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The maximum multiplicity and the two largest multiplicities of eigenvalues in a Hermitian matrix whose graph is a tree

Abstract: The maximum multiplicity of an eigenvalue in a matrix whose graph is a tree, M_1 , was understood fully (from a combinatorial perspective) by C.R. Johnson, A. Leal-Duarte (Linear Algebra and Multilinear Algebra 46 (1999) 139-144). Among the possible multiplicity lists for the eigenvalues of Hermitian matrices whose graph is a tree, we focus upon \overline{M}_2 , the maximum value of the sum of the two largest multiplicities when the largest multiplicity is M_1 . Upper and lower bounds are given for \overline{M}_2 . Using a combinatorial algorithm, cases of equality are computed for \overline{M}_2 . 5

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1 Introduction

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Let T be a tree on $n \geq 2$ vertices. We denote by $S(T)$ the collection of all n -by- n complex Hermitian matrices whose graph is T . No restriction is placed upon the diagonal entries of matrices in $S(T)$.

For convenience, when $A \in S(T)$, we place in non-increasing order the multiplicities of the eigenvalues of A . We refer to such a list of multiplicities as the *unordered multiplicity list* and we denote it by $(m_1(A), m_2(A), \dots, m_{k(A)}(A))$, where $k(A)$ is the number of distinct eigenvalues of A . So, $m_j(A)$ is the j th largest multiplicity of an eigenvalue in the multiplicity list of A . 20

Definition 1.1. Let $\mathcal{L}(T)$ be the set of all positive integer lists (unordered multiplicity lists) (p_1, p_2, \dots, p_s) satisfying:

- (1) $p_1 \geq p_2 \geq \dots \geq p_s \geq 1$;
- (2) $\sum_{i=1}^s p_i = n$;
- (3) There is an $A \in S(T)$ with $(m_1(A), m_2(A), \dots, m_{k(A)}(A)) = (p_1, p_2, \dots, p_s)$.

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For $j \geq 1$, we denote by

$$M_j(T) = \max_{(p_1, p_2, \dots, p_s) \in \mathcal{L}(T)} (p_1 + \dots + p_j).$$

It is well known that $M_1(T)$ is equal to the path cover number $P(T)$, the smallest number of nonintersecting induced paths of T that cover all the vertices of T ; this is the same as $\max(p - q)$, where p is the number of paths remaining when q vertices have been removed from T in such a way as to leave only induced paths [3]. 30

Remark 1.2. In [7] a combinatorial algorithm was given to compute $M_2(T)$. It is easy to see that if $(p_1, p_2, \dots, p_s) \in \mathcal{L}(T)$ then

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- (1) $p_1 \leq M_1(T)$.
 (2) $p_1 + p_2 \leq M_2(T)$.
 (3) $p_1 + p_2 \geq 2$, $p_2 \neq 0$ (because if T is a tree and $A \in \mathcal{S}(T)$ then the largest and the smallest eigenvalues of A have multiplicities one. So, each list in $\mathcal{L}(T)$ has at least two 1's, [4]).
 5 (4) Using the definition of $M_1(T)$, there exists $(p_1, p_2, \dots, p_s) \in \mathcal{L}(T)$ such that $p_1 = M_1(T)$.

Given $M_1(T)$ and $M_2(T)$, we cannot say there exists a list $(p_1, p_2, \dots, p_s) \in \mathcal{L}(T)$ such that $p_1 = M_1(T)$ and $p_2 = M_2(T) - M_1(T)$. For example, [7], the double star $D_{3,3}$ has $M_1(D_{3,3}) = 4$, $M_2(D_{3,3}) = 6$ but $(4, 2, 1, 1) \notin \mathcal{L}(D_{3,3})$ (we can prove this using the Parter-Wiener theorem [5]). $M_1(D_{3,3}) = 4$ because $(4, 1, 1, 1, 1) \in \mathcal{L}(D_{3,3})$, for example, consider the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

$M_2(D_{3,3}) = 6$ because $(3, 3, 1, 1) \in \mathcal{L}(D_{3,3})$, for example, consider the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

So, it is important to know when given $M_2(T)$, we can say that there is a list $(p_1, p_2, \dots, p_s) \in \mathcal{L}(T)$ such that $p_1 = M_1(T)$ and $p_2 = M_2(T) - M_1(T)$.

Let $\overline{M}_2(T)$ (or simply \overline{M}_2) denote the maximum value of the sum of the two largest integers among the lists $(p_1, p_2, \dots, p_s) \in \mathcal{L}(T)$, when $p_1 = M_1(T)$, i.e.,

$$\overline{M}_2(T) = \max_{(M_1(T), p_2, \dots, p_s) \in \mathcal{L}(T)} (M_1(T) + p_2).$$

Using the definition of $M_2(T)$, we have $\overline{M}_2(T) \leq M_2(T)$. In this paper we give upper and lower bounds for \overline{M}_2 and in some cases, a method for calculating \overline{M}_2 .

10 2 Assignments

Let T be a tree on $n \geq 2$ vertices. If $A \in \mathcal{S}(T)$ and v is a vertex of T then $A(v)$ denotes the principal submatrix of A resulting from deleting row and column associated with v , and $m_A(\lambda)$ denotes the multiplicity of eigenvalue λ of matrix A . The Parter theorem, [8], indicates that if $A \in \mathcal{S}(T)$ and $m_A(\lambda) \geq 2$, then there is at least one vertex v of T , of degree at least 3, such that $m_{A(v)}(\lambda) = m_A(\lambda) + 1$. Moreover, v may be chosen
 15 so that λ is an eigenvalue of at least three principal submatrices of A associated with branches of T at v . So, we refer to any vertex v of degree greater or equal to 3 as a *high-degree vertex*, or HDV. The Parter theorem was refined by Wiener [9] and more fully in [5]. A vertex v of T is a *Parter vertex* for $A \in \mathcal{S}(T)$ and λ when $m_A(\lambda) \geq 1$ and $m_{A(v)}(\lambda) = m_A(\lambda) + 1$. The Parter theorem guarantees the existence of at least one Parter HDV

for any multiple eigenvalue. If a principal submatrix of A associated with some branch at v again has λ as a multiple eigenvalue, then this theorem may again be applied to that branch. Parter vertices for λ may be removed in this fashion until (fully) fragmenting T into many subtrees when λ occurs as an eigenvalue in such a submatrix associated with the subtree at most once. Such a set of Parter vertices is called a *fully fragmented Parter set* for λ , and it is known that each successive Parter vertex is also a Parter vertex for A and λ in the original tree. 5

If X is a set or collection (or graph), then $|X|$ denotes the cardinality of (number of vertices in) X . If V is a set of vertices and X is a graph then $V \cap X$ denotes the set of vertices in both V and X . If X is a tree then $\mathcal{P}(X)$ denotes the collection of all subtrees of X , including X .

Definition 2.1. [7] (Assignment) Let T be a tree on $n \geq 2$ vertices and let

$$\left(p_1, p_2, \dots, p_k, 1^{n-\sum_{i=1}^k p_i}\right)$$

be a non-increasing list of positive integers, with $\sum_{i=1}^k p_i \leq n$. The notation 1^l denotes that the last l entries of the list are 1. Note that some of the p_i 's may be 1. An assignment \mathcal{A} of $\left(p_1, p_2, \dots, p_k, 1^{n-\sum_{i=1}^k p_i}\right)$ to T is a collection $\mathcal{A} = ((\mathcal{A}_1, V_1), \dots, (\mathcal{A}_k, V_k))$ of k collections \mathcal{A}_i of subtrees of T and k collections V_i of vertices of T , with the following properties. 10

- (1) (Specification of Parter vertices) For each integer i between 1 and k ,
 - (1a) Each subtree in \mathcal{A}_i is a connected component of $T - V_i$. 15
 - (1b) $|\mathcal{A}_i| = p_i + |V_i|$.
 - (1c) For each vertex $v \in V_i$, there exists a vertex x adjacent to v such that x is in one of the subtrees in \mathcal{A}_i .
- (2) (No overloading) We require that no subtree S of T is assigned more than $|S|$ integers; define

$$c_i(S) = |\mathcal{A}_i \cap \mathcal{P}(S)| - |V_i \cap S|,$$

the difference between the number of subtrees contained in S and the number of Parter vertices in S for the i th integer. So, we require that

$$\sum_{i=1}^k \max(0, c_i(S)) \leq |S|, \text{ for each } S \in \mathcal{P}(T).$$

If this condition is violated at any subtree, then that subtree is said to be overloaded.

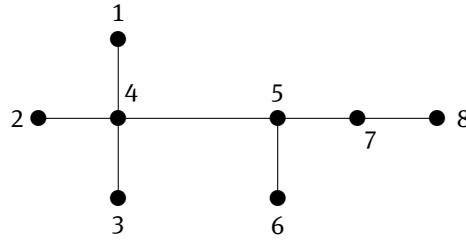
Definition 2.2. [7] A collection $\mathcal{A} = ((\mathcal{A}_1, V_1), \dots, (\mathcal{A}_k, V_k))$ of k collections \mathcal{A}_i of subtrees of T and k collections V_i of vertices of T is: 20

- (1) an assignment candidate of $\left(p_1, p_2, \dots, p_k, 1^{n-\sum_{i=1}^k p_i}\right)$ to T when \mathcal{A} satisfies condition 1, but not necessarily 2 of Definition 2.1.
- (2) a near-assignment of $\left(p_1, p_2, \dots, p_k, 1^{n-\sum_{i=1}^k p_i}\right)$ to T when \mathcal{A} satisfies conditions 1a, 1b, 2, but not necessarily 1c of Definition 2.1. 25
- (3) a near-assignment candidate of $\left(p_1, p_2, \dots, p_k, 1^{n-\sum_{i=1}^k p_i}\right)$ to T when \mathcal{A} satisfies conditions 1a, 1b, but not necessarily 1c or 2 of Definition 2.1.

In [7] a simplification of assignments of the list $(p_1, p_2, 1^l)$ is considered.

Lemma 2.3. (Overloading Lemma) If T is a tree and \mathcal{A} is an assignment candidate (or a near-assignment candidate) of the list $(p_1, p_2, 1^l)$ to T , but \mathcal{A} is not an assignment (or a near-assignment, respectively), then there must exist a single vertex in T that is overloaded by \mathcal{A} . 30

Example 2.4. Let T be the following tree



and let $(3, 2, 1^3)$ be a list.

If we consider $\mathcal{A} = ((\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2))$ where

$$\mathcal{A}_1 = T - \{4, 5\}, \mathcal{A}_2 = T - \{5\}, V_1 = \{4, 5\} \text{ and } V_2 = \{5\},$$

then \mathcal{A}_1 has 5 connected components and \mathcal{A}_2 has 3 connected components. So, $|\mathcal{A}_1| = 5$ and $|\mathcal{A}_2| = 3$.

\mathcal{A} is an assignment candidate of $(3, 2, 1^3)$ to T but not an assignment because the subtree $\{6\}$ of T satisfies

$$\max(0, c_1(\{6\})) + \max(0, c_2(\{6\})) = 1 + 1 = 2 > 1 = |\{6\}|.$$

If we consider $\mathcal{A}' = ((\mathcal{A}'_1, V'_1), (\mathcal{A}'_2, V'_2))$, where

$$\mathcal{A}'_1 = T - \{4\}, \mathcal{A}'_2 = T - \{5\} \quad V'_1 = \{4\} \text{ and } V'_2 = \{5\}$$

then \mathcal{A}'_1 has 4 connected components and \mathcal{A}'_2 has 3 connected components. So, $|\mathcal{A}'_1| = 4$ and $|\mathcal{A}'_2| = 3$.

5 \mathcal{A}' satisfies condition 1 of Definition 2.1.

If $S = \{1\}$ or $S = \{2\}$ or $S = \{3\}$, then

$$\max(0, c_1(S)) + \max(0, c_2(S)) = 1 + 0 = |S|.$$

If $S = \{4\}$ or $S = \{5\}$ or $S = \{7\}$ or $S = \{8\}$, then

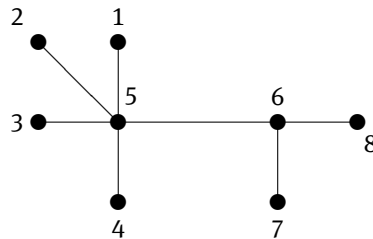
$$\max(0, c_1(S)) + \max(0, c_2(S)) = 0 + 0 < |S| = 1.$$

If $S = \{6\}$ then

$$\max(0, c_1(S)) + \max(0, c_2(S)) = 0 + 1 = |S|.$$

Using Lemma 2.3, \mathcal{A}' is an assignment of $(3, 2, 1^3)$ to T .

Example 2.5. Let T be the following tree



and let $(2, 2, 1^4)$ be a list.

If we consider $\mathcal{A} = ((\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2))$, where

$$\mathcal{A}_1 = T - \{5, 6, 7, 8\}, \mathcal{A}_2 = T - \{6\}, V_1 = \{5, 6\} \text{ and } V_2 = \{6\}$$

10 then \mathcal{A}_1 has 4 connected components and \mathcal{A}_2 has 3 connected components. So, $|\mathcal{A}_1| = 4$ and $|\mathcal{A}_2| = 3$.

\mathcal{A} is a near-assignment of $(2, 2, 1^4)$ to T (to prove condition 2 of Definition 2.1 use Lemma 2.3) but not an assignment because $6 \in V_1$ and there is not a vertex of T adjacent to 6 in a subtree of \mathcal{A}_1 .

Using the Overloading Lemma (Lemma 2.3), another important result appears.

Lemma 2.6. *Let T be a tree. Then*

there exists a near-assignment of the list $(p_1, p_2, 1^l)$ to T if and only if there exists an assignment of the list $(p_1, p_2, 1^l)$ to T .

Proof Suppose there exists a near-assignment $\mathcal{A} = ((\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2))$ of the list $(p_1, p_2, 1^l)$ to T . If \mathcal{A} satisfies 5 1c of Definition 2.1, then \mathcal{A} is an assignment of $(p_1, p_2, 1^l)$ to T .

Suppose that \mathcal{A} does not satisfy 1c. Then V_1 or V_2 does not satisfy 1c. Suppose, without loss of generalization that V_1 does not satisfy 1c. So, there exists a vertex $v_1 \in V_1$ such that there is not a vertex x adjacent to v_1 in a subtree of \mathcal{A}_1 .

Since $|\mathcal{A}_1| = p_1 + |V_1|$, remove v_1 from V_1 and remove a subtree R_1 from \mathcal{A}_1 . We obtain $\mathcal{A}'_1 = \mathcal{A}_1 \setminus R_1$ 10 and $V'_1 = V_1 \setminus \{v_1\}$. Since $|\mathcal{A}'_1| = p_1 + |V'_1|$, we conclude that $\mathcal{A}' = ((\mathcal{A}'_1, V'_1), (\mathcal{A}_2, V_2))$ is a near-assignment candidate of the list $(p_1, p_2, 1^l)$ to T .

If \mathcal{A}' is not a near-assignment, by Lemma 2.3, there must exist a single vertex y in T that is overloaded by \mathcal{A}' . Using the fact that \mathcal{A} is a near-assignment, $y = v_1$. But v_1 does not belong to \mathcal{A}'_1 . Consequently, $S = \{v_1\}$ satisfies condition 2 of Definition 2.1. Contradiction. Therefore, \mathcal{A}' is a near-assignment. 15

If \mathcal{A}' satisfies 1c of Definition 2.1, then \mathcal{A}' is an assignment of $(p_1, p_2, 1^l)$ to T . If \mathcal{A}' does not satisfy 1c of Definition 2.1, repeat the process.

Repeating this process we obtain an assignment because $p_1, p_2 \geq 1$ and in each process we have a collection of subtrees of T satisfying condition 1a of Definition 2.1.

Conversely, the proof is trivial. □ 20

Definition 2.7. *If $A \in \mathcal{S}(T)$ and S is a subgraph of T then*

- (1) $A[S]$ denotes the principal submatrix of A lying on rows and columns associated with the vertices of S .
- (2) $A(S)$ denotes the principal submatrix of A resulting from deleting rows and columns associated with the vertices of S .

Using the interlacing theorem for Hermitian matrices [2], if x is a vertex of T (tree) and λ is an eigenvalue of $A \in \mathcal{S}(T)$, then there is a simple relation between $m_{A(x)}(\lambda)$ and $m_A(\lambda)$:

$$m_{A(x)}(\lambda) = m_A(\lambda) - 1 \quad \text{or} \quad m_{A(x)}(\lambda) = m_A(\lambda) \quad \text{or} \quad m_{A(x)}(\lambda) = m_A(\lambda) + 1.$$

Definition 2.8. [7] *Let T be a tree on $n \geq 2$ vertices. We call an assignment $\mathcal{A} = ((\mathcal{A}_1, V_1), \dots, (\mathcal{A}_k, V_k))$ of 25 $(p_1, p_2, \dots, p_k, 1^{n-\sum_{i=1}^k p_i})$ to T realizable if there exists a matrix $B \in \mathcal{S}(T)$ with unordered multiplicity list $(p_1, p_2, \dots, p_k, 1^{n-\sum_{i=1}^k p_i})$, such that, for each i between 1 and k , if s_i is the eigenvalue of B associated with p_i , i.e, $m_B(s_i) = p_i$, then:*

- (1) For each subtree R of T in \mathcal{A}_i , $m_{B[R]}(s_i) = 1$.
- (2) For each connected component Q of $T - V_i$ that is not in \mathcal{A}_i , $m_{B[Q]}(s_i) = 0$. 30
- (3) For each $x \in V_i$, x is a Parter vertex for B and s_i .

Remark 2.9. *Note that if $C \in \mathcal{S}(T)$ is a matrix that satisfies conditions 1 and 2 of Definition 2.8, then for each i between 1 and k , $m_C(s_i) = p_i \geq 1$.*

Using the interlacing theorem for Hermitian matrices, if $x \in V_i$, then $m_{C(x)}(s_i)$ is equal to

$$m_C(s_i) - 1 \quad \text{or} \quad m_C(s_i) \quad \text{or} \quad m_C(s_i) + 1.$$

By conditions 1 and 2 of Definition 2.8, $m_{C(V_i)}(s_i) = |\mathcal{A}_i|$. But \mathcal{A} is an assignment, so, $|\mathcal{A}_i| = p_i + |V_i|$. Thus,

$$m_{C(x)}(s_i) = m_C(s_i) + 1.$$

Therefore, C satisfies Definition 2.8. □

Using the last remark, we can rewrite Definition 2.8.

Definition 2.8 Let T be a tree on $n \geq 2$ vertices. We call an assignment $\mathcal{A} = ((\mathcal{A}_1, V_1), \dots, (\mathcal{A}_k, V_k))$ of $(p_1, p_2, \dots, p_k, 1^{n-\sum_{i=1}^k p_i})$ to T realizable if there exists a matrix $B \in \mathcal{S}(T)$ with unordered multiplicity list $(p_1, p_2, \dots, p_k, 1^{n-\sum_{i=1}^k p_i})$, such that, for each i between 1 and k , if s_i is the eigenvalue of B associated with p_i , i.e, $m_B(s_i) = p_i$, then:

- (1) For each subtree R of T in \mathcal{A}_i , $m_{B[R]}(s_i) = 1$.
- (2) For each connected component Q of $T - V_i$ that is not in \mathcal{A}_i , $m_{B[Q]}(s_i) = 0$.

Definition 2.10. If T is a tree on $n \geq 2$ vertices, \mathcal{A} is a realizable assignment of $(p_1, p_2, \dots, p_k, 1^{n-\sum_{i=1}^k p_i})$ to T and $B \in \mathcal{S}(T)$ is a matrix that satisfies Definition 2.8, then we say that B realizes the assignment \mathcal{A} .

10 There are assignments that are not realizable. For instance see Example 2.3 in [7]. However when we study the list $(p_1, p_2, 1^l)$ we have the following result.

Theorem 2.11. [7] Given a tree T on $n = p_1 + p_2 + l$ vertices, a near-assignment of the list (p_1, p_2, l^1) to T , $\mathcal{A} = ((\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2))$, and any distinct real numbers α and β , then there exists $A \in \mathcal{S}(T)$ satisfying the following conditions:

If R is a connected component of $T - V_1$, then

$$\alpha \text{ is an eigenvalue of } A[R] \text{ if and only if } R \in \mathcal{A}_1.$$

Similarly, if S is a connected component of $T - V_2$, then

$$\beta \text{ is an eigenvalue of } A[S] \text{ if and only if } S \in \mathcal{A}_2.$$

15 Using Lemma 2.6, Theorem 2.11 and the new version of Definition 2.8 we obtain the following result.

Theorem 2.12. Given a tree T on $n = p_1 + p_2 + l$ vertices, a near-assignment \mathcal{A} of the list (p_1, p_2, l^1) to T , and any distinct real numbers α and β , then

- (1) there exists a realizable assignment \mathcal{B} of $(p_1, p_2, 1^l)$ to T .
- (2) there exists $A \in \mathcal{S}(T)$ that realizes the assignment \mathcal{B} with $m_A(\alpha) = p_1$ and $m_A(\beta) = p_2$.

20 Therefore, we immediately have as a consequence:

Corollary 2.13. For any tree T , if there exists a near-assignment of the list $(M_1(T), p_2, 1^l)$ to T , then

$$\overline{M}_2(T) \geq M_1(T) + p_2.$$

3 Upper and lower bounds for \overline{M}_2

In this section, using the reduction theorem for M_2 , [7], we directly compute \overline{M}_2 for particular trees. For other kind of trees, we give bounds on \overline{M}_2 .

In [7], the authors directly computed M_2 for generalized stars (for the notion of generalized star see [6]).

25 **Definition 3.1.** [6] Let T be a tree and x_0 be a vertex of T . A generalized star T with central vertex x_0 is a tree such that $T - \{x_0\}$ is a union of paths (arms), each one of them is adjacent to x_0 by an endpoint.

Proposition 3.2. [7] Let T be a generalized star on $n \geq 2$ vertices, with f arms of length 1 and g arms of length at least 2. Then:

- (A) If $g \geq 2$, then $M_2(T) = f + 2g - 2$.

- (B) If $g \leq 1$ and T is not a path, then $M_2(T) = f + g$.
- (C) If T is a path, then $M_2(T) = 2$.

Definition 3.3. [7] (*Peripheral HDV, peripheral arm*) Given a tree T and a high-degree vertex v , v is a peripheral HDV of T if and only if there is a branch of T at v that contains all the other high-degree vertices in T . A peripheral arm of a tree T is a branch of T at a peripheral HDV such that the branch does not itself contain any HDV. 5

Definition 3.4. Throughout this section, we will consider a peripheral HDV v in a tree T .

The subtree of T consisting of v and its peripheral arms will be called S - however, if v is the only HDV in T , we will let S be v and all but one of its peripheral arms (chosen arbitrarily). The point is that S should be a generalized star containing everything except a single branch of T at v .

Let w be the one vertex adjacent to v that is not in S . We denote by $(T - S)_{+w} K_1$ the tree obtained from $T - S$ 10 by putting a vertex adjacent to w .

Theorem 3.5. [7] (M_2 Reduction Theorem) Let T be a tree and v a peripheral HDV, with S as defined earlier in this section. Suppose that S has f arms of length 1 and g arms of length at least 2. Then:

- (A) If $g \geq 2$, then $M_2(T - S) = M_2(T) - f - 2g + 2$.
- (B) If $g \leq 1$, then $M_2((T - S)_{+w} K_1) = M_2(T) - f - g + 1$. 15

In [1] a class of trees was introduced that contains the generalized stars, the superstars.

Definition 3.6. [1] Let T be a tree and x_0 be a vertex of T . A superstar T with central vertex x_0 is a tree such that $T - \{x_0\}$ is a union of paths.

The focus of this section is to directly compute \overline{M}_2 for a subclass of superstars.

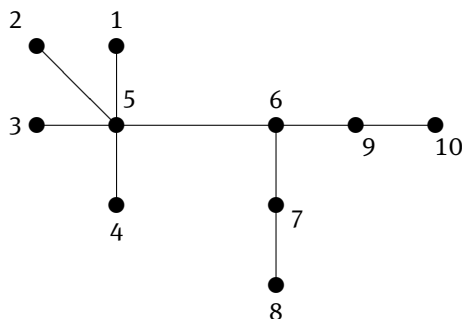
Definition 3.7. Let T be a superstar with central vertex x_0 . A small pincer of T is a path, P , of $T - \{x_0\}$ such 20 that:

- (1) P is adjacent to x_0 by a vertex u of degree two in P .
- (2) At least one path of $P - u$ is a vertex.

Definition 3.8. Let T be a superstar with central vertex x_0 . T is a small superstar if all paths of $T - \{x_0\}$ are small pincers or are adjacent to x_0 by an endpoint (arms). 25

Example 3.9. The superstar T of Example 2.4 is a small superstar with central vertex 4. The superstar T of Example 2.5 is a small superstar with central vertex 5. All stars and generalized stars are small superstars.

The following superstar is not a small superstar



Definition 3.10. Let T be a tree and \mathcal{A} an assignment of $(M_1(T), p_2, 1^1)$ to T . 30

- (1) We refer to \mathcal{A} as an \overline{M}_2 assignment to T .
- (2) If $M_1(T) + p_2 = \overline{M}_2(T)$, we refer to \mathcal{A} as an \overline{M}_2 -maximal assignment to T .

Remark 3.11. Let T be a tree and $\mathcal{A} = ((\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2))$ an \overline{M}_2 assignment of $(M_1(T), p_2, 1^1)$ to T . Because $M_1(T) = |\mathcal{A}_1| - |V_1|$,

- (1) All components of $T - V_1$ are in \mathcal{A}_1 .
- (2) We can assume that if $v \in V_1$ then v is a HDV.
- 5 (3) Since all components of $T - V_1$ are paths, if v is a peripheral HDV of degree greater or equal to 4 in T then $v \in V_1$.
- (4) If v is a peripheral HDV, $v \in V_1$ and all peripheral arms have length 1 then they are in \mathcal{A}_1 and no one is in \mathcal{A}_2 (see Lemma 2.3).

Remark 3.12. Let T be a tree and $\mathcal{A} = ((\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2))$ an \overline{M}_2 -maximal assignment of $(M_1(T), p_2, 1^1)$ to T . Because $\overline{M}_2(T) = |\mathcal{A}_1| - |V_1| + |\mathcal{A}_2| - |V_2|$,

- (1) All components of $T - V_2$ with more than one vertex are in \mathcal{A}_2 .
- (2) We can assume that if $v \in V_2$ then v is a HDV.
- (3) All components of $T - V_2$ with one vertex that are not components of $T - V_1$ are in \mathcal{A}_2 .
- (4) If v is a peripheral HDV, $v \in V_1$ and all peripheral arms have length 1 then using Remark 3.11, 4, we conclude that $v \notin V_2$.
- 15 (5) If v is a peripheral HDV, $v \in V_1$ and all peripheral arms have length 1, except one, then there is an \overline{M}_2 -maximal assignment of $(M_1(T), p_2, 1^1)$ to T such that $v \notin V_2$.

Remark 3.13. In some proofs we construct an \overline{M}_2 -maximal (or simply an \overline{M}_2) assignment of $(M_1(T), p_2, 1^1)$ to T . for some integer p_2 . In these cases, first we construct an \overline{M}_2 assignment of $(M_1(T), p_2, 1^1)$ to T , $\mathcal{A} = ((\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2))$ by putting the elements in \mathcal{A}_1 and in V_1 , next we put the elements in \mathcal{A}_2 and in V_2 , using Remarks 3.11 and 3.12. This construction is in such a way that $M_1(T) = |\mathcal{A}_1| - |V_1|$ and $M_1(T) + p_2 = |\mathcal{A}_1| - |V_1| + |\mathcal{A}_2| - |V_2|$. After using Lemma 2.3 we conclude condition 2 of Definition 2.1 and by Corollary 2.13, we say that $\overline{M}_2(T) \geq M_1(T) + p_2$.

Proposition 3.14. Let T be a small superstar on $n \geq 2$ vertices, with f arms of length 1, g arms of length at least 2 and h small pincers, with $f + g \geq 2$ or $h \geq 2$. Then:

- (A) If $g \geq 2$, then $\overline{M}_2(T) = 2h + f + 2g - 2$.
- (B) If $g \leq 1$ and T is not a path, then $\overline{M}_2(T) = 2h + f + g$.
- (C) If T is a path, then $\overline{M}_2(T) = 2$.

Proof Let x be the central vertex of T . If S is a small pincer of T , by Theorem 3.5,

$$M_2((T - S) +_x K_1) = M_2(T) - 1.$$

Since T has h small pincers,

$$M_2(T') = M_2(T) - h,$$

where T' is obtained from T by removing all small pincers and by putting h vertices adjacent to x . Consequently, T' is a generalized star with $f + h$ arms of length 1 and g arms of length at least 2. Using Proposition 3.2

$$M_2(T') = \begin{cases} f + h + 2g - 2 & \text{if } g \geq 2 \\ f + h + g & \text{if } g \leq 1 \text{ and } T' \text{ is not a path} \\ 2 & \text{if } T' \text{ is a path.} \end{cases}$$

Therefore,

$$M_2(T) = \begin{cases} f + 2h + 2g - 2 & \text{if } g \geq 2 \\ f + 2h + g & \text{if } g \leq 1 \text{ and } T' \text{ is not a path} \\ 2 + h & \text{if } T' \text{ is a path.} \end{cases}$$

Note that if T' is a path with $h = 2$ and $f = g = 0$ then T is not a path and $M_2(T) = M_2(T') + h = 2 + 2 = 4 = f + 2h + g$. By hypothesis, if $h < 2$ then $f + g \geq 2$. In this case, if T' is a path then $h = 0$ and $f + g = 2$. Consequently, T is a path.

So, we conclude that

$$M_2(T) = \begin{cases} f + 2h + 2g - 2 & \text{if } g \geq 2 \\ f + 2h + g & \text{if } g \leq 1 \text{ and } T \text{ is not a path} \\ 2 & \text{if } T \text{ is a path.} \end{cases}$$

Since $\overline{M}_2(T) \leq M_2(T)$, we have

- (A) If $g \geq 2$, then $\overline{M}_2(T) \leq 2h + f + 2g - 2$.
- (B) If $g \leq 1$ and T is not a path, then $\overline{M}_2(T) \leq 2h + f + g$.
- (C) If T is a path, then $\overline{M}_2(T) \leq 2$.

Conversely, since T is a tree,

$$M_1(T) = \begin{cases} f + h + g - 1 & \text{if } f + g \geq 2 \\ h + 1 & \text{if } f + g \leq 1. \end{cases}$$

We are going to construct an \overline{M}_2 assignment of $(M_1(T), p_2, 1^l)$, for some integer p_2 , to T (see Remark 3.13). 5

Case 1 If $f + g \geq 2$, we put the central vertex of T in V_1 and we put the $f + h + g$ paths obtained by removing the central vertex of T in \mathcal{A}_1 .

If $g \geq 2$, we put the central vertex of T in V_2 and we put the $h + g$ paths of length at least 2, obtained by removing the central vertex of T in \mathcal{A}_2 . So, $|\mathcal{A}_1| - |V_1| = f + g + h - 1 = M_1(T)$. Using Remark 3.13, $\overline{M}_2(T) \geq f + h + g - 1 + h + g - 1 = f + 2h + 2g - 2$. 10

If $g \leq 1$, we put the central vertex of each small pincer of T in V_2 , we put the $2h + 1$ subtrees obtained by removing the central vertex of all small pincers of T in \mathcal{A}_2 . Since $|\mathcal{A}_1| - |V_1| = f + g + h - 1 = M_1(T)$, using Remark 3.13, $\overline{M}_2(T) \geq f + h + g - 1 + 2h + 1 - h = f + 2h + g$.

Note that if T is a path and $f + g \geq 2$ then $f + g = 2$ and $h = 0$. Thus, if $g = 2$, then $\overline{M}_2(T) \geq f + 2h + 2g - 2 = 2$ and if $g \leq 1$, then $\overline{M}_2(T) \geq f + 2h + g = 2$. 15

Case 2 If $f + g \leq 1$ then $h \geq 2$ and T is not a path. We put the central vertex of each small pincer of T in V_1 and we put the $2h + 1$ subtrees obtained by removing the central vertex of all small pincers of T in \mathcal{A}_1 . We put the central vertex of T in V_2 and we put the $f + h + g$ paths obtained by removing the central vertex of T in \mathcal{A}_2 . Since $|\mathcal{A}_1| - |V_1| = h + 1 = M_1(T)$, by Remark 3.13, $\overline{M}_2(T) \geq h + 1 + f + g + h - 1 = f + g + 2h$.

Consequently, 20

- (A) If $g \geq 2$, then $\overline{M}_2(T) \geq 2h + f + 2g - 2$.
- (B) If $g \leq 1$ and T is not a path, then $\overline{M}_2(T) \geq 2h + f + g$.
- (C) If T is a path, then $\overline{M}_2(T) \geq 2$.

Therefore,

- (A) If $g \geq 2$, then $\overline{M}_2(T) = 2h + f + 2g - 2$. 25
- (B) If $g \leq 1$ and T is not a path, then $\overline{M}_2(T) = 2h + f + g$.
- (C) If T is a path, then $\overline{M}_2(T) = 2$. □

Proposition 3.15. *Let T be a tree and v a peripheral HDV, with S as defined earlier in this section. Suppose that S has 3 arms of length 1 and 0 arms of length at least 2 and $T \neq S$. Then*

$$\overline{M}_2(T - S) + 2 \leq \overline{M}_2(T) \leq \overline{M}_2(T - S) + 3.$$

Proof Let $\mathcal{A} = ((\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2))$ be an \overline{M}_2 -maximal assignment to T . We are going to construct an \overline{M}_2 assignment to $T - S$, $\mathcal{A}' = ((\mathcal{A}'_1, V'_1), (\mathcal{A}'_2, V'_2))$ (see Remark 3.13). Note that $M_1(T - S) = M_1(T) - 2$. Because v has degree 4, by Remark 3.11, 3, $v \in V_1$, the peripheral arms of S are in \mathcal{A}_1 and no one is in \mathcal{A}_2 . Using Remark 3.12, 4, $v \notin V_2$. So, let F be the component of $T - V_2$ containing S . By Remark 3.12, 1, F is in \mathcal{A}_2 . 30

Let $\mathcal{A}'_1 = \mathcal{A}_1 \setminus \{\text{the peripheral arms of } S\}$, $V'_1 = V_1 \setminus \{v\}$, $V'_2 = V_2$ and

$$\mathcal{A}'_2 = \begin{cases} \mathcal{A}_2 \setminus \{F\} & \text{if } \mathcal{A}_2 \neq \{F\} \\ T - S & \text{if } \mathcal{A}_2 = \{F\} \end{cases}.$$

By Remark 3.13, $\mathcal{A}' = ((\mathcal{A}'_1, V'_1), (\mathcal{A}'_2, V'_2))$ is an \overline{M}_2 assignment to $T - S$ and $\overline{M}_2(T - S) \geq \overline{M}_2(T) - 2 - 1 = \overline{M}_2(T) - 3$.

Let $\mathcal{A} = ((\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2))$ be an \overline{M}_2 -maximal assignment to $T - S$. We are going to construct an \overline{M}_2 assignment to T , $\mathcal{A}' = ((\mathcal{A}'_1, V'_1), (\mathcal{A}'_2, V'_2))$ (see Remark 3.13). Note that $M_1(T) = M_1(T - S) + 2$. Let w be the vertex of $T - S$ adjacent to v in T . If $w \notin V_2$ then let R be the component of $(T - S) - V_2$ containing w and let P be the component of $T - V_2$ containing S .

Let $\mathcal{A}'_1 = \mathcal{A}_1 \cup \{\text{the peripheral arms of } S\}$, $V'_1 = V_1 \cup \{v\}$, $V'_2 = V_2$ and

$$\mathcal{A}'_2 = \begin{cases} (\mathcal{A}_2 \setminus \{R\}) \cup \{P\} & \text{if } R \in \mathcal{A}_2 \text{ and } w \notin V_2 \\ \mathcal{A}_2 & \text{otherwise} \end{cases}.$$

By Remark 3.13, $\mathcal{A}' = ((\mathcal{A}'_1, V'_1), (\mathcal{A}'_2, V'_2))$ is an \overline{M}_2 assignment to T and $\overline{M}_2(T) \geq \overline{M}_2(T - S) + 2$. \square

Proposition 3.16. *Let T be a tree and v a peripheral HDV, with S as defined earlier in this section. Suppose that S has 1 arm of length 1 and 1 arm of length at least 2 (or T has 2 arms of length 1 and 0 arms of length at least 2) and $T \neq S$. Then*

$$\overline{M}_2(T - S) + 1 \leq \overline{M}_2(T) \leq \overline{M}_2(T - S) + 2.$$

Proof Let $\mathcal{A} = ((\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2))$ be an \overline{M}_2 -maximal assignment to T . We are going to construct an \overline{M}_2 assignment to $T - S$, $\mathcal{A}' = ((\mathcal{A}'_1, V'_1), (\mathcal{A}'_2, V'_2))$ (see Remark 3.13). Note that $M_1(T - S) = M_1(T) - 1$.

Using Remark 3.11, 1, if v is in V_1 , then the peripheral arms of $S - v$ are in \mathcal{A}_1 . Using Remark 3.12, 5, without loss of generality, we can assume that $v \notin V_2$. Let F be the component of $T - V_2$ containing S . By Remark 3.12, 1 and 3, F is in \mathcal{A}_2 .

Let $\mathcal{A}'_1 = \mathcal{A}_1 \setminus \{\text{the peripheral arms of } S\}$, $V'_1 = V_1 \setminus \{v\}$, $V'_2 = V_2$ and

$$\mathcal{A}'_2 = \begin{cases} \mathcal{A}_2 \setminus \{F\} & \text{if } \mathcal{A}_2 \neq \{F\} \\ T - S & \text{if } \mathcal{A}_2 = \{F\} \end{cases}.$$

By Remark 3.13, $\mathcal{A}' = ((\mathcal{A}'_1, V'_1), (\mathcal{A}'_2, V'_2))$ is an \overline{M}_2 assignment to $T - S$ and $\overline{M}_2(T - S) \geq \overline{M}_2(T) - 1 - 1 = \overline{M}_2(T) - 2$.

If v is not in V_1 , since v has degree 3 in T , then $w \in V_1$. By Remark 3.11, 1, S is in \mathcal{A}_1 . By Remark 3.12, 3, we can assume, without loss of generality, that $v \in V_2$ and the peripheral arms of S are in \mathcal{A}_2 .

Let $\mathcal{A}'_1 = \mathcal{A}_1 \setminus \{S\}$, $V'_1 = V_1$, $V'_2 = V_2 \setminus \{v\}$, $\mathcal{A}'_2 = \mathcal{A}_2 \setminus \{\text{the peripheral arms of } S\}$.

By Remark 3.13, $\mathcal{A}' = ((\mathcal{A}'_1, V'_1), (\mathcal{A}'_2, V'_2))$ is an \overline{M}_2 assignment to $T - S$ and $\overline{M}_2(T - S) \geq \overline{M}_2(T) - 1 - 1 = \overline{M}_2(T) - 2$.

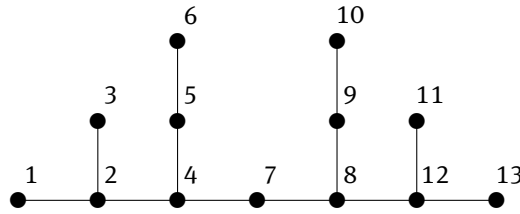
Let $\mathcal{A} = ((\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2))$ be an \overline{M}_2 -maximal assignment to $T - S$. We are going to construct an \overline{M}_2 assignment to T , $\mathcal{A}' = ((\mathcal{A}'_1, V'_1), (\mathcal{A}'_2, V'_2))$ (see Remark 3.13). Note that $M_1(T) = M_1(T - S) + 1$. Let w be the vertex of $T - S$ adjacent to v in T . If $w \notin V_2$ then let F be the component of $(T - S) - V_2$ containing w and let P be the component of $T - V_2$ containing S .

Let $\mathcal{A}'_1 = \mathcal{A}_1 \cup \{\text{the peripheral arms of } S\}$, $V'_1 = V_1 \cup \{v\}$, $V'_2 = V_2$ and

$$\mathcal{A}'_2 = \begin{cases} (\mathcal{A}_2 \setminus \{F\}) \cup \{P\} & \text{if } F \in \mathcal{A}_2 \text{ and } w \notin V_2 \\ \mathcal{A}_2 & \text{otherwise} \end{cases}.$$

By Remark 3.13, $\mathcal{A}' = ((\mathcal{A}'_1, V'_1), (\mathcal{A}'_2, V'_2))$ is an \overline{M}_2 assignment to T and $\overline{M}_2(T) \geq \overline{M}_2(T - S) + 1$. \square

Example 3.17. *Let T be the following tree*



Let H be the subtree obtained from T by removing vertices 11, 12, 13. By Proposition 3.16,

$$\overline{M}_2(H) + 1 \leq \overline{M}_2(T) \leq \overline{M}_2(H) + 2.$$

Since H is a small superstar with central vertex 4, by Proposition 3.14, $\overline{M}_2(H) = 4 + 0 + 4 - 2 = 2$. So,

$$5 \leq \overline{M}_2(T) \leq 6.$$

4 An algorithm for \overline{M}_2

The purpose of this section is to find simple reductions of the initial tree in such a way that we know the effect of each reduction on \overline{M}_2 . The process may be continued until a small superstar, for which \overline{M}_2 is known (Proposition 3.14), or until a subtree for which \overline{M}_2 has bounds (Section 3). 5

Definition 4.1. (Peripheral SHDV, peripheral super path) Let T be a tree that is not a small superstar. A peripheral superstar high degree vertex (SHDV) v of T is an HDV vertex of T such that

- [(1) there is a unique subtree of $T - v$, R , that contains high-degree vertices;
- [(2) $T - R$ is a small superstar;
- [(3) if $w \in R$ and w is adjacent to v , then w does not satisfy 1, 2. 10

A peripheral super path of T at v (v is a SHDV) is a path of $(T - R) - v$. There are two kinds of peripheral super paths of T at v (SHDV): peripheral arms and small pincers.

Example 4.2. Consider the tree T of Example 3.17.

The vertices 4 and 8 are peripheral superstar high degree vertices.

The vertex 2 is not a peripheral superstar high degree vertex because it is adjacent to vertex 4 and this vertex 15 satisfies conditions 1 and 2 of Definition 4.1.

The subtree of T generated by vertices 1, 2, 3 is a peripheral super path of T at 4, but it is not a peripheral arm of T at 4 (it is a small pincer).

Definition 4.3. Throughout this section, we will consider a peripheral SHDV v in a tree T that is not a small superstar. The subtree of T consisting of v and its peripheral super paths will be called Q . Let w be the one 20 vertex adjacent to v that is not in Q .

Remark 4.4. Let T be a tree and $\mathcal{A} = ((\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2))$ an \overline{M}_2 assignment of $(M_1(T), p_2, 1^1)$ to T . Because $M_1(T) = |\mathcal{A}_1| - |V_1|$,

- (1) All components of $T - V_1$ are in \mathcal{A}_1 .
- (2) We can assume that if $v \in V_1$ then v has degree greater than two in T . 25
- (3) Since all components of $T - V_1$ are paths, if v is a peripheral SHDV of degree greater or equal to 4 in T then $v \in V_1$ or there is at most one peripheral arm adjacent to v and the central vertex of each small pincer adjacent to v is in V_1 .
- (4) If v is a peripheral SHDV, $v \in V_1$ and all peripheral super paths adjacent to v have length 1 then they are in \mathcal{A}_1 and no one is in \mathcal{A}_2 (see Lemma 2.3). 30

Remark 4.5. Let T be a tree and $\mathcal{A} = ((A_1, V_1), (A_2, V_2))$ an \overline{M}_2 -maximal assignment of $(M_1(T), p_2, 1^1)$ to T . Because $\overline{M}_2(T) = |A_1| - |V_1| + |A_2| - |V_2|$,

- (1) All components of $T - V_2$ with more than one vertex are in A_2 .
- (2) We can assume that if $v \in V_2$ then v has degree greater than two in T .
- 5 (3) All components of $T - V_2$ with one vertex that are not components of $T - V_1$ are in A_2 .
- (4) If v is a peripheral SHDV, $v \in V_1$ and all peripheral super paths adjacent to v have length 1, then using Remark 4.4, 4, we conclude that $v \notin V_2$.
- (5) If v is a peripheral SHDV, $v \in V_1$ and all peripheral super paths adjacent to v have length 1, except one, then there is an \overline{M}_2 -maximal assignment of $(M_1(T), p_2, 1^1)$ to T such that $v \notin V_2$.

10 **Remark 4.6.** In some proofs we construct an \overline{M}_2 -maximal (or simply an \overline{M}_2) assignment of $(M_1(T), p_2, 1^1)$ to T . for some integer p_2 . In these cases, first we construct an \overline{M}_2 assignment of $(M_1(T), p_2, 1^1)$ to T , $\mathcal{A} = ((A_1, V_1), (A_2, V_2))$ by putting the elements in A_1 and in V_1 , next we put the elements in A_2 and in V_2 , using Remarks 4.4 and 4.5. This construction is in such a way that $M_1(T) = |A_1| - |V_1|$ and $M_1(T) + p_2 = |A_1| - |V_1| + |A_2| - |V_2|$. After using Lemma 2.3 we conclude condition 2 of Definition 2.1 and by Corollary 2.13,
15 we say that $\overline{M}_2(T) \geq M_1(T) + p_2$.

Proposition 4.7. Let T be a tree that is not a small superstar and v a peripheral SHDV, with Q as defined earlier in this section. Suppose that Q has $h \geq 1$ small pincers and the degree of v in T is greater than 4. Let H be the graph obtained from T by removing one small pincer of Q . Then

$$\overline{M}_2(H) = \overline{M}_2(T) - 2.$$

Proof By Proposition 3.16, $\overline{M}_2(H) \geq \overline{M}_2(T) - 2$.

Let $\mathcal{A} = ((A_1, V_1), (A_2, V_2))$ be an \overline{M}_2 -maximal assignment to H . We are going to construct an \overline{M}_2 assignment to T , $\mathcal{A}' = ((A'_1, V'_1), (A'_2, V'_2))$ (see Remark 4.6). Note that $M_1(T) = M_1(H) + 1$. Since the degree of v in T is greater than 4, we conclude that the degree of v in H is greater than 3. By Remark 4.4, 3 and Remark 4.5, 1,
20 3, we have $v \in V_1 \cup V_2$.

Suppose that $v \in V_1 \cap V_2$. Let P be the small pincer $T - H$.

Let $\mathcal{A}'_1 = A_1 \cup \{P\}$, $V'_1 = V_1$, $V'_2 = V_2$ and $\mathcal{A}'_2 = A_2 \cup \{P\}$.

By Remark 4.6, $\mathcal{A}' = ((A'_1, V'_1), (A'_2, V'_2))$ is an \overline{M}_2 assignment to T and $\overline{M}_2(T) \geq \overline{M}_2(H) + 2$.

Suppose that $v \in V_1 \setminus V_2$. Let x be the central vertex of the small pincer, P , of $T - H$.

25 Let $\mathcal{A}'_1 = A_1 \cup \{P\}$, $V'_1 = V_1$, $\mathcal{A}'_2 = A_2 \cup \{\text{the peripheral arms of } P \text{ at } x\}$ and $V'_2 = V_2 \cup \{x\}$.

By Remark 4.6, $\mathcal{A}' = ((A'_1, V'_1), (A'_2, V'_2))$ is an \overline{M}_2 assignment to T and $\overline{M}_2(T) \geq \overline{M}_2(H) + 2$.

Suppose that $v \in V_2 \setminus V_1$. Let x be the central vertex of the small pincer, P , of $T - H$.

Let $V'_1 = V_1 \cup \{x\}$, $\mathcal{A}'_1 = A_1 \cup \{\text{the peripheral arms of } P \text{ at } x\}$, $\mathcal{A}'_2 = A_2 \cup \{P\}$ and $V'_2 = V_2$.

By Remark 4.6, $\mathcal{A}' = ((A'_1, V'_1), (A'_2, V'_2))$ is an \overline{M}_2 assignment to T and $\overline{M}_2(T) \geq \overline{M}_2(H) + 2$.

30 Consequently, $\overline{M}_2(T) = \overline{M}_2(H) + 2$. □

Lemma 4.8. Let T be a tree that is not a small superstar. Suppose that v is a peripheral SHDV in T with Q , w as defined earlier in this section. Then, there exists an \overline{M}_2 -maximal assignment to T , $\mathcal{A} = ((A_1, V_1), (A_2, V_2))$, in which $v \in V_1 \cup V_2$.

Moreover,

- 35 (1) If v has at least two peripheral arms of length at least 2, then there exists an \overline{M}_2 -maximal assignment, $\mathcal{A}' = ((A'_1, V'_1), (A'_2, V'_2))$ in which $v \in V'_1 \cap V'_2$.
- (2) If v has at most one peripheral arm of length at least 2 and w has degree two in T , then there exists an \overline{M}_2 -maximal assignment, $\mathcal{A}'' = ((A''_1, V''_1), (A''_2, V''_2))$ such that v is in exactly one V''_1 or V''_2 .
- (3) If Q has f peripheral arms of length 1 and $g \leq 1$ peripheral arms of length at least 2, $f + g > 2$ and
40 $\mathcal{A}''' = ((A'''_1, V'''_1), (A'''_2, V'''_2))$ is an \overline{M}_2 -maximal assignment to T , then $v \in V'''_1$.

Proof Let $\mathcal{A} = ((\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2))$ be an \overline{M}_2 -maximal assignment to T in which $v \notin V_1 \cup V_2$. Suppose that Q has f peripheral arms of length 1 and g peripheral arms of length at least 2. We are going to construct an \overline{M}_2 -maximal assignment to T , $\mathcal{B} = ((\mathcal{B}_1, U_1), (\mathcal{B}_2, U_2))$ (see Remark 4.6).

If $f + g \geq 2$, then by Remark 4.4, 1, the component, R , of $T - V_1$ containing v is in \mathcal{A}_1 . Note that the peripheral arms of Q might be in R . 5

Let $\mathcal{B}_1 = (\mathcal{A}_1 \setminus \{R\}) \cup \{\text{two peripheral arms of } Q\}$, $U_1 = V_1 \cup \{v\}$, $\mathcal{B}_2 = \mathcal{A}_2$ and $U_2 = V_2$.

By Remark 4.6 and the cardinality of \mathcal{B} , $\mathcal{B} = ((\mathcal{B}_1, U_1), (\mathcal{B}_2, U_2))$ is an \overline{M}_2 -maximal assignment to T in which $v \in U_1$.

If $f + g \leq 1$, by Remark 4.4, 3 and Remark 4.5, 1 and 3, the central vertex of each small pincer of Q is in $V_1 \setminus V_2$. By Remark 4.5, 1, the component, R , of $T - V_2$ containing v , is in \mathcal{A}_2 . 10

Let $\mathcal{B}_1 = \mathcal{A}_1$, $U_1 = V_1$, $\mathcal{B}_2 = (\mathcal{A}_2 \setminus \{R\}) \cup \{\text{two peripheral super paths of } Q\}$ and $U_2 = V_2 \cup \{v\}$.

By Remark 4.6 and the cardinality of \mathcal{B} , $\mathcal{B} = ((\mathcal{B}_1, U_1), (\mathcal{B}_2, U_2))$ is an \overline{M}_2 -maximal assignment to T in which $v \in U_2$. So, there exists an \overline{M}_2 -maximal assignment to T $\mathcal{A} = ((\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2))$ in which $v \in V_1 \cup V_2$.

(1) By what we just proved, there exists an \overline{M}_2 -maximal assignment to T ,

$$\mathcal{A} = ((\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2)),$$

in which $v \in V_1 \cup V_2$. Suppose without loss of generality that $v \in V_1 \setminus V_2$. We are going to construct an \overline{M}_2 -maximal assignment to T , $\mathcal{A}' = ((\mathcal{A}'_1, V'_1), (\mathcal{A}'_2, V'_2))$, in which $v \in V'_1 \cap V'_2$. (see Remark 4.6). By 15
Remark 4.5, 1 and 3, the component, R , of $T - V_2$ containing v , is in \mathcal{A}_2 . Note that the peripheral arms of Q might be in R .

Let $\mathcal{A}'_1 = \mathcal{A}_1$, $V'_1 = V_1$, $V'_2 = V_2 \cup \{v\}$ and

$$\mathcal{A}'_2 = (\mathcal{A}_2 \setminus \{R\}) \cup \{\text{two peripheral arms of length at least two of } Q\}.$$

Since $|\mathcal{A}_2| - |V_2| = |\mathcal{A}'_2| - |V'_2|$, by Remark 4.6 and the cardinality of \mathcal{A}' , $\mathcal{A}' = ((\mathcal{A}'_1, V'_1), (\mathcal{A}'_2, V'_2))$ is an \overline{M}_2 -maximal assignment to T , in which $v \in V'_1 \cap V'_2$.

(2) By what we just proved, there exists an \overline{M}_2 -maximal assignment,

$$\mathcal{A} = ((\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2)),$$

in which $v \in V_1 \cup V_2$. Suppose $v \in V_1 \cap V_2$. We are going to construct an \overline{M}_2 -maximal assignment 20
to T , $\mathcal{A}'' = ((\mathcal{A}''_1, V''_1), (\mathcal{A}''_2, V''_2))$, in which $v \in V''_1 \setminus V''_2$. (see Remark 4.6) Using Remark 4.4, 1, each peripheral super path of Q is in \mathcal{A}_1 . By Remark 4.5, 1, the longer arm of Q and the small pincers of Q are in \mathcal{A}_2 and there is not a peripheral arm of length 1 of Q in \mathcal{A}_2 . By Remark 4.5, 2, $w \notin V_2$. Let R be the component of $T - V_2$ containing w and let F be the component of $T - ((V_2 \setminus \{v\}) \cup \{\text{the central vertex of each small pincer of } Q\})$ containing v and w . By Remark 4.5, 1, $F \in \mathcal{A}_2$. 25

Let $\mathcal{A}''_1 = \mathcal{A}_1$, $V''_1 = V_1$,

$$\mathcal{A}''_2 = (\mathcal{A}_2 \setminus \{\text{the peripheral super paths of length at least two of } Q, R\}) \cup \{\text{the peripheral arms of each small pincer of } Q, F\}$$

and $V''_2 = (V_2 \setminus \{v\}) \cup \{\text{the central vertex of each small pincer of } Q\}$.

If Q does not have a longer arm or $R \notin \mathcal{A}_2$ then $|\mathcal{A}''_2| - |V''_2| > |\mathcal{A}_2| - |V_2|$. This is impossible because \mathcal{A} is an \overline{M}_2 -maximal assignment to T . So, $v \notin V_1 \cap V_2$.

If Q has a longer arm and $R \in \mathcal{A}_2$ then $|\mathcal{A}_2| - |V_2| = |\mathcal{A}''_2| - |V''_2|$. By Remark 4.6 and using the cardinality of \mathcal{A}'' , $\mathcal{A}'' = ((\mathcal{A}''_1, V''_1), (\mathcal{A}''_2, V''_2))$ is an \overline{M}_2 -maximal assignment to T , in which v in $V''_1 \setminus V''_2$. 30

(3) Let $\mathcal{A}''' = ((\mathcal{A}'''_1, V'''_1), (\mathcal{A}'''_2, V'''_2))$ be an \overline{M}_2 -maximal assignment to T . By Remark 4.4, 1, each peripheral super path of Q belongs to \mathcal{A}'''_1 and $v \in V'''_1$. □

Lemma 4.9. *Let T be a tree that is not a small superstar. Suppose that v is a peripheral SHDV in T with Q , w as defined earlier in this section. Suppose that Q has f peripheral arms of length 1 and $g \leq 1$ peripheral arms of length at least 2 and the degree of w in T is 2. Then, there exists an \overline{M}_2 -maximal assignment to T , 35
 $\mathcal{A} = ((\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2))$, in which:*

- (1) If $f + g \geq 1$, then $v \in V_1$ and the central vertex of each small pincer of Q belongs to V_2 .
 (2) If $f + g = 0$, then $v \in V_2$ and the central vertex of each small pincer of Q belongs to V_1 .

Proof

- (1) By 2 of Lemma 4.8, let $\mathcal{A} = ((\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2))$ be an \overline{M}_2 -maximal assignment to T such that v is exactly
 5 one V_1 or V_2 .
 If $f + g > 1$, since v is a peripheral SHDV and $w \notin V_1$ (the degree of w in T is 2), by Remark 4.4, 1, each peripheral super path of Q belongs to \mathcal{A}_1 and $v \in V_1$. In this case, because $v \notin V_2$ and \mathcal{A} is an \overline{M}_2 -maximal assignment to T , we conclude that the central vertex of each small pincer of Q is in V_2 and the peripheral arms of each small pincer of Q are in \mathcal{A}_2 .
 10 Suppose that $f + g = 1$ and $v \in V_2$. then by Remark 4.4, 1, the central vertex of each small pincer of Q is in V_1 and the peripheral arms of each small pincer of Q are in \mathcal{A}_1 . By Remark 4.5, 1 and 3, the peripheral super paths of Q are in \mathcal{A}_2 . Since w has degree two in T , we can assume that $w \notin V_1 \cup V_2$. We are going to construct an \overline{M}_2 -maximal assignment to T , $\mathcal{A}' = ((\mathcal{A}'_1, V'_1), (\mathcal{A}'_2, V'_2))$, in which $v \in V_1$ and the central vertex of each small pincer of Q is in V_2 (see Remark 4.6). Let R be the component of $T - V_1$ containing v , w . By Remark 4.4, 1, $R \in \mathcal{A}_1$. Let P be the component of $T - V_2$, containing w . Since $P \neq R$, by Remark 4.5, 1 and 3, $P \in \mathcal{A}_2$. Let B be the component of $T - ((V_2 \setminus \{v\}) \cup \{\text{the central vertex of each small pincer of } Q\})$, containing v and w . Let C be the component of $T - (V_1 \cup \{v\})$, containing w . Note that $B \neq C$.

Let

$$\mathcal{A}'_1 = (\mathcal{A}_1 \setminus \{\text{the peripheral arms of each small pincer of } Q, R\}) \cup \{C, \text{ the peripheral super paths of } Q\},$$

$$V'_1 = (V_1 \setminus \{\text{the central vertex of each small pincer of } Q\}) \cup \{v\},$$

$$\mathcal{A}'_2 = (\mathcal{A}_2 \setminus \{\text{the peripheral super paths of } Q, P\}) \cup \{\text{the peripheral arms of each small pincer of } Q, B\}$$

and $V'_2 = (V_2 \setminus \{v\}) \cup \{\text{the central vertex of each small pincer of } Q\}$.

- 20 Since $|\mathcal{A}'_1| - |V'_1| = |\mathcal{A}_1| - |V_1|$ and $|\mathcal{A}'_2| - |V'_2| = |\mathcal{A}_2| - |V_2|$ and by Remark 4.6, we get an \overline{M}_2 -maximal assignment to T , $\mathcal{A}' = ((\mathcal{A}'_1, V'_1), (\mathcal{A}'_2, V'_2))$, where $v \in V'_1$ and the central vertex of each small pincer of Q belongs to V'_2 .

- (2) By 2 of Lemma 4.8, let $\mathcal{A} = ((\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2))$ be an \overline{M}_2 -maximal assignment to T such that v is exactly one V_1 or V_2 . Since $f + g = 0$, v is a peripheral SHDV and $w \notin V_1$, if $v \in V_1$ then by Remark 4.4, 1, the peripheral super paths of Q are in \mathcal{A}_1 . Let F be the component of $T - V_1$ containing w . By Remark 4.4, 1, $F \in \mathcal{A}_1$. Let H be the component of $T - (V_1 \setminus \{v\}) \cup \{\text{the central vertex of each small pincer of } Q\}$ containing w and v . Let

$$\mathcal{A}'_1 = (\mathcal{A}_1 \setminus \{\text{the peripheral super paths of } Q, F\}) \cup \{\text{the peripheral arms of each small pincer of } Q, H\},$$

$V'_1 = (V_1 \setminus \{v\}) \cup \{\text{the central vertex of each small pincer of } Q\}$. Since $|\mathcal{A}'_1| - |V'_1| = |\mathcal{A}_1| - |V_1| + 1$ we conclude that \mathcal{A} is not an \overline{M}_2 -maximal assignment to T . Impossible. Consequently, $v \notin V_1$ and $v \in V_2$.

- 25 Therefore, the central vertex of each small pincer of Q belongs to V_1 . \square

Theorem 4.10. (\overline{M}_2 Reduction Theorem) *Let T be a tree that is not a small superstar and v a peripheral SHDV, with Q , w as defined earlier in this section. Suppose that Q has f peripheral arms of length 1, g peripheral arms of length at least 2 and h small pincers. Then:*

- (A) If $g \geq 2$, then $\overline{M}_2(T - Q) = \overline{M}_2(T) - f - 2g - 2h + 2$.
 30 (B) If $g \leq 1$ and the degree of w in T is 2, then $\overline{M}_2((T - Q) +_w K_1) = \overline{M}_2(T) - f - g - 2h + 1$, where $(+_w K_1)$ means that we put a vertex adjacent to w .
 (C) If $g \leq 1$, the degree of w in T is greater than 2 and $f + g > 2$ then

$$\overline{M}_2((T - Q) +_w S_4) = \overline{M}_2(T) - f - g - 2h + 3,$$

where S_4 is the star with 3 arms of length 1 and $(+wS_4)$ means that S_4 is adjacent to w by the central vertex.

Proof Part A: Let $\mathcal{A} = ((A_1, V_1), (A_2, V_2))$ be an \overline{M}_2 -maximal assignment to $T - Q$. We are going to construct an \overline{M}_2 assignment to T , $\mathcal{A}' = ((A'_1, V'_1), (A'_2, V'_2))$ (see Remark 4.6).

Let $\mathcal{A}'_1 = \mathcal{A}_1 \cup \{\text{the peripheral super paths of } Q\}$, $V'_1 = V_1 \cup \{v\}$, $\mathcal{A}'_2 = \mathcal{A}_2 \cup \{\text{the peripheral super paths of length at least two of } Q\}$ and $V'_2 = V_2 \cup \{v\}$.

Since $M_1(T) = M_1(T - Q) + f + g + h - 1$, by Remark 4.6, this creates an \overline{M}_2 assignment to T , $\mathcal{A}' = ((A'_1, V'_1), (A'_2, V'_2))$ and $\overline{M}_2(T) \geq \overline{M}_2(T - Q) + f + g + h - 1 + g + h - 1 = \overline{M}_2(T - Q) + f + 2g + 2h - 2$.

Conversely, by Lemma 4.8, 1, there exists an \overline{M}_2 -maximal assignment to T , $\mathcal{A} = ((A_1, V_1), (A_2, V_2))$, in which v is in $V_1 \cap V_2$. We are going to construct an \overline{M}_2 assignment to $T - Q$, $\mathcal{A}' = ((A'_1, V'_1), (A'_2, V'_2))$. By Remarks 4.4, 1 and 4.5, 1, each of the $f + g + h$ peripheral super paths of Q might be in \mathcal{A}_1 and each of the $g + h$ peripheral super paths of length at least 2 of Q might be in \mathcal{A}_2 .

Let $\mathcal{A}'_1 = \mathcal{A}_1 \setminus \{\text{the peripheral super paths of } Q\}$, $V'_1 = V_1 \setminus \{v\}$, $\mathcal{A}'_2 = \mathcal{A}_2 \setminus \{\text{the peripheral super paths of length at least two of } Q\}$ and $V'_2 = V_2 \setminus \{v\}$.

Since $M_1(T - Q) = M_1(T) - f - g - h + 1$, by Remark 4.6, this creates an \overline{M}_2 assignment to $T - Q$, $\mathcal{A}' = ((A'_1, V'_1), (A'_2, V'_2))$ and $\overline{M}_2(T - Q) \geq \overline{M}_2(T) - f - g - h + 1 - g - h + 1 = \overline{M}_2(T) - f - 2g - 2h + 2$. So, we have $\overline{M}_2(T - Q) = \overline{M}_2(T) - f - 2g - 2h + 2$.

Part B: Let $\mathcal{A} = ((A_1, V_1), (A_2, V_2))$ be an \overline{M}_2 -maximal assignment to $(T - Q) +_w K_1$. We are going to construct an \overline{M}_2 assignment to T , $\mathcal{A}' = ((A'_1, V'_1), (A'_2, V'_2))$ (see Remark 4.6). Let R be the component of $((T - Q) +_w K_1) - V_1$ containing K_1 and let U be the component of $((T - Q) +_w K_1) - V_2$ containing K_1 . Since degree of w in T is 2, without loss of generality, by Remarks 4.4, 2, and 4.5, 2, we can assume that $w \in R \cap U$. Consequently, $R \neq K_1$ and $U \neq K_1$. By Remarks 4.4, 1 and 4.5, 1, R is in \mathcal{A}_1 and U is in \mathcal{A}_2 .

Suppose that $f + g \geq 1$. Let P be the component of $T - (V_1 \cup \{v\})$ containing w and let H be the component of $T - (V_2 \cup \{\text{the central vertex of each small pincer of } Q\})$ containing w and v .

Let $\mathcal{A}'_1 = (\mathcal{A}_1 \setminus \{R\}) \cup \{\text{the peripheral super paths of } Q, P\}$, $V'_1 = V_1 \cup \{v\}$, $\mathcal{A}'_2 = (\mathcal{A}_2 \setminus \{U\}) \cup \{\text{the peripheral arms of each small pincer of } Q, H\}$ and $V'_2 = V_2 \cup \{\text{the central vertex of each small pincer of } Q\}$.

Since $M_1(T) = M_1((T - Q) +_w K_1) + f + g + h - 1$, by Remark 4.6, this creates an \overline{M}_2 assignment to T , $\mathcal{A}' = ((A'_1, V'_1), (A'_2, V'_2))$ and $\overline{M}_2(T) \geq \overline{M}_2((T - Q) +_w K_1) + f + g + h - 1 + 2h - h = \overline{M}_2((T - Q) +_w K_1) + f + g + 2h - 1$.

Suppose that $f + g = 0$. Let B be the component of $T - (V_2 \cup \{v\})$ containing w and let C be the component of $T - (V_1 \cup \{\text{the central vertex of each small pincer of } Q\})$ containing w and v .

Let $\mathcal{A}'_1 = (\mathcal{A}_1 \setminus \{R\}) \cup \{\text{the peripheral arms of each small pincer of } Q, C\}$, $V'_1 = V_1 \cup \{\text{the central vertex of each small pincer of } Q\}$, $V'_2 = V_2 \cup \{v\}$ and $\mathcal{A}'_2 = (\mathcal{A}_2 \setminus \{U\}) \cup \{\text{the peripheral super paths of } Q, B\}$.

Since $M_1(T) = M_1((T - Q) +_w K_1) + h$, by Remark 4.6, this creates an \overline{M}_2 assignment to T , $\mathcal{A}' = ((A'_1, V'_1), (A'_2, V'_2))$ and $\overline{M}_2(T) \geq \overline{M}_2((T - Q) +_w K_1) + 2h - h + h - 1 = \overline{M}_2((T - Q) +_w K_1) + f + g + 2h - 1$.

Conversely, suppose that $f + g \geq 1$. By Lemma 4.9, 1, there exists an \overline{M}_2 -maximal assignment to T , $\mathcal{A} = ((A_1, V_1), (A_2, V_2))$, in which v is in V_1 and the central vertex of each small pincer of Q is in V_2 . We are going to construct an \overline{M}_2 assignment to $(T - Q) +_w K_1$, $\mathcal{A}' = ((A'_1, V'_1), (A'_2, V'_2))$ (see Remark 4.6). By Remarks 4.4, 1 and 2, and 4.5, 1 and 3, each of the $f + g + h$ peripheral super paths of Q might be in \mathcal{A}_1 , the peripheral arms of each small pincer of Q might be in \mathcal{A}_2 and $w \notin V_1 \cup V_2$. Let R be the component of $T - V_1$ containing w and let P be the component of $T - V_2$ containing v and w . By Remarks 4.4, 1 and 4.5, 1, $R \in \mathcal{A}_1$ and $P \in \mathcal{A}_2$. Let R' be the component of $((T - Q) +_w K_1) - (V_1 \setminus \{v\})$ containing w and K_1 , and let P' be the component of $(T - Q) +_w K_1 - (V_2 \setminus \{\text{the central vertex of each small pincer of } Q\})$ containing w and K_1 .

Let $\mathcal{A}'_1 = (\mathcal{A}_1 \setminus \{R, \text{the peripheral super paths of } Q\}) \cup \{R'\}$, $V'_1 = V_1 \setminus \{v\}$, $\mathcal{A}'_2 = (\mathcal{A}_2 \setminus \{P, \text{the peripheral arms of each small pincer of } Q\}) \cup \{P'\}$ and $V'_2 = V_2 \setminus \{\text{the central vertex of each small pincer of } Q\}$.

Since $M_1((T - Q) +_w K_1) = M_1(T) - f - g - h + 1$, by Remark 4.6, this creates an \overline{M}_2 assignment to $(T - Q) +_w K_1$, $\mathcal{A}' = ((A'_1, V'_1), (A'_2, V'_2))$ and $\overline{M}_2((T - Q) +_w K_1) \geq \overline{M}_2(T) - f - g - h + 1 - 2h + h = \overline{M}_2(T) - f - g - 2h + 1$.

Suppose that $f + g = 0$. By Lemma 4.9, 2, there exists an \overline{M}_2 -maximal assignment to T , $\mathcal{A} = ((A_1, V_1), (A_2, V_2))$, in which v is in V_2 , the central vertex of each small pincer of Q is in V_1 and $w \notin V_1 \cup V_2$. We are going to construct an \overline{M}_2 assignment to $(T - Q) +_w K_1$, $\mathcal{A}' = ((A'_1, V'_1), (A'_2, V'_2))$. By Remarks 4.4, 1 and

4.5, 1 and 3, each of the h small pincers of Q might be in \mathcal{A}_2 and the peripheral arms of each small pincer of Q might be in \mathcal{A}_1 . Let R be the component of $T - V_1$ containing v , w and let P be the component of $T - V_2$ containing w . By Remarks 4.4, 1 and 4.5, 1 and 3, $R \in \mathcal{A}_1$ and $P \in \mathcal{A}_2$. Let P' be the component of $((T - Q) +_w K_1) - (V_2 \setminus \{v\})$ containing wand K_1 , and let R' be the component of $((T - Q) +_w K_1) - (V_1 \setminus \{$
 5 $\{$ the central vertex of each small pincer of $Q\})$ containing w and K_1 .

Let $\mathcal{A}'_1 = (\mathcal{A}_1 \setminus \{R, \text{ the peripheral arms of each small pincer of } Q\}) \cup \{R'\}$, $V'_1 = V_1 \setminus \{$ the central vertex of each small pincer of $Q\}$, $V'_2 = V_2 \setminus \{v\}$ and $\mathcal{A}'_2 = (\mathcal{A}_2 \setminus \{P, \text{ the peripheral super paths of } Q\}) \cup \{P'\}$.

Since $M_1((T - Q) +_w K_1) = M_1(T) - h$, by Remark 4.6, this creates an \overline{M}_2 assignment to $(T - Q) +_w K_1$, $\mathcal{A}' = ((\mathcal{A}'_1, V'_1), (\mathcal{A}'_2, V'_2))$ and $\overline{M}_2((T - Q) +_w K_1) \geq \overline{M}_2(T) - 2h + h - h + 1 = \overline{M}_2(T) - f - g - 2h + 1$.

10 So, we have $\overline{M}_2((T - Q) +_w K_1) = \overline{M}_2(T) - f - g - 2h + 1$.

Part C: Let $\mathcal{A} = ((\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2))$ be an \overline{M}_2 -maximal assignment to $(T - Q) +_w S_4$. We are going to construct an \overline{M}_2 assignment to T , $\mathcal{A}' = ((\mathcal{A}'_1, V'_1), (\mathcal{A}'_2, V'_2))$. Let x be the central vertex of S_4 .

By Lemma 4.8, 3 and by Remark 4.4, 1, $x \in V_1$ and the peripheral arms of S_4 are in \mathcal{A}_1 . By Remark 4.5, 4, $x \notin V_2$. Let R be the component of $((T - Q) +_w S_4) - V_2$ containing S_4 . By Remark 4.5, 1, R is in \mathcal{A}_2 . Let R' be
 15 the component of $T - (V_2 \cup \{$ the central vertex of each small pincer of $Q\})$ containing v .

Let $\mathcal{A}'_1 = (\mathcal{A}_1 \setminus \{$ the peripheral arms of $S_4\}) \cup \{$ the peripheral super paths of $Q\}$, $V'_1 = (V_1 \setminus \{x\}) \cup \{v\}$, $V'_2 = V_2 \cup \{$ the central vertex of each small pincer of $Q\}$ and $\mathcal{A}'_2 = (\mathcal{A}_2 \setminus \{R\}) \cup \{$ the peripheral arms of each small pincer of $Q, R'\}$.

Since $M_1(T) = M_1((T - Q) +_w S_4) + f + g + h - 3$, by Remark 4.6, this creates an \overline{M}_2 assignment to T ,
 20 $\mathcal{A}' = ((\mathcal{A}'_1, V'_1), (\mathcal{A}'_2, V'_2))$ and $\overline{M}_2(T) \geq \overline{M}_2((T - Q) +_w S_4) + f + g + h - 3 + 2h - h = \overline{M}_2(T - Q +_w S_4) + f + g + 2h - 3$.

Conversely, let $\mathcal{A} = ((\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2))$ be an \overline{M}_2 -maximal assignment to T . By Lemma 4.8 3, v is in V_1 .

If $v \in V_2$ then by Remark 4.5, 1, the longer arm and the small pincers of Q are in \mathcal{A}_2 . By Remark 4.4, 1, each of the $f + g + h$ peripheral super paths of Q might be in \mathcal{A}_1 . If $w \notin V_2$, then let F be the component of $T - V_2$ containing w . Let H be the component of $T - ((V_2 \setminus \{v\}) \cup \{$ the central vertex of each small pincer of $Q\})$
 25 containing v .

Let $\mathcal{B}_1 = \mathcal{A}_1$, $U_1 = V_1$, $\mathcal{B}_2 = (\mathcal{A}_2 \setminus \{F, \text{ the longer arm and the small pincers of } Q\}) \cup \{$ the peripheral arms of each small pincer of $Q, H\}$ and $U_2 = (V_2 \setminus \{v\}) \cup \{$ the central vertex of each small pincer of $Q\}$.

By Remark 4.6, this creates an \overline{M}_2 assignment to T , $\mathcal{B} = ((\mathcal{B}_1, U_1), (\mathcal{B}_2, U_2))$. Using the cardinality of \mathcal{B} we conclude that $g = 1$, $w \notin V_2$ and $F \in \mathcal{A}_2$.

30 We are going to construct, $\mathcal{A}' = ((\mathcal{A}'_1, V'_1), (\mathcal{A}'_2, V'_2))$, an \overline{M}_2 assignment to $(T - Q) +_w S_4$. Let x be the central vertex of S_4 . Let R' be the component of $((T - Q) +_w S_4) - (V_2 \setminus \{v\})$ containing x .

Let $\mathcal{A}'_1 = (\mathcal{A}_1 \setminus \{$ the peripheral super paths of $Q\}) \cup \{$ the peripheral arms of $S_4\}$, $V'_1 = (V_1 \setminus \{v\}) \cup \{x\}$, $\mathcal{A}'_2 = (\mathcal{A}_2 \setminus \{F, \text{ the longer arm and the small pincers of } Q\}) \cup \{R'\}$ and $V'_2 = V_2 \setminus \{v\}$.

Since $M_1((T - Q) +_w S_4) = M_1(T) - f - g - h + 3$, by Remark 4.6, this creates an \overline{M}_2 assignment to $(T - Q) +_w S_4$,
 35 $\mathcal{A}' = ((\mathcal{A}'_1, V'_1), (\mathcal{A}'_2, V'_2))$ and $\overline{M}_2((T - Q) +_w S_4) \geq \overline{M}_2(T) - f - g - h + 3 - 1 - 1 - h + 1 + 1 = \overline{M}_2(T) - f - g - 2h + 3$.

If $v \notin V_2$, using the maximality of $|\mathcal{A}_2| - |V_2|$, then the central vertex of each small pincer of Q is in V_2 . We are going to construct an \overline{M}_2 assignment to $(T - Q) +_w S_4$, $\mathcal{A}' = ((\mathcal{A}'_1, V'_1), (\mathcal{A}'_2, V'_2))$. By Remarks 4.4, 1 and 4.5, 1 and 3, each of the $f + g + h$ peripheral super paths of Q might be in \mathcal{A}_1 and the peripheral arms of each small pincer of Q might be in \mathcal{A}_2 . Let R be the component of $T - V_2$ containing v . Let R' be the component of
 40 $((T - Q) +_w S_4) - V_2$ containing x (x is the central vertex of S_4).

Let $\mathcal{A}'_1 = (\mathcal{A}_1 \setminus \{$ the peripheral super paths of $Q\}) \cup \{$ the peripheral arms of $S_4\}$, $V'_1 = (V_1 \setminus \{v\}) \cup \{x\}$, $\mathcal{A}'_2 = (\mathcal{A}_2 \setminus \{R, \text{ the peripheral arms of each small pincer of } Q \cup \{R'\})$ and $V'_2 = V_2 \setminus \{$ the central vertex of each small pincer of $Q\}$.

Since $M_1((T - Q) +_w S_4) = M_1(T) - f - g - h + 3$, by Remark 4.6, this creates an \overline{M}_2 assignment to $(T - Q) +_w S_4$,
 45 $\mathcal{A}' = ((\mathcal{A}'_1, V'_1), (\mathcal{A}'_2, V'_2))$ and $\overline{M}_2((T - Q) +_w S_4) \geq \overline{M}_2(T) - f - g - h + 3 - 2h + h = \overline{M}_2(T) - f - g - 2h + 3$.

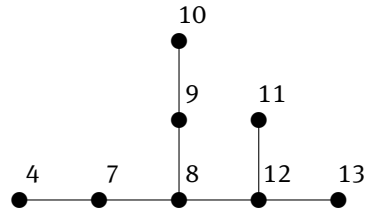
Consequently, we have $\overline{M}_2((T - Q) +_w S_4) = \overline{M}_2(T) - f - g - 2h + 3$. \square

Example 4.11. Let T be the tree of Example 3.17. Let Q be the subtree of T generated by vertices 1, 2, 3, 4, 5, 6. Since Q is a small superstar (T is not a small superstar) with 1 arm of length 1, 1 small pincer, and 7 is a vertex

of T with degree 2, by Theorem 4.10,

$$\overline{M}_2(T) = \overline{M}_2((T - Q) +_w K_1) + 2,$$

where w is the vertex 7. So, $(T - Q) +_w K_1$ (that is a small superstar with central vertex 8) is the tree



By Proposition 3.14,

$$\overline{M}_2(((T - Q) +_w K_1) - J) = 2 + 4 - 2 = 2.$$

Therefore,

$$\overline{M}_2(T) = 6.$$

□

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