The maximum multiplicity among eigenvalues of matrices with a given graph cannot generally be expressed in terms of the degrees of the vertices (even when the graph is a tree). Given are best possible lower and upper bounds, and characterization of the cases of equality in these bounds. A by-product is a sequential algorithm to calculate the exact maximum multiplicity by simple counting.

Key words. Eigenvalues, multiplicity, symmetric matrix, tree, vertex degrees.

AMS subject classifications. 15A18, 15A57, 05C50, 05C05, 05C07

1. Introduction. Throughout let T be a tree on n vertices, whose degrees are $d_1 \geq d_2 \geq \dots \geq d_k \geq 3 > d_{k+1} \geq \dots \geq d_n$, and let $\mathcal{S}(T)$ be the set of all real symmetric matrices whose graph is T . The diagonal entries of matrices in $\mathcal{S}(T)$ are unconstrained by T . For any n -by- n matrix A , $m_A(\lambda)$ denotes the (algebraic) multiplicity of λ as an eigenvalue of A , and $m(A) = \max_{\lambda \in \mathbb{C}} m_A(\lambda)$, the maximum multiplicity of an eigenvalue of A . Then, we let $m(T) = \max_{A \in \mathcal{S}(T)} m(A)$, the maximum multiplicity of an eigenvalue among matrices whose graph is T .

It was shown in [2] that $m(T)$ equals the *path cover number*, $p(T)$, of T , the smallest number of vertex disjoint paths of T that cover all the vertices of T . It was also shown in [2] that $m(T) = \max[p - q]$ over all ways in which q vertices may be removed from T so as to leave p paths (i.e. the subgraph of T induced by the other $n - q$ vertices consists of p paths, an isolated vertex counting as a path). However, neither of these formulae for $m(T)$, though polynomially computable from T , is expressible in terms of overt parameters of T , such as the vertex degrees; in fact, there can be no formula for $m(T)$ strictly in terms of the vertex degrees. There are trees with the same degree sequence but different path cover number. For example, the following trees have degree sequence 3, 3, 2, 1, 1, 1, 1, but, $m(T_1) = 3$ and $m(T_2) = 2$.



*Received by the editors on 12 December 2000. Accepted for publication on 28 February 2002.
 Handling Editor: Bryan L. Shader.

[†]Department of Mathematics, College of William and Mary, P.O. Box 8795, Williamsburg, VA 23187-8795, USA (crjohnso@math.wm.edu).

[‡]Dep. de Matemática, Fac. de Ciências e Tecnologia da Univ. Nova de Lisboa, 2829-516 Quinta da Torre, Portugal (cls@fct.unl.pt). Research supported in part by Fundação para a Ciência e a Tecnologia, Portugal, through the research grant SFRH/BD/899/2000. Part of the research was done while visiting the College of William and Mary.

Our purpose here is to give simple, tight bounds for $m(T)$ in terms of vertex degrees only, and then to indicate for which trees these bounds are exact. For this purpose, let $H = H(T)$ be the subgraph of T induced by the k vertices of degree at least 3 in T . If H consists of a collection of disjoint vertices (possibly empty), we call T *segregated*; in general $e = e(H)$ is the number of edges present in H , so that $e = 0$ is equivalent to T being segregated.

2. Main results. Our principal result is Theorem 2.1 whose proof consists of the pendant lemmas. Given a graph G we denote by $\nu(G)$ the vertex set of G and by $\deg_G(v)$ the degree of a vertex v in G .

THEOREM 2.1. *Let T be a tree. Then,*

$$1 + \sum_{i=1}^k (d_i - 2) - e \leq m(T) \leq 1 + \sum_{i=1}^k (d_i - 2).$$

Equality occurs in the right hand inequality if and only if T is segregated. Equality occurs in the left hand inequality if and only if $\nu(H) = \emptyset$ or $\deg_T(v) \geq \deg_H(v) + 2$ for all vertices v of H .

As usual, given a graph G with vertex set $\nu(G)$ and $U \subset \nu(G)$, $G - U$ denotes the subgraph of G induced by the vertex set $\nu(G) \setminus U$. Denote by p_U the number of components of $G - U$ and by q_U the cardinality of U . If $U = \{u\}$ and G is a tree, the components of $G - U$ (or $G - u$) are called *branches* of G at u .

As we will see, the lower bound presented in Theorem 2.1 is $p_{\nu(H)} - q_{\nu(H)}$.

LEMMA 2.2. *Let F be a forest with c components (trees). Let F' be any subtree of F with vertex set $\nu(F') = \{u_1, \dots, u_q\}$, $q \geq 1$. Then, $F - \nu(F')$ has*

$$c + \sum_{i=1}^q [\deg_F(u_i) - \deg_{F'}(u_i)] - 1$$

components.

Proof. For each $u_i \in \nu(F')$, define $\delta(u_i) = \deg_F(u_i) - \deg_{F'}(u_i)$, the number of vertices in F , but not in F' , that are adjacent to u_i . Since F is acyclic and F' is a subtree of one component F'' of F , $\sum_{i=1}^q \delta(u_i)$ counts the (distinct) branches of F'' that are left as components when $\nu(F')$ is removed from F . Since F'' was a component of F , there are then $\sum_{i=1}^q \delta(u_i) - 1$ additional components, $c + \sum_{i=1}^q \delta(u_i) - 1$ in all, after the removal of $\nu(F')$ from F . \square

LEMMA 2.3. *Let F be a forest with c components. Consider a nonempty subset U of $\nu(F)$ and F' the subgraph of F induced by U . Assume that $U = \{u_1, \dots, u_q\}$ and F' has $e(F')$ edges. Then, the forest $F - U$ has*

$$p_U = c + \sum_{i=1}^q [\deg_F(u_i) - 1] - e(F')$$

components.

Proof. Assume that F' has s components (and hence $q - s$ edges), and that $\{u'_1, \dots, u'_k\}$ are the vertices of a component F'' of F' . Then by Lemma 2.2, the

removal of F'' from F will increase the number of components by $\left[\sum_{i=1}^k [\deg_F(u'_i) - \deg_{F''}(u'_i)] - 1\right]$. Thus the removal of all s components of F' from F will increase the number of components to

$$\begin{aligned} p_U &= c + \sum_{i=1}^q [\deg_F(u_i) - \deg_{F'}(u_i)] - s \\ &= c + \sum_{i=1}^q \deg_F(u_i) - \sum_{i=1}^q \deg_{F'}(u_i) - s \\ &= c + \sum_{i=1}^q [\deg_F(u_i) - 1] + q - 2e(F') - s. \end{aligned}$$

Since $e(F') = q - s$ the result follows. \square

By Lemma 2.3, we have $p_{\nu(H)} = 1 + \sum_{i=1}^k (d_i - 1) - e$. Since $q_{\nu(H)} = k$ it follows that

$$(2.1) \quad p_{\nu(H)} - q_{\nu(H)} = 1 + \sum_{i=1}^k (d_i - 2) - e.$$

LEMMA 2.4. *Let F be a forest with c components. Let p be the number of components when q vertices each of degree at most 2 are removed from F . Then, $p - q \leq c$. Equality occurs if and only if all the vertices removed from F have degree 2 and no two such vertices are adjacent.*

Proof. By Lemma 2.3, the removal of q vertices u_1, \dots, u_q from F increase the number of components by $\sum_{i=1}^q [\deg_F(u_i) - 1] - e(F')$ where F' is the subgraph of F induced by u_1, \dots, u_q . Since all removed vertices from F have degree at most 2, the number of remaining components is $p \leq c + q$, with equality if and only if all the vertices removed from F have degree 2 and no two such vertices are adjacent. \square

Given a tree T , let M be a subset of $\nu(T)$ such that $p_M - q_M = \max[p_I - q_I]$ with $I \subseteq \nu(T)$. From Lemma 2.4, there is no vertex of degree 1 in M . Lemma 2.4 also implies that we can assume M contains no vertices of degree 2. In other words, there is a subset M of $\nu(H)$ such that $p_M - q_M = \max[p_I - q_I]$ with $I \subseteq \nu(T)$ i.e., $p_M - q_M = m(T)$.

LEMMA 2.5. *Let T be a tree. Suppose that $\nu(H) \neq \emptyset$ and $\deg_T(v) - \deg_H(v) \geq 2$ for all vertices v of H . Then, $p_I - q_I \leq p_{\nu(H)} - q_{\nu(H)}$ for all $I \subseteq \nu(H)$.*

Proof. Suppose to the contrary, i.e., that there exist $I \subseteq \nu(H)$ such that $p_I - q_I > p_{\nu(H)} - q_{\nu(H)}$. Then, there exist I' and I'' such that $I \subseteq I' \subset I'' \subseteq \nu(H)$, with $I'' = I' \cup \{v\}$ and $v \notin I'$, such that $p_{I'} - q_{I'} > p_{I''} - q_{I''}$. Since $q_{I'} = q_{I''} - 1$, $p_{I'} - q_{I'} > p_{I''} - q_{I''}$ implies that $p_{I'} \geq p_{I''}$. But, since $T - I'$ has $p_{I'}$ components and v is a vertex of one component T' of $T - I'$ with $\deg_{T'}(v) \geq 2$ (remember that $\delta(v) \geq 2$), by Lemma 2.2 the removal of v from T' gives at least 1 additional components. Then, $p_{I''} \geq p_{I'} + 1$ gives a contradiction. \square

LEMMA 2.6. *Let T be a tree. Then, $m(T) \geq 1 + \sum_{i=1}^k (d_i - 2) - e$, with equality if and only if $\nu(H) = \emptyset$ or $\deg_T(v) \geq \deg_H(v) + 2$ for all vertices v of H .*

Proof. First, observe that if $\nu(H) = \emptyset$ i.e., T is a path, then $k = 0$, $e = 0$ and $1 + \sum_{i=1}^k (d_i - 2) - e = 1$, which is the path cover number when T is a path. Suppose now that $\nu(H) \neq \emptyset$ and assume that $\deg_T(v) \geq \deg_H(v) + 2$ for all vertices v of H . Consider a vertex set M such that $p_M - q_M = m(T)$ with $M \subseteq \nu(H)$. Since $M \subseteq \nu(H)$ and $\deg_T(v) \geq \deg_H(v) + 2$ for all vertices v of H then, from Lemma 2.5, $p_M - q_M \leq p_{\nu(H)} - q_{\nu(H)}$. Then, $m(T) = p_{\nu(H)} - q_{\nu(H)}$ and, by (2.1), $p_{\nu(H)} - q_{\nu(H)} = 1 + \sum_{i=1}^k (d_i - 2) - e$. Therefore, $m(T) = 1 + \sum_{i=1}^k (d_i - 2) - e$. Thus, if $\deg_T(v) \geq \deg_H(v) + 2$ for all vertices v of H , then equality occurs.

It suffices to prove that if there is a vertex v in $\nu(H)$ such that $\delta(v) = \deg_T(v) - \deg_H(v) \leq 1$, then $m(T) > 1 + \sum_{i=1}^k (d_i - 2) - e$. Consider the set $H^* = \nu(H) \setminus \{v\}$ of vertices of T . Then, $T - H^*$ has p_{H^*} components and $q_{H^*} = q_{\nu(H)} - 1$. Since $v \notin H^*$, v is a vertex of one component T' of $T - H^*$ and, the removal of v from T' gives $\delta(v) - 1$ “additional” components. Then, $p_{\nu(H)} = p_{H^*} + \delta(v) - 1$ and since $\delta(v) \leq 1$, $p_{H^*} \geq p_{\nu(H)}$. Thus, $p_{H^*} - q_{H^*} \geq p_{\nu(H)} - q_{H^*}$ and since $q_{H^*} = q_{\nu(H)} - 1$, we have $p_{H^*} - q_{H^*} > p_{\nu(H)} - q_{\nu(H)}$. Then, $m(T) > p_{\nu(H)} - q_{\nu(H)}$ and by (2.1), $p_{\nu(H)} - q_{\nu(H)} = 1 + \sum_{i=1}^k (d_i - 2) - e$. Therefore, $m(T) > 1 + \sum_{i=1}^k (d_i - 2) - e$. \square

LEMMA 2.7. *Let T be a tree. Then $m(T) \leq 1 + \sum_{i=1}^k (d_i - 2)$, with equality occurring if and only if T is segregated.*

Proof. If $\nu(H) = \emptyset$ then T is a path, and hence a segregated tree. In this case, $k = 0$ and, $1 + \sum_{i=1}^k (d_i - 2) = 1$, which is the path cover number when T is a path. Suppose now that $\nu(H) \neq \emptyset$. If T is segregated, for all vertices v of H , $\deg_H(v) = 0$ and $\deg_T(v) \geq 3 > \deg_H(v) + 2$. From Lemma 2.6, since $e = 0$ it follows that $m(T) = 1 + \sum_{i=1}^k (d_i - 2)$. Thus, if T is segregated, equality occurs.

It suffices to prove that if T is a tree that is not segregated then, $m(T) < 1 + \sum_{i=1}^k (d_i - 2)$. Consider a vertex set M such that $p_M - q_M = m(T)$ and $M \subseteq \nu(H)$. From Lemma 2.3, $p_M - q_M = 1 + \sum_{v \in M} [\deg_T(v) - 2] - e(M)$ where $e(M)$ is the number of edges of the subgraph of T induced by the q_M vertices in M . If $M = \nu(H)$, since T is not segregated, there are, at least, two adjacent vertices of T in M which implies $e(M) > 0$. Therefore $p_M - q_M < 1 + \sum_{v \in \nu(H)} [\deg_T(v) - 2]$ i.e., $p_M - q_M < 1 + \sum_{i=1}^k (d_i - 2)$. If $M \subset \nu(H)$, $e(M) \geq 0$ and then, $p_M - q_M \leq 1 + \sum_{v \in M} [\deg_T(v) - 2]$. Since $M \subset \nu(H)$ and $\deg_T(v) \geq 3$ for all $v \in \nu(H)$, $\sum_{v \in M} [\deg_T(v) - 2] < \sum_{v \in \nu(H)} [\deg_T(v) - 2]$, so that $p_M - q_M < 1 + \sum_{i=1}^k (d_i - 2)$. \square

The proof of Theorem 2.1 is now complete.

3. An algorithm. We close by giving a simple algorithm to compute $m(T)$ by computing the path cover number of T .

LEMMA 3.1. *Let T be a tree and let M be a subset of $\nu(H)$ such that $p_M - q_M = m(T)$. Then, there is no vertex of degree at least 3 in $T - M$.*

Proof. Let $T_{(M)} = T - M$ and suppose that $v \notin M$ and $\deg_{T_{(M)}}(v) \geq 3$. Consider the set $M' = M \cup \{v\}$ of vertices of T and the graph $T_{(M')} = T - M'$ ($= T_{(M)} - v$). From Lemma 2.2, $T_{(M')}$ has $p_{M'} = p_M + \deg_{T_{(M)}}(v) - 1$ components and $q_{M'} = q_M + 1$. Since $\deg_{T_{(M)}}(v) \geq 3$, $p_{M'} \geq p_M + 2$, and $p_{M'} - q_{M'} \geq p_M + 2 - q_{M'} = p_M - q_M + 1$. Thus, $p_{M'} - q_{M'} > p_M - q_M$ gives a contradiction. \square

LEMMA 3.2. *Let T be a tree and let M be a subset of $\nu(H)$ such that $p_M - q_M = m(T)$. If $v \in \nu(H)$ is such that $\delta(v) = \deg_T(v) - \deg_H(v) \geq 3$, then $v \in M$.*

Proof. Let $T_{(M)} = T - M$ and suppose that $v \notin M$ and $\delta(v) \geq 3$. Remember that $M \subseteq \nu(H)$ thus, $\deg_{T_{(M)}}(v) \geq \delta(v) \geq 3$. From Lemma 3.1, $\deg_{T_{(M)}}(v) \geq 3$ gives a contradiction. Therefore, $v \in M$. \square

LEMMA 3.3. *Let T be a tree and let M be a subset of $\nu(H)$ such that $p_M - q_M = m(T)$. For any $v \in \nu(H)$ such that $\delta(v) = \deg_T(v) - \deg_H(v) = 2$, there is a subset M' of $\nu(H)$ such that $M \subseteq M'$, $v \in M'$ and $p_{M'} - q_{M'} = m(T)$.*

Proof. Let $T_{(M)} = T - M$ and suppose that $v \in \nu(H)$ with $\delta(v) = 2$. If $v \in M$, let $M' = M$. Suppose now that $v \notin M$. Recall that $M \subseteq \nu(H)$; then $\deg_{T_{(M)}}(v) \geq \delta(v)$. Consider the set $M' = M \cup \{v\}$ and graph $T_{(M')} = T - M'$ ($= T_{(M)} - v$). From Lemma 2.2, $T_{(M')}$ has $p_{M'} = p_M + \deg_{T_{(M)}}(v) - 1$ components and $q_{M'} = q_M + 1$. Then, $p_{M'} - q_{M'} = p_M - q_M + \deg_{T_{(M)}}(v) - 2$.

By Lemma 3.1, $\deg_{T_{(M)}}(v) < 3$, hence $\deg_{T_{(M)}}(v) = 2$ in which case $p_{M'} - q_{M'} = p_M - q_M$. Therefore, $v \in M'$ and $p_{M'} - q_{M'} = m(T)$. \square

We note that Lemmas 3.2 and 3.3 indicate how to inductively compute a subset M' of $\nu(H)$ such that $p_{M'} - q_{M'} = m(T)$. By Lemma 2.4, given any tree T , there is a subset M of $\nu(H)$ such that $p_M - q_M = m(T)$ and, by Lemmas 3.2 and 3.3, there is always a subset M' of $\nu(H)$, containing all vertices v of T with $\delta(v) \geq 2$, such that $M \subseteq M'$ and $p_{M'} - q_{M'} = m(T)$.

ALGORITHM 3.4. *Given a tree T , consider the subgraph H of T induced by the vertices of degree at least 3. Remove from T all vertices v of H such that $\delta(v) \geq 2$.*

This algorithm is applied to the initial tree T and then to each of the resulting components and so on. Let M' be the set of vertices removed via repeated application of Algorithm 3.4. By Lemmas 3.2 and 3.3, all vertices v of T with $\delta(v) \geq 2$ are in M' . If there is a vertex of degree at least 3 in $T - M'$, again by Lemmas 3.2 and 3.3, the application of the Algorithm 3.4 to each of the components of $T - M'$ (each one is a tree) allows us to include new vertices in M' in route to maximizing $p_{M'} - q_{M'}$. By Lemma 3.1, the process stops when there is no vertex of degree at least 3. The set of removed vertices from T gives M' , the cardinality of M' gives $q_{M'}$ and the number of components of $T - M'$ (each of which is a path) gives $p_{M'}$. Finally, the maximum multiplicity of an eigenvalue among matrices whose graph is T , $m(T)$, is equal to $p_{M'} - q_{M'}$.

REFERENCES

- [1] R. Horn and C. R. Johnson. *Matrix Analysis*. Cambridge University Press, New York, 1985.
- [2] A. Leal-Duarte and C. R. Johnson. The Maximum Multiplicity of an Eigenvalue in a Matrix Whose Graph is a Tree. *Linear and Multilinear Algebra*, 46:139–144, 1999.
- [3] A. Leal-Duarte and C. R. Johnson. On the Minimum Number of Distinct Eigenvalues for a Symmetric Matrix Whose Graph is a Given Tree. *Mathematical Inequalities and Applications*, To appear.