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ACTIVE PORTFOLIO MANAGEMENT USING THE BLACK-LITTERMAN MODEL

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Abstract

In this project, we are studying the application of the Black-Litterman model for active portfolio management. We focus on three elements that are important for a real-world application of the model. First, we present a robust construction of the variance-covariance matrix that is used as input for the model. Second, we present a rigorous explanation of the model as it was presented initially by Black and Litterman, which is important to highlight the driving mechanisms inside the model. Third, we describe how to adjust the model to an application in an active setting, where the investor is evaluated relative to a benchmark. Empirical evidence for the active case is limited to date. Working with a range of benchmarks (relevant for various types of investors), we derive a portfolio that includes multiple asset classes and present the results obtained under the application of the model. Finally, a simulation exercise was developed to understand how the model reacts to its inputs.

Key Words: Covariance; Implied returns; views; confidence

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Section 1. Introduction

How to allocate one's wealth in a variety of assets and portfolio optimization are topics widely discussed in literature and crucial for investment banks. The major drawback of the Markowitz optimization (developed in 1952) is that it can lead to corner solutions, which is highly undesirable for portfolio managers. They develop accurate estimations of expected returns to aim for a more diversified portfolio. Thus, this optimization method is not often used in practice. Black and Litterman (1992) introduced the Black-Litterman (BL) asset allocation model, which has the advantage of considering the forward-looking investor's views for the estimation of the expected excess returns. The views are investors' opinions on how the market or specific assets will perform in the future. In addition to the opinions regarding the market, investors need to specify their confidence about them; this confidence level will be vital to understand how much the final portfolio weights will tilt away from the market portfolio. All in all, the main idea of this model is to start the portfolio construction with a neutral reference point which is the market equilibrium, that will then suffer deviations due to the specific investors' views and their respective confidence. Black and Litterman argue that their model leads to a more diversified portfolio.

However, applying this model in the active portfolio management, where one evaluates the weights' deviations relative to a given benchmark portfolio, comes along with some problems. The initial BL model aims to maximize the Sharpe ratio of a portfolio-average return earned in excess of the risk-free rate per unit of volatility or total risk. On the contrary, under an active management environment, it is the investor's intention to maximize the information ratio, which involves to maximize the portfolio's alpha (active return) per unit of tracking error. As a result, if the benchmark portfolio is not chosen to be the Global Minimum Variance portfolio, the portfolio will move away from the benchmark even if no views are specified. This is due to the mismatch of the defined objective functions defined in the original BL model and in the active management framework, that leads the investor to believe that he/she can take advantage of alpha opportunities that in fact do not exist.

The goal of the project is to implement the BL framework in a portfolio of indexes given by Caixagest with a major focus on the active portfolio management. Caixagest is the asset management subsidiary of the group Caixa General Depósitos (CGD) and is specialized in the mutual fund management and portfolio fund management, having a huge portfolio of different funds diversified by global financial markets. In this Work Project, we present several ways of implementing the model under analysis always followed by an explanation of the results based on portfolio statistics. In the end, Caixagest should have an essential basis for deciding if this asset allocation method is an appropriate one to suggest to its investors and under which hypothesis it wants to work with. Moreover, Caixagest should be able to apply the BL model for every type of investor as, in the active framework, different benchmarks that mirror different risk appetites (defensive, balanced and dynamic) are considered. It is crucial for Caixagest to understand how the views impact the different risk profiles in the active management framework, so that they can give the most accurate advice to each type of investor.

It is important to mention that this project is supported and has it basis in many academic papers, which constituted a great help in order to implement the model in the portfolio of indexes that we are working with. In a first step, this paper focus on the estimation methods of the variance-covariance matrix since this is an important input of the BL model and in portfolio theory in general. Second, we introduce the initial BL model as it was developed by Black and Litterman with a detailed explanation of all it inputs. Moreover, we implement the BL model for an active portfolio management environment. An exhausting analysis of the results implying the BL method is followed. Our Work Project also introduces a simulation exercise in which the goal is to understand how the model, particularly the weights allocation, reacts when the inputs change and some constraints are added to the initial setting. In this simulation, we focused on the analysis of the reaction of the active and, consequently, portfolio weights to different types of returns- the first ones result from the implementation of the initial BL framework and the second ones from considering the active reformulation of the BL returns. Which model is better to work with and gives more accurate results when compared with the initial unconstrained BL framework is not trivial, in the sense that both cases have pros and cons. One can point out that using the initial BL returns leads to a higher active return for the types of investor analysed. However, when considering the more constrained portfolio, the correlation between the active weights and the returns of the assets included in the views is negative, which goes against the fact that if an investor specifies a positive view in a certain asset, the correspondent active weight should move in the same direction. On the other hand, a positive correlation between the weights previously mentioned is positive when one considers the reformulation of the returns that will be presented in the BL framework. Furthermore, it is important to notice that when doing a comparison between the final weights of the portfolio and the tangency, the correlation is higher when one considers the initial BL returns, which allows to have a notion of if the portfolio obtained is moving towards the tangency.

We conclude our report with further research suggestions.

Section 2. Theoretical context

As referred in section 1, the aim of this project is to apply the active BL model to a real portfolio of assets that was given to us by the company. In order to fulfill this purpose, we closely followed the work that was performed by some investment banks that are already using the model to allocate weights to the assets included in a portfolio. The project was also supported and has it basis in many academic papers that study, for instance, the implications, advantages and disadvantages of using this model under active management. This project came associated with some business challenges in the sense that we needed to adapt the initial model developed by Black and Litterman in the early 1990s for the type of Caixagest's investors and to its investment strategy. Thus, the main goal of using this model was to obtain a more diversified portfolio with not much concentration in just a few assets, allowing to a higher diversification of risk. As a result, we needed to reformulate the original framework in which the model was built and to study the inclusion of some constraints on it, such as a risk target per type of investor or account for the fact that no short selling is allowed. Finally, another challenge that we needed to face was based on the difficulty of building some of the inputs needed to implement the model, an issue discussed in existing literatures.

Section 3. Literature review

Starting by the estimation of the variance-covariance matrix, our project has as support and is closely related with some papers that analyse the difference between the diverse methods that can be used in order to obtain a robust estimation of this important input to the BL model. Furthermore, some papers that specifically discuss the three methods that will be studied and implemented in this project were also considered, namely the historical method, the Exponentially Weighted Moving Average (EWMA) and the Generalized Auto Regressive Conditional Heteroskedasticity (GARCH) method.

Some basic properties of covariances, variances and correlations are described by Carol Alexander (2007). He presents the different methods of calculating variances and covariances, namely the EWMA and the equally weighted average methods, explaining all the different inputs that each model requires; the GARCH model is briefly mentioned as well. In the paper, he argues about the forecast (predictive) power of each model and presents the advantages and drawbacks of each one. The criticisms go mainly to the historical formula, also called the equally weighted average method, which is presented with many pitfalls, such as the fact that extreme events are just as important to current estimates whether they occurred yesterday or very far in the past. All in all, the paper suggests that the historical method is obsolete and is not appropriate anymore to model volatility, since other methods such as the EWMA and the GARCH method are available.

Laplante, Desrochers and Préfontaine (2008) provide research not only on the GARCH model but also on other methods that serve as risk predictors in financial applications. The authors put forward the importance of having good estimates of the variances and covariances for financial portfolios: according to them, using forecasts that do not provide a reasonable estimate of variances and covariances close to their actual value can result in a wrong portfolio mix. In the paper, they first examine and compare three different variance and covariance estimation methods, namely the JP Morgan's EWMA, the historical mean, the random walk and a GARCH (1,1). Then, the authors analyse and measure the effectiveness of all methods in forecasting the standard deviation of a minimum variance portfolio using several econometric statistics, such as the root-mean square error, mean absolute error and mean absolute percentage error. Their results provide confirmatory evidence that both the GARCH (1,1) and the EWMA preform way better than the historical estimate. Their findings also suggest that the GARCH is the best method to estimate the variances and covariances as all the indicators computed based on this model are the lowest ones. These results were supported by many authors, such as Brailsford and Faff (1996) and West and Cho (1997). Nevertheless, Alexander and Leigh (1997) studied three models in order to forecast the covariance matrix, and concluded that the EWMA preforms better than the GARCH model. Thus, the literature shows no consensus on which method to systematically use to model volatilities.

When one mentions the GARCH model, it is important to refer to Engle (2001) who is prominent in the literature on modelling variances for financial applications. His paper focuses on the Autoregressive Conditional Heteroskedasticity (ARCH) and GARCH models. The need to use a GARCH model comes from the presence of a "volatility clustering" and high degree of autocorrelation in the "riskiness of financial returns". In the paper, all the inputs of the GARCH are described when modelling the variance and the implementation of a GARCH (1,1) is explained as well. The author encouraged other ones as Bollerslev (1986), who only focuses on the GARCH model and goes beyond a simple explanation of the model by developing it in detail. The author finally gives an empirical example, related to inflation rate uncertainty, very helpful to fully understand the dynamics of the GARCH model. As one can see, there is ample support for the claim that a GARCH (1,1) is very capable to model volatility.

It is very important to have a robust variance covariance matrix to input in the BL model, which is an asset allocation method used to overcome the problem of unintuitive, highly-concentrated and input-sensitive portfolios. Input sensitivity is a well-documented problem with mean-variance optimization and is the most likely reason that more portfolio managers do not use the Markowitz optimization, in which return is maximized for a given level of risk.

In addition to describing the intuition behind the BL Model, the paper of Idzorek (2005) consolidates critical insights contained in the various works on the BL model and focuses on the details of actually combining market equilibrium expected returns with "investor views" to generate a new vector of expected returns. At this time, relative few works on the model were published and thus, the aim was to provide instructions by describing the model and the process of building the required inputs in very detail, which enables the reader to implement this complex model. Furthermore, a new method for determining Ω was introduced: it allows the specification of the investor view confidence as a percentage, where the model will back out the proper value of Ω to reach the confidence level specified.

In their paper, Benzschawel and Su (2015) focus mainly on asset allocation across major fixed income sectors. The paper points out the major shortcomings of mean-variance optimization and thus, explains the benefits of extending vesting duration, investing in a diversified portfolio of asset classes, and optimization methods. The authors explain deeply the mean-variance optimization because it comprises the final stage of the BL model. Furthermore, they allocate a great part of the paper to the description of the inputs used in the model and various equivalent formulations and their implications. Besides the focus on the mechanics of inputting views into the model and controlling the relationship between investors' views and market returns, the authors provide a step-by-step formulation of the model, resolve ambiguities and use examples to demonstrate the effects of parameter choices on resulting optimizations. Finally, they compare performance of model-based estimates of sector returns, the effects of risk and equilibrium parameters and how investors' view uncertainty affects allocations.

Walters (2014) focus on the canonical model including full derivations from the underlying principles using both Theil's Mixed Estimation model and Bayes Theory and a full description of this model. The author describes the fact that the BL model uses a Bayesian approach to combine the investor's views with the market equilibrium vector of expected returns to form a new, mixed estimate of expected returns. He also describes the theory behind the BL model and its inputs including all the various formulas.

Since its publication in 1990, the BL asset allocation model has gained wide application in many financial institutions as it provides the flexibility to combine the market equilibrium with additional market views provided by the investor. Thus, the user inputs any number of views and statements about the expected returns of arbitrary portfolios, and the model combines those views with equilibrium, producing a set of expected returns for the assets that are included in the portfolio and the resulting portfolio weights. As the investors usually want to beat a specific benchmark, the initial model was also extended to an active setting by some practitioners.

In the article developed by the Goldman Sachs' Investment Management Research Team, He and Litterman (1999) present examples to illustrate the difference between the traditional mean-variance optimization process (which can produce results that are extreme and not intuitive) and the BL one. Moreover, they show that the optimal portfolios that result from the BL framework have intuitive properties, such as the fact that the unconstrained optimal portfolio is the market equilibrium portfolio plus a weighted sum of portfolios representing an investor's views. In addition, the weight on a portfolio representing a view is positive when the view is more bullish than the one implied by the equilibrium and other views, being this weight higher as the investor becomes more bullish and confident in the view. This article is also meaningful in the sense that it tests the inclusion of constraints in the optimal portfolio: a risk constraint (the investor aims to maximize the expected return while keeping the volatility of the portfolio under a given level), a budget constraint (forcing the sum of the total portfolio weights to be one) and a beta constraint (forces the beta of the portfolio with respect to the market portfolio to be one).

In the original BL paper, the asset universe to be optimized was constituted of equity in a framework that did not include active management. However, it can be extended to a wider class of assets and to an active management setting; that is what Maggiar (2009) studies in his thesis. Thus, he focusses on the study of the application of the BL model to an active fixed-income portfolio[1] management. Maggiar presents a derivation of the model in a general setting and then moves on to apply it to active management, in which one desires to maximize the return of a portfolio with respect to a benchmark. Furthermore, this paper discusses the advantages of the BL model as being a great tool in allocating the portfolio's weights and an improvement over the classical mean-variance optimization process developed by Markowitz. In particular, Maggiar points out that the highest advantages of this model are the fact that it allocates risk according to the managers' confidence in their views and offers a great flexibility in the way in which they can be expressed. However, despite its many qualities and advantages

[1] Multi-currency fixed-income portfolio comprising bonds, inflation-linked bonds and currencies

over the allocation methods, the model still suffers from some constraints, namely the assumptions on the distribution of the returns, the parameters that one needs to specify and the imposed linearity of the expression of the views are all factors which limit its optimal use. Maggiar also discusses one of the crucial points of the BL model- the covariance matrix, as previously mentioned- by presenting a "state-dependent model to try to more accurately capture the expected covariance matrix". Finally, this paper adds to the previous literature the introduction of some risk management tools to assess the benefits of using the BL model in an active management setting, by adapting the method developed by Fusai and Meucci (2003).

Moreover, Silva, Lee and Pornrojnangkool (2009) give special attention to the fact that the straight application of the BL framework in active investment management can lead to unintended trades and risk taking, which leads to a riskier portfolio than desired. Thus, focusing on the mismatch between the Sharpe ratio (SR) optimization used in the initial BL framework and the information ratio (IR) optimization that constitutes the goal of the active management, they propose a solution to fix this problem that leads to portable alpha implementation of active portfolios. These resulting active portfolios reflect the real investment insights given by the investors. Furthermore, after explaining the pitfalls of the traditional Minimum-Variance optimization and providing a review of the BL framework, both theoretical and empirical results are provided in order to show how one can go from the original BL setting to the one that includes active management.

Section 4. Data

4.1. Data treatment

At the very beginning, Caixagest team gave us a broad list of assets (Indices divided by asset class) that are used on clients' portfolio allocation and were previously used by the Caixagest team to perform some studies on the BL model. This broad list of assets includes US stock indices, international indices, fixed income instruments from all over the world and an index that tracks commodities price evolution. The list assets whose allocations we are studying in this project are displayed in the following figure:

Figure 1: Assets included in the portfolio and respective description

Asset	Ticker	Description
MSCI North America	GDDUNA Index	MSCI[2] daily total return gross North America (USD)
MSCI Japan	GDDLJN Index	MSCI daily total return gross Japan (local currency)
MSCI Euro	GDDLEURO Index	MSCI daily total return gross Euro
MSCI UK	GDDLUK Index	MSCI daily total return gross UK (local currency)
MSCI Swiss	GDDLSZ Index	MSCI daily total return gross Switzerland (local currency)
MSCI Australia	GDDLAS Index	MSCI daily total return gross Australia (local currency)
MSCI Hong Kong	GDDLHK Index	MSCI daily total return gross Hong Kong (local currency)
MSCI EM Asia	MSDEMASG Index	MSCI emerging markets Asia daily gross total return (EUR)
MSCI EM Latin America	MSDEMLAG Index	MSCI emerging markets Latin America daily gross total
MSCI EM EMEA	MSDEEMEG Index	MSCI daily gross total return emerging markets in Europe & Middle East & Africa (EUR)
Treasury Value U.S.	LUATTREH Index	Barclays US aggregate total treasury value hedged (EUR)

[2] A MSCI index is a measurement of stock market performance in a particular area. It tracks the performance of the stocks included in the index. MSCI stands for Morgan Stanley Capital International

Corporate investment U.S.	LUACTREH Index	The corporate bond index measures the investment grade fixed rate and taxable corporate bond market; it includes USD denominated securities publicly issued by US and Non-US industrial
JPM Euro	JNEUI1R3 Index	European sovereign debt up to 3 years (short maturities)
Euro-Aggregate Treasury	LEATTREU Index	Barclays euro treasury index an index that contains 13 Euro countries (all maturities)
Euro-floating rate	LEF1TREU Index	Benchmark that measures the investment grade, fixed rate bond market, including treasuries, governments (EUR)
Euro Aggregate Corporate	LECPTREU Index	Barclays Euro aggregate: Corporate bonds index is a rules based benchmark measuring investment grade, EUR denominated, fixed rate, and corporate only. Only bonds with a maturity of 1 year and above are eligible
EM hard currency aggregate	BEHSTREH Index	This index is a flagship hard currency Emerging Markets debt benchmark that includes fixed and floating-rate US dollar-denominated debt issued from sovereign and corporate EM issuer
EM currency government total return	EMLCTREU Index	Barclays Emerging Markets local currency government total return index value unhedged (EUR)
US Corporate Yield	LF98TRUU Index	Barclays high yield (fixed rate corporate bond market): middle rated bonds by Moody's, Fitch and S&P (USD Denominated)
European high yield	LP01TREH Index	Barclays Pan-European high yield index measures the market of non-investment grade, fixed rate, corporate

		bonds (inclusion is based on the currency of issuer (EUR))
US TIPS Index	JPILUILA Index	The index includes all the publicly issued US Treasury inflation protected securities that have at least one year remaining to maturity and are rated investment grade
Euro linker securities	JPILEILA Index	It is an inflation-linked bond index (bonds where the principal is indexed to inflation)
Bloomberg commodities index EU	BCOMEU Index	The index is composed of futures contracts on 22 physical commodities; it reflects the return of underlying commodity futures price movements only and is quoted in EUR

For each class of assets, price indices were downloaded from Bloomberg terminal. They contain the last price or closing price at which the index is traded on a given trading day. Special attention was given to the treatment of equities' data as the equities' price indices can have two variants: gross and net. Gross indices do not include any tax effect, while net indices account for tax deduction. To decide which one to use, we studied the correlations between the gross and net price indices and concluded that they are highly correlated meaning that using one or another would be indifferent and would not bias the model. The fact that the gross price indices had more data for all assets than the net price indices, lead us to choose to work with the first ones. As for fixed income and commodities' data, special attention was not needed as we only had one index per asset in this two classes of assets.

When it came to decide which risk-free rate to use, many options were available: in most literature, it is common practise to input treasury bills as the risk-free asset, however investors prefer to use a broad range of money market instruments as a risk-free rate. As Caixagest works with European investors, Euribor rates were the most plausible option to use in the BL model as a proxy for the risk-fee rate. Having made this choice, we downloaded the EUCV01W index[3] data from Bloomberg terminal, aligning the time frame with the one of the assets used in the portfolio and imposing the same periodicity. It is important to make clear that, to back out the risk-free rates, we had to carefully treat the data, as this index gives a yield to maturity of a zero coupon bond (ZCB). Using this yield, we backed out the ZCB's price, and used the last week bond prices to obtain the risk-free rate observed today. Having a weekly risk-free rate supposes that investors re-invest the risk-free rate weekly. It is also important to notice that having an interest rate of such short maturity, interest rate risk or default risk are negligible.

The time frame chosen was from the 5th December 2009 to 2nd September 2016. It is very important to mention that it was chosen in function of the missing data; as a matter of fact, only this time frame had all the gross prices for all the assets. As for the data periodicity, weekly data was chosen in detriment of daily data, as the later contains a lot of random fluctuations and noise. Furthermore, monthly data was not considered as the time frame was very short, and using it could bring inaccuracy to our model as the data to work with would decrease. All in all, using weekly data seemed appropriate to compensate for the short time frame, to have more data to work with and to avoid noise and make the trend clearer.

4.2. Returns and descriptive statistics

Theoretically, a random variable *x,* is lognormal if its logarithm *ln(x)* is normally distributed. In the financial world, most academics and practitioners agree in assuming that the stock prices follow a lognormal distribution, therefore is common to use a log transformation to compute the stocks' returns. Making this transformation results in approximately normal returns, easier to work with. Nevertheless, it is important to keep in mind that historical returns on stocks

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[3] Euribor index, giving a yield to maturity of zero coupon bonds, with a weekly periodicity

empirically exhibit more frequent large negative deviations from the mean than it would be predicted from a normal distribution, and the actual tails of returns' distribution are fatter. Thus, we computed weekly returns for each single asset using the log transformation and also the weekly excess returns using the risk-free rate previously mentioned. In addition, cumulative returns by asset class are displayed in graph 1 and some descriptive statistics, presented in figure 2, were computed, such as the average returns, standard deviation and SR (weekly and annual), kurtosis and skewness.

Section 5. The Variance-Covariance Matrix: An input to the Black-Litterman model

5.1. The importance of the covariance Matrix

The covariance matrix is used as an input for the BL model, and at a later stage as an input for the Mean-Variance optimization process. A variance-covariance matrix is a square matrix that contains the variances and covariances associated with several variables (in this case 23 assets). The diagonal elements of the matrix contain the variances of the assets and the off-diagonal elements contain the covariance between all possible pairs of the 23 assets. It is important to underline that the variance-covariance matrix is symmetric because the covariance between asset 1 and 2, for instance, is the same as the covariance between asset 2 and 1. Therefore the covariance for each possible pair of assets is displayed twice in the matrix.

At this stage, it is relevant to highlight the key role that this matrix plays on some financial applications, especially on the portfolio theory. When talking about portfolio theory, it is necessary to talk about diversification, where the covariance matrix appears to be vital as it studies the co-movements between assets and helps the investor to choose how much of an asset should he/she buy/sell relatively to another, so that the portfolio can be diversified. A positive covariance means that assets generally move in the same direction and a negative one means that assets move in opposite directions. When constructing the portfolio, it is critical to attempt to reduce its overall risk by including assets that are negatively correlated with each other.

As we can see, the variance-covariance matrix is an extreme important input for the BL model and, because it may improve the accuracy of the model, three different estimations were performed to then decide which one to input in the model. First, we estimated the covariance matrix with historical data, second, we used an EWMA covariance matrix and finally, we computed it using a GARCH estimation.

5.2. Methods of estimation of the Covariance matrix

5.2.1. Historical Method

Estimating the covariance matrix with historical data is essentially the simplest solution. It is a method that uses the joint history of all the 23 assets beginning on the 25th December of 2009 until 2nd September 2016. It is relevant to mention that the historical covariance estimation was a two-step process: before calculating the covariance, we calculated the diagonal elements of the variance-covariance matrix. Theoretically, the diagonal elements, as previously mentioned, correspond to the variance of each asset. The variance, which is a measure of variability of the set of data, mathematically speaking is the average squared deviation from the mean of the data set. The formula used to compute those diagonal elements of the covariance matrix is the following:

$$\sigma_x^2 = \frac{\sum_{i=1}^{n}(\mu_{xi} - E(\mu))^2}{n}, \quad (1)$$

where σ_x^2 is the historical variance of asset x, μ_{xi} is the return of the asset at time i, $E(\mu)$ is the average of the asset returns and n corresponds to the number of observations.

As the covariance studies the co-movements between the different pairs of assets, the following formula was used in its computation:

$$\sigma_{xy} = \frac{1}{n}\sum_{i=1}^{n}(x_i - E(x))(y_i - E(y)), \quad (2)$$

where σ_{xy} is the covariance between assets *x* and *y*, x_i is the return of asset *x* at time i, $E(x)$ is the average of the asset *x* returns, y_i is the return of asset y at time i, $E(y)$ is the average of the asset *y* returns and n is, once again, the number of observations (number of weeks).

It is important to highlight that this method assumes that the same relationship between assets will continue in the future, which in practice would not be the case; this represents a big disadvantage of this method and can possibly bias the model. Also, due to the fact that there is some time dependency present in the data, the historical estimation brings another big disadvantage as it assigns equal weights for each return when computing the variances and covariances. Treating past data, without adding the effect of time dependency, is limited when inferring volatilities and correlations. Nevertheless, the historical estimation of the variance-covariance matrix is useful for benchmark and comparing purposes.

Moving to our specific data and assets included in the portfolio, in the excel file, we computed both the variance and the covariance using equations (1) and (2), respectively. For the sake of displaying more information to the user, in addition to the covariance matrix estimate, we decided to separate the variances and the correlations between assets in two different matrices, so that the co-movements between assets could be seen at first glance. We computed a weekly and annual covariance matrix assuming 52 weeks in one year.

To fill the gaps that the previous estimation has, we decided to compute variances and covariances with the EWMA method, to then compare its effectiveness to the historical method.

5.2.2. The exponentially weighted moving average

The EWMA method improves the simple variance calculated based on the historical method, where basically the yesterday's returns have the same importance on the variance of the asset as the return observed two months ago. As a matter of fact, the EWMA estimation assigns more weight to the most recent returns, as to reflect some time-dependency of the data. For instance, when computing the variances, the most recent returns have more importance than the returns observed in the past. Brooks (2014) underlines the two main advantages of the EWMA estimation method compared to the historical volatility estimation: the first is that "*Volatility will be more affected by recent events, which have more weight assigned, than by past events"*, and secondly, "*the effect on volatility of a single given event declines at an exponential rate as weights attached to recent events fall*".

To better understand this estimation method, it is important to decompose the EWMA formulas - both the variance and covariance. Thus, using this estimation method, the variance can be obtained with the following equation:

$$\sigma_{i,t}^2 = \sigma_{i,t-1}^2 . \lambda + (1-\lambda) . r_{i,t-1}^2, \quad (3)$$

where $\sigma_{i,t}^2$ is the today's variance, $\sigma_{i,t-1}^2$ is the last week's variance, $r_{i,t-1}^2$ is the lagged squared returns (last week's return) and λ is the decay factor.

Secondly, the covariance can be obtained with

$$\sigma_{ij,t} = (1-\lambda) . r_{i,t-1} . r_{j,t-1} + \lambda . \sigma_{ij,t-1}, \quad (4)$$

where $\sigma_{ij,t}$ is the covariance at time t between asset *i* and *j* (today's covariance), $r_{i,t-1}$ is the lagged return of asset *i* (last week return), $r_{j,t-1}$ is the lagged return of asset *j* (last week return) and $\sigma_{ij,t-1}$ is the lagged covariance between asset *i* and *j* (last week's variance).

As one can observe in both formulas, the variance and covariance are a function of the lagged variance (covariance) and the lagged returns. In addition, this model depends heavily on a parameter usually called the decay factor. Depending on the value assigned to this decay factor, recent observations will become more or less heavily weighted. Going deeper into the EWMA model, one can basically decompose it in two different terms. The first one, $\sigma_{i,t-1}^2.\lambda$, that determines the persistence in the volatility; for instance, if the volatility was high yesterday it will still be high today, meaning that the last observed volatility will have an effect on the next volatility to be observed. The closer λ is to 1, the more persistent volatility will be. The second term, $(1-\lambda)\ r_{i,t-1}^2$, determines the intensity of reaction to market events - the bigger λ, the less the volatility reacts to market information contained in yesterday's return[4]. It is important to mention that the same reasoning applies to the covariance formula. Because the decay factor used in many academic papers is 0.94, and Brooks (2014) also recommends it, we decided to use the same value for the parameter. Before describing our procedure, it is relevant to understand the major pitfall of this EWMA model: as mentioned by Carol Alexander (2007), when using EWMA to estimate variances and covariances, one uses the same value of lambda for all the variances and covariances of assets that belong to different asset classes. This means that we apply the same lambda for equity, fixed income and commodities indices. In practice, the different asset classes do not have the same reactions to shocks on returns, for instance, neither the same volatility persistence. Nevertheless, as also mentioned by the same author, the same lambda for all indices should be used to guarantee the matrix to be positive semidefinite. In our excel file, instead of using the squared returns, we used the squared of the difference between the returns and the average of the returns[5]. Having all the necessary data, we computed the weekly variances for each asset recursively using equation (3) and subsequently, we computed the weekly covariances using equation (4) for each possible pair of assets.

[4] Carol Alexander (2007)

[5] The goal was to correct the estimation of the variance-covariance matrix for the cases that the mean of the return of the asset was not exactly zero

In many papers volatility is effectively modelled by a GARCH model. For instance, Jingjing Bai (2011) mentions in his work that this model is "widely acknowledge to be able to generate more realistic long term forecasts than exponentially weighted moving averages". Adding a GARCH estimation to compare it with the previous variance estimations would only be beneficial to completely be sure about the best covariance matrix to input in the Black-Litterman model. Nevertheless, it is important to bear in mind that due to the short sample, it is not obvious to get a robust variance covariance matrix by the GARCH method.

5.2.3. GARCH

The GARCH process is an econometric model widely used to estimate volatility. Being an autoregressive process, the GARCH (1,1) will make the conditional variance depend, in our case, on last week's variance (1) and on last week's square returns (1). As mentioned by Brooks (2014), the GARCH (1,1) model will be sufficient to capture the volatility of the data. Moreover, in most academic papers, it is rarely extended to model volatility; therefore, it was appropriate to use a GARCH of type (1,1) to model volatility. To have a deeper understanding of the GARCH model, one must look to the formula:

$$\sigma_{i,t}^2 = \gamma V_L + \alpha\mu_{i,t-1}^2 + \beta\sigma_{i,t-1}^2 \quad (5)$$

where $\sigma_{i,t}^2$ is the today's variance, V_L is the long Run Variance, γ is the parameter (weight) associated to the long run variance, $\mu_{i,t-1}^2$ is the lagged squared returns (last week's return), α is the parameter (weight) associated to the lagged squared returns, $\sigma_{i,t-1}^2$ is the lagged variance (last week's variance) and β is the parameter associated to the lagged variance.

As one can see, the GARCH equation contains **three different components** and each one has a different parameter (weights) assigned to it. The first component is the long run variance and represents the essential difference between the GARCH and the EWMA. By inputting this parameter, one expects the variance to be pulled towards the long run variance. Furthermore, one can see a parameter associated with the lagged squared return and another one associated with the lagged variance.

For the practical estimation of the GARCH (1,1) model several steps are required. As referred by *Jingjing Bai (2011)*, the GARCH model can be implemented via the maximum likelihood method. As a matter of fact, the first step of this process is to estimate the different parameters by maximizing a log-likelihood function equal to the one depicted below.

$$log\ L(\gamma;\delta) = -\frac{1}{2}\left(T.log(2\pi) + \sum_{i=1}^{T}(\log(h_t) + \frac{r_t^2}{h_t})\right), \quad (6)$$

where h_t is the conditional variance, r_t^2 is the return square and T is the number of observations. Before discussing its application, it is important to mention the requirements to apply the log-likelihood function (LLF). For the LLF to work, assuming normality is vital, and besides that, the residuals must be white-noise. The log-returns can be negatively skewed and demonstrate some excess kurtosis (characteristics that should also be observed in log returns), and still be well illustrated by a GARCH. Having all the requirements filled, all the necessary computations could be done. After obtaining a value for the density function, one must maximize it by changing the three different parameters (weights) - gamma, alpha and beta (from the GARCH model) - subject to the following constraint: $\alpha + \beta < 1$ (7). Finally, one can compute the covariance matrix.

Moving to the excel file, we followed the steps described above**.** We performed some tests on the residuals to check if they were in line with the requirements needed to apply the density function. Thus, we computed the returns' residuals, the standardized residuals and the necessary statistics (Kurtosis, Skewness, Average[6]). Finally, we studied the serial correlation of the errors, using the NumXL tool, which showed that the errors were in fact a white-noise process.

[6] The average of the residuals was zero for all of the assets

After being sure that the density function could be used, we calculated a value for the LLF for each asset, using equation (6). Having the correspondent LLF value for each asset, we maximized them by changing the value of the parameters gamma, alpha and beta, that were used to calculate the conditional variance (equation (5)). It is important not to forget to constrain the maximization problem with the constraint given in formula (7). After having all the parameters estimated and the right values for the conditional variances, we calculated the covariances using the EWMA correlation matrix computed previously, as calculating covariances through a GARCH process requires very heavy and complex computations[7].

5.3. Invertibility

Not much attention is usually given to matrix invertibility, but it is highly important to explain it in detail. Invertibility is one of the main characteristics needed in the variance-covariance matrix for portfolio allocation models. For instance, to compute BL returns, an inverted covariance matrix is needed. For the matrix to be invertible, some rules must be respected, namely the matrix has to be squared, positive semi-definite and non-singular[8].

By nature, a covariance matrix is squared (23 x 23), therefore this requirement was filled. A matrix is positive semi-definite if and only if $x^t A x \geq 0$, which was verified in all the matrices estimates. It is important to note that in the case of the covariance matrix, as long as the assets have non-zero variance, the matrix will be positive definite[9]: all the assets' variances were non-zero, which made the matrix positive definite and, therefore, invertible. Nevertheless, to be sure about the invertibility, we calculated the determinant of each matrix estimate and they were

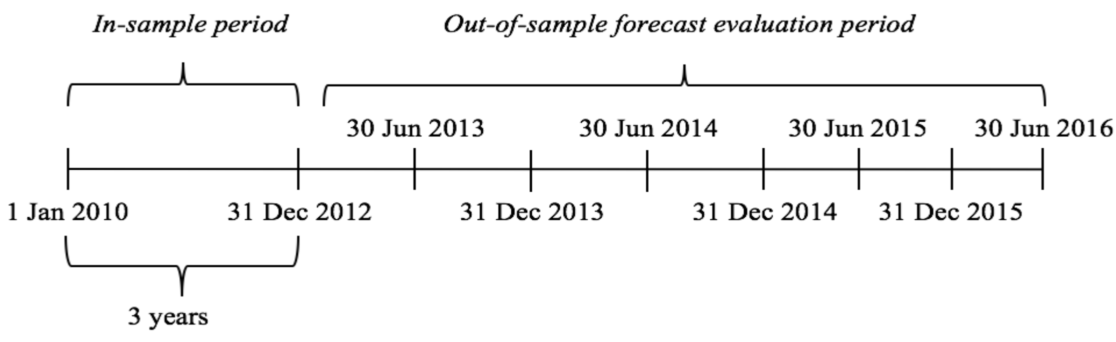
[7] As a matter of fact in most literature a bivariate GARCH or a DCC-GARCH are needed to compute the covariances of each pair of assets
[8] A square matrix is non-singular if and only if its determinant is non-zero
[9] F.Bouttier (2003)

non-zero determinants, which confirmed that the covariance matrices were invertible.

The purpose of modelling volatility with three different methods is to understand what model gives a more accurate volatility prediction. Understanding the forecast ability of each method is vital to know which covariance estimate to input in the model.

5.4. Analysis of the Covariance estimations

Back tests for each of the three different methods of estimating volatilities were necessary to assess the forecast ability of each method. To conduct those back tests, we decided to use an in-sample analysis. In-sample analysis means that one estimates the volatility using all available data, and then compares the fitted values with the actual realizations. To proceed, one must split the data series into the in-sample estimation period used for the initial parameter estimation, and an out-of-sample forecast evaluation period used to evaluate the prediction/forecasting performance. Theoretically, in-sample refers to the data that one has and out-of-sample the data that one does not have and wants to forecast. In our work, we divided the data as showed below:



It is relevant to mention that two different variance back tests were computed. First we have a non-overlapping back test and second, we computed an overlapping back test. In the following paragraphs, both procedures will be described and the indicators chosen to evaluate the performance of the three different estimation methods will be explained as well.

5.4.1. Non-overlapping analyses of the estimations

In this first analysis that was performed, the goal was to obtain the errors of the difference between the verified volatility in the market and the one that is obtained using the three methods explained previously, namely the historical one, the EWMA and the GARCH (1,1). The estimation of the variances started in the 31st December 2012 and for that purpose all the past data was used (history). As we are working with three different methods, the same procure was followed for all of them. Then, a prediction of the volatility for the next six months (corresponding to the 30th June 2013) was obtained and the errors of the difference between the actual volatility that was verified in this date and the one estimated by the different methods under analysis were stored. Thus, in every step of this analysis, a comparison between the actual volatility (computed only using the six months' data correspondent to the period that is being analysed) and the forecasted one was done every six months in order to obtain the errors of the different estimations that will be used in the indicators that allow to see which estimation method performs better ad fit better to our data.

After storing all the differences, one can test the forecasts using the mean square error (MSE), the mean absolute error (MAE) and the mean absolute percentage error (MAPE). The formulas of the three different indicators are depicted below:

$$\text{MSE} = \frac{1}{N}\sum_{s=1}^{N}(y_{t+s} - \hat{y}_{t+s})^2 \ (8)$$

$$\text{MAE} = \frac{1}{N}\sum_{s=1}^{N}|y_{t+s} - \hat{y}_{t+s}| \ (9)$$

$$\text{MAPE} = \frac{1}{N}\sum_{s=1}^{N}\left|\frac{y_{t+s}-\hat{y}_{t+s}}{y_{t+s}}\right| \ (10)$$

where y_{t+s} is the actual value at time $t+s$, $\hat{y}_{t+s}$ is the prediction for time $t+s$ and N is the number of predictions.

In this first variance analysis, the indicators were conclusive: the most effective predictive method was the EWMA method, as all the econometric indicators were the lowest ones compared to the other methods (GARCH and historical variance), as one can see in figure 3.

5.4.2. Overlapping analyses of the estimations

The second analysis followed a very similar procedure as the first one but differs from the latter in the time lapse chosen. In the first analysis, we chose a time lapse of six months between estimations, while in this analysis we reduce that time lapse to one month. Basically, in the previous one, each time we predict a new volatility, six months are added to the in-sample data; meaning that we only obtained six predictions. Having only six predictions and therefore six errors to analyse, the choice of the most performing model could be biased as it depended on very few observations. Moreover, doing the predictions every six months could mean that we would be giving a great importance to the 31st December and the 30th of June. Therefore, we decided to reduce the time lapse between estimations to increase the number of predictions (overlapping analysis[10]). By this means, we first estimated variance for the 31st December 2012 using all the past data, then we predicted the volatility for the next 6 months (30th of June 2013), using the estimator of the 31st December 2012, just as we did on the first analysis; but instead of adding six months to the in-sample period, we only add one month: we now estimated the variance for the 30th of January 2013 and used this value to predict the variance for the 31st of July 2013. This procedure was followed until the end of the data set, always storing the errors of the difference between the actual variance and the predicted one for each one of the three

[10] Overlapping analysis means that estimations periods will overlap one another; the new estimation period will overlap the previous estimation period

methods. This resulted in much more predictions, and therefore in a much more robust analysis. Once again, all the relevant econometric indicators that measure prediction effectiveness were computed for the three methods: MSE, MAE and MAPE.

In this second analysis, the results were not in line with the ones obtained in the first analysis. Overall, the GARCH performed better than the other methods. As a matter of fact, all the econometric indicators were the lowest ones for the GARCH (1,1) compared to the other methods (EWMA and historical variance), as one can see on figure 4.

After carefully reviewing our results, we decided to use the covariance matrix estimated by the EWMA. After all, the correlations inputted in the covariance matrix estimated with the GARCH method were the ones obtained with the EWMA. Basically, the variance-covariance matrix was obtained with the variances estimated by the GARCH and the correlations used were the ones obtained from the EWMA method; thus, it resulted in a not so robust estimation. Moreover, in the GARCH model one needs to estimate more parameters and, therefore, bears greater model risk. EWMA was found to have good estimates and preformed much better results than a simple historical estimation.

To simplify the users' comprehension, we made the excel user friendly by giving the option to choose any covariance matrix estimate that the user wants to work with. The user is also given the option to choose a particular time frame that he would want to study. For instance, the users are given the option to choose the start and end dates (month and year) for the estimation of the variance-covariance matrix, in order to analyse a timeframe that may be relevant. Both the variance and correlation matrix of each method are displayed in two separate matrices so that the user can see the behaviour and the co-movements of the assets at first glance.

Section 6. The Black-Litterman model

6.1. Limitations of the Markowitz Optimization – An introduction to the Black-Litterman Framework

An essential element of asset allocation theory is the Mean-Variance Optimization[11] (also known as Modern Portfolio Theory), which holds that the optimal portfolio is the one that maximizes the expected returns with respect to a certain level of risk, given by the variance or volatility of those returns. Consequently, the optimal portfolio is obtained through an efficient diversification of the underlying assets, considering the correlations among the asset classes involved in the portfolio. The mean variance optimizer will assign more weight to the assets that have smaller estimated variances in returns, larger estimated expected returns or negative estimated correlations with other assets in the portfolio, which allows for portfolio diversification and risk reduction.

However, common criticisms[12] to the Mean-Variance framework state that the model is very sensitive to small changes in expected returns because it requires the forecasting of absolute returns for all the available assets and, furthermore, it assigns extreme positions in some assets that are not intuitive from an investor's perspective and a weight of zero to some other asset classes. This leads to the so-called corner solutions, which have as consequence an undesirable portfolio concentration among only a few assets. As a remedy, constraints on positions are often imposed as an alternative, however it is often criticized as ad hoc - when enough constraints are imposed, one can almost pick any desired portfolio without giving too much attention to the optimization process itself. Moreover, one can also point out as a pitfall of this procedure the fact that the optimal allocation weights are not stable over time, and it can result in large

[11] Markowitz (1952 and 1959)

[12] Some criticisms were presented, for instance, by Su, Yong and Benzschavel (2015) and Silva, Lee and Pornrojnangkool (2009)

portfolio turnovers and high transaction costs. Finally, Markowitz's Optimization is based on a backward looking in its implementation, meaning that it only uses historical data (returns, variances and correlations), not allowing the incorporation of expectations regarding future returns.

To overcome the pitfalls inherent to the Mean-Variance Optimization, Black and Litterman (1991 and 1992) developed a new framework to construct optimal portfolios, in which the optimization is forward looking allowing investors to input their own views[13] regarding future performance of the market and the assets (the optimizer blends the views with an equilibrium portfolio to output the new optimized portfolio). Thus, if one investor has no views, he should just hold the market portfolio. Another advantage of the Black and Litterman framework is that it provides a starting point for the estimation of the asset returns, which is the equilibrium market portfolio. In other words, the model uses a shrinkage approach to improve the final asset allocation: uses reverse optimization to generate a stable distribution of returns from the market portfolio, giving robustness and more flexibility to the model. Consequently, this approach provides a much greater stability to changes in the inputs.

In summary, the BL model is a portfolio optimization model, which smoothly incorporates the expression of particular views on the market.

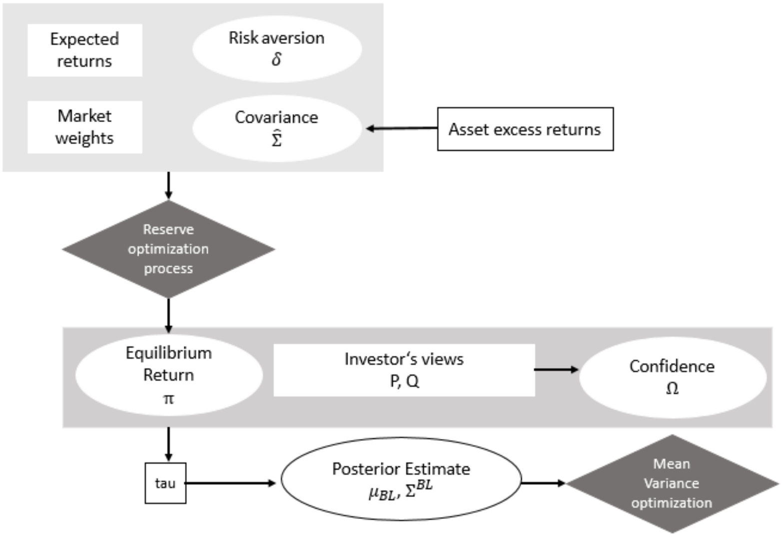
[13] Views do not need to be in all of the assets and they can be relative or absolute (see section 6.5)

6.2. Overview

As referred in the previous point, the BL optimization model overcomes major shortcomings of the mean-variance optimization and results in a combining portfolio optimization model with forward-looking investor's views on the market (Benzschawel and Su, 2015). Figure 5 illustrates an overview of the BL model and all its relevant inputs.

The model begins with the upper right box, which explains the reverse optimization process. The starting inputs are the market weights, a risk aversion parameter, expected excess returns, which are assumed to be the historical average risk-adjusted returns, and finally, an estimated covariance-variance matrix.

Figure 5: Overview of the BL model



Based on Benzschawel and Su, 2015

The equilibrium returns are determined using reverse optimization, which will be explained in a later section. In a next step, investor can input their views on the market in the BL model (middle portion of Figure 5). Investors usually do not have expertise in every single asset in the portfolio and thus, it is a useful implication of the model that the views do not need to include all assets. The model incorporates investors' views on asset returns via a projection matrix, a view matrix, and a confidence matrix.[14]

The expected excess returns can be obtained with the actual BL formula by

$$\mu_{BL} = [(\tau\Sigma)^{-1} + P^T\Omega^{-1}P]^{-1}[(\tau\Sigma)^{-1}\pi + P^T\Omega^{-1}Q]. \quad (11)$$

[14] A detailed description of the method whereby investors' views and confidence are expressed is presented in a later section of this report

In a final stage, the model uses a Mean-Variance Optimization, but with the BL inputs, in order to find the optimal weights.

6.3. Equilibrium Returns

The BL model uses the equilibrium portfolio as a neutral starting point in which the equilibrium returns clear the market. This means that if all investors hold the same belief, the demand for these assets will exactly equal the outstanding supply (Black and Litterman, 1992). Based on this neutral reference point, the investors can express their views about the market. This reference portfolio has been chosen to be the market capitalization weighted portfolio. By using reserve optimization, one can identify the excess returns implied by the equilibrium portfolio. Usually, the optimization process is used to find the optimal weights given the expected excess returns and the covariance matrix of the portfolio. However, in the reverse optimization process, one is given the variance-covariance matrix and the market weights in order to calculate the expected returns.

Black and Litterman (1992) start the reverse optimization from the utility function given by

$$U = w_{mkt}^T \pi - \left(\frac{\delta}{2}\right) w_{mkt}^T \Sigma w_{mkt},$$

where w_{mkt} is the vector of market equilibrium weights invested in each asset, π is the vector of equilibrium excess returns for each asset, δ is the risk aversion parameter, and Σ is the covariance matrix[15] of the excess returns for the assets. By taking the first derivative of the utility function with respect to the weights, in the absence of any constraints on the weights and setting the derivative to zero, one will find the maximum of the utility function:

$$\frac{\partial U}{\partial w} = \pi - \delta \Sigma w_{mkt} = 0$$

[15] In our case, the covariance matrix of the excess returns was estimated using the EWMA method

Solving now for π, results in the following formula to compute **expected excess returns**:

$$\pi = \delta \Sigma w_{mkt}. \text{ (12)}$$

Alternatively, the equilibrium excess returns can be obtained with the CAPM, where the implied excess returns relate to the market portfolio via

$$\pi = \beta \times \mu_M,$$

where μ_M is the excess return on the market portfolio and β is the vector of asset class betas determined relative to the market's excess returns. Beta is calculated as

$$\beta = \frac{Cov(r, r^T, w_{mkt})}{\sigma_{mkt}^2} = \frac{\Sigma w_{mkt}}{\sigma_{mkt}^2},$$

where σ_{mkt}^2 represents the variance of the returns of the market portfolio. Substituting beta in the equation for the implied excess returns leads to the reverse optimization formula (Su and Benzschawel, 2015):

$$\pi = \frac{\mu_{mkt}}{\sigma_{mkt}^2} \times \Sigma w_{mkt} = \delta \Sigma w_{mkt}.$$

Given the market weights and the calculation of the covariance matrix, one needs to specify the risk aversion parameter. The risk aversion parameter describes the trade-off between expected risks and expected returns. Satchell and Scowcroft (2000) suggest to calculate δ with

$$\delta = \frac{\mu_{mkt}}{\sigma_{mkt}^2},$$

where μ_{mkt} describes the vector of expected excess return of the market portfolio and σ_{mkt}^2 is the variance of the excess market returns.

To calculate the risk aversion parameter based on the CAPM[16], one first needs to find an appropriate benchmark to represent the market. In a next step, one computes the expected returns with the following formula:

$$E(r) = r_f + \beta * (r_{mkt} - r_f).$$

[16] Su and Benzschawel (2015)

The portfolio returns equal the expected returns multiplied by the market cap. Finally, one divides the excess return by the standard deviation of the market caps to reach the risk aversion parameter. With the above mentioned formula, one implements the risk parameter to reach the implied returns.

An issue arises when using the unadjusted historical returns to compute the implied excess returns, as they generate highly volatile sector weights. A technique exists for dampening those fluctuations, called the Bayes-Stein estimator (Jorion, 1986). This estimator performs better than the historical average no matter how one will shrink the individual sector's sample mean towards the group mean $\bar{\mu}$.

The Bayes-Stein estimator is defined as

$$\mu^B = (1-\alpha)\hat{\mu} + \alpha\bar{\mu}\mathbf{1} \text{ (13)}$$

in which $\mathbf{1}$ is a $N \times 1$ vector of 1's (assuming $N \geq 3$), and α is a shrinkage factor where $0 < \alpha < 1$ and equals to

$$\hat{\alpha} = \frac{N+2}{(N+2) + T(\hat{\mu} - \bar{\mu}\text{x}1)'^{\hat{\Sigma}^{-1}}(\hat{\mu} - \bar{\mu} \times 1)}, \text{(14)}$$

where N is the number of assets, T the number of observations and $\bar{\mu}$ is the average excess return of the sample mean-variance portfolio according to optimal mean-variance portfolio weights:

$$\bar{\mu} = \hat{\mu}_1\widehat{w}_1 + \hat{\mu}_2\widehat{w}_2 + \cdots + \hat{\mu}_N\widehat{w}_N \text{ (15)}$$

Since, in practice it is common to fix the risk parameter between 2 and 5, we came to the conclusion to do our further calculations with a risk aversion parameter of 2.5, which is the average risk aversion parameter[17], in order to obtain more stable results.

[17] Bodie, Kane and Marcus (2013)

6.4. The Black Litterman formula

The BL model of expected excess returns can be expressed as stated in equation (11):

$$\mu_{BL} = [(\tau\Sigma)^{-1} + P^T\Omega^{-1}P]^{-1}[(\tau\Sigma)^{-1}\pi + P^T\Omega^{-1}Q],$$

where μ_{BL} is an *Nx1* vector of expected excess returns, τ is a scaling parameter, Σ is a *NxN* variance-covariance matrix, P is a *KxN* matrix, whose elements in each row represent the weight of each asset in each of the K view portfolios, Ω is the matrix that represents the confidence in each view, and Q is a *Kx1* vector of expected returns of the K view portfolios.

In this form of the BL formula for the expected excess returns, one can see that the returns are a weighted combination of the market equilibrium returns and the expected return implied by the investor's views. The two weighted matrices are

$$w_\pi = [(\tau\Sigma)^{-1} + P^T\Omega^{-1}P]^{-1}(\tau\Sigma)^{-1}$$

$$w_q = [(\tau\Sigma)^{-1} + P^T\Omega^{-1}P]^{-1}P^T\Omega^{-1}P.$$

Specifically, the phrases $(\tau\Sigma)^{-1}$ and $P^T\Omega^{-1}P$ represent the investor's confidence in the estimates of the market equilibrium and the views, respectively. With a low confidence level in the views, the investor will hold a portfolio which is close to the ones implied by the market equilibrium. Conversely, a high confident investor will invest in a portfolio which deviates from the market portfolio.[18]

By rearranging the BL formula[19], it appears even more clearly the function of π as a neutral starting point:

$$\mu_{BL} = \pi + \Sigma P^T \times \left(P\Sigma P^T + \frac{\Omega}{\tau}\right)^{-1} \times (Q - P\pi). \quad (16)$$

[18] Fabozzi, Focardi and Kolm (2006)

[19] We refer to Appendix A for the reformulation of μ_{BL} and Σ^{BL}

The BL vector of expected excess returns is the sum of the equilibrium excess returns plus an additional term, representing the views of investors. In the case when the investor does not have any views about the market, the expected returns implied by the BL model are equal to the equilibrium returns, which allows the investor to hold the market portfolio. However, if the active manager has views about the market, the BL approach combines the information from the equilibrium and pushes the optimal portfolio away from the market portfolio in the direction of the investor's views.

The estimated variance-covariance matrix refers to the expected excess return from historical data, but it does not correspond to the variance of the actual returns computed by the BL model. Thus, in literature it is suggested to include the BL variance-covariance matrix in the optimization process when calculating the optimal expected excess returns (Benzschawel and Su, 2015). The new variance-covariance matrix is defined as

$$\Sigma^{BL} = (1+\tau)\Sigma - \tau^2\Sigma P'(P\tau\Sigma P' + \Omega)^{-1} + P\Sigma. \quad (17)$$

6.5. Investors' Views: P, Q and Ω Matrix

One advantage of the BL model is that investors can input their own views, which may differ from the implied equilibrium returns. The model allows such views to be expressed in either absolute or relative terms. However, Idzorek (2005) argues that relative views are more close to the way investment managers feel about different assets. The model does not require the specification of views on all assets and investors are rarely fully confident about their views. The higher the confidence on a view is, the more the model pushes the portfolio towards the investors views. Inversely, the less confident an investor is about his/her views, the more the weights are going to shift towards the equilibrium weights.

A view is expressed as a statement that the expected return of a portfolio has a normal distribution with mean equal to Q and a standard deviation of ε. The investors' views can be represented formally as

$$P\mu = Q + \varepsilon, \ \ \epsilon \sim N(0, \Omega), (18)$$

where ε is an unobservable normally distributed random vector with zero mean and covariance Ω, which represents a diagonal covariance matrix. The investor's views are matched to specific assets by matrix P, the "pick" matrix, whose elements in each row represent the weight of each asset in each of the views, as mentioned before. Each expressed view results in a *1xN* row vector. Thus, *K* views result in a *KxN* P matrix:

$$P = \begin{bmatrix} p_{1,1} & \cdots & p_{1,N} \\ \vdots & \ddots & \vdots \\ p_{K,1} & \vdots & p_{K,N} \end{bmatrix}.$$

Constructing the P matrix, the weights of a relative view will sum up to zero, while for an absolute view the sum will be one. The outperforming assets have a positive value in this matrix, while for the underperforming assets' the value is negative. There are two different ways to compute the weights within the view. He and Litterman (1999) and Idzorek (2005) use a market capitalization weighed scheme, whereas Satchell and Scowcroft (2000) use an equal weighted scheme in their examples. In the equally weighted scheme, the weighting of each individual asset is proportional to one divided by the number of respective assets outperforming or underperforming. Using the market weighted scheme, one considers the market capitalization. In this scheme the weightings are proportional to the asset's market capitalization divided by the total market capitalization of either the outperforming or underperforming assets of that particular view. Within the P matrix, a zero is displayed when a certain asset is not included in the view.

The Q vector states investors' expected returns regarding the market performance, which are different from the implied equilibrium returns in a *Kx1* vector. The uncertainty of the views results in a random, unknown, independent, normally distributed error term vector (ε) with a mean of zero and covariance matrix, Ω. Thus, a view can be expressed as

$$Q + \varepsilon = \begin{bmatrix} Q_1 \\ \vdots \\ Q_k \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_k \end{bmatrix}.$$

Usually, the error term of the Q matrix will have a value other than zero, which means that the investor is not 100% confident in his/her views. The error term vector does not directly enter the BL formula, however the variance of each error term from Ω, which represents the uncertainty of the views, enters in the BL model (Idzorek, 2005). The larger the variance of the error term, the greater the uncertainty of the view. In case the investor is fully confident in his/her view, Ω will be a matrix of zeros and thus, the investor should hold the market portfolio. Contrary, if the investor is not confident about his/her view, Ω would compose only ones on the diagonal. This leads to a portfolio, completely tilt to the investor's views.

Ω is a diagonal covariance matrix with zeros in all of the off-diagonal positions, defined by He and Litterman (1999). The off diagonal elements of Ω are zeros because the model assumes that the views are independent of one another:

$$\Omega = \mathrm{diag}(\mathrm{P}\Sigma P^T) = \begin{bmatrix} P_1 \Sigma P_1^T & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & P_N \Sigma P_N^T \end{bmatrix}, (19)$$

where P_i represents the i-th row of the view matrix.

Omega may also be zero on the diagonal if the investor is certain of a view. This means that Ω may or may not be invertible (Walters, 2004). However, at a practical level we can require that the weights are bigger than zero, so that Ω is invertible.

This method expresses the uncertainty of investors' views as the variance of the view portfolio. In other words, it assumes that investors who have the same views should have the same level of confidence about their views, given the same variance-covariance matrix. However, in practice this is not the case since investors are more or less prone to specific views (Barber and

Odean, 1998). Further, in literature it is argued that not only the variance of the views is the only source of information which influences the uncertainty of views, but also additional information, such as historical accuracy and the score of the model.

Another approach to calculate Ω is formulated by Meucci (2005). He suggested an approach that the Q matrix is not restricted to be a diagonal matrix. Moreover, it depends on a scalar c. He consequently writes:

$$\Omega = \left(\frac{1}{c} - 1\right) \mathrm{P}\Sigma P^T. \text{ (20)}$$

The scalar c varies according to the absolute confidence in the investor's skills. Setting c equal to one neutralizes the volatility of the views and hence, confers complete confidence to the investor's skills, whereas a c close to zero produces an infinitely disperse distribution of the views, denying the investor any skill. A value of c = 1/2 corresponds to an equal confidence in the investor and in the market.

The big advantage of Meucci’s formulation of Ω is that the calculation of the expected excess returns can be simplified, such as the prior returns and the views are weighted equally in posteriors returns:

$$\mu_{BL} = \pi + \Sigma P^T \times \left(P\Sigma P^T + \frac{\Omega}{\tau}\right)^{-1} \times (Q - P\pi)$$
$$= \pi + \Sigma P^T \times (P\tau\Sigma P^T + P\tau\Sigma P^T)^{-1} \times (Q - P\pi)$$
$$= \frac{1}{2}(P^{-1}Q + \pi). \text{ (21)}$$

This stems from the fact that one has scaled the variance of both the returns and views by the same parameter, τ. Furthermore, the variance-covariance matrix can be rewritten since it is a scaled version of the original variance that depends on tau, but no longer in the projection on the P matrix:

$$\Sigma^{BL} = (1 + \tau)\Sigma - \tau^2 \Sigma P^T (P\tau\Sigma P^T + \Omega)^{-1} + P\Sigma$$

$$(1 + \tau)\Sigma - \tau^2 \Sigma P^T (P\tau\Sigma P^T + P\tau\Sigma P^T)^{-1} + P\Sigma$$

$$= \left(1 + \frac{\tau}{2}\right) \Sigma. \ (22)$$

Tau can be interpreted as a scalar weighting the confidence in the equilibrium expected excess returns estimation. When Ω is calculated using the method of He and Litterman (1999), the actual value of the scalar tau becomes irrelevant. Changing tau to a much higher value will lead to changes in the value of the diagonal elements of Ω: Setting tau close to zero will push the equilibrium portfolio and conversely, a high value of tau will yield the weight towards the investor's views (Maggiar, 2005). Since tau is cancelled out in the formula of the excess return, the new combined return vector is unaffected of changes of tau.

In literature, there are different conclusions regarding the value of tau: Satchell and Scowcroft (2000) argue that tau should be set to one or not set at all, because tau only appears in the Black Litterman formula as a factor of Ω and could thus be inserted into delegating the estimation of τ to an estimation of $\frac{\Omega}{\tau}$. Black and Litterman (1992) reason that tau should be close to zero and takes a value between 0.01 and 0.05, since the uncertainty in the mean is much smaller than the uncertainty in the return itself. In active management, however, it is often the case that τ is set such that the information ratio does not exceed 2.0 (Bevan and Winkelman, 1998). This leads in practice often to values of τ between 0.5 and 0.7.

6.6. Black Litterman excess returns and optimal weights

After defining all the inputs for the BL model, one can compute the expected excess returns given by the model. In case the active manager inputs views, the expected returns will deviate away from the starting point, the implied returns, consistent with the views being expressed by the investor. Since each individual return is linked to the other returns by the covariance matrix, all expected excess returns implied by the BL model will change, no matter if all indexes are included in the views or not.

Finally, the optimal weights can be calculated by rearranging the formula for the implied returns and substitute π for μ_{BL}, which represents the vector of excess returns implied by the model. This is the solution for the maximization problem: $max\ w'\mu_{BL} - \frac{\delta}{2} w^T \Sigma \mathrm{w}$:

$$w^{BL} = (\delta\Sigma)^{-1}\mu^{BL}. \ (23)$$

However, one needs to keep in mind that those weights have no constraints at all, meaning that the sum of the weights may not be one. In the absence of views that differ from the implied equilibrium return, investors should hold the market portfolio. Only those weights, whose corresponding index is included in one of the investor's views, will deviate from the market weights. This is one big criticism of the BL model. Logically, critics argue that the weight of assets that are highly correlated with the assets of the views should change.[20]

In order to satisfy the condition that the sum of the weights equal to one, the following formula should be used to calculate the optimal weights:

$$w^{BL} = \delta^{-1}\Sigma^{-1}\left(\mu^{BL} - 1\left(\frac{({\mu^{BL}}^T\Sigma^{-1}1) - \delta}{1^T\Sigma^{-1}1}\right)\right). \ (24)$$

6.6. Caixagest – Initial Black Litterman model

For the calculation of the implied excess returns we used the covariance matrix, which was already estimated, and the market weights. Those market weights were given to us by Caixagest and are displayed in figure 6. Furthermore, it is necessary to calculate the risk aversion parameter. In the formula that allows to obtain this parameter, μ_{mkt} describes the vector of expected excess returns of the market portfolio. Thus, we would like to note that one should subtract the risk-free rate of the 2nd September 2016 from the summary of the historical average annual returns, since this is the rate we expect to get in the future. In figure 7 of the appendix,

[20] Idzorek (2002)

one can see a summary of the implied excess returns given by the different risk aversion parameters, denoted by δ. The high risk aversion parameter of 9.19 using the historical method is, among others, caused by the small sample size. Moreover, we would like to point out that the implied returns calculated based on the CAPM and using historical returns are the same since we used the excess returns multiplied by the market cap as a benchmark for the market. Thus, the risk aversion parameter is the same and, subsequently, the implied returns are equal. Graph 2 in the appendix represents the difference in the implied returns and the expected returns based on historical average returns. It shows that the implied returns are more stable than the ones on basis of historical returns. The reverse-optimized returns are less volatile than the expected returns using historical data because existing market conditions have a greater impact on these estimates (Benzschawel and Su, 2015).

After researching about the actual market situation, we built our own views[21] and eleven mini-portfolios[22]. Having eleven views is a quite extreme assumption since investors usually have only a few views on the market performance and are rarely confident in their views. We used the standard error to obtain the standard deviation over the mean of implied excess returns, calculated with the following formula:

$$Standard\ error = \sqrt{\frac{P\Sigma P^T}{N}}. \quad (25)$$

Finally, our Q vector equals the standard error multiplied by the simulation of the standard error, which works as a kind of weight that is assigned to the standard error of the view. These simulations allow us to avoid too extreme results in the Q matrix. The user can differentiate in the excel file between equally and market weighted scheme. Furthermore, the order of the views in the projection matrix and the view matrix is not important.

[21] Views were builted having as basis Black Rock's Global Investment Outlook

[22] For more details, we refer to figure 8

The user has the possibility to choose between the He and Litterman and the Meucci approach to calculate Ω. However, in the case of Meucci's formulation, we would like to note that it is necessary that the P matrix is invertible in order to calculate the expected excess returns[23]. In our case, the P matrix is not invertible, and therefore, we continued our calculations with the He and Litterman formulation of Ω. Further research should be undertaken in order to find a user friendly solution when applying the Meucci approach.

In addition, in the excel file, we give the user the possibility to choose between the different values of tau. However, one should keep in mind that when using Ω defined by He and Litterman (1999), tau will not influence the model anymore and thus, does not change the results. Furthermore, one should note that the final optimized weights will still change when using Σ^{BL}, since the variance-covariance matrix depends on tau.

6.6.1. Simulation and Model Analysis

To analyze and have a better sense on how the views given by the investor influence and change the final portfolio weights, a simulation exercise was performed. In this simulation set, the P matrix only includes 4 views (2 absolute ones with long positions and 2 relative views where one index has the long position and the other has the short one) and the indexes that are in each view were randomly selected among the 23 that compose the investor's portfolio. Furthermore, in each P matrix we never use the same index in order to avoid contradictory views. Regarding the Q matrix, we decided to simulate the weight that will be assigned to the standard error of each view, corresponding to a value between 0 and 0.5 to obtain more stable and not too extreme active weights, because usually the investor is not that confident in the views that he decides to input in the model.

[23] Maggiar (2009)

Subsequently, 100 different vectors of expected excess returns and optimal weights (without any constraint and with the constraint that the weights add up to one) were computed. The expected excess returns implied by the model and the optimal weights were computed given by the formulas 23 and 24. However, we did not use the variance-covariance matrix given by the BL model for calculating the optimal weights, but the estimated one. This is due to several reasons[24]: first, the BL assumption that only the weights of those indexes that are included in a view change is not considered when using the BL variance-covariance matrix. Thus, each weight would change no matter if the manager has a view on that asset or not. Second, in practice, active managers have their own estimation methods to calculate the variance-covariance matrix and thus, prefer to use their estimated one.

In our simulation, the weights without any constraint, represented by w^{BL}, only deviate from the market weights in case the corresponding index is included in the view. This means that in each simulation six weights (since we have in addition to two absolute views, two relative views, each including two positions) should change with regard to the view. The results show that in all 100 simulations, the BL portfolio has a higher SR than the portfolio using the market weights as inputs. In the simulation, we recognized that the risk of the BL portfolio is in all scenarios less than the one that results from investing in the market portfolio. However, we cannot conclude that the expected return computed with the BL formula is higher than the one from the market. Indeed, in our simulation, there was not even one case where the expected portfolio returns of the BL model is higher than the market portfolio returns.

Having a closer look at the weights with the constraint that the weights add up to one $\mathrm{w}_{\mathrm{c}}^{\mathrm{BL}}$, we will observe that they change, even if its corresponding index in not included in the views. Using $\mathrm{w}_{\mathrm{c}}^{\mathrm{BL}}$ leads to a violation of the BL assumption. Although this property of the model is ignored, the BL portfolio performs better than the market portfolio. In all 100 scenarios, the SR

[24] More details regarding this issue will be presented in section 7.6

is higher than the one from the market portfolio. In addition, the risk is smaller in the BL framework compared to the market portfolio, however, the expected excess returns are in all scenarios smaller than in the market portfolio.

Using w^{BL} and w_c^{BL}, the difference of the portfolio performances is minimal: the average return and risk of the BL portfolios of all 100 scenarios are the same for the portfolio calculated with w^{BL} and w_c^{BL} (0.61% and 4.87% respectively). In addition, we compared the results of the simulation by introducing several statistics. Figure 9, shows the average, absolute weights, the average of the largest three weights and of the smallest three weights. From the same figure, one can see that within the BL framework, the weights have more extreme values than the one from the market. Subsequently, the market weights are in absolute terms smaller than w^{BL} and w_c^{BL}.

To get a more detailed analysis of the two calculations of optimal BL weights, we randomly chose two scenarios, 14 and 66, where the assets included in the views are presented in figure 8.

In simulation 14, the two absolute views state to go long on the EM currency government total return and on Euro linker securities, and as a result, the weights deviate from the market weight by 17% and 24%, respectively. Further, one relative view states to short JPM Euro and to go long MSCI Hong Kong. The BL framework takes those views into account and pushes the portfolio towards the investor's views. Thus, the deviation from the market weight is 13% and -13%, respectively. In a fourth view, it states to short the US Corp yield and to go long on Euro Aggregate Corporate. Also in this case, the weighs deviate from the market weighs by -30% and 30%. The absolute views in simulation 66 say to go long on EM hard cry Agg and MSCI Hong Kong. The deviation from the market weights is -29% and -13%, respectively. The first relative view states that one should invest 19% more in Treasury Values US than the market weigh of 9.2% and to go short on EM cry Gov TR with a deviation from the market weight of

-19%. Furthermore, the second relative view states that one should go long on US Corp yield and have a short position on Euro-floating rate. Also in this case, the weighs deviate from the market weighs by -20% and 20%. Graph 3 summarizes the findings for simulation 14 and 66. It shows that only those weights change whose corresponding asset is included in the views. The deviation of the other optimal BL weights from the market weights equals zero, which means that the investor should stay with the market weights if he/she does not have a view on that index.

Although there might be no deviation in some weights from the market weights because the indexes are not included in the views, which implies that the investor should hold the market portfolio, the expected excess returns will always change. Figure 10 in the appendix represents the deviation of the BL returns from the implied returns from simulation 14 and 66. It shows that in any cases the BL returns change compared to the equilibrium returns. This is due to the fact that returns are linked via the variance-covariance matrix and thus, also returns of those assets, which were not included in the views, will change.[25]

Using w_c^{BL}, the BL property is not valid anymore. The drawback of including constraints is that the weights change no matter if the index is included in a view or not. In graph 4, one can see that also those weights deviate from the market weight whose index was not included in the view. However, the views are still reflected in the BL model because those indexes included in the views have the highest deviation from the market weights.

Summing up, the simulation shows that the BL portfolio not only results in a more diversified portfolio, but also lead to a higher SR compared to a portfolio using market weights. However, the optimal weights implied by the model might be not optimal in a sense that investor only desire long only portfolios. In practice, it is often the case that investors do have a budget

[25] Fabozzi, Focardi and Kolm (2006)

restriction, which does not allow short selling. Adding further constraints, just as a long only portfolio constraint, may lead to more concentrated investments in a few assets and does not completely include the investor's views in the BL model.

In conclusion, all three ways of calculating weights have drawbacks as inputs for the prediction of returns. Using the market weights is not looking-forward and does not consider the big advantage of the Black Litterman model that investor's views can impact the estimation of the expected excess returns. Building a portfolio using the BL model, the weights, w^{BL}, are calculated without any constraints. However, in nature weights should be sum up to one and thus, we implemented an portfolio analysis using the constraint BL weights, w_c^{BL}. These weights, however, can include short positions on some indexes, but Caixagest manages portfolios without short selling and therefore, a portfolio with only positive weights is desired.

Section 7. Black Litterman and Active Management

7.1. Black Litterman framework

The BL framework used so far is focused on maximizing the classical mean-variance problem, that is, maximizing the SR of a portfolio- average return earned in excess of the risk-free rate per unit of volatility or total risk. However, many investors do not manage their portfolio with this objective in mind. Usually, they have a benchmark that they wish to outperform and in a typical portfolio delegation problem, the investor assigns the management of assets to a portfolio manager, who is given the task to manage the benchmark. Thus, in this case, when outperformance is observed for the active portfolio, the important thing to notice is that the added value in terms of return is aligned with the risks undertaken. This is especially important when performance fees are involved, in the sense that there is an incentive to bet on more risk. In other words, it induces the manager to optimize the portfolio in an excess return space only, ignoring the investor's overall portfolio risk.

In this context of active management, it does not make sense to use the equilibrium portfolio as a starting point anymore; instead, one should use the benchmark portfolio because this is the one that the investor is trying to beat. This change of starting point can be explained by the fact that if the investor only has access to public information (no views were specified), the best one can do is to hold the benchmark portfolio itself.

In the active management, the goal is to maximize the IR instead of the SR, which means that one wants to maximize the portfolio's alpha (active return), defined as the return of the active portfolio in excess of the benchmark portfolio with respect to a certain level of tracking error (TE)- active risk-, defined as the standard deviation of alpha. Thus,

$$IR = \frac{\alpha}{TE} = \frac{R_p - R_b}{\sqrt{Var(R_p - R_b)}} \quad (26)$$

where R_p is the return of the portfolio and R_b is the return of the benchmark.

For example, this objective is reflected in Bevan and Winkelmann (1998): “After finding expected returns, we then set target risk levels. Since we construct our optimal portfolio relative to a benchmark, we consider all of our risk measures as risks relative to the benchmark. The two risks that we care most about are the tracking error and the Market Exposure.”

7.2. Formulation of the new model

In the case of the active management problem, we will use the following notation:

W_b:	Vector of benchmark weights
W_a:	Vector of active weights
W:	Vector of portfolio weights ($W = W_a + W_b$)
W_{GMV}[26]:	Vector of the Global Minimum Variance portfolio weights
μ:	Vector of expected excess returns
μ_{GMV}:	Expected excess returns of the Global Minimum Variance Portfolio ($\mu_{GMV}=\mu^T W_{GMV}$)
λ:	Active risk aversion parameter
δ:	Absolute risk aversion parameter

One needs to remember that the objective function of active management in the presence of a benchmark is to maximize the total return of the portfolio with a penalty on the square of the TE. This means that the objective function is different in this case. As previously mentioned,

[26] The Global Minimum Variance Portfolio (GMV) is the efficient portfolio which has minimum variance: $W_{GMV}=\arg min_w\, w^T\Sigma w\ s.t.\ w^T\mathbf{1}=1$
From this problem results that $w_{GMV}\frac{1^T\Sigma^{-1}}{1^T\Sigma^{-1}1}$.

the goal was to express the expected excess returns of assets in order to use them in the following problem:

$$argmax_w w^T\mu - \frac{\delta}{2} w^T \Sigma w \quad (27)$$

$$s.t.\ w^T\mathbf{1} = 1$$

$$other\ constraints$$

Thus, in the case of active management, we have a new optimization problem aligned with the new problem that we are trying to solve:

$$arg\ max_W\ W_a^T\mu - \frac{\lambda}{2} W_a^T \Sigma W_a \quad (28)$$

$$or\ arg\ max_W\ (w - w_b)^T\mu - \frac{\lambda}{2}(w - w_b)^T\Sigma(w - w_b)$$

$$s.t. constraints$$

The difference between the two problems is based on the fact that the first one is built in a way that aims to maximize the expected excess returns over the risk-free of a portfolio with a penalty for its volatility (standard deviation), which means that the goal is to maximize the SR, while the objective of the second one is to maximize the expected excess returns over the benchmark (maximize the return of the active weights) with a penalty not on the volatility, but on the square of the TE, which is the same as maximizing the IR, as previously mentioned.

At this point, it is important to understand what can differ when we use this new optimization problem compared with the previous one without the benchmark. Thus, we will study the particular case where we have the constraint $w_a^T\mathbf{1} = 0$[27], which is a self-financing condition meaning that there is no investment or withdrawal of money; the purchase of a new asset must be financed by the sale of an old one (allowing for short and long position in the final portfolio). The optimization problem with this constraint is

[27] This analysis has as basis Silva, Lee and Pornrojnangkool (2009) and Maggiar (2009)

$$arg\ max_W\ W_a^T \mu - \frac{\lambda}{2} W_a^T \Sigma W_a \quad (29)$$

$$s.t.\ w_a^T \mathbf{1} = 0$$

and the respective Lagrangian is given by

$$L(w_a, \theta) = W_a^T \mu - \frac{\lambda}{2} W_a^T \Sigma W_a - \theta W_a$$

where θ is the Lagrangian multiplier.

Taking the first derivative of the Lagrangian with respect to W_a and then setting it to zero gives

$$\frac{\partial L}{\partial w_a} = 0 \iff \mu - \lambda \Sigma W_a - \theta \mathbf{1} = 0 \iff$$

$$W_a = \frac{1}{\lambda} \Sigma^{-1}(\mu - \theta \mathbf{1}) \quad (30)$$

Substituting to W_a into the budget constraint in the objective function gives

$$\frac{1}{\lambda}(\mu^T \Sigma^{-1} 1 - \theta 1^T \Sigma^{-1} 1) = 0$$

meaning that

$$\theta = \frac{1^T \Sigma^{-1}}{1^T \Sigma^{-1} 1} \mu = w_{GMV}^T \mu = \mu_{GMV}$$

Finally, substituting θ in the formula that gives W_a gives

$$W_a = \frac{1}{\lambda} \Sigma^{-1}(I - \mathbf{1} w_{GMV}^T)\mu.$$

Alternatively, the equation can be expressed as

$$W_a = \frac{1}{\lambda} \Sigma^{-1}(\mu - \mu_{GMV}). \quad (31)$$

In optimizing the IR, the process makes pairwise comparisons between the expected excess returns and the expected excess return of the GMV portfolio. In the case where the expected excess returns are different, we will have active positions. Thus, long positions will be taken in the assets that are expected to outperform the GMV portfolio and short positions in the ones that underperform it. Hence, the final vector of optimal active weights results from the combination of these pair trades.

Making comparisons with the excess returns of the GMV portfolio to reach the active weights may be seen a good approach as starting point because it has been reported in the literature that it "has as large an out-of-sample Sharpe Ratio as other efficient portfolios"[28]. For instance, it was also shown that using monthly stock index return data for G7 countries Jorion (1985) convincingly argues that "(...) benefits from diversification are more likely to accrue from a reduction in risk." In the data set he examined, the global minimum variance portfolio had the best out of sample performance. It outperformed classical tangent portfolio, the tangent portfolio constructed using the Bayes-Stein estimator for the vector of mean returns, and the value weighted and equally weighted portfolios. Using simulation methods, Jorion (1986) also showed that this conclusion is robust if the sample size is not large. Finally, Jagannathan and Ma (2002) also prove that tangency portfolios, whether constrained or not, do not perform as well as the GMV portfolios in terms of out-of-sample Sharpe Ratio.

7.3. Problems of applying Black-Litterman in active management

As previously explained, the original BL model was derived with the objective of maximizing the expected return of a portfolio (over the risk-free rate) for a certain level of risk measured by the volatility or standard deviation and under the mean-variance equilibrium framework. Thus, because of the framework that was the basis to obtain the original returns of the BL model defined as stated in equation (11)

$$\mu_{BL} = \pi + \tau\Sigma P^T(\tau P\Sigma P^T + \Omega)^{-1}(Q - P\pi)$$
$$= \pi + V,$$

[28] Jagannathan and Ma (2002)

the strict application of these expected returns in the case of active management can potentially lead to unintentional trades, even in the case where one does not have any views, if the equilibrium portfolio is not chosen to be the GMV portfolio.

Now, in the case of the active management, the aim is to maximize the IR and not the SR as in the initial framework. We will consider the case where there are not any investment views, such that the vector of expected excess returns is the equilibrium. Consequently, in this information-less scenario the action that makes sense is just to hold the benchmark portfolio and not make active trades, because, accordingly to the BL model, we should just deviate from the initial weights (in this case the benchmark ones) if any view regarding the future performance of the market is specified. However, it can be shown that the IR maximization will lead to active trades even when no views are specified.

As suggested by Black and Litterman (1992), a reverse optimization procedure should be used to derive the benchmark-implied equilibrium excess return, Π, from where we obtain that

$$\Pi = \delta \Sigma w_B. \quad (32)$$

The optimal vector of active positions given equilibrium assumptions and no active views can be obtained by substituting μ by Π in the equation that allows to obtain the active weights defined previously. Through this procedure, we obtain that

$$w_{a,\Pi} = \frac{1}{\lambda}\,\Sigma^{-1}(I - \mathbf{1}w_{GMV}^{T})\Pi = \frac{1}{\lambda}\,\Sigma^{-1}(I - \mathbf{1}w_{GMV}^{T})(\delta\Sigma w_B)$$

$$= \frac{\delta}{\lambda}(\Sigma w_B - w_{GMV}^{T}\Sigma w_B) = \frac{\delta}{\lambda}(w_B - \Sigma^{-1}w_{GMV}^{T}\Sigma w_B 1) \quad (33)$$

where $w_{GMV}^{T}\Sigma w_B$ is a scalar equal to the covariance of the GMV portfolio and the benchmark portfolio, denoted by B. Adding the investment budget condition of

$$1^T w_B = 1 \iff 1^T(\Pi(\delta\Sigma)^{-1}) = 1 \iff$$

$$\iff \frac{1}{\delta} 1^T \Sigma^{-1} \Pi = 1$$

one can see that

$$w_{GMV}^T \Sigma w_B = \frac{1^T \Sigma^{-1}}{1^{\mathrm{T}} \Sigma^{-1} 1} \Sigma w_B = \frac{1^T \Sigma^{-1} \Sigma w_B}{1^{\mathrm{T}} \Sigma^{-1} 1} = \frac{1^T w_B}{1^{\mathrm{T}} \Sigma^{-1} 1} = \frac{1}{1^{\mathrm{T}} \Sigma^{-1} 1} = \sigma_{GMV}^2.$$

In fact, it was shown that the covariance between the benchmark portfolio and the GMV portfolio corresponds to the variance of the GMV portfolio.

Plugging in this result in the equation that gives us the active weights, we accomplish the following

$$w_{a,\Pi} = \frac{\delta}{\lambda}\left(w_B - \sigma_{GMV}^2 \Sigma^{-1} 1\right) = \frac{\delta}{\lambda}\left(w_B - \frac{\Sigma^{-1} 1}{1^{\mathrm{T}} \Sigma^{-1} 1}\right) = \frac{\delta}{\lambda}\left(w_B - w_{GMV}\right). \quad (34)$$

This equation[29] shows that unless the investor holds the GMV portfolio as the benchmark portfolio, using the BL model will generate a set of active trades even if no views are specified in the model.

This inconsistent result is due to the mismatch between the objectives in the optimization problems defined in the case of the original BL model and the one defined for the active management case. In the first one, the goal is to maximize the SR and in the latter, is to maximize the IR. Moreover, it is important to recall that the vector of equilibrium excess returns defined by Π is the result of a reverse-optimization procedure using the benchmark weights under the SR optimization.

As a consequence, when we are in the IR-criteria set and even with no specified views regarding the market behavior, any discrepancies in pairs of asset returns are seen as “alpha opportunities” and therefore a portion of the total tracking error budget will be allocated to it, even though the trades embed absolutely no alpha information at all.

In summary, the problem of applying the BL approach in active management is based on the mismatch between the defined objective functions that leads the investor to believe that there are alpha opportunities (a positive IR is available) and, consequently, he incurs in a set of active weights that will only allow to obtain a portfolio that is more volatile than the initial benchmark.

[29] This analysis was developed by Silva, Lee and Pornrojnangkool (2009)

7.4. Black-Litterman model and Active Management

Implementing the BL model in active management corresponds to substituting the expected excess returns in equation (30) that allows us to obtain the vector of active weights by the excess returns obtained in the BL framework.

Consequently, one obtains

$$W_a = \frac{1}{\lambda}\, \Sigma^{-1}(I - \mathbf{1}w_{GMV}^T)(\pi + V) = w_{a,\Pi} + w_{a,\mathrm{V}}, \quad (35)$$

where $w_{a,\Pi}$ corresponds to the case of using the implied excess returns to optimize the IR. However, we saw previously that using these excess returns as inputs will lead to active trades even in the case of no views. In the case of $w_{a,\mathrm{v}}$, an active trade will occur based on the deviation of any investment views on portfolios (P) from the ones that are implied by the equilibrium (PΠ).

In the case that no investment views were specified, meaning that V (deviations from implied returns) is equal to zero, one will stay with the case explained in the previous section. On the other case, where V is different than zero, the active deviations from the implied returns will depend directly on the confidence that the investor puts in the views.

7.5. Solution: New vector of excess returns

As we have been analyzing, the problem in applying the initial BL framework in the case of an active management setting comes from the mismatch between the optimization problems that are involved (the implied returns were backed out using a SR optimization and not an IR optimization). To fix this problem one should make both problems consistent- derive the new vector of implied returns using an IR optimization problem similar to the one used to construct the active management problem. Thus, in this case, the optimization problem from which we can derive Π corresponds to

$$0 = \arg max_{w_a} (w_a + w_B)^T \Pi - \frac{\lambda}{2} w_a^T \Sigma w_a \quad (36)$$

$$s.t\ constraints$$

Solving this problem can be simplified if one chooses any Π whose elements are all the same, implying that the first term of the equation drops out as it becomes constant (because of the constraint $w_a^T \mathbf{1} = 0$), and the TE minimization is the part that is left to solve. Clearly, the TE is minimized when the active weights are equal to zero. Furthermore, any vector of implied returns will solve the optimization problem and to ensure that no active trades are taken when no views are specified, all assets should be expected to have the same expected excess return. Thus, the more intuitive solution is to set the vector of implied returns to zero, meaning that all assets are expected to yield a risk-free rate of return as prior.

In other words, in the initial BL setting, we were looking to the expected excess returns over the risk-free rate, while in this active framework we are interested in the expected excess returns over the benchmark. Thus, in this new model, the equilibrium vector should be null[30], meaning that when no view is specified an investor should not hold any active position and the expected excess return over the benchmark will be the implied return, which was set to zero.

By setting π to zero, the vector of BL active returns is

$$\mu_{BL}^a = ((\tau\Sigma)^{-1} + P^T \Omega^{-1} P)^{-1} (P^T \Omega^{-1} Q) \quad (37)$$

and the active return and risk are computed as follows

$$\alpha = (\mu_{BL}^a - \mu_b) w_{BL}^a \quad (38)$$

$$TE = {w_{BL}^a}^T \Sigma w_{BL}^a. \quad (39)$$

[30] Result showed by Silva, Lee and Pornrojnangkool (2009)

7.6. Implementation- General case

Similar to the initial model, one can consider the corresponding active management problem

$$\arg max_{w_a} w_a^T \mu_{BL}^a - \frac{\lambda}{2} w_a^T \Sigma w_a \quad (40)$$

$$s.t. constraints.$$ [31]

This optimization problem can be substituted by a maximization of the expected alpha subject to a constraint on the TE.

Regarding the constraints that can be inputted in the problem, one may consider bounding the weights on each asset or even in specific classes of assets or now allow for short positions in the final portfolio weights. We will study in particular the inclusion of these constraints in the initial model.

It is important to make some clarifications regarding the variance-covariance matrix that should be used in the optimization problem defined previously. The BL model allows us to derive a new vector of expected returns and a new variance-covariance matrix. However, as mentioned by Maggiar (2009), most authors and practitioners do not use the variance-covariance matrix defined by the model; therefore, they use as input their prior covariance matrix. This is the approach that will be followed in our case, meaning that the two main issues in portfolio optimization will be treated separately: expected returns on the one side and covariance matrix on the other.

7.7. Qualitative views

Another way of specifying views in the BL model is in a qualitative way. A qualitative view

[31] We will discuss in 6.8 the constraints that we will include in the model

can be expressed, for instance, as going long one portfolio (the bearish one) and going short another (the bullish one). According to Grinolds's (1989) "fundamental law" of active management, the maximum Information Ratio attainable[32] is given by

$$IR \approx IC\ x\ \sqrt{BR},\ (41)$$

where IR is the Information Ratio, IC is the Information Coefficient and BR is the Strategy's Breadth.

The Information Coefficient (number between 0 and 1) captures the degree of confidence that the investor has in a specific view or, in other words, the probability of a view to happen; measures the correlation of the manager's forecast with the actual returns (basically, how good the forecasts are) and the Breath is the number of securities that can be traded and how frequently they can be traded (is also defined as the number of independent forecasts of exceptional return made per year for a strategy). The fundamental law[33] has been quite influential in active quantitative portfolio management because it offers a guideline as to how good asset managers should be at forecasting, how many bets they should make, or both to generate alpha.

Using this structure, and because the IR is the ratio between α and the TE, one can rewrite this expression in order of the active risk, which leads us to

$$\alpha = IR * TE\ \ (42)$$

and the investor's expected alphas, q, can also be rewritten as

$$q_i = \sigma_{ai} IC_i \sqrt{BR_i},\ (43)$$

where $\sigma_{ai} = \sqrt{diag\ (P\Sigma P^{T})_i} \equiv \Omega$ is the tracking error of the i-th view. Thus, the diagonal of Ω represents the volatility of the tracking error of each view. This means that the higher the volatility of one view, the riskier, the more it is going to be dampened by Ω.

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32 Ignoring transaction costs, restrictions on trading and other real world considerations

33 Andrew Ang (2014)

7.8. Constraints added to the initial setting

To obtain more intuitive results and also more adapted to investors' reality, one can add constraints to the active weights obtained through the BL model as well as to the final weights that should be invested (which in this active setting correspond to the sum of the active weight to the benchmark, for each asset). Two constraints will be imposed in the final weights for this model to become investible in practice, namely:

- A long only constraint, meaning that no short positions are allowed ($w_{BL} = w_a + w_B \geq 0$). There is no close formula for the computation of the weights under this assumption, which means that the weights need to be obtained using SOLVER[34] in excel;
- A maximum weight for each class of assets is imposed, meaning that one has a vector of boundaries that need to be respected. In this case, an optimization using SOLVER will also need to be performed;
- A risk constraint where one can set a target for the TE based in each specific type of investor;
- Finally, by the portfolio theory, the sum of the invested weights need to add up to 1 and the sum of the active weights add up to 0 (in this case there are long and short positions based on the views inputted in the model).

7.9. Risk Analysis

A key component of the BL model is the views that an investor has regarding the market that lead to active positions. Thus, it is important to understand the risks that are undertaken in

[34] SOLVER framework will be explained later

consequence of these positions. First, it is important to analyze the coherence of the expressed views and, secondly, one should analyze the importance of each view in the total active risk.

- **Views consistency**

To assess the coherence and consistency of the views, one must check whether the views actually include investor's opinion and that he/she has not expressed contradictory views, meaning that the views should be analyzed individually and globally. This can be done by making pairwise analysis of the views based on their covariance matrix.

Individual assessment: The views are expressed as a linear combination of the assets involved in the portfolio through matrix P. As previously mentioned, the variance-covariance of the views is given by $P\Sigma P^T$(the diagonal terms represent the volatilities of the tracking error for each view and the other elements represent the covariance between the views). Thus, the coherence of the views is achieved by looking at their correlation matrix[35]. There are three possible cases that can be mentioned, namely

1. High positive correlation, meaning that the views are consistent with the historical data of the market. If most of the views are positively correlated, there is not so many diversification opportunities and the implied alphas should move in the same way.
2. High negative correlation, reflecting that the investor is betting against the market data and, in this case, the implied alphas should move in opposite ways.
3. Low correlation between the views allow for diversification opportunities since the movements in one of the views portfolio are unlikely to affect the other.

[35] Obtained through the varience-covarance matrix of the views

Global assessment: The second step is to check the views overall consistency. This can be achieved by computing Mahalanobis distance[36]. For the case of active management, this distance, denoted by M(Q), for the optimized alphas provided by the BL model, $\alpha_{BL}(Q)$, is given by

$$M(Q)^2 = \alpha_{BL}(Q)^T\Sigma^{-1}\alpha_{BL}(Q) \quad (44)$$

and $M(Q)^2$approximately follows a chi-square distribution with N degrees of freedom where N is the number of assets. Thus, a probability can be assigned to vector Q such that

$$\mathbb{P}(Q) = 1 - F_{\chi^2}(M(Q)^2, N) \quad (45)$$

where $F_{\chi^2}(\,.\,, N)$ is the cumulative probability of the chi-squared distribution with N degrees of freedom.

The manager sets a probability threshold[37] under which they reject the current vector Q for being too extreme. In the case where the threshold is not overcome, the manager must readjust the vector Q of forecasts. To decide which forecast to modify, one should analyze the sensibility of the probability $\mathbb{P}(Q)$ to Q, obtained though the derivative. Once the derivative given by

$$\frac{\partial\mathbb{P}(Q)}{\partial Q} = -2f_{\chi^2}(M(Q)^2, N)(P\Sigma P^T + \Omega)^{-1}P\alpha_{BL}(Q), \quad (46)$$

the manager needs to find the entry with the highest absolute value that corresponds to the forecast that will have the highest influence on $\mathbb{P}(Q)$. If that entry is positive, then the forecast for that view must be decreased, and the other way around. One should repeat this procedure until the threshold defined is satisfied.

The same procedure can be applied to the vector of Q to assess the overall consistency of the prior forecasts. In this case,

$$M'(Q)^2 = Q^T(P\Sigma P^T)^{-1}Q \quad (47)$$

and the correspondent probability is

[36] Introduced by Fusai and Meucci (2003)

[37] For instance, Q must be revised if $\mathbb{P}(Q)$ is higher than 0,95

$$\mathbb{P}'(Q) = 1 - F_{\chi^2}(M'(Q)^2, K) \quad (48)$$

being K the number of expressed views.

- **Risk Contributions**

To assess the risks inherent to active positions on the assets (w_a) and on view portfolios (w_{v_a}) two tools can be used, namely the Marginal Contribution to Tracking Error[38] (MCTE), which represents the sensibility of the tracking error regarding an increase of an asset's weight, henceforth the derivative of the tracking error with respect to its active weight and, secondly, the Percentage Contribution to Tracking Error[39] (PCTE), which provides the relative contribution of each asset to the total tracking error.

The vector of MCTEs (for the active weights) is

$$MCTE_i = \frac{\partial \sigma_a}{\partial w_{a_i}} = \frac{\Sigma w_{a_i}}{\sigma_a} \quad (49)$$

where i represents the i^{th} asset.

Recalling that $w_a^T \Sigma w_a = \sigma_a^2$ and multiplying the previous formula by w_a^T leads us to

$$w_a^T MCTE = \frac{w_a^T \Sigma w_a}{\sigma_a} = \sigma_a$$

$$\text{i. e.} \sum_i w_{a_i} MCTE_i = \sigma_a.$$

From the previous formula, one can see that the tracking error can be written as a weighted sum of the assets' MCTEs where the weights are each asset active weight. Also rearranging the formula, one obtain

[38] also called Absolute Marginal Contribution to Tracking Error (ACTE)

[39] also called Relative Marginal Contribution to Tracking Error (RCTE)

$$\sum_i \frac{w_{a_i} MCTE_i}{\sigma_a} = 1,$$

where in the left-hand side is the relative contribution of the asset to the total tracking error defined previously as PCTE. Thus, the PCTE for i^{th} asset is given by

$$PCTE_i = \frac{W_{a_i} MCTE_i}{\sigma_a} \quad (50)$$

The PCTEs can also be seen as a relative change in tracking error given a relative change in active weights.

Finally, these two indicators can also be calculated for the case of the view's active weights, w_{v_a}, by changing by replacing w_a in the previous formulas by w_{v_a} leading us with

$$MCTE_v = \frac{P\Sigma P^T w_{v_a}}{\sigma_a} \quad (51)$$

and

$$PCTE_{v_j} = \frac{w_{v_{a_j}} MCTE_{v_j}}{\sigma_a}. \quad (52)$$

7.10. Caixagest portfolio - Active Management

In the specific case on which we have been working on, there are three types of available benchmarks that correspond to different types of investors giving their preference for risk (in an increasing way)- a defensive one, more risk averse compared to the others, a balanced one, and a more risk lover, corresponding to a dynamic profile. These different types of investors will have a benchmark with different levels of risk, being the dynamic investor the one whose benchmark will provide the higher level of risk. The weights assigned to each asset can be observed in figure 11 and the corresponding risk[40] associated to each profile.

[40] Each level of risk was computed using the EWMA variance-covariance matrix with excess returns

It is important to notice that the defensive investor assigns the highest weights in his/her benchmark to the indexes that have the lower annual standard deviation- JPM Euro and Euro-floating rate with a standard deviation of 0.0147 and 0.0082, respectively. The weights assigned to these two assets will decrease as the investor becomes more risk lover.

To obtain the final weights that should be invested in each asset that the portfolio contains, one needs the benchmark weights (already introduced) and the active weights that should be taken according to the specified views regarding the market. Furthermore, as previously mentioned in the variation of the active model, also active views will appear if the active BL returns[41] differ from the returns provided by the GMV portfolio (presented in figure 12 joined with the active Black Litterman returns).

Starting with the GMV portfolio, the weight allocation is quite similar to what was explained earlier: the highest weights were assigned to the assets that have the lowest risk and, in this case, a weight of 93.14% was allocated to the Euro-floating rate, which is reasonable, because it can be seen as a risk-free asset. The return obtained with this portfolio is around 0.00004% with a risk of 0.106%.

Moreover, to get the active weights, one needs to specify an active risk aversion coefficient for each type of investor or a certain level of desired tracking error. Thus, our choice was to assign a TE of 1% to the defensive investor, of 1,5% to the balanced one and, finally, 2% for the case of a dynamic investor (an increasing tracking error proportional to the risk preference of each investor). Because this corresponds to including just a risk constraint in the active weights, one can scale the solution of the unconstrained optimization problem to the desired risk level and this can be done by taking the unconstrained optimal weights and multiply it by the ratio between the target level and the volatility of the unconstrained optimal portfolio[42]. After

[41] Obtained setting Π to zero

[42] Computed using a risk aversion parameter of 2.5

obtaining the vector of active weights using this procedure, one can find the final BL weights by adding the active weights to the benchmark ones. At this stage, it is to keep in mind that all the computations (including the TE) were done using the EWMA variance-covariance matrix of excess returns and not the one adjusted for the BL case, as also described in section 6.6.

Figure 13 exhibits the weights of the final portfolio that just includes the risk constraint and no constraint regarding long and short positions and also presents some summary statistics. The active return (given by α) are 0.0205%, 0.0328% and 0.0447% for the defensive, balanced and dynamic investors, respectively. The final IR will be the same in the three cases because the increase in active return was proportional to the increase in the active risk undertaken. This corresponds to a value of 1.5176%.

Another important analysis that was performed is the comparison of the IR that one would obtain using the covariance matrix under the BL framework and the initial one. Thus, in the first case, one would obtain an IR of 1.5154% against the 1.5176% referred, which corresponds to a difference of 0.00217%.

Variations:

At this stage, it is important to include some variations/constraints in the initial BL framework, namely the construction of another vector q, the long only constraint and the introduction of boundaries in the maximum weight allowed per asset class.

1. Qualitative q

Using a qualitative vector Q (prediction on alphas) obtained through the tracking error of each view, the breadth (that was set equal to 4, to obtain an IR of 0.5[43]) and the Information Coefficient (the degree of confidence in each view was set in the value of 0.25). Thus, a new vector Q was obtained, and a new correspondent vector of Black-Litterman active returns. We

[43] Typical value of IR used in this procedure, Andrew And (2014)

decided not to move on using this procedure because we consider that the values assigned to each one of the referred components are quite subjective and is very investor specific. However, out of curiosity, in figure 14 we present the qualitative Q vector and the correspondent vector of returns (α_{BL}).

2. Long only portfolio

In this case, there is no close formula to reach the vector of active weights. Thus, an optimization using SOLVER will be performed, in which the goal is to maximize the active returns of the portfolio (given by α), changing the vector of the active weights (w_a) subject to the desired target for the TE, the sum of the active weights must be zero and each weight of the final portfolio must be higher or equal to zero (long only constraint). Figure 15 displays the results obtained with this procedure as well as the respective summary statistics. For the case of a defensive investor, the alpha obtained was approximately 0.012%, against the return of 0.015% for the unconstrained portfolio (given the same level of active risk). This leads to a decrease of 0.0035 in the value of the IR. Regarding the balanced investor, the decrease in IR was lower corresponding to 0.0027 (0.019% against the alpha of 0.023% in the initial framework). Finally, a decrease of 0.0037 occurred in the riskier case.

3. Boundaries

Finally, it is left to adjust the final portfolio weights subject to the lower and the upper limits for each asset class[44] that were given to us. The procedure is very similar to the one applied before, however two more constraints should be included in the SOLVER optimization- one for the lower limits and another one for the upper limits. In figure 16, one can observe the

[44] Namely, Equities, Fixed Income US and EUR, Sovereign Emerging Markets, Credit High Yield Inflation linked and commodities

differences regarding the active return, risk and the corresponding IR. As expected, putting constraints in the initial portfolio[45] reduces the return that an investor can achieve with the strategy. Comparing the unconstrained portfolio with the more constrained one (long only and with boundaries), one can observe a reduction of 0.3852% in the IR for the defensive investor, of 0.3458% for the balanced one, and in the case of the dynamic investor, this reduction was approximately 0.3669%.

However, it is necessary to point out some pitfalls of inputting these constraints: the final portfolio is not very diversified in the sense that one should invest only in a few assets (corner solutions) in order to obtain the desired risk level. This may happen because, under the optimization process, we are performing a mean-variance optimization using the returns given by the BL active model and probably the results (the weights that one should invest in the assets) are not consistent with the views of the investor regarding the future performance of the market.

7.10.1. Risk analysis

At this stage, it is important to assess the coherence and the consistency of the views inputted in the model. To assess the individual coherence of the views, one should analyze its correlation matrix, as previously mentioned. Two thresholds need to be defined to have a sense of whether we have present a high correlation value or a low one, in absolute value. For the first case, a value of 0.65 was chosen, meaning that a correlation value above this value is considered high. Regarding the second case, the value chosen was 0.25. Thus, we observe more values with high positive correlation, which leads us to believe that the views are consistent with the historical data of the market, not allowing for much diversification opportunities. Furthermore, the views

[45] Denominated by "Unconstrained" in figure 16

are mainly little correlated, suggesting that movements in one of the views are unlikely to affect the other.

Regarding the overall consistency of the views (global assessment), Mahalanobis distance was computed and a value of 0.0002 was obtained; the probability $\mathbb{P}(Q)$ was approximately 1, a value higher than 0.95[46], meaning that there is no need to adjust the vector of the views or, in other words, Q is not too extreme. The same procedure can be applied to vector Q to evaluate the overall consistency of the prior forecasts. In this case, a value of 0.59 was obtained for $M'(Q)^2$ and the correspondent probability, $\mathbb{P}'(Q)$ is, once again, around 1, leading us to the same conclusion.

Moreover, it is essential to assess the risks inherent to active positions on the assets, as well as on portfolio's weights. Keeping this objective in mind, two statistics were computed- MCTE and PCTE. Starting by the case of the active weights, the $MCTE$ statistic (sensibility of the tracking error to the active weights) allows us to conclude that the MSCI Europe (GDDLEURO Index) is the asset to which the TE is more sensible (MCTE= 0.0035), followed by the MSCI North America (GDDUNA index) with a value of 0.025 and the Corporate investment U.S. (LUACTREH Index) with 0.016[47]. Analyzing the relative contribution of each asset to the total tracking error, given by the PCTE, MSCI U.K. comes in first place with a value of approximately 0.207, followed by the MSCI Europe and, once again, the MSCI North America, with values of 0.174 and 0.130, respectively.

Considering now the total BL weights, the results change per type of investor. Considering a defensive investor, the highest MCTEs belong to the MSCI EM EMEA (0.179) and to the MSCI Japan (0.178), while the MSCI U.K. (PCTE= 0.207) is, once again, the asset that has the highest relative contribution to the total tracking error of the portfolio. For the cases of the balanced

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[46] The usual threshold that is used (Q must be revised if $\mathbb{P}(Q)$ falls under 0,95)

[47] Results are consistent for the three types of investors

and dynamic investors, MSCI Japan presents the highest MCTE with a value of approximately 0.186 for the balanced investor and 0.189 for the dynamic one, whereas, for the case of the PCTE, the correspondent place belongs to the MSCI North America.

7.10.2. Simulation

To analyze and have a better sense on how the views given by the investor influence and change the final portfolio weights, a simulation exercise was performed, as in the non-active BL model. This simulation is based on the same assumptions regarding the P and the Q matrices as the one presented in section 5.7.1. 100 simulations with the presented structure were performed and some specific cases were studied, namely:

1. Use the BL active returns to obtain the active weights and the final portfolio weights (adding the benchmark weights to the previous ones) for the case of the unconstrained portfolio given a target level for the tracking error for each type of investor;
2. Use the BL active returns to obtain the portfolio weights in the case of a long only portfolio and, afterwards, include the boundaries;
3. Use the initial BL returns, without setting π equal to zero, also in the case of a long only final portfolio and with restrictions on the total weights that should be invested in each asset class;

Simulations 2 and 3 have the goal of analyzing which portfolio performs better in the case of betting a benchmark- if it is the one obtained using the initial BL returns or the one that uses the reformulated returns obtained setting the implied returns equal to zero.

It is also important to notice that a comparison between the weights obtained with the BL model (always computed for the case of the two returns previously mentioned) and the ones that one can get with the tangency portfolio[48] was performed, with the objective of understanding if the BL weights are moving towards the tangency. The tangency portfolio was computed using the following formula:

$$W_T = \frac{\Sigma^{-1}(\mu - R_f)}{i^T \Sigma^{-1}(\mu - R_f)}. \quad (53)$$

To simplify the computations, we assumed a value of zero for the risk free. Furthermore, the EWMA variance-covariance matrix of excess returns was used, as in the BL model.

Furthermore, we also computed the weights that one can get if no views are specified in the model and some statistics were computed in order to get a sense of the difference between the weights obtained with the simulated views and the ones that one gets in the case of no views[49]. Finally, a comparison between the weights of the portfolio for each investor with the correspondent benchmark and the market portfolio was also performed, in order to analyze if they are correlated and to have a sense of how much the final weights deviate, on average, from the benchmark or, in the second case, from the market.

Analysis 1.

To analyze the results given by the simulation, some performance statistics regarding the portfolio behavior (IR and the final portfolio return) were computed as well as statistics regarding the active weights, such as the average of the active weights in absolute value, the average of the top three active weights and, finally, the average of the last three active weights

[48] The tangency portfolio was computed only using the initial BL returns in order to do a comparison of the wights of the portfolio with the same tangency weights

[49] The weights for the case of no views were obtained by setting the returns equal to the implied ones for the case of the non-active setting. As occurred in the case of the tangency portfolio, the portfolio weights (for both returns) will be compared with the no views weights obtained using the implied returns

(to assess the sensibility of the model to the views). This scanning was also done for the three types of investors that we have been working with, using different levels of tracking error that go from 1% to 5%. However, for simplification purposes, we will make only reference to the typical values (1% for the defensive investor, 1.5% for the balanced one and 2% for the risk lover investor)[50]. In figure 17, we present the referred statistics and, as expected because the level of the risk, the defensive investor will have the lowest value for the average of the absolute active weights (2.511%), while the dynamic investor will have the highest one (5.021%). In terms of the top three active weights, the same behavior is observed, and the values are 8.643%, 12.964% and 17.285% for the defensive, balanced and dynamic investors, respectively.

Analysis 2.

Using, for instance, the active returns (obtained setting π equal to zero) to get the final weights that one should invest in each asset that composes the portfolio, an average correlation between the returns and the active weights can be obtained. Thus, we obtained a correlation of 0.204 for the case of a defensive investor, 0.301 for a balanced one and 0.397 for the defensive investor, considering all the restrictions on the final weights mentioned. One can see that the values of the correlation are positive, meaning that it is expected that the active weights increase when the return is higher and one has a positive view on a specific asset which will result in a positive active weight (they move in the same direction). Comparing the previous values with the average correlation for the case that only includes the returns and weights of the assets contained in the views, the average values obtained are even higher: 0.357 of correlation for the defensive investor and 0.477 and 0.585 for the balanced and dynamic ones, respectively. It is important to notice that the values for the case of a long only portfolio without restrictions on the weights do not differ that much when compared with the previous ones. The fact that the

[50] One can observe all analysis performed in the excel file

correlation between the active weights and the returns of the assets that are included in the views is positive is an important result that comes from this simulation exercise, in the sense that under the BL model, it is expected that if an investor has a positive view in a certain asset, the deviation from the implied returns (that were set to zero) will be positive and the other way around, which corresponds to our initial expectations regarding the behavior of the model. Not considering each investor in specific, an average correlation of 0.473 was obtained between the returns and the active weights when one considers just the assets for which a view was set in the model.

Analyzing now the average IR that one can get using the BL model with boundaries on the final weights, for the case of a defensive investor, that value corresponds to 0.0123 (average active return of 0.012% and a TE of 0.01), a value of 0.0124 for the balanced investor (alpha is equal to 0.019% and the TE is 0.015) and, finally, the riskier investor can get an IR of approximately 0.0117 for a return of 0.023% and a TE of 0.02. The values of the IR for the case of a long only portfolio are, as expected, a little bit higher than the ones mentioned as less restrictions are included in the original framework of the BL model.

Regarding the average active weights, and once again analyzing the more constrained portfolio, the average absolute active weight has a value of 2.610% (for instance, the average of top three is 7.364% and for the last three is -8.199%), 3.227% and 3.211% for the defensive, balanced and dynamic investors, respectively.

In figure 18 one can have a sense of all the statistics computed and the respective values for each case of investor, such as the average top and bottom three active and final portfolio weights.

Analysis 3.

Studying now the case where one uses the initial BL returns, the average values for the active and final portfolio weights are quite similar to the ones described in the previous analysis. However, some importance should be given to the average correlations between the returns and the active weights and to the IR (and consequently the value of alpha[51]) obtained with this strategy.

Thus, for the correlation in which all assets are included in the computations, the average values are also positive (returns and active weights move in the same direction) with values of 0.140 for the defensive investor, 0.2442 for the balanced one and 0.3761 for the dynamic. The average correlation in this case is lower than the one obtained using the active BL returns, because in this last case we "delete" the effect of the implied returns from the computation of the BL returns in each simulation, meaning that, by construction, a positive correlation is expected. Regarding the average correlation that only includes the assets for which we have a view, the defensive and balanced investors have, in this case, a negative relation between returns and active weights of approximately -0.200 and -0.074, respectively. This negative correlation is also verified in the case where we do not include constraints on the total weights that should be invested in each asset class. The fact that we do not obtain a positive correlation when using the initial BL returns can be viewed as a downside of using these type of returns in the BL framework. As a matter of fact, if one has a positive view in a certain asset, the active weight assigned to it by the model should be positive (the final portfolio weight will deviate positively from the implied returns in order to account for the expectation of the investor).

For the dynamic investor, a positive value equal to 0.056 was obtained. It corresponds to a positive correlation, however is very close to zero, and it can be explained by the fact that this investor is more close to the market weights.

Regarding the IR, one can get a higher return for the same level of risk with the model that uses the initial BL returns for both the setups analyzed, namely the one that only assumes that the

[51] One should recall that the levels of TE are the same

final weights should be higher than zero and the one that includes the constraints on the total weights that should be invested per asset class. Hence, the average IR for the defensive investor (case with boundaries) is 0.1186 (increased 0.1063 when compared with the one obtained for the case of active returns), the balanced investor obtained a value of 0.1193 (increased 0.1069 when compared with the other model) and for the dynamic one the increase was around 0.1030, corresponding to an IR of 0.1147. Regarding the average active return, it is 0.119%, 0.179% and 0.229% for the three types of investors under analysis being the last the dynamic one.

Figure 19 presents all the computed statistics and respective values for the case where one chooses to use the initial BL returns in the model instead of the active ones.

As previously mentioned, a comparison of these portfolios with the tangency one was performed. Firstly, it is important to have a sense of the average values obtained when one decides to invest in the tangency portfolio. Assuming the initial BL returns for the computation of this portfolio, the average return obtained was of 0.69% and the average absolute weights of this portfolio is 12.32% (the top three correspond to 40.41% and the bottom three to -26.83%). At this point, one is able to study the difference between the weights obtained using the BL framework and the ones obtained through the tangency portfolio. Studying specifically the more constrained model (maximum weight that should be invested in each asset class) using the initial BL returns, the average difference between the mentioned weights is 12.31% for the defensive investor and around 11% for the other two types of investors (these values almost do not change when one considers just the long only portfolio). To test if the final portfolio is moving towards the tangency, the correlation between both weights was computed. Thus, for the defensive investor, the average correlation is approximately equal to 0.3251, a value that increases to 0.4780 for the balanced one and for the case of the dynamic investor it is equal to 0.4161. If one choses to use the active returns, the average of the difference between the BL and the tangency weights increases and consequently the average correlation decreases for all the investors. Thus, the correlation between the two weights is around 0.2082, 0.2976 and 0.267 for the three investors, starting by the more risk averse.

One can also analyze the difference between the portfolio weights where views are specified with the case where the investor does not have views regarding the future performance of the market (the BL returns correspond to the implied returns in this case). Studying in particular the same example previously analyzed for the tangency portfolio- portfolio with boundaries-, and starting by the initial BL returns, the average difference (in absolute value) between the portfolio weights and the weights obtained with no views is equal to 1.30% for the defensive investor, 1.89% for the balanced one and 1.46% for the dynamic investor. Regarding the average correlation between these weights, it is very high, corresponding to a value of almost 1 (perfectly correlated). Observing now the case where one uses the active returns as input to obtain the portfolio weights, the average of the absolute difference between the portfolio weights and the ones where we do not have any views increases comparing with the previous framework (is approximately 3% for all the investors). Moreover, as expected, the correlation between them decreases to 0.6721 for the case of the defensive investor, 0.6305 and 0.6933 for the balanced and dynamic ones, respectively. It is important to point out that when looking to the average correlation between the active weights and the active weights with the case of no views, the defensive and balanced investors have a negative value (-0.3300 and -0.1903, respectively).

Moving to the comparison between the portfolio weights and the benchmark ones, the correlation between them are in order of 0.8 for the investors under analysis and under the case that all the restrictions were taken into the model and one considers the initial BL returns. The same behavior is observed when one switches to the returns obtained setting the implied returns to zero, meaning that these two frameworks do not vary that much in what concerns the deviations and correlations with the benchmark portfolio.

Finally, doing the same analysis but for the case of the market weights and considering the initial BL returns, the correlation has a value of 0.2095 for the defensive investor and increases to 0.8189 for the dynamic one, meaning that the more risk lover investor is more close to the market behavior. Moving to the framework where one uses the active returns in the computation of the portfolio weights, the correlation between the market and the final portfolio decreases when compared with the previous case for values of 0.0561 for the defensive investor and 0.7494 for the dynamic one.

As previous mentioned, in order to obtain more information, one should look at figures 18 and 19.

In summary, from the simulation exercise one can conclude that the correlation between the active weights and the returns considering just the assets that are included in the views is more reasonable when one uses the returns obtained setting the implied returns to zero, in the sense that the model accounts by construction for the fact that is an investor has a positive view in a certain asset the active weight will also be positive. Regarding the correlation between the tangency portfolio and the final portfolio weights, it is higher when one uses the initial BL returns, result that is also verified when one compares the portfolio weights with the weights obtained when an investor does not has any views regarding the market. Moving to the comparison with the market portfolio, this correlation also lower for the case where one uses the active returns. Finally, it is also important to understand which model will allow to obtain a higher value of IR (or active return for a certain level of TE). Thus, the framework that uses the initial BL returns results in a higher values of active return (alpha).

In appendix B, it is presented an analysis of specific simulation cases that can be interesting to analyze if one wants to have a deeper sense of the results obtained with this exercise.

Section 8. Conclusion

In this work project, we studied all the theoretical concepts behind the BL model and exposed in detail its inputs and implementation. It is important to mention the value added of the work project to Caixagest: the improvement of the estimation of the variance-covariance matrix was clear. Caixagest was working with an historical rolling window to estimate the variance-covariance matrix, which did not properly capture the behavior of volatility and did not have a robust predictive power. Therefore, the development of an EWMA and a GARCH method, that allow to capture more accurately the behavior of volatility, was a major improvement. In terms of predictive power, no clear conclusion between the EWMA and the GARCH method was obtained, nevertheless the GARCH is not as robust as the EWMA method as the correlations inputted in the variance-covariance matrix estimated by the GARCH were the ones obtained with the EWMA. Therefore, the GARCH variance-covariance matrix lacks of robustness and, therefore, the variance-covariance matrix obtained through the EWMA was the chosen to input into the model.

A rigorous explanation of the original BL model was important to clear the concepts and served as a departure point for the development of the BL model under the active management framework, where the investor is evaluated relative to a benchmark. Some important conclusions regarding the original model can be taken: the implementation of the model as it was initially built does not result in an appropriate solution for Caixagest. Although the simulation shows that the optimal BL weights result in a more diversified portfolio and the portfolio performance is better when compared to a portfolio using market weights, one should keep in mind that the optimal BL weights are calculated without any constraints. Adding any other further constraints that are given from Caixagest and are aligned with its investment policy, will lead to more concentrated investments in a few assets and the investor's views will not completely be included in the BL model.

As Caixagest is an asset manager and the chase for active return on investments is very important, the implementation of the BL model in the active framework was crucial. In this section, the value-added is evident as we expose reformulations that should be done to the initial BL model in order to surpass the fragilities of implementing it under an active framework. Thus, the implementation of the model under the active management could lead to unintended trades and to excessive risk taking, as mentioned in section 7. The findings relative to this problem, mentioned in some of the most prominent literature, lead us to use the remedy of setting the vector of implied returns to zero.

However, as the results obtained with this framework were not that conclusive regarding the practical benefits of applying the BL model considering all the specificities of Caixagest's portfolio[52], in section 8, a simulation exercise was developed to assess how the model reacts to the views that are inputted in the model. At this point, it was also interesting to make comparison between the two ways of computing returns under the BL model- one is with the initial formula present by Black and Litterman and discussed in section 6 and the other is setting the implied returns equal to zero.

It was important to analyze the correlations that exist between the portfolio weights obtained under the previous mentioned returns with the tangency portfolio and the market portfolio, for instance, because it is expected that an investor will move towards the tangency and away from benchmark even if views are opposite. At a second level, an analysis regarding the correlation between the behavior of the active weights and the returns was also performed.

[52] For instance, no short-selling is allowed and there are boundaries that should be respected in terms of the maximum and minimum weights that can be invested in each asset class

Both frameworks studied have pros and cons when implemented in the particular case of Caixagest's portfolio, and this is the main reason why we do not specify a clear approach to follow. Using the initial BL returns will lead to a higher SR on average, however the correlation between the active weights and the returns are not returning the expected comportment: on average, a negative correlation is observed between these two vector, while the opposite should happen. Thus, this is not verified using the reformulation of the results (by construction), however the active return is lower in this case. In the statistics regarding the final active and portfolio weights, both cases present similar values which does not allow to have a clear conclusion of what should be used to obtain the final portfolio.

This work is relevant for future researchers and practitioners that intend to implement this model, as it describes a direct implementation of the model in the real world and exposes all the process and its fragilities. These fragilities come from the fact that some of the model inputs are not well described in literature. Therefore, it is important to leave some recommendations about the implementation of the BL model. The practitioner should be aware that some topics still need to be deepened: firstly, it is important to mention that more complex methods of modelling volatility are available and are worth being mentioned, as there is not a "best model" to estimate volatilities. As a matter of fact, it is very difficult to fully capture the future behavior of volatility. Moreover, volatility has different characteristics depending on the asset class studied and many authors have shown that the predictive ability of a model depends on the type of asset studied and on the sample size. For instance, there are some models that can capture accurately the volatility of commodities and not that accurately the equities' volatility. Thus, we recommend one not to gird to the three methods described in our work project and broaden the implementation of estimating variance-covariance matrices to other methods as any method has its upsides and downsides. The ARCH, ARMA and the implied standard deviation models, that use volatility implied by the pricing in the options market, are just some other methods of modelling volatility. Furthermore, one input of the BL model that is not very well defined in literature is Ω. Meucci (2005) describes Ω, which represents the uncertainty of the views, as a non-diagonal matrix. This formulation of Ω leads to a much simpler calculation of expected excess returns implied by the model. However, computing the BL expected returns in this way requires the P matrix to be invertible. This is not always the case and thus, further research should be undertaken in order to find a user friendly solution when applying the Meucci approach. Additionally, it might be interesting to investigate the impact of τ when calculating Ω. Consequentially, a higher value of tau will lead to changes in the value of the diagonal elements of Ω and might push the portfolio towards the investor's views.

All in all, this work project was a result of the combination of the theoretical knowledge that was gleaned from the academic literature and the practical implementation of the model at Caixagest. This work project brings value to Caixagest as it improves the mean-variance optimization approach of portfolio selection and solves for the problems described previously in the active management framework. Despite having much literature on BL available, there is still many research too be done regarding the application of the model to different assets classes and studying the effect of including different constraints in the initial framework.

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