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RESEARCH ARTICLE

Minimal polynomial decomposition of plane curve branch truncation using a semigroup based algorithm

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Abstract

Let Y be an irreducible plane curve germ with branch ζ and s characteristic exponents. We introduce a class of truncation sequences of ζ having finite support. For a given $(\widetilde{\zeta_i})_{i=1,\ldots,s}$ from this class, we explicitly compute the convex hull of the minimal polynomial f_i for each germ of plane curve Y_i, with branch $\widetilde{\zeta_i}$. We investigate the relationships between the semigroup of the Y_i's, as well as the induced canonical valuations. Additionally, we provide methods for selecting truncation sequences that yield topologically equivalent approximations Y_s of Y. The sequence $(\widetilde{\zeta_i})_{i=1,\ldots,s}$ of ζ provides a unique decomposition of each polynomial f_i. Given that the minimal polynomial f_i can be written as a power of f_{i-1} plus a tail δ_i, our first decomposition theorem studies properties of the tail. The second decomposition theorem characterizes the decomposition of δ_i and enables its explicit computation. To conclude, a pseudocode algorithm is presented along with an example.

Keywords Singularities · Curves · Semigroup

1 Introduction

Describing a set using equations provides a straightforward mathematical and computational method for determining whether a point belongs to that set. This type of representation has garnered significant interest due to its applications in computational fields such as geometric modelling. There are several methods to obtain an equation-based representation of an algebraic variety, including Gröbner bases (see [4]), resultants (see [6]), interpolation techniques (see [14]) and syzygies (see [7]).

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For the authors, the main interest in obtaining an equation for a germ of a plane curve lies in studying equisingular versal deformations and microlocal versal deformations.

Section 2 is dedicated to establishing notations and definitions, as well as reviewing key concepts related to plane curve germs, specifically characteristic exponents, semigroup and the connection between valuation and semigroup. Additionally, we present an essential known result for later use. Since the authors aim to generalize the results of this article to quasi-ordinary hypersurfaces, the definition of characteristic exponents used here is the one applicable to the quasi-ordinary case, rather than the classical definition.

Let Y be an irreducible plane curve germ with s characteristic exponents. Inspired by the proof of Theorem 3.9 of [19], in Sect. 3, we introduce the notion of a sequence of admissible truncations of a branch of Y, along with the induced parameterizations ι_i, for $i = 1, \ldots, s$, and valuations ϑ_{ι_i}. These truncations have finite support, and each one corresponds to the branch of an irreducible plane curve Y_i. The valuations ϑ_{ι_i} are induced by polynomial parameterizations, making them suitable for computational implementation. In Theorem 7, we explicitly compute the convex hull, N_i, of the support of the minimal polynomial, f_i, for each branch in the sequence of truncations.

Since the Weierstrass Division Algorithm is extensively used in Sect. 4, Sect. 4.1 is dedicated to reviewing the Algorithm and some of its properties. Lemma 11 examines the behavior of the support of the quotient and the reminder when applying the Weierstrass Division Algorithm to divide by a Weierstrass polynomial. This lemma, along with Theorem 7, plays an important role in proving statement (D) of Theorem 15.

In Sect. 4.2 we present the decomposition results, Theorems 12 and 15. Theorem 12 states that the minimal polynomial f_i, for $i \in \{1, \ldots, s\}$, can be obtained by adding a polynomial tail, δ_i, with certain properties, to a power of f_{i-1}. Furthermore, Theorem 12 establishes a relationship between $\vartheta_{\iota_j}(f_{i-1})$, for $j = i, \ldots, s$, and the generators of the semigroup of Y_j. Lemma 14 provides relations between the valuations ϑ_{ι_i}. An important aspect of obtaining these relations is the decomposition given in statement *(B)* of Lemma 14, which is also presented in the proof of Theorem 3.9 of [19]. Theorem 15 refines the information about the tail δ_i and provides a method for computationally obtaining δ_i.

Based upon the elimination procedure outlined in Sect. 5.1, we present an algorithm in Sect. 5.2, Algorithm 1, to obtain f_i from f_{i-1}, by computing δ_i. This relies heavily on Theorem 15. We compute δ_i as described in statement (B) and by applying the constraints of statements (A), (C) and (D). This involves identifying points with non-negative integers entries on a compact section of a hyperplane, whose orthogonal complement is spanned by a vector with entries corresponding to the generators of the semigroup associated to Y_i. This section provides a detailed explanation of how this Algorithm works, along with pseudocode. Lemma 18 offers a simple method to verify if statement (D) is satisfied, which can also be easily implemented computationally. Algorithm 1 requires the computation of valuations, which are performed using only the ϑ_{ι_i}'s. Corollary 20 presents an upper limit for the number of iterations needed to compute δ_i. We conclude with an Example whose computations were carried out using an implementation of Algorithm 1 coded by the authors in *Mathematica*.

2 Semigroup of the germ of an irreducible plane curve

In this section we review important concepts related to plane curve germs, such as characteristic exponents, associated semigroup, parametrization and its induced valuation. We begin by establishing some notations and definitions. Singletons are identified with their elements when unambiguous. Let $\mathbb{N}_0$ [$\mathbb{N}$] be the set of non-negative integers [positive integers], $\mathbb{Q}_{\geq 0}$ [$\mathbb{Q}_{>0}$] be the set of non-negative rationals [positive rationals] and $\mathbb{R}_{\geq 0}$ the set of non-negative real numbers. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Q}^n_{\geq 0}$, define $|\alpha| = \sum_{i=1}^n \alpha_i$.

We denote the polynomial ring, the convergent power series ring and the formal power series ring in the variables $x_1, \ldots, x_n$ with complex coefficients by, respectively, $\mathbb{C}[x_1, \ldots, x_n]$, $\mathbb{C}\{x_1, \ldots, x_n\}$ and $\mathbb{C}[[x_1, \ldots, x_n]]$. Let $f(x_1, \ldots, x_n) = \sum_{|\alpha|\geq 0} a_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in \mathbb{C}[[x_1, \ldots, x_n]]$. Define the *support* of f, denoted by $\operatorname{Supp}(f)$, as

$$\operatorname{Supp}(f) = \{\alpha \in \mathbb{N}_0^n : a_\alpha \neq 0\}.$$

By convention, the support of the null series is the empty set. Let $\operatorname{ord}(f)$ denote the *order* of f, defined as the infimum of the set

$$\{|\alpha| : \alpha \in \operatorname{Supp}(f)\}.$$

For a polynomial $g \in \mathbb{C}[x_1, \ldots, x_n]$, the degree of g is denoted by $\deg(g)$. Given $p \in \mathbb{C}[[x_1, \ldots, x_n]][y]$ non-zero, we define *degree of p in y* as the greatest power of y that appears in p with non-zero coefficient, and it is denoted by $\deg_y(p)$. If p is the zero polynomial, we set $\deg_y(p) = -\infty$.

Let A be a subset of $\mathbb{C}[x_1, \ldots, x_n]$ or $\mathbb{C}\{x_1, \ldots, x_n\}$ or $\mathbb{C}[[x_1, \ldots, x_n]]$. Denote by, when it has meaning, $(A)\mathbb{C}[x_1, \ldots, x_n]$, $(A)\mathbb{C}\{x_1, \ldots, x_n\}$ and $(A)\mathbb{C}[[x_1, \ldots, x_n]]$ the ideal generated by A in the ring $\mathbb{C}[x_1, \ldots, x_n]$, $\mathbb{C}\{x_1, \ldots, x_n\}$ and $\mathbb{C}[[x_1, \ldots, x_n]]$, respectively.

Throughout this paper, for simplicity, we will refer to plane curve germs at the origin of $\mathbb{C}^2$ simply as plane curves. Let Y be a singular irreducible plane curve. We denote a representative of Y by $\{f(x, y) = 0\}$, where $f \in (x, y)\mathbb{C}\{x, y\}$ and is of order $k > 1$. The series $f(x, y)$ can be expressed as $\sum_{i\geq k} f_i(x, y)$, where f_i is either null or a homogeneous polynomial of degree i, for $i \geq k$. Since k is the order of f, f_k is non-zero. The multiplicity of Y, denoted by $m_0(Y)$, is defined as $\operatorname{ord}(f)$ (it does not depend on the choice of the representative). After applying a linear change of coordinates, we can assume that $f(0, y)$ is non-zero and has order k. Under these circumstances, the Weierstrass Preparation Theorem allows us to assume that f is a Weierstrass polynomial in y of degree k, that is

$$f(x, y) = y^k + \sum_{j=0}^{k-1} a_j(x) y^j \in \mathbb{C}\{x\}[y],$$

where $a_j(0) = 0$ for all $j \in \{0, \dots, k-1\}$. Again, after applying a linear change of coordinates, it can be assumed that, for all $j \in \{0, \dots, k-1\}$, the order in x of $a_j(x)$ is strictly greater than $k - j$.

The roots of the irreducible Weierstrass polynomial f are elements of the ring $\mathbb{C}\{x^{1/k}\}$. These roots of f are called *branches* of Y. In a natural way, the notions of order and support can be extended to the elements of $\mathbb{C}\{x^{1/k}\}$ and the notion of degree to the elements of $\mathbb{C}[x^{1/k}]$. Let ζ be a branch of Y. There are $\lambda_1, \dots, \lambda_s \in \mathrm{Supp}(\zeta)$, the *characteristic exponents of* Y, such that

(*C*1) $\lambda_1 < \lambda_2 < \cdots < \lambda_s$.
(*C*2) For all $i \in \{1, \dots, s\}$, λ_i does not belong to $\mathbb{Z} + \sum_{\ell=1}^{i-1} \mathbb{Z}\lambda_\ell$.
(*C*3) If $\lambda \in \mathrm{Supp}(\zeta)$ then λ belongs to $\mathbb{Z} + \sum_{\lambda_\ell \leq \lambda} \mathbb{Z}\lambda_\ell$.

In fact, $\zeta \in \mathbb{C}\{x^{1/k}\}$ is the branch of an irreducible curve if and only if there are $\lambda_1, \dots, \lambda_s \in \mathrm{Supp}(\zeta)$ such that (*C1*), (*C2*) and (*C3*) hold. Our choice of coordinates for Y implies that $\lambda_1 > 1$. Given a branch ζ of Y, we can compute all the branches of Y by evaluating $\zeta(\theta x^{1/k})$, where θ runs through all k-th roots of the unity. Due to our choice of representative,

$$f(x, y) = \prod_{j=1}^{k} \left(y - \zeta_j \left(x^{1/k} \right) \right). \tag{1}$$

Under these circumstances, we say that f is the *minimal polynomial* of Y.

Let $M_0 = \mathbb{Z}$ and $M_i = M_{i-1} + \mathbb{Z}\lambda_i$, $i = 1, \dots, s$. For $i = 1, \dots, s$, let k_i be the smallest positive integer such that $k_i\lambda_i \in M_{i-1}$. Then $k = \prod_{j=1}^{s} k_j$. By default, zero will belong to all semigroups mentioned in this paper. We define $\Gamma(\lambda_1, \dots, \lambda_s)$ as the semigroup generated by k,

$$\gamma_1 = k\lambda_1 \text{ and } \gamma_{j+1} = k_j\gamma_j + k\lambda_{j+1} - k\lambda_j \text{ for } j = 1, \dots, s-1. \tag{2}$$

The semigroup $\Gamma(\lambda_1, \dots, \lambda_s)$ is the semigroup of an irreducible curve with characteristic exponents $\lambda_1, \dots, \lambda_s$. For $i = 0, \dots, s$, define $\Gamma_i(\lambda_1, \dots, \lambda_s) = k\mathbb{N}_0 + \sum_{j=1}^{i} \mathbb{N}_0\gamma_j$, a subsemigroup of $\Gamma(\lambda_1, \dots, \lambda_s)$.

In the following proposition, we present some properties that will be useful in Sect. 4. Statement *(B)* also plays a key role in the initial step of the elimination procedure outlined in Sect. 5.1.

Proposition 1 *The following statements hold.*

(*A*) *For all* $i \in \{0, \dots, s\}$, $kM_i = k\mathbb{Z} + \sum_{j=1}^{i} \mathbb{Z}\gamma_j$.
(*B*) *Let* $i \in \{1, \dots, s\}$ *and* $\ell \in \{1, \dots, k_i\}$. *Then* $\ell\gamma_i \in \Gamma_{i-1}(\lambda_1, \dots, \lambda_s)$ *if and only if* $\ell = k_i$.

Proof See [3]. □

Let ζ be a branch of Y. After a change of coordinates of the type

$$(x, y) \to (x, y - \varphi(x)),\ \varphi \in \mathbb{C}\{x\},$$

we can assume that $\mathrm{Supp}(\zeta)$ contains no positive integers and therefore, we can write $\zeta = x^{\lambda_1} H(x^{1/k})$, where $H \in \mathbb{C}\{x\}$ and $H(0) \neq 0$ (see Proposition 2.4 of [12]). Branch ζ induces a parameterization ι of Y by setting $x = t^k$. By *(C3)*, we can write the parametrization ι as

$$x = t^k,\ y = c_1 t^{k\lambda_1} + \varphi_1(t) + c_2 t^{k\lambda_2} + \varphi_2(t) + \cdots + c_s t^{k\lambda_s} + \psi(t), \tag{3}$$

where

- $(P1)$ $c_1, \ldots, c_s \in \mathbb{C}^*, \varphi_1, \ldots, \varphi_{s-1} \in \mathbb{C}[t]$, $\psi \in \mathbb{C}\{t\}$ and $\varphi_1(0) = \cdots = \varphi_{s-1}(0) = \psi(0) = 0$;
- $(P2)$ for all $i \in \{1, \ldots, s-1\}$, $\mathrm{ord}(\varphi_i) > k\lambda_i$, $\deg(\varphi_i) < k\lambda_{i+1}$ and $\mathrm{Supp}(\varphi_i) \subset kM_i$;
- $(P3)$ $\mathrm{ord}(\psi) > k\lambda_s$ and $\mathrm{Supp}(\psi) \subset kM_s$.

Parameterization ι induces a map $\vartheta_\iota : \mathbb{C}[[x, y]] \to \mathbb{N}_0 \cup \{+\infty\}$ defined by

$$\vartheta_\iota(g) = \mathrm{ord}(\iota^* g).$$

This map is a valuation, that is, $\vartheta_\iota(0) = +\infty$,

$$\vartheta_\iota(gh) = \vartheta_\iota(g) + \vartheta_\iota(h), \tag{4}$$

$$\vartheta_\iota(g + h) \geq \min\{\vartheta_\iota(g), \vartheta_\iota(h)\}. \tag{5}$$

Zariski proved in Theorem 3.9 of [19] that

$$\vartheta_\iota(\mathbb{C}[[x, y]]) = \Gamma(\lambda_1, \ldots, \lambda_s) \cup \{+\infty\}. \tag{6}$$

This equality will play an important role in Sect. 5, namely in the elimination process used in Algorithm 1.

3 Truncation and Newton polytope of the equation

In this section, we adopt the approach employed by Zariski in the proof of Theorem 3.9 of [19] and perform a suitable truncation of a branch of an irreducible curve. This truncation corresponds to a root of a polynomial. Subsequently, we compute the convex hull of the support of that polynomial.

Definition 2 Let ϱ be a non-negative rational and

$$\zeta(x^{1/k}) = \sum_{\alpha \in \mathbb{Q}_{\geq 0}} c_\alpha x^\alpha$$

a branch of Y. We say that ϱ is an admissible exponent of Y if $\varrho \in M_s$ and $\lambda_s < \varrho$. Take $\widetilde{\zeta}_0 \in \mathbb{C}\{x^{1/k}\}$ equal to zero. Given ϱ, an admissible exponent of Y, we say that a sequence $(\widetilde{\zeta}_i)_{i=1,\dots,s}$ of elements of $\mathbb{C}[x^{1/k}]$ is an admissible sequence of truncations of Y with respect to ϱ if

$$\operatorname{Supp}(\widetilde{\zeta}_s) = \operatorname{Supp}(\zeta) \cap \{\alpha \in \mathbb{Q}_{\geq 0} : \alpha < \varrho\} \tag{7}$$

and, for each $i \in \{1, \dots, s\}$, there is $\psi_i \in \mathbb{C}[x^{1/k}]$ such that

$(T1)$ $\psi_i(x^{1/k}) = \sum_{\alpha \in A_i} c_\alpha x^\alpha$, where $A_i \subseteq \{\alpha \in \operatorname{Supp}(\widetilde{\zeta}_s) : \alpha > \lambda_i\} \backslash \operatorname{Supp}(\widetilde{\zeta}_{i-1})$.
$(T2)$ $\operatorname{Supp}(\psi_i) \subset M_i$.
$(T3)$ $\widetilde{\zeta}_i(x^{1/k}) = \widetilde{\zeta}_{i-1}(x^{1/k}) + c_{\lambda_i} x^{\lambda_i} + \psi_i(x^{1/k})$.

Remark 3 Chosen ϱ, the support of $\widetilde{\zeta}_s$ is always the same, independently of our choice for $\widetilde{\zeta}_1, \dots, \widetilde{\zeta}_{s-1}$.

Example 4 Let ϱ be an admissible exponent of Y,

$$A = \operatorname{Supp}(\zeta) \cap \{\alpha \in \mathbb{Q}_{\geq 0} : \alpha < \varrho\},$$

$A_s = \{\alpha \in A : \alpha > \lambda_s\}$ and, for $i \in \{1, \dots, s-1\}$,

$$A_i = \{\alpha \in A : \lambda_i < \alpha < \lambda_{i+1}\}.$$

The intersection of the sets $A_i, A_j, i, j = 1, \dots, s$ and $i \neq j$, is the empty set. Also, due to our choice of coordinates, $A \setminus \{\lambda_1, \dots, \lambda_s\} = \cup_{i=1}^{s} A_i$. Statement $C3$) (see Sect. 2) implies $A_i \subseteq M_i$, for all $i \in \{1, \dots, s\}$. Set $\lambda_{s+1} = \varrho$, $\zeta_{\lambda_1,\varrho}$ equal to zero and

$$\psi_i\left(x^{1/k}\right) = \sum_{\alpha \in A_i} c_\alpha x^\alpha$$

if A_i is non-empty, $\psi_i(x^{1/k}) = 0$ otherwise. Then

$$\zeta_{\lambda_{i+1},\varrho}\big(x^{1/k}\big) = \zeta_{\lambda_i,\varrho}\big(x^{1/k}\big) + c_{\lambda_i} x^{\lambda_i} + \psi_i\big(x^{1/k}\big),\ i = 1, \dots, s,$$

is an admissible sequence of truncations of Y with respect to ϱ.

Example 5 Let Y be an irreducible plane curve with characteristic exponents $7/3, 5/2$ and branch

$$\zeta = c_1 x^{7/3} + c_2 x^{5/2} + c_3 x^{8/3} + c_4 x^{17/6} + c_5 x^{19/6} + c_6 x^{11/3} + c_7 x^{23/6} + c_8 x^{25/6} + c_9 x^{31/6} + c_{10} x^{37/6}$$

where $c_1, c_2 \in \mathbb{C}^*$, and $c_i \in \mathbb{C}$, $i \in \{3, 4, \dots, 10\}$. Note that $k_1 = 3$, $k_2 = 2$ and

$$\operatorname{Supp}(\zeta) \cap M_1 = \{7/3, 8/3, 11/3\}.$$

The following sequences of truncations are all admissible:

$$(c_1 x^{7/3}, \zeta),\ (c_1 x^{7/3} + c_3 x^{8/3}, \zeta),\ (c_1 x^{7/3} + c_6 x^{11/3}, \zeta),\ (c_1 x^{7/3} + c_3 x^{8/3} + c_6 x^{11/3}, \zeta).$$

The admissible sequence of truncations presented in Example 4, for this case, is $(c_1x^{7/3}, \zeta)$.

We keep the notations and the system of coordinates established in the previous section. Let ι be as defined in (3) and ζ be a branch of Y that induces the parametrization ι. Set $e_0 = 1$ and, for $i = 1, \ldots, s$, let us define $e_i = k_i e_{i-1}$. Note that $e_s = k$. From now on, we assume an admissible sequence of truncations of Y with respect to a ϱ, $(\widetilde{\zeta_i})_{i=1,\ldots,s}$, has been chosen. For $i \in \{1, \ldots, s\}$, the truncation $\widetilde{\zeta_i}$ is the branch of an irreducible curve Y_i with characteristic exponents $\lambda_1, \ldots, \lambda_i$, hence having multiplicity e_i and semigroup $\Gamma(\lambda_1, \ldots, \lambda_i)$. Additionally, $\widetilde{\zeta_i} \in \mathbb{C}[x^{1/e_i}]$. Define ι_i as the polynomial parametrization induced by $\widetilde{\zeta_i}$.

Let $e_i, \gamma_1^{(i)}, \ldots, \gamma_i^{(i)}$ be the generators of $\Gamma(\lambda_1, \ldots, \lambda_i)$. For $j = 1, \ldots, i$, $\gamma_j^{(i)}$ is defined as follows, based on (2):

$$\gamma_1^{(i)} = e_i\lambda_1 \text{ and } \gamma_{j+1}^{(i)} = k_j\gamma_j^{(i)} + e_i\lambda_{j+1} - e_i\lambda_j \text{ for } j = 1, \ldots, i-1.$$

The next proposition, for $j \le i$, gives us a relation between $\Gamma_j(\lambda_1, \ldots, \lambda_i) = e_i\mathbb{N}_0 + \sum_{\ell=1}^{j} \mathbb{N}_0\gamma_\ell^{(i)}$ and $\Gamma(\lambda_1, \ldots, \lambda_j)$.

Proposition 6 *For all $i \in \{1, \ldots, s\}$ and $j \in \{1, \ldots, i\}$,*

$$\Gamma_j(\lambda_1, \ldots, \lambda_i) = \frac{e_i}{e_j}\Gamma(\lambda_1, \ldots, \lambda_j). \tag{8}$$

Proof See [3]. □

From now on f_i will represent the minimal polynomial of Y_i. The following Theorem computes the convex hull of the support of f_i.

Theorem 7 *Let $i \in \{1, \ldots, s\}$, μ_i be the degree of $\widetilde{\zeta_i}$ and N_i the convex hull of Supp(f_i). Then N_i equals the convex hull of $\{(0, e_i), (e_i\lambda_1, 0), (e_i\mu_i, 0)\}$, that is,*

$$N_i = \{(\alpha, \beta) \in \mathbb{R}^2_{\ge 0} : k_1\alpha + (k_1\lambda_1)\beta \ge e_i(k_1\lambda_1) \wedge e_i\alpha + (e_i\mu_i)\beta \le e_i(e_i\mu_i)\}. \tag{9}$$

Proof Let $C = \{(0, e_i), (e_i\lambda_1, 0), (e_i\mu_i, 0)\}$. From (1),

$$C \subseteq \text{Supp}(f_i).$$

Therefore, the definition of convex hull implies (see Sect. 0 of [20])

$$\text{Conv}(C) \subseteq \text{Conv}(\text{Supp}(f_i)).$$

The set Conv(C) is equal to

$$\{(\alpha, \beta) \in \mathbb{R}^2_{\ge 0} : e_i\alpha + e_i\lambda_1\beta \ge e_i(e_i\lambda_1) \wedge e_i\alpha + e_i\mu_i\beta \le e_i(e_i\mu_i)\}.$$

All that is left is to prove that $\mathrm{Conv}(\mathrm{Supp}(f_i)) \subseteq N_i$. The branches $(\widetilde{\zeta_i})_1, \dots, (\widetilde{\zeta_i})_{e_i}$, of Y_i, have the same support. For each $j \in \{1, \dots, e_i\}$, the support of $y - (\widetilde{\zeta_i})_j$ is equal to

$$\widetilde{C} = \left(\mathrm{Supp}\left(\widetilde{\zeta_i}\right) \times 0\right) \cup \{(0, 1)\}.$$

Since (1) holds for f_i,

$$\mathrm{Supp}(f_i) \subseteq \sum_{j=1}^{e_i} \widetilde{C}.$$

Notice that, since the set $\mathrm{Supp}(\widetilde{\zeta_i})$ is totally ordered with minimum λ_1 and maximum μ_i, then

$$\mathrm{Conv}(\widetilde{C}) = \left\{(\alpha, \beta) \in \mathbb{R}^2_{\geq 0} : \alpha + \lambda_1\beta \geq \lambda_1 \wedge \alpha + \mu_i\beta \leq \mu_i\right\}. \tag{10}$$

For $j = 1, \dots, e_i$, let $(\alpha_j, \beta_j), (\widetilde{\alpha}_j, \widetilde{\beta}_j) \in \widetilde{C}$. Set

$$P = (\rho, \eta) = \nu \sum_{j=1}^{e_i} (\alpha_j, \beta_j) + (1 - \nu) \sum_{j=1}^{e_i} \left(\widetilde{\alpha}_j, \widetilde{\beta}_j\right),$$

where $\nu \in \mathbb{R}$. The point P belongs to the convex hull of $\sum_{j=1}^{e_i} \widetilde{C}$ and let us prove that it belongs to N_i. We have

$$e_i\rho + e_i\lambda_1\eta = e_i\nu \sum_{j=1}^{e_i} (\alpha_j + \lambda_1\beta_j) + e_i(1 - \nu) \sum_{j=1}^{e_i} \left(\widetilde{\alpha}_j + \lambda_1\widetilde{\beta}\right)$$

and

$$e_i\rho + e_i\mu_i\eta = e_i\nu \sum_{j=1}^{e_i} (\alpha_j + \mu_i\beta_j) + e_i(1 - \nu) \sum_{j=1}^{e_i} \left(\widetilde{\alpha}_j + \mu_i\widetilde{\beta}\right).$$

We apply (10) to the two previous equalities to conclude that P belongs to N_i. Therefore,

$$\mathrm{Supp}(f_i) \subseteq \sum_{j=1}^{e_i} \widetilde{C} \subseteq \mathrm{Conv}\left(\sum_{j=1}^{e_i} \widetilde{C}\right) \subseteq N_i$$

and we conclude that $\mathrm{Conv}\,(\mathrm{Supp}(f_i)) \subseteq N_i$. $\square$

4 Truncation minimal polynomial decomposition

This section is split into two subsections and is dedicated to proving the existence of a unique decomposition of the minimal polynomials associated to the truncations $\widetilde{\zeta}_1, \ldots, \widetilde{\zeta}_s$. Section 4.2 presents the key decomposition results, Theorems 12 and 15, which are subsequently applied in Algorithm 1. The proofs of these theorems rely on the Weierstrass Division Algorithm and Sect. 4.1 is dedicated to proving some technical results around it.

4.1 Weierstrass division algorithm

The Weierstrass Division Algorithm plays an important role in proof of Lemma 14 and Theorem 15, so we need to get (re)acquainted with it (see Theorem 6.2.6 of [10] and its proof for the formal version). We are only interested in the case where we divide an element of $\mathbb{C}[[x]][y]$ by a Weierstrass polynomial in y. Given $m \in \mathbb{N}$ and $w(x, y) = \sum_{\ell \geq 0} c_\ell(x) y^\ell \in \mathbb{C}[[x]][y]$, define $r_m(w)(x, y) = \sum_{\ell=0}^{m-1} c_\ell(x) y^\ell$ and

$$h_m(w)(x, y) = \sum_{\ell \geq m} c_\ell(x) y^{\ell-m} = \frac{1}{y^m}(w(x, y) - r_m(w)(x, y)).$$

Note that $w(x, y) = h_m(w)(x, y) y^m + r_m(w)(x, y)$, $r_m(w) \in \mathbb{C}[[x]][y]$ and, in our case, $h_m(w) \in \mathbb{C}[[x]][y]$. Let $g \in \mathbb{C}[[x, y]]$, $p \in \mathbb{C}[[x]][y]$ a Weierstrass polynomial of degree m. Weierstrass Division Theorem states that there are uniquely determined $q \in \mathbb{C}[[x, y]]$ and $v \in \mathbb{C}[[x]][y]$ such that $g(x, y) = q(x, y) p(x, y) + v(x, y)$ and $\deg_y(v) \leq m-1$. We will refer to q and v as the *quotient* and *remainder*, respectively. From its proof, we know that q verifies $h_m(qp)(x, y) = h_m(g)(x, y)$ and

$$q(x, y) = h_m(g)(x, y) + h_m(-q r_m(p))(x, y). \tag{11}$$

The later equality results from the former with some computations, that we will not replicate. Obviously, if $\deg_y(g) \leq m - 1$ then q is zero. Define the $\mathbb{C}$-linear map

$$H_m(g)(x, y) = h_m(-g r_m(p))(x, y).$$

Let H_m^0 be the identity map and, for $\ell \in \mathbb{N}$, H_m^ℓ be H_m composed with itself ℓ times. We rewrite (11) as

$$q(x, y) = H_m^0(h_m(g))(x, y) + H_m(q)(x, y). \tag{12}$$

The rest of this subsection focuses mostly on proving some technical results around the Weierstrass Division Algorithm that are needed for future proofs. Some known properties of this Algorithm, essential for future proofs, are presented in the following lemma:

Lemma 8 *Let $g, p \in \mathbb{C}[[x]][y]$ be such that p is a Weierstrass polynomial of degree m and $deg_y(g) \geq m$. Let q and v be, respectively, the quotient and the remainder of the division of g by p. Then $q, v \in \mathbb{C}[[x]][y]$ and the following statements hold:*

(A) *There is $\sigma \in \mathbb{N}_0$ such that*

$$q(x, y) = \sum_{\ell=0}^{\sigma} H_m^{\ell}(h_m(g))(x, y). \tag{13}$$

(B) *$deg_y(q) = deg_y(g) - m$.*
(C) *if g and p both belong to $\mathbb{C}[x, y]$ then $q \in \mathbb{C}[x, y]$.*

The following Lemma is a technical Polyhedral Geometry result needed for some proofs.

Lemma 9 *Let $\nu \in \mathbb{N}$, $s \geq 2$, $i \in \{2, \ldots, s\}$ and $j \in \{1, \ldots, i-1\}$. Then Supp(f_j^{ν}) is contained in the convex hull of $\{(0, \nu e_j), (\nu e_j \lambda_1, 0), (\nu e_j \mu_j, 0)\}$, that is, in the set*

$$\widetilde{N}_j = \{(\alpha, \beta) \in \mathbb{R}^2_{\geq 0} : k_1\alpha + (k_1\lambda_1)\beta \geq \nu e_j(k_1\lambda_1) \wedge e_j\alpha + (e_j\mu_j)\beta \leq \nu e_j(e_j\mu_j)\}.$$

The set Supp(f_j^{ν}) is also a subset of

$$\{(\alpha, \beta) \in \mathbb{R}^2_{\geq 0} : \alpha + \mu_i\beta \leq \nu e_j \mu_i\}.$$

Furthermore, $\widetilde{N}_j$ is a subset of N_i if and only if $\nu = k_{j+1} \cdots k_i$.

Proof The Multinomial formula implies

$$\text{Supp}(f_j^{\nu}) \subseteq \sum_{\ell=1}^{\nu} \text{Supp}(f_j).$$

Since Supp(f_j) is contained in its convex hull then Supp$(f_j) \subseteq N_j$, hence

$$\sum_{\ell=1}^{\nu} \text{Supp}(f_j) \subseteq \sum_{\ell=1}^{\nu} N_j.$$

Let $(\alpha_\ell, \beta_\ell) \in N_j$, $\ell = 1, \ldots, \nu$. By (9), inequalities

$$k_1 \sum_{\ell=1}^{\nu} \alpha_\ell + (k_1\lambda_1) \sum_{\ell=1}^{\nu} \beta_\ell = \sum_{\ell=1}^{\nu} (k_1\alpha_\ell + (k_1\lambda_1)\beta_\ell) \geq \nu e_j(k_1\lambda_1) \tag{14}$$

and

$$e_j \sum_{\ell=1}^{\nu} \alpha_\ell + (e_j\mu_j) \sum_{\ell=1}^{\nu} \beta_\ell = \sum_{\ell=1}^{\nu} \big(e_j\alpha_\ell + (e_j\mu_j)\beta_\ell\big) \leq \nu e_j(e_j\mu_j) \tag{15}$$

hold. From inequalities (14) and (15) we conclude that

$$\sum_{\ell=1}^{\nu} N_j \subseteq \widetilde{N}_j.$$

If $\widetilde{N}_j$ is contained in N_i then the points $(0, \nu e_j)$ and $(0, e_i)$ must be the same, which implies $\nu = k_{j+1} \cdots k_i$. Assume that $\nu = k_{j+1} \cdots k_i$. Then $\widetilde{N}_j$ is the convex hull of $\{(0, e_i), (e_i\lambda_1, 0), (e_i\mu_j, 0)\}$. Since $\mu_j < \mu_i$ then $\{(0, e_i), (e_i\lambda_1, 0), (e_i\mu_j, 0)\}$ is a subset of N_i, and we conclude that $\widetilde{N}_i \subset N_i$.

To prove the proposition's last statement, it suffices to prove that $\widetilde{N}_j$ is contained in $\{(\alpha, \beta) \in \mathbb{R}^2_{\geq 0} : \alpha + \mu_i\beta \leq \nu e_j\mu_i\}$. Let $(\alpha, \beta) \in \widetilde{N}_j$. There are τ_1, τ_2, τ_3 non-negative real numbers such that $\tau_1 + \tau_2 + \tau_3 = 1$ and

$$(\alpha, \beta) = \tau_1(0, \nu e_j) + \tau_2(\nu e_j\lambda_1, 0) + \tau_3(\nu e_j\mu_j, 0). \tag{16}$$

For this characterization of convex hull see Sect. 0 of [20]. From (16), we have

$$\alpha + \mu_i\beta = \tau_2\nu e_j\lambda_1 + \tau_3\nu e_j\mu_j + \tau_1\nu e_j\mu_i.$$

By definition, $\lambda_1 < \mu_i$ and $\mu_j < \mu_i$. Hence,

$$\tau_2\nu e_j\lambda_1 + \tau_3\nu e_j\mu_j + \tau_1\nu e_j\mu_i \leq (\tau_1 + \tau_2 + \tau_3)\nu e_j\mu_i$$

and we conclude that $\alpha + \mu_i\beta \leq \nu e_j\mu_i$. $\square$

Let $i \in \{1, \dots, s\}$ and $\nu = (\beta_0, \dots, \beta_{i-1}) \in \mathbb{N}_0^i$. Define $N_\nu^{(i)}$ as the set of the elements of $\mathbb{R}^2_{\geq 0}$ that verify

$$k_1\eta + (k_1\lambda_1)\rho \geq e_i(k_1\lambda_1) - (k_1\lambda_1)\sum_{\ell=0}^{i-1}\beta_\ell e_\ell$$

as well

$$\eta + \mu_i\rho \leq e_i\mu_i - \mu_i\sum_{\ell=0}^{i-1}\beta_\ell e_\ell.$$

Note that if $\nu = (0, \dots, 0)$ then $N_\nu^{(i)} = N_i$.

Proposition 10 *Let* $\tau \in \mathbb{N}_0$, $i \in \{1, \dots, s\}$, $j \in \{1, \dots, i\}$, $\nu = (\beta_0, \dots, \beta_{i-1}) \in \mathbb{N}_0^i$ *and* $A \subseteq Supp(f_{j-1}^{\beta_{j-1}})$. *Then*

$$N_\nu^{(i)} - (0, \tau e_{j-1}) \subseteq N_{\underline{\nu}}^{(i)} \tag{17}$$

and

$$N_{\nu}^{(i)} + A \subseteq N_{\overline{\nu}}^{(i)}, \tag{18}$$

where $\underline{\nu} = (\beta_0, \ldots, \beta_{j-2}, \beta_{j-1} + \tau, \beta_j, \ldots, \beta_{i-1})$ *and* $\overline{\nu} = (\beta_0, \ldots, \beta_{j-2}, 0, \beta_j, \ldots, \beta_{i-1})$.

Proof Let $(\eta, \rho) \in N_{\nu}^{(i)}$. We have

$$\begin{aligned} k_1\eta + (k_1\lambda_1)(\rho - \tau e_{j-1}) &\geq e_1(k_1\lambda_1) - (k_1\lambda_1)\sum_{\ell=0}^{i-1}\beta_\ell e_\ell - (k_1\lambda_1)\tau e_{j-1} \\ &= e_1(k_1\lambda_1) - (k_1\lambda_1)\sum_{\ell=0}^{i-1}\underline{\beta}_\ell e_\ell, \end{aligned}$$

where $\underline{\beta}_\ell = \beta_\ell$, for $\ell \neq j-1$, and $\underline{\beta}_{j-1} = \beta_{j-1} + \tau$. Also,

$$\eta + \mu_i(\rho - \tau e_{j-1}) \leq e_i\mu_i - \mu_i\sum_{\ell=0}^{i-1}\beta_\ell e_\ell - \mu_i\tau e_{j-1} = e_i\mu_i - \mu_i\sum_{\ell=0}^{i-1}\underline{\beta}_\ell e_\ell.$$

We conclude that (17) hold. We now prove (18) similarly. Let $(\widetilde{\eta}, \widetilde{\rho}) \in A$. Then,

$$k_1(\eta + \widetilde{\eta}) + (k_1\lambda_1)(\rho + \widetilde{\rho}) \geq e_1(k_1\lambda_1) - (k_1\lambda_1)\sum_{\ell=0}^{i-1}\beta_\ell e_\ell + k_1\widetilde{\eta} + (k_1\lambda_1)\widetilde{\rho}.$$

From Lemma 9, we know that

$$k_1\widetilde{\eta} + (k_1\lambda_1)\widetilde{\rho} \geq (k_1\lambda_1)\beta_{j-1}e_{j-1}.$$

Therefore,

$$e_1(k_1\lambda_1) - (k_1\lambda_1)\sum_{\ell=0}^{i-1}\beta_\ell e_\ell + k_1\widetilde{\eta} + (k_1\lambda_1)\widetilde{\rho} \geq e_1(k_1\lambda_1) - (k_1\lambda_1)\sum_{\ell=0}^{i-1}\overline{\beta}_\ell e_\ell$$

with $\overline{\beta}_\ell = \beta_\ell$, for $\ell \neq j-1$, and $\overline{\beta}_{j-1} = 0$. Lemma 9 also tells us that $\widetilde{\eta} + \mu_i\widetilde{\rho} \leq \beta_{j-1}e_{j-1}\mu_i$, which allows us to conclude that

$$\eta + \widetilde{\eta} + \mu_i(\rho + \widetilde{\rho}) \leq e_i\mu_i - \mu_i\sum_{\ell=0}^{i-1}\beta_\ell e_\ell + \mu_i\beta_{j-1}e_{j-1} = e_i\mu_i - \mu_i\sum_{\ell=0}^{i-1}\overline{\beta}_\ell e_\ell.$$

□

Lemma 11 *The following statements hold:*

(A) *Let $w, p \in \mathbb{C}[[x]][y]$, with p a Weierstrass polynomial in y with degree m. Then*

$$Supp(H_m(w)) \subseteq [(Supp(w) + Supp(p)) - (0, m)] \cap \mathbb{R}^2_{\geq 0}.$$

(B) *Let $\tau \in \mathbb{N}$, $i \in \{1, \dots, s\}$, $j \in \{1, \dots, i\}$ and $g \in \mathbb{C}[x, y]$. Suppose that there is $\nu = (\beta_0, \dots, \beta_{i-1}) \in \mathbb{N}_0^i$ such that $\beta_0 = \cdots = \beta_{j-1} = 0$ and $Supp(g) \subseteq N_\nu^{(i)}$. Let $q, v \in \mathbb{C}[x, y]$ be, respectively, the quotient and the remainder of the division of g by f_{j-1}^τ. Then $Supp(v)$ is a subset of $N_\nu^{(i)}$ and $Supp(q)$ is a subset of $N_{\overline{\nu}}^{(i)}$, where $\overline{\nu} = (0, \dots, 0, \tau, \beta_j, \dots, \beta_{i-1})$.*

Proof The proof of statement (A) is pretty straightforward. Let

$$\mathcal{H}_m = \{(\eta, \rho) \in \mathbb{R}^2_{\geq 0} : \rho \geq m\}.$$

Since $\text{Supp}(wr_m(p)) \subseteq \text{Supp}(w) + \text{Supp}(r_m(p))$, the definition of h_m implies that

$$\text{Supp}(h(wr_m(p))) \subseteq [(\text{Supp}(w) + \text{Supp}(r_m(p))) \cap \mathcal{H}_m] - (0, m).$$

Statement (A) is a direct result from the fact that $\text{Supp}(r_m(p)) \subseteq \text{Supp}(p)$ and

$$[(\text{Supp}(w) + \text{Supp}(p)) \cap \mathcal{H}_m] - (0, m) = [(\text{Supp}(w) + \text{Supp}(p)) - (0, m)] \cap \mathbb{R}^2_{\geq 0}.$$

We now proceed to prove statement (B). Note that if $\deg_y(g) \leq \tau e_{j-1} - 1$ then $q = 0$ and the result trivially holds. Let $m = \tau e_{j-1}$. We know from statement (A) of Lemma 8 that there is $\sigma \in \mathbb{N}_0$ such that

$$q = \sum_{\ell=0}^{\sigma} H_m^{(\ell)}(h_m(g)).$$

Hence, $\text{Supp}(q) \subseteq \cup_{\ell=0}^{\sigma} \text{Supp}(H_m^{(\ell)}(h_m(g)))$. We have immediately that

$$\text{Supp}(H_m^{(0)}(h_m(g))) = (\text{Supp}(g) - (0, m)) \cap \mathbb{R}^2_{\geq 0}.$$

Since $\text{Supp}(g) \subseteq N_\nu^{(i)}$, from (17) of Proposition 10, we obtain that

$$\text{Supp}(H_m^{(0)}(h_m(g))) \subseteq N_{\overline{\nu}}^{(i)}.$$

Assume that, for $\ell \in \{0, \dots, \sigma - 1\}$, $\text{Supp}(H_m^{(\ell)}(h_m(g))) \subseteq N_{\overline{\nu}}^{(i)}$. We apply statement (A) to obtain

$$\text{Supp}(H_m^{(\ell+1)}(h_m(g))) \subseteq [(\text{Supp}(H_m^{(\ell)}(h_m(g))) + \text{Supp}(f_{j-1}^\tau)) - (0, m)] \cap \mathbb{R}^2_{\geq 0}.$$

Taking into account (18) of Proposition 10,

$$\text{Supp}(H_m^{(\ell)}(h_m(g))) + \text{Supp}(f_{j-1}^\tau) \subseteq N_\nu^{(i)},$$

and consequently, by (17) of Proposition 10, $\mathrm{Supp}(H_m^{(\ell+1)}(h_m(g))) \subseteq N_{\overline{\nu}}^{(i)}$. We conclude that $\mathrm{Supp}(q) \subseteq N_{\overline{\nu}}^{(i)}$. From the equality $v = g - qf_{j-1}^{\tau}$, due to

$$\mathrm{Supp}(qf_{j-1}^{\tau}) \subseteq \mathrm{Supp}(q) + \mathrm{Supp}(f_{j-1}^{\tau}) \subseteq N_{\nu}^{(i)},$$

we obtain that $\mathrm{Supp}(v) \subseteq N_{\nu}^{(i)}$. □

4.2 Decomposition results

In this Subsection we present two theorems that form the core of Algorithm 1. The first theorem, Theorem 12 states that we can express f_i as the sum of $f_{i-1}^{k_i}$ with a tail that satisfies certain properties. It also provides the values of the induced valuations at f_i. These valuation values enable us to perform an elimination procedure, that is used in Algorithm 1. The second Theorem, Theorem 15, asserts that we can uniquely decompose our tail as a sum of products of the polynomials $x, y, \ldots, f_{i-1}$. This decomposition is implemented in Algorithm 1 and allows us to compute the tail.

Let ι be as defined in (3), and let ι_i and f_i, for $i = 1, \ldots, s$ be as described in Sect. 3. For $i \in \{1, \ldots, s\}$, let $e_i, \gamma_1^{(i)}, \ldots, \gamma_i^{(i)}$ be the generators of the semigroup $\Gamma(\lambda_1, \ldots, \lambda_i)$. Define $f_0(x, y) = y$.

Theorem 12 *The following statements hold:*

(*A*) *For all $i \in \{1, \ldots, s-1\}$ and $j \in \{i+1, \ldots, s\}$, $\vartheta_{\iota_j}(f_i) = \gamma_{i+1}^{(j)}$.*
(*B*) *For all $i \in \{1, \ldots, s\}$, there is $\delta_i \in \mathbb{C}[x, y]$ such that $\mathrm{Supp}(\delta_i) \subset N_i \setminus \{(0, e_i)\}$, $\vartheta_{\iota_i}(\delta_i) = k_i\gamma_i^{(i)}$ and $f_i(x, y) = f_{i-1}^{k_i}(x, y) + \delta_i(x, y)$.*

Proof Let i, j be as in statement (*A*). By construction $\widetilde{\zeta}_i$ is a root of f_i. Let $\widetilde{\zeta}_i^{(\ell)}$, $\ell \in \{1, \ldots, e_i\}$, be the roots of f_i, that is,

$$f_i(x, y) = \prod_{\ell=1}^{e_i}(y - \widetilde{\zeta}_i^{(\ell)}(x^{1/e_i})).$$

Definition 2 allows us to write

$$\widetilde{\zeta}_j(x^{1/e_j}) = \widetilde{\zeta}_i(x^{1/e_i}) + c_{\lambda_{i+1}}x^{\lambda_{i+1}} + \widetilde{\psi}_j(x^{1/e_j}), \tag{19}$$

where $\widetilde{\psi}_j \in \mathbb{C}[x^{1/e_j}]$ and $\mathrm{ord}(\widetilde{\psi}_j) > \lambda_{i+1}$. Hence,

$$\iota_j^* f_i(t) = \prod_{\ell=1}^{e_i}\left[(\widetilde{\zeta}_i(t^{e_j/e_i}) - \widetilde{\zeta}_i^{(\ell)}(t^{e_j/e_i})) + c_{\lambda_{i+1}}t^{e_j\lambda_{i+1}} + \widetilde{\psi}_j(t)\right]. \tag{20}$$

Set $w = t^{e_j/e_i}$. We can assume that $\widetilde{\zeta}_i = \widetilde{\zeta}_i^{(1)}$. Let $\ell \in \{2, \ldots, n\}$. There is $\theta_\ell \in \mathbb{C}[w]$ and $\widetilde{\ell} \in \{1, \ldots, i\}$ such that $\theta_\ell(0) \neq 0$ and

$$\widetilde{\zeta}_i(w) - \widetilde{\zeta}_i^{(\ell)}(w) = w^{e_i\lambda_{\widetilde{\ell}}}\theta_\ell(w) = t^{e_j\lambda_{\widetilde{\ell}}}\theta_\ell(w). \tag{21}$$

Furthermore, as ℓ runs through $\{2,\dots,n\}$, the order of the subtractions in (21) takes all the values of the set $\{e_j\lambda_{\widetilde{\ell}} : \widetilde{\ell} \in \{1,\dots,i\}\}$. Equality (20) is equivalent to

$$\iota_j^* f_i(t) = f_i(w^{e_i}, \widetilde{\zeta}_i(w)) + c_{\lambda_{i+1}} t^{e_j\lambda_{i+1}} \prod_{\ell=2}^{e_i} \left(\widetilde{\zeta}_i(w) - \widetilde{\zeta}_i^{(\ell)}(w)\right) + R(t). \tag{22}$$

Set

$$\varepsilon(t) = c_{\lambda_{i+1}} t^{e_j\lambda_{i+1}} \prod_{\ell=2}^{e_i} \left(\widetilde{\zeta}_i(w) - \widetilde{\zeta}_i^{(\ell)}(w)\right).$$

Note that $f_i(w^{e_i}, \widetilde{\zeta}_i(w)) = 0$. Taking into account that $\lambda_i < \lambda_{i+1}$ and the order of $\widetilde{\psi}_j(t)$ is strictly bigger than $e_j\lambda_{i+1}$, the order of $R(t)$ is strictly bigger than the order of $\varepsilon(t)$. To prove statement (A) we just have to prove that the order of $\varepsilon(t)$ equals $\gamma_{i+1}^{(j)}$. Given $\ell \in \{1,\dots,i\}$, there are $k_\ell - 1$ roots of f_ℓ that coincide with $\widetilde{\zeta}_i$ up to the term, but excluding, $c_{\lambda_\ell} x^{\lambda_\ell}$. Therefore, there are

$$(k_\ell - 1)k_{\ell+1}\cdots k_i = \frac{e_i}{e_{\ell-1}} - \frac{e_i}{e_\ell}$$

roots of f_i that coincide with $\widetilde{\zeta}_i$ up to the term, but excluding, $c_{\lambda_\ell} x^{\lambda_\ell}$. The total number of roots of f_i that are not equal to $\widetilde{\zeta}_i$ is

$$\sum_{\ell=1}^{i} \left(\frac{e_i}{e_{\ell-1}} - \frac{e_i}{e_\ell}\right) = \frac{e_i}{e_0} - 1 = e_i - 1.$$

We conclude that the order of $\varepsilon(t)$ is

$$(k_1 - 1)k_2\cdots k_i e_j\lambda_1 + (k_2 - 1)k_3\cdots k_i e_j\lambda_2 + \cdots + (k_i - 1)e_j\lambda_i + e_j\lambda_{i+1} \tag{23}$$

Expression (23) equals $\gamma_{i+1}^{(j)}$ (see Section 2 of [9]) and statement (A) is proved.

We now proceed to prove statement (B). Once again, we turn to Definition 2 to write

$$\widetilde{\zeta}_i^{(\ell)}\left(x^{1/e_i}\right) = \widetilde{\zeta}_{i-1}^{(\ell_1)}(x^{1/e_{i-1}}) + c_{\lambda_i}^{(\ell_2)} x^{\lambda_i} + \psi_i^{(\ell_2)}\left(x^{1/e_i}\right),$$

where $\ell \in \{1,\dots,e_i\}$, $\ell_1 \in \{1,\dots,e_{i-1}\}$, $\ell_2 \in \{1,\dots,k_i\}$ and $c_{\lambda_i}^{(\ell_2)}$ is a non-zero complex number. Note that to obtain all roots of f_i, ℓ_1 must run through all the values of $\{1,\dots,e_{i-1}\}$. Hence,

$$f_i(x,y) = \prod_{\ell=1}^{e_i} \left(y - \widetilde{\zeta}_i^{(\ell)}\left(x^{1/e_i}\right)\right)$$

$$= \prod_{\ell=1}^{k_i} \prod_{\ell_1=1}^{e_{i-1}} \left(y - \widetilde{\zeta}_i^{(\ell_1)} \left(x^{1/e_{i-1}} \right) \right) + \delta_i(x, y).$$

But

$$\prod_{\ell=1}^{k_i} \prod_{\ell_1=1}^{e_{i-1}} \left(y - \widetilde{\zeta}_i^{(\ell_1)} \left(x^{1/e_{i-1}} \right) \right) = \prod_{\ell=1}^{k_i} f_{i-1}(x, y) = f_{i-1}^{k_i}(x, y).$$

By Lemma 9, the support of $f_{i-1}^{k_i}$ is contained in N_i. Therefore, the support of

$$\delta_i(x, y) = f_i(x, y) - f_{i-1}^{k_i}(x, y)$$

is contained in N_i. Since f_i and $f_{i-1}^{k_i}$ are Weierstrass polynomials in y of degree e_i, $(0, e_i)$ does not belong to the support of δ_i. We must have

$$\vartheta_{\iota_i}(\delta_i) = \vartheta_{\iota_i}(f_{i-1}^{k_i}) = k_i \gamma_i^{(i)}.$$

Otherwise,

$$\vartheta_{\iota_i}(f_i) = \min\{\vartheta_{\iota_i}(\delta_i), \vartheta_{\iota_i}(f_{i-1}^{k_i})\} \neq +\infty.$$

□

Remark 13 Statement (A) of Theorem 12 tell us that $\widetilde{\zeta}_i$ is an i-semi-root of f_j (see [2] or [9]).

Note that if $(\alpha, \beta) \in N_i \cap \mathbb{N}_0^2$, then $\operatorname{ord}(x^\alpha y^\beta) = e_i$ if and only if $(\alpha, \beta) = (0, e_i)$. One immediate consequence of this fact is that $\operatorname{Supp}(\delta_i) \subset N_i \setminus \{(0, e_i)\}$ implies $\operatorname{ord}(\delta_i) > e_i$. The following Lemma is needed for the proof of Theorem 15.

Lemma 14 *For all $i \in \{1, \ldots, s-1\}$, the following statements hold:*

- (A) *Let $\alpha, \beta \in \mathbb{N}_0$. Let $a(x, y), b(x, y) \in \mathbb{C}[[x, y]]$ be non-zero such that $\vartheta_{\iota_i}(a)$ and $\vartheta_{\iota_i}(b)$ are elements of $\Gamma_{i-1}(\lambda_1, \ldots, \lambda_i)$. If $\vartheta_{\iota_i}(a f_{i-1}^{\alpha}) = \vartheta_{\iota_i}(b f_{i-1}^{\beta})$ then $\alpha - \beta$ is a multiple of k_i.*
- (B) *Let $n \in \mathbb{N}$ and $g \in \mathbb{C}[[x]][y]$ such that $\deg_y(g) = n$ and $e_{i-1} \leq n < e_i$. Then there are uniquely determined $a_\ell \in \mathbb{C}[[x]][y]$, $\ell \in \{0, \ldots, k_i - 1\}$, such that*

$$g(x, y) = \sum_{\ell=0}^{k_i - 1} a_\ell(x, y) f_{i-1}^{\ell}(x, y) \tag{24}$$

and $\deg_y(a_\ell) < e_{i-1}$. Furthermore, $\vartheta_{\iota_i}(g) \in \Gamma(\lambda_1, \ldots, \lambda_i)$ and

$$\vartheta_{\iota_j}(g) = \frac{e_j}{e_i} \vartheta_{\iota_i}(g).$$

Proof We will prove the Lemma by induction on i. For $i = 1$, the proof of both statements is straightforward. Let us begin by proving statement (A). We can assume $\alpha \geq \beta$. Equality $\vartheta_{\iota_i}(af_{i-1}^{\alpha}) = \vartheta_{\iota_i}(bf_{i-1}^{\beta})$ is equivalent to

$$(\alpha - \beta)\gamma_i^{(i)} = \vartheta_{\iota_i}(b) - \vartheta_{\iota_i}(a). \tag{25}$$

Let q and $r \in \{0, \ldots, k_i - 1\}$ be the unique non-negative integers such that

$$\alpha - \beta = qk_i + r. \tag{26}$$

We apply (26) to (25) and obtain

$$r\gamma_i^{(i)} = \vartheta_{\iota_i}(b) - \vartheta_{\iota_i}(a) - qk_i\gamma_i^{(i)}. \tag{27}$$

By hypothesis, $\vartheta_{\iota_i}(b) - \vartheta_{\iota_i}(a) \in e_i\mathbb{Z} + \sum_{j=1}^{i-1}\mathbb{Z}\gamma_j^{(i)}$, which is a subset of e_iM_{i-1} by statement (A) of Proposition 1. Also, by Proposition 1, we know that $k_i\gamma^{(i)}$ is an element of e_iM_{i-1}. We conclude that $r\gamma_i^{(i)} \in e_iM_{i-1}$. We apply (2) to this fact and obtain that $r\lambda_i \in M_{i-1}$. Since $r \in \{0, \ldots, k_i - 1\}$, the definition of k_i implies $r = 0$.

We now prove statement *(B)*. Let d, τ be the unique non-negative integers such that $n = d\,e_{i-1} + \tau$ and $\tau < e_{i-1}$. Since $e_{i-1} \leq n < e_i$ then $1 \leq d < k_i$. The fact that f_{i-1} is a Weierstrass polynomial with $\deg_y(f_{i-1}) = e_{i-1}$ and coefficients in $\mathbb{C}[x]$ implies that f_{i-1}^d is a Weierstrass polynomial with $\deg_y(f_{i-1}^d) = de_{i-1}$ and coefficients in $\mathbb{C}[x]$. Let $a_d, r_d \in \mathbb{C}[[x]][y]$ be, respectively, the quotient and the reminder of the division of g by f_{i-1}^d. Then, $\deg_y(r_d) < d\,e_{i-1}$ and, by statement (B) of Lemma 8, $\deg_y(a_d) = n - d\,e_{i-1} = \tau$. Let $\rho \in \{1, \ldots, d\}$. Suppose that there are uniquely determined $a_\ell \in \mathbb{C}[[x]][y]$, $\ell \in \{\rho, \ldots, d\}$, and $r_\rho \in \mathbb{C}[[x]][y]$ such that $\deg_y(r_\rho) < \rho e_{i-1}$,

$$g(x, y) = r_\rho(x, y) + \sum_{\ell=\rho}^{d} a_\ell(x, y) f_{i-1}^{\ell}(x, y) \tag{28}$$

and $\deg_y(a_\ell) < e_{i-1}$. If $\rho = 1$ then $a_0(x, y) = r_1(x, y)$, otherwise, set $a_{\rho-1}, r_{\rho-1} \in \mathbb{C}[[x]][y]$ as the quotient and the reminder, respectively, of the division of r_ρ by $f_{i-1}^{\rho-1}$. Therefore, $\deg_y(r_{\rho-1}) < (\rho - 1)e_{i-1}$ and

$$\deg_y(a_{\rho-1}) = \deg_y(r_\rho) - (\rho - 1)e_{i-1} < e_{i-1}.$$

Thus, iterating the division process, equality (24) holds.

By the induction hypothesis, for $a_\ell \neq 0$, there is $i_\ell \leq i - 1$ such that $\vartheta_{\iota_{i_\ell}}(a_\ell) \in \Gamma(\lambda_1, \ldots, \lambda_{i_\ell})$. Furthermore,

$$\vartheta_{\iota_i}(a_\ell) \in \frac{e_i}{e_{i_\ell}}\Gamma(\lambda_1, \ldots, \lambda_{i_\ell}).$$

By Proposition 6,

$$\vartheta_{\iota_i}(a_\ell) \in \Gamma_{i_\ell}(\lambda_1, \ldots, \lambda_i).$$

Since $i_\ell \leq i-1$ then $\Gamma_{i_\ell}(\lambda_1, \ldots, \lambda_i)$ is a subsemigroup of $\Gamma_{i-1}(\lambda_1, \ldots, \lambda_i)$, and we conclude that $\vartheta_{\iota_i}(a_\ell) \in \Gamma_{i-1}(\lambda_1, \ldots, \lambda_i)$.

Let

$$\mathcal{C} = \{\vartheta_{\iota_i}(a_\ell f_{i-1}^\ell) : a_\ell \neq 0 \text{ and } \ell \in \{0, \ldots, d\}\} \text{ and } \rho = \min \mathcal{C}. \tag{29}$$

Statement *(A)* implies that, if $\sigma, \tau \in \mathcal{C}$ are such that $\vartheta_{\iota_i}(a_\sigma f_{i-1}^\sigma) = \vartheta_{\iota_i}(a_\tau f_{i-1}^\tau)$ then $\sigma = \tau$. Let τ be the element of $\mathcal{C}$ such that

$$\vartheta_{\iota_i}(a_\tau f_{i-1}^\tau) = \rho. \tag{30}$$

Therefore, $\vartheta_{\iota_i}(g) = \rho$. Since $\vartheta_{\iota_i}(a_\tau)$ belongs to $\Gamma_{i-1}(\lambda_1, \ldots, \lambda_i)$, which is a subsemigroup of $\Gamma(\lambda_1, \ldots, \lambda_i)$. Then, also taking into account statement (A) of Theorem 12,

$$\vartheta_{\iota_i}(g) = \vartheta_{\iota_i}(a_\tau) + \tau\gamma_i^{(i)} \in \Gamma(\lambda_1, \ldots, \lambda_i).$$

For all $j \in \{i, \ldots, s\}$ and $\ell \in \{0, \ldots, d\}$, the induction hypothesis implies that

$$\vartheta_{\iota_j}(a_\ell f_{i-1}^\ell) = \vartheta_{\iota_j}(a_\ell) + \ell\vartheta_{\iota_j}(f_{i-1}) = \frac{e_j}{e_{i_\ell}}\vartheta_{\iota_{i_\ell}}(a_\ell) + \ell\gamma_i^{(j)}.$$

Applying Proposition 6, we obtain that

$$\gamma_i^{(j)} = \frac{e_j}{e_i}\gamma_i^{(i)}.$$

Apply the induction hypothesis once more to obtain

$$\frac{e_j}{e_{i_\ell}}\vartheta_{\iota_{i_\ell}}(a_\ell) = \frac{e_j}{e_i}\frac{e_i}{e_{i_\ell}}\vartheta_{\iota_{i_\ell}}(a_\ell) = \frac{e_j}{e_i}\vartheta_{\iota_i}(a_\ell).$$

Thus,

$$\vartheta_{\iota_j}\left(a_\ell f_{i-1}^\ell\right) = \frac{e_j}{e_i}\vartheta_{\iota_i}(a_\ell) + \frac{e_j}{e_i}\ell\gamma_i^{(i)} = \frac{e_j}{e_i}\vartheta_{\iota_i}\left(a_\ell f_{i-1}^\ell\right).$$

We conclude that

$$\vartheta_{\iota_j}(g) = \frac{e_j}{e_i}\rho = \frac{e_j}{e_i}\vartheta_{\iota_i}(g).$$

□

We now refine the decomposition presented in statement (B) of Theorem 12 in Theorem 15. To simplify writing, we endow $\mathbb{R}^n$ with the partial order

$$(a_1, \dots, a_n) \leq (b_1, \dots, b_n) \Leftrightarrow a_i \leq b_i,\ \forall i \in \{1, \dots, n\}.$$

Let $i \in \{1, \dots, s\}$, $\nu = (\alpha, \beta_0, \dots, \beta_{i-1}) \in \mathbb{N}_0^{i+1}$. Define

$$\Psi_\nu(x, y) = x^\alpha y^{\beta_0} \prod_{\ell=1}^{i-1} f_\ell^{\beta_\ell}. \tag{31}$$

Theorem 15 *Let $i \in \{1, \dots, s\}$. There are uniquely determined $\sigma \in \mathbb{N}$, non-zero complex numbers $c_1, \dots, c_\sigma$ and $\nu_\ell = (\alpha^{(\ell)}, \beta_0^{(\ell)}, \dots, \beta_{i-1}^{(\ell)}) \in \mathbb{N}_0^{i+1}$, $\ell \in \{0, \dots, \sigma\}$, such that*

- (A) *For all $\ell \in \{1, \dots, \sigma\}$, $(\beta_0^{(\ell)}, \beta_1^{(\ell)}, \dots, \beta_{i-1}^{(\ell)}) \leq (k_1 - 1, k_2 - 1, \dots, k_i - 1)$.*
- (B) *$f_i(x, y) = f_{i-1}(x, y)^{k_i} + \sum_{\ell=1}^{\sigma} c_\ell \Psi_{\nu_\ell}(x, y)$.*
- (C) *For all $\ell \in \{1, \dots, \sigma\}$, $\vartheta_{\iota_i}(\Psi_{\nu_\ell}) \geq k_i \gamma_i^{(i)}$.*
- (D) *For all $\ell \in \{1, \dots, \sigma\}$, Supp $\left(\Psi_{\nu_\ell}\right) \subseteq N_i$.*

Proof Assume $i = 1$. The Theorem holds since $\mathrm{Supp}(\delta_1) \subseteq N_1 \setminus \{(0, k_1)\}$ and, for all $(\alpha, \beta) \in N_1 \cap \mathbb{N}^2$, $\vartheta_{\iota_1}(x^\alpha y^\beta) \geq k_1(k_1\lambda_1) = k_1\gamma_1^{(1)}$. Assume $i \geq 2$. Let

$$\delta_i(x, y) = \sum_{\ell=0}^{k_i-1} a_\ell(x, y) f_{i-1}^\ell(x, y) \tag{32}$$

be as in statement (B) of Lemma 14, $r_{d+1}(x, y) = \delta_i(x, y)$, $\tau \in \{1, \dots, d+1\}$, $a_{d+1}(x, y) = 0$ and

$$\delta_i(x, y) = r_\tau(x, y) + \sum_{\ell=\tau}^{d} a_\ell(x, y) f_{i-1}^\ell(x, y)$$

be as in (28). From statement (B) of Lemma 14, we know that $d \leq k_i - 1$ and the decomposition (32) is uniquely determined. Assume that, for all $\ell \in \{\tau, \dots, d+1\}$, $\mathrm{Supp}(a_\ell) \subseteq N_{\nu_\ell}^{(i)}$, where $\nu_\ell = (0, \dots, 0, \ell) \in \mathbb{N}_0^i$, and $\mathrm{Supp}(r_\ell) \subseteq N_i$. Since we obtain $r_\tau = a_{\tau-1} f_{i-1}^{\tau-1} + r_{\tau-1}$ by dividing r_τ by $f_{i-1}^{\tau-1}$, using the Weierstrass Division Algorithm, statement (B) of Lemma 11 and the fact that $\mathrm{Supp}(r_\tau) \subseteq N_i$ imply $\mathrm{Supp}(r_{\tau-1}) \subseteq N_i$ and $\mathrm{Supp}(a_{\tau-1}) \subseteq N_{\nu_{\tau-1}}^{(i)}$. We conclude that, for all $\ell \in \{0, \dots, d\}$, $\mathrm{Supp}(a_\ell) \subseteq N_{\nu_\ell}^{(i)}$. We now iterate this reasoning.

Let $j \in \{1, \dots, i-1\}$. Suppose that there are uniquely determined $\widetilde{d} \in \mathbb{N}$, $\overline{a}_\ell \in \mathbb{C}[x, y]$ and $\overline{\nu}_\ell = (0, \dots, 0, \beta_j^{(\ell)}, \dots, \beta_{i-1}^{(\ell)}) \in \mathbb{N}_0^i$, for $\ell \in \{1, \dots, \widetilde{d}\}$, such that

$$\delta_i(x, y) = \sum_{\ell=0}^{\widetilde{d}} \overline{a}_\ell(x, y) \prod_{\tau=j}^{i-1} f_\tau^{\beta_\tau^{(\ell)}}, \tag{33}$$

$\overline{\nu}_\ell \leq (k_1 - 1, k_2 - 1, \dots, k_i - 1)$, $\mathrm{Supp}(\overline{a}_\ell) \subseteq N^{(i)}_{\overline{\nu}_\ell}$ and $\deg_y(\overline{a}_\ell) < e_j$. Assume $j = 1$. Let $(\eta, \rho) \in \mathrm{Supp}(\overline{a}_\ell)$. Then,

$$\mathrm{Supp}\left(x^\eta y^\rho \prod_{\tau=1}^{i-1} f_\tau^{\beta_\tau^{(\ell)}}\right) \subseteq N^{(i)}_{\nu_\ell} + \sum_{\sigma=1}^{i-1} \mathrm{Supp}(f_{i-1}^{\beta_{i-1}^{(\ell)}})$$

which means, by (18) of Proposition 10, that

$$\mathrm{Supp}\left(x^\eta y^\rho \prod_{\tau=1}^{i-1} f_\tau^{\beta_\tau^{(\ell)}}\right) \subseteq N_i.$$

We conclude that statements (B) and (D) hold. Statement (A) is a direct consequence of the fact that, for $\ell \in \{1, \dots, \widetilde{d}\}$, $\deg_y(\overline{a}_\ell) < e_1 = k_1$. Assume $j \geq 2$. Let $\ell \in \{0, \dots, \widetilde{d}\}$ and $\underline{\nu}^{(\ell)}_\tau = (0, \dots, 0, \tau, \beta^{(\ell)}_j, \dots, \beta^{(\ell)}_{i-1})$, $\tau \in \mathbb{N}$. Since $\deg_y(\overline{a}_\ell) < e_j$, by statement (B) of Lemma 14, we have the uniquely determined decomposition

$$\overline{a}_\ell(x, y) = \sum_{\tau=0}^{d_\ell} \underline{a}^{(\ell)}_\tau(x, y) f^\tau_{j-1}(x, y), \tag{34}$$

for some non-negative integer d_ℓ strictly smaller than k_j. Obviously, if $\deg_y(\overline{a}_\ell) < e_{j-1}$ then $d_\ell = 0$. Let $r^{(\ell)}_{d_\ell+1} = \overline{a}_\ell$. For $\tau \in \{0, \dots, d_\ell\}$, we know from the proof of statement (B) of Lemma 14, that $\underline{a}^{(\ell)}_\tau$ verifies the equality $r^{(\ell)}_{\tau+1} = \underline{a}^{(\ell)}_\tau f^\tau_{j-1} + r^{(\ell)}_\tau$ obtained by applying the Weierstrass Division Algorithm. Note that if $\tau = 0$ then $\underline{a}^{(\ell)}_\tau = r^{(\ell)}_1$ and $r^{(\ell)}_0$ is zero. Since $\mathrm{Supp}(\overline{a}_\ell) \subseteq N^{(i)}_{\overline{\nu}_\ell}$, once again by statement (B) of Lemma 11, we conclude that $\mathrm{Supp}(r^{(\ell)}_\tau) \subseteq N^{(i)}_{\overline{\nu}_\ell}$ and $\mathrm{Supp}(\underline{a}^{(\ell)}_\tau) \subseteq N^{(i)}_{\underline{\nu}^{(\ell)}_\tau}$.

All that is left is to prove statement (C). Let ρ be as in (29) and τ as in (30). Since $\vartheta_{\iota_i}(\delta_i) = \rho$, then

$$\vartheta_{\iota_i}\left(a_\tau f^\tau_{i-1}\right) = \vartheta_{\iota_i}\left(f^{k_i}\right).$$

By statement *(A)* of Lemma 14, we conclude that $\tau = 0$, that is $\vartheta_{\iota_i}(a_0) = k_i \gamma^{(i)}_i$ and, for all $\ell \in \{1, \dots, d\}$, $\vartheta_{\iota_i}(a_\ell f^\ell_{i-1}) > k_i \gamma^{(i)}_i$. Suppose that, in (33), we also have

$$\vartheta_{\iota_i}\left(\overline{a}_\ell \prod_{\tau=j}^{i-1} f_\tau^{\beta_\tau^{(\ell)}}\right) \geq k_i \gamma^{(i)}_i. \tag{35}$$

Assume $j = 1$ and let $(\eta, \rho) \in \mathrm{Supp}(\overline{a}_\ell)$. Since $\deg_y(\overline{a}_\ell) < e_1$, then, by statement (A) of Lemma 14,

$$\vartheta_{\iota_1}(\overline{a}_\ell) = \min\{\vartheta_{\iota_1}(x^\eta y^\rho) : (\eta, \rho) \in \mathrm{Supp}(\overline{a}_\ell)\}.$$

Hence, $\vartheta_{\iota_1}(x^\eta y^\rho) \geq \vartheta_{\iota_1}(\overline{a}_\ell)$. From statement (B) of Lemma 14, we have

$$\vartheta_{\iota_i}(x^\eta y^\rho) = \frac{e_i}{e_1}\vartheta_{\iota_1}(x^\eta y^\rho)$$

and

$$\vartheta_{\iota_i}(\overline{a}_\ell) = \frac{e_i}{e_1}\vartheta_{\iota_1}(\overline{a}_\ell).$$

Consequently, $\vartheta_{\iota_i}(x^\eta y^\rho) \geq \vartheta_{\iota_i}(\overline{a}_\ell)$. Therefore,

$$\vartheta_{\iota_i}\left(x^\eta y^\rho \prod_{\tau=1}^{i-1} f_\tau^{\beta_\tau^{(\ell)}}\right) = \vartheta_{\iota_i}(x^\eta y^\rho) + \vartheta_{\iota_i}\left(\prod_{\tau=1}^{i-1} f_\tau^{\beta_\tau^{(\ell)}}\right) \geq \vartheta_{\iota_i}(\overline{a}_\ell) + \vartheta_{\iota_i}\left(\prod_{\tau=1}^{i-1} f_\tau^{\beta_\tau^{(\ell)}}\right).$$

Inequality (35) allows us to conclude that

$$\vartheta_{\iota_i}(\overline{a}_\ell) + \vartheta_{\iota_i}\left(\prod_{\tau=1}^{i-1} f_\tau^{\beta_\tau^{(\ell)}}\right) \geq k_i \gamma_i^{(i)}.$$

Hence, statement (C) holds. Assume $j \geq 2$. Taking into account that $\deg_y(\overline{a}_\ell) < e_j$, statement (A) of Lemma 14 tell us that, for $\tau_1, \tau_2 \in \{0, \ldots, d_\ell\}$, if $\vartheta_{\iota_j}(\underline{a}_{\tau_1}^{(\ell)} f_{j-1}^{\tau_1}) = \vartheta_{\iota_j}(\underline{a}_{\tau_2}^{(\ell)} f_{j-1}^{\tau_2})$ then $\underline{a}_{\tau_1}^{(\ell)} = \underline{a}_{\tau_2}^{(\ell)} = 0$ or $\tau_1 = \tau_2$. Consequently, there is one and only one $\widetilde{\tau} \in \{0, \ldots, k_j\}$ such that $\vartheta_{\iota_j}(\overline{a}_\ell) = \vartheta_{\iota_j}(\underline{a}_{\widetilde{\tau}}^{(\ell)} f_{j-1}^{\widetilde{\tau}})$. Furthermore, for all $\tau \in \{0, \ldots, d_\ell\}$, $\vartheta_{\iota_j}(\underline{a}_\tau^{(\ell)} f_{j-1}^{\tau}) \geq \vartheta_{\iota_j}(\underline{a}_{\widetilde{\tau}}^{(\ell)} f_{j-1}^{\widetilde{\tau}})$. Following the same reasoning as in the case where $j = 1$, but now using statement (B) of Lemma 14, we obtain the inequality $\vartheta_{\iota_i}(\underline{a}_\tau^{(\ell)} f_{j-1}^{\tau}) \geq \vartheta_{\iota_i}(\overline{a}_\ell)$, for all $\tau \in \{0, \ldots, d_\ell\}$, which in conjunction with (35) allow us to conclude that

$$\vartheta_{\iota_i}\left(\underline{a}_\tau^{(\ell)} f_{j-1}^{\tau} \prod_{\tau=j}^{i-1} f_\tau^{\beta_\tau^{(\ell)}}\right) \geq k_i \gamma_i^{(i)}.$$

□

5 Computational implementation

In this section, we present Algorithm 1, which computes f_i from $f_{i-1}^{k_i}$. This algorithm is based on the two theorems stated in Sect. 4.2. Algorithm 1 constructs δ_i, as described in statement (B) of Theorem 12, by applying the decomposition outlined in statement (B) of Theorem 15, using a valuation based elimination procedure. We detail the elimination procedure in Sect. 5.1, while Sect. 5.2 is dedicated to presenting and elaborating on Algorithm 1.

5.1 Elimination procedure

We begin with two technical lemmas that validate the elimination procedure.

Lemma 16 *The following statements hold:*

(A) *For all* $i \in \{1, \ldots, s\}$ *and* $j \in \{1, \ldots, i\}$,

$$\gamma_j^{(i)} > e_{j-1}e_i.$$

(B) *Assume* $s \geq 2$. *For all* $i \in \{2, \ldots, s\}$ *and* $j \in \{1, \ldots, i-1\}$,

$$e_{j-1}\gamma_i^{(i)} - e_{i-1}\gamma_j^{(i)} > 0.$$

Proof We prove statement (A) by induction on j. Since $\gamma_1^{(i)} = e_i\lambda_1$ and $\lambda_1 > 1$ then $\gamma_1^{(i)} > e_i$. Now let $j \in \{2, \ldots, i\}$. Statement $C1$) implies $\lambda_j - \lambda_{j-1} > 0$. By (2),

$$\gamma_j^{(i)} = k_{j-1}\gamma_{j-1}^{(i)} + e_i(\lambda_j - \lambda_{j-1}).$$

The induction hypothesis as well the fact that $\lambda_j - \lambda_{j-1} > 0$ imply that

$$\gamma_j^{(i)} > k_{j-1}e_{j-2}e_i = e_{j-1}e_i.$$

To prove statement (B), we rewrite the expression $e_{j-1}\gamma_i^{(i)} - e_{i-1}\gamma_j^{(i)}$ in the following manner:

$$\begin{aligned} e_{j-1}\gamma_i^{(i)} - e_{i-1}\gamma_j^{(i)} &= e_{j-1}\gamma_i^{(i)} - e_{i-1}\gamma_j^{(i)} + \sum_{\ell=j+1}^{i-1} \frac{e_{j-1}}{e_{\ell-1}} e_{i-1}\gamma_\ell^{(i)} - \sum_{\ell=j+1}^{i-1} \frac{e_{j-1}}{e_{\ell-1}} e_{i-1}\gamma_\ell^{(i)} \\ &= \sum_{\ell=j+1}^{i} \frac{e_{j-1}}{e_{\ell-1}} e_{i-1}\gamma_\ell^{(i)} - \sum_{\ell=j}^{i-1} \frac{e_{j-1}}{e_{\ell-1}} e_{i-1}\gamma_\ell^{(i)} \\ &= \sum_{\ell=j+1}^{i} \frac{e_{j-1}}{e_{\ell-1}} e_{i-1}\gamma_\ell^{(i)} - \sum_{\ell=j+1}^{i} \frac{e_{j-1}}{e_{\ell-2}} e_{i-1}\gamma_{\ell-1}^{(i)} \\ &= \sum_{\ell=j+1}^{i} \frac{e_{j-1}}{e_{\ell-1}} e_{i-1} \left(\gamma_\ell^{(i)} - \frac{e_{\ell-1}}{e_{\ell-2}}\gamma_{\ell-1}^{(i)}\right) \\ &= \sum_{\ell=j+1}^{i} \frac{e_{j-1}}{e_{\ell-1}} e_{i-1} \left(\gamma_\ell^{(i)} - k_{\ell-1}\gamma_{\ell-1}^{(i)}\right). \end{aligned}$$

For all $i = 1, \ldots, s$, $e_i > 0$ since, by definition, $k_i > 0$. Furthermore, from (2) we conclude that $\gamma_\ell^{(i)} - k_{\ell-1}\gamma_{\ell-1}^{(i)} > 0$ for all $\ell \in \{j+1, \ldots, i\}$. Therefore,

$$\sum_{\ell=j+1}^{i} \frac{e_{j-1}}{e_{\ell-1}} e_{i-1} \left(\gamma_\ell^{(i)} - k_{\ell-1}\gamma_{\ell-1}^{(i)}\right) > 0.$$

□

Lemma 17 *Let* $m(z, x_0, \ldots, x_{i-1}) = z + \sum_{\ell=0}^{i-1} e_\ell x_\ell$, *for* $(z, x_0, \ldots, x_{i-1}) \in \mathbb{R}^{i+1}$ *and* $i \geq 2$ *an integer. Consider the set*

$$\mathcal{R} = \left\{ (z, x_0, \ldots, x_{i-1}) \in \mathbb{R}^{i+1} : (z, x_0, \ldots, x_{i-1}) \geq (0, \ldots, 0) \wedge e_i z + \sum_{\ell=0}^{i-1} \gamma_{\ell+1}^{(i)} x_\ell \geq k_i \gamma_i^{(i)} \right\}.$$

Then $m(\mathcal{R})$ *admits minimum, equal to* e_i, *and this value is reached only at* $(0, \ldots, 0, k_i)$.

Proof Note that $m(0, \ldots, 0, k_i) = k_i e_{i-1} = e_i$. Let $(H, h_0, \ldots, h_{i-1}) \in \mathbb{R}^{i+1}$ such that

$$(H, h_0, \ldots, h_{i-1}) + (0, \ldots, 0, k_i) \in \mathcal{R}.$$

Then $h_\ell \geq 0$, for all $\ell \in \{0, \ldots, i-2\}$, $H \geq 0$, $h_{i-1} + k_i \geq 0$ and

$$(h_{i-1} + k_i)\gamma_i^{(i)} + e_i H + \sum_{\ell=0}^{i-2} \gamma_{\ell+1}^{(i)} h_\ell \geq k_i \gamma_i^{(i)}. \tag{36}$$

Inequality (36) is equivalent to

$$h_{i-1} \geq -\frac{e_i}{\gamma_i^{(i)}} H - \sum_{\ell=0}^{i-2} \frac{\gamma_{\ell+1}^{(i)}}{\gamma_i^{(i)}} h_\ell. \tag{37}$$

Hence,

$$\begin{aligned} m(H, h_0, \ldots, h_{i-2}, h_{i-1} + k_i) &= (h_{i-1} + k_i)e_{i-1} + H + \sum_{\ell=0}^{i-2} e_\ell h_\ell \\ &\geq e_i - e_{i-1}\frac{e_i}{\gamma_i^{(i)}} H - e_{i-1} \sum_{\ell=0}^{i-2} \frac{\gamma_{\ell+1}^{(i)}}{\gamma_i^{(i)}} h_\ell + H + \sum_{\ell=0}^{i-2} e_\ell h_\ell \\ &= e_i + \left(1 - e_{i-1}\frac{e_i}{\gamma_i^{(i)}}\right) H + \sum_{\ell=0}^{i-2} \left(e_\ell - \frac{\gamma_{\ell+1}^{(i)}}{\gamma_i^{(i)}} e_{i-1}\right) h_\ell. \end{aligned}$$

From statement (B) of Lemma 16 we conclude that, for all $\ell \in \{0, \ldots, i-2\}$,

$$e_\ell - \frac{\gamma_{\ell+1}^{(i)}}{\gamma_i^{(i)}} e_{i-1} > 0,$$

and, from statement (A), that

$$1 - e_{i-1}\frac{e_i}{\gamma_i^{(i)}} > 0.$$

Hence, if $H > 0$ or there is $\ell \in \{0, \dots, i-2\}$ such that $h_\ell > 0$, we conclude that

$$m(H, h_0, \dots, h_{i-2}, h_{i-1} + k_i) > e_i.$$

Assume that $H = h_0 = \cdots = h_{i-2} = 0$. Then inequality (37) implies $h_{i-1} \geq 0$. If $h_{i-1} > 0$, we have

$$m(0, \dots, 0, h_{i-1} + k_i) = h_{i-1} e_{i-1} + e_i > e_i.$$

We conclude that the result holds. □

Let $i \in \{1, \dots, s-1\}$. Given $\mu \in \Gamma(\lambda_1, \dots, \lambda_i)$, there is $\nu = (\alpha, \beta_0, \dots, \beta_{i-1}) \in \mathbb{N}_0^{i+1}$ such that

$$\mu = \alpha e_i + \sum_{\ell=0}^{i-1} \beta_\ell \gamma_{\ell+1}^{(i)}. \tag{38}$$

Note that $\vartheta_{\iota_i}(\Psi_\nu) = \mu$ (see 31). Define $f_{i,0}(x, y) = f_{i-1}^{k_i}(x, y)$. Let $j \geq 0$. If $j \geq 1$, assume that we have constructed a sequence $f_{i,\ell}$, $\ell \in \{0, \dots, j\}$, of non-zero polynomials such that

(*i*) $+\infty \neq \vartheta_{\iota_i}(f_{i,\ell}) > \vartheta_{\iota_i}(f_{i,\ell-1})$, for all $\ell \in \{1, \dots, j\}$.
(*ii*) $\operatorname{ord}(f_{i,\ell}) = e_i$, for all $\ell \in \{0, \dots, j\}$.

Let $\mu = \vartheta_{\iota_i}(f_{i,j})$ and J_μ be a non-empty finite set whose elements $\nu \in \mathbb{N}_0^{i+1}$ verify equality (38). If $j = 0$, also assume that the elements of J_μ verify $\beta_{i-1} = 0$. This assumption can be made due to $\mu = k_i \gamma_i^{(i)}$ and statement (B) of Proposition 1. Define $\widetilde{\delta}_{i,j+1}(x, y)$ as

$$\sum_{\nu \in J_\mu} c_\nu \Psi_\nu(x, y),$$

where $c_\nu \in \mathbb{C}$, and $f_{i,j+1}(x, y) = f_{i,j}(x, y) + \widetilde{\delta}_{i,j+1}(x, y)$. There are $u, \theta_\nu \in \mathbb{C}[t]$, $\nu \in J_\mu$, such that $u(0) \neq 0$, $\theta_\nu(0) \neq 0$, $\iota_i^* f_{i,j} = t^\mu u(t)$ and $\iota_i^* \Psi_\nu = t^\mu \theta_\nu(t)$. Then

$$\begin{aligned} \frac{1}{t^\mu} \iota_i^* f_{i,j+1} = u(t) + \sum_{\nu \in J_\mu} c_\nu \theta_\nu(t) = \\ = u(0) + \sum_{\nu \in J_\mu} c_\nu \theta_\nu(0) + u(t) - u(0) + \sum_{\nu \in J_\mu} c_\nu (\theta_\nu(t) - \theta_\nu(0)). \end{aligned}$$

Note that

$$u(t) - u(0) + \sum_{\nu \in J_\mu} c_\nu (\theta_\nu(t) - \theta_\nu(0)) \in (t)\mathbb{C}[t].$$

Since $u(0)$ and all the $\theta_\nu(0)$ are non-zero, the linear affine variety of $\mathbb{C}^{|J_\mu|}$, defined by

$$u(0) + \sum_{\nu \in J_\mu} c_\nu \theta_\nu(0) = 0, \tag{39}$$

does not contain the origin. Apply condition (39) to $f_{i,j+1}$. Then

$$\vartheta_{\iota_i}(f_{i,j+1}) \geq \mu + 1 > \vartheta_{\iota_i}(f_{i,j}).$$

Note that $\operatorname{ord}(f_{i-1}^{k_i}) = e_i$. To prove that $f_{i,j+1}$ is non-zero, it suffices to prove that, for all $\nu \in J_\mu$, $\operatorname{ord}(\Psi_\nu) > e_i$. For all $\ell \in \{0, \ldots, i-1\}$, $\operatorname{ord}(f_\ell) = e_\ell$. Therefore,

$$m(\alpha, \beta_0, \ldots, \beta_{i-1}) = \alpha + \sum_{\ell=0}^{i-1} e_\ell \beta_\ell$$

equals the order of Ψ_ν. If $j = 0$, then $\beta_{i-1} = 0$ and $\mu = \vartheta_{\iota_i}(f_{i,0}) = k_i \gamma_i^{(i)}$. Lemma 17, in this case, allows us to conclude that $m(\alpha, \beta_0, \ldots, 0) > e_i$. Assume now that $j \geq 1$. Since $\mu = \vartheta_{\iota_i}(f_{i,j}) > \vartheta_{\iota_i}(f_{i,0}) = k_i \gamma_i^{(i)}$, Lemma 17 implies $m(\alpha, \beta_0, \ldots, \beta_{i-1}) > e_i$.

5.2 Decomposition algorithm

The following lemma facilitates the implementation of the restriction stated in statement (D) of Theorem 15 within our algorithm.

Lemma 18 *Let $i \in \{1, \ldots, s\}$ and $\nu = (\alpha, \beta_0, \ldots, \beta_{i-1}) \in \mathbb{N}_0^{i+1}$. Then $Supp(\Psi_\nu) \subseteq N_i$ if and only if $(\alpha, \sum_{\ell=0}^{i-1} \beta_\ell e_\ell) \in N_i$.*

Proof If $\beta_0 = \cdots = \beta_{i-1} = 0$ then the Lemma holds. Assume $\beta_\ell \neq 0$ for some $\ell \in \{0, \ldots, i-1\}$. Then $y^{\beta_0} f_1^{\beta_1} \cdots f_{i-1}^{\beta_{i-1}}$ is Weierstrass polynomial in y of degree $\sum_{\ell=0}^{i-1} \beta_\ell e_\ell$ and $P = (\alpha, \sum_{\ell=0}^{i-1} \beta_\ell e_\ell)$ belongs to the support of Ψ_ν. Then $\operatorname{Supp}(\Psi_\nu) \subseteq N_i$ implies that P belongs to N_i. Let us prove the other implication. The point P belongs to N_i if and only if

$$k_1 \alpha + (k_1 \lambda_1) \sum_{\ell=0}^{i-1} \beta_\ell e_\ell \geq e_i (k_1 \lambda_1) \tag{40}$$

and

$$\alpha_0 + \mu_i \sum_{\ell=0}^{i-1} \beta_\ell e_\ell \leq e_i \mu_i. \tag{41}$$

Given (η, ρ) an element of $\operatorname{Supp}(\Psi_\nu)$, there are $(\eta_\ell, \rho_\ell) \in \operatorname{Supp}(f_\ell^{\beta_\ell})$ such that $(\eta, \rho) = (\alpha, \beta_0) + \sum_{\ell=1}^{i-1} (\eta_\ell, \rho_\ell)$. We prove that (η, ρ) verifies the inequalities found in (9). By Lemma 9, for all $\ell \in \{1, \ldots, i-1\}$,

$$k_1\eta_\ell + (k_1\lambda_1)\rho_\ell \geq \beta_\ell e_\ell (k_1\lambda_1).$$

Hence,

$$k_1\eta + (k_1\lambda_1)\rho = k_1\alpha + (k_1\lambda_1)\beta_0 + \sum_{\ell=1}^{i-1}(k_1\eta_\ell + (k_1\lambda_1)\rho_\ell) \geq k_1\alpha + (k_1\lambda_1)\sum_{\ell=0}^{i-1}\beta_\ell e_\ell.$$

Inequality (40) implies that $k_1\eta + (k_1\lambda_1)\rho \geq e_i(k_1\lambda_1)$. Lemma 9 tell us that, for all $\ell \in \{1, \ldots, i-1\}$, $\eta_\ell + \mu_i\rho_\ell \leq \beta_\ell e_\ell \mu_i$. Hence,

$$\eta + \mu_i\rho = \alpha + \mu_i\beta_0 + \sum_{\ell=1}^{i-1}(\eta_\ell + \mu_i\rho_\ell) \leq \alpha + \mu_i\sum_{\ell=0}^{i-1}\beta_\ell e_\ell$$

and from inequality (41) we conclude that $\eta + \mu_i\rho \leq e_i\mu_i$. □

We now proceed to explain Algorithm 1. As input for this algorithm we need:

- The polynomials $f_0, \ldots, f_{i-1}$,
- The list CE_{i-1} of the characteristic exponents $\lambda_1, \ldots, \lambda_{i-1}$,
- The list SG_{i-1} of the generators of the semigroup associated to characteristic exponents CE_{i-1},
- The list $E_{i-1} = \{e_0, e_i, \ldots, e_{i-1}\}$,
- The parametrization ι_{i-1} given by x_{i-1} and y_{i-1},
- The characteristic exponent λ_i, the positive integer k_i and the polynomial $\psi_i(t)$.

Set $x_0(t) = t$, $y_0(t) = 0$, CE_0 as the empty list and $E_0 = SG_0 = \{1\}$.

Lines 1 to 7: We update the lists CE_{i-1}, SG_{i-1} and E_{i-1}. We compute ι_i and store it in x_i and y_i. If we are computing f_1 then Line 4 must be

$$\textbf{Int } \gamma_1 = k_1 * CE_1(1);$$

Lines 8 to 10: The lists LS_1 and LS_2 store vectors needed to apply Lemma 18. The lists VE and VC contain variables needed for future computations.

Lines 11 to 12: We initiate the elimination procedure outlined in Sect. 5.1. The variable n stores the valuation we are currently applying the elimination procedure.

Lines 13 to 14: We start a loop that will only end when we obtain g such that $\vartheta_{\iota_i}(g) = +\infty$. For line 14 we assume that we have access to a procedure, denoted by **intregion**, that computes all $(i+1)$-tuples with entries non-negative integers, on a compact region of a hyperplane, and we store those tuples on the list RE. The equality

$$VE \cdot SG_i == n$$

defines the hyperplane accordingly to the current valuation we are eliminating. Inequalities

$$VE \cdot LS_1 \geq SG_i(1) * E_i(2) * CE_i(1)$$

and

$$VE \cdot LS_2 \leq SG_i(1) * SG_i(1) * E_i(2) * \mathrm{Deg}(y_i(t))$$

come from Lemma 18. We apply statement (A) of Theorem 15 in inequality

$$(VE \setminus \{\alpha\}) \leq \left\{ \frac{E_i(2)}{E_i(1)} - 1, \ldots, \frac{E_i(i+1)}{E_i(i)}. - 1 \right\}$$

Lines 15 to 18: In this loop, we update the decomposition of δ_i, as in statement (B) of Theorem 15, from the elements of the list RE. We also update the list VC with the coefficients of the decomposition introduced in each cycle of the loop.

Lines 19 to 20: We solve the linear non-homogeneous equation, with variables the elements of VC, obtained by requiring that the coefficient of t^n in $\iota_i^* g$ is zero. For computational reasons, we are assuming that the solution of this equation is given in rule form, so we apply it to g.

Algorithm 1 Decomposition Procedure

```
Require: $k_i, \lambda_i, \psi_i, x_{i-1}, y_{i-1}, E_{i-1}, SG_{i-1}, CE_{i-1}, f_0, \ldots, f_{i-1}$
Ensure: $f_i$
1: list $CE_i = CE_{i-1} \cup \{\lambda_i\}$
2: list $E_i = E_{i-1} \cup k_i * \{E_{i-1}(i)\}$
3: list $SG_i = k_i \cdot SG_{i-1}$
4: int $\gamma_i = \frac{E_{i-1}(i)}{E_{i-1}(i-1)} * SG_i(i) + SG_i(1) * \lambda_i - SG_i(1) * CE_i(i-1)$
5: list $SG_i = SG_{i-1} \cup \{\gamma_i\}$
6: poly $x_i(t) = x_{i-1}(t^{k_i})$
7: poly $y_i(t) = y_{i-1}(t^{k_i}) + c_i t^{SG_i(1) * \lambda_i} + \psi_i(t)$
8: list $VE = \{\alpha, \beta_0, \ldots, \beta_{i-1}\}$; list $VC = \{\}$
9: list $LS_1 = \{E_i(2)\} \cup (E_i(2) * CE_i(1)) * E_i$
10: list $LS_2 = \{SG_i(1)\} \cup \mathbf{deg}(y_i(t)) * E_i$
11: poly $g(x, y) = f_{i-1}^{k_i}(x, y)$
12: int $n = \mathbf{ord}(g(x_i(t), y_i(t)))$; int $j = 1$
13: while $n \neq +\infty$ do
14:     list $RE = \mathbf{intregion}(VE \cdot SG_i == n \wedge$
                $VE \cdot LS_1 \geq SG_i(1) * E_i(2) * CE_i(1) \wedge$
                $VE \cdot LS_2 \leq SG_i(1) * \mathbf{deg}(y_i(t)) \wedge$
                $(VE \setminus \{\alpha\}) \leq \{\frac{E_i(2)}{E_i(1)} - 1, \ldots, \frac{E_i(i+1)}{E_i(i)} - 1\} \wedge VE \geq \{0, \ldots, 0\})$
15:     for $\ell = 1$ to length$(RE)$ do
16:         poly $g(x, y) = g(x, y) + \epsilon_{j,\ell} x^{E(\ell)(1)} \prod_{m=2}^{i+1} f_{m-2}^{E(\ell)(m)}$
17:         list $VC = VC \cup \{\epsilon_{j,\ell}\}$
18:     end for
19:     rule S=linsolve(coef($g(x_i(t), y_i(t)), n$)==0,$VC$)
20:     poly g(x,y)=applyrule($S$,$g(x, y)$)
21:     int $n = \mathbf{ord}(g(x_i(t), y_i(t)))$; int $j = j + 1$
22: end while
23: poly $f_i(x, y) = g(x, y)$
```

The elimination process in the algorithm starts with valuation $k_i\gamma_i^{(i)}$ and each iteration is associated to a strictly increasing valuation. From Proposition 19 it is quite easy to obtain an upper bound for the number of iterations needed to obtain f_i.

Proposition 19 *Let* $i \in \{1, \dots, s\}$ *and* $g \in \mathbb{C}[x, y]$ *such that* $Supp(g) \subseteq N_i$. *Then*

$$Supp(\iota_i^* g) \subseteq [e_i(e_i\lambda_1), e_i(e_i\mu_i)].$$

Proof We begin by proving the result for the monomial case. Given a monomial $x^\alpha y^\beta$,

$$\mathrm{Supp}(\iota_i^* x^\alpha y^\beta) \subseteq \left\{ e_i\alpha + \sum_{\ell=1}^{\beta} a_\ell : a_1, \dots, a_\beta \in \mathrm{Supp}(\iota_i^* y) \right\}.$$

Note that $e_i\lambda_1 = \min \mathrm{Supp}(\iota_i^* y)$ and $e_i\mu_i = \max \mathrm{Supp}(\iota_i^* y)$. Hence,

$$\min \mathrm{Supp}(\iota_i^* x^\alpha y^\beta) = e_i\alpha + (e_i\lambda_1)\beta, \ \max \mathrm{Supp}(\iota_i^* x^\alpha y^\beta) = e_i\alpha + (e_i\mu_i)\beta.$$

If $(\alpha, \beta) \in N_i$ then, by the inequalities in (9), we must have

$$\mathrm{Supp}(\iota_i^* x^\alpha y^\beta) \subseteq [e_i(e_i\lambda_1), e_i(e_i\mu_i)] \tag{42}$$

We conclude that the proposition holds from (42) and the fact that

$$\mathrm{Supp}(\iota_i^* g) \subseteq \cup_{(\alpha,\beta)\in \mathrm{Supp}(g)} \mathrm{Supp}(\iota_i^* x^\alpha y^\beta).$$

□

Corollary 20 *An upper bound for the number of iterations of the* ***While*** *loop initiated in line* 13 *of Algorithm* 1 *is*

$$e_i(e_i\mu_i) - k_i\gamma_i^{(i)} + 1.$$

In the following example, the computations were performed using an implementation, coded by the authors, of Algorithm 1 in *Mathematica*.

Example 21 We return to the branch

$$\zeta = c_1x^{7/3} + c_2x^{5/2} + c_3x^{8/3} + c_4x^{17/6} + c_5x^{19/6} + c_6x^{11/3} + c_7x^{23/6} + c_8x^{25/6} + c_9x^{31/6} + c_{10}x^{37/6}$$

of Example 5. Let us consider the admissible sequence of truncations

$$\widetilde{\zeta}_1 = c_1x^{7/3} + c_3x^{8/3}, \ \widetilde{\zeta}_2 = \zeta.$$

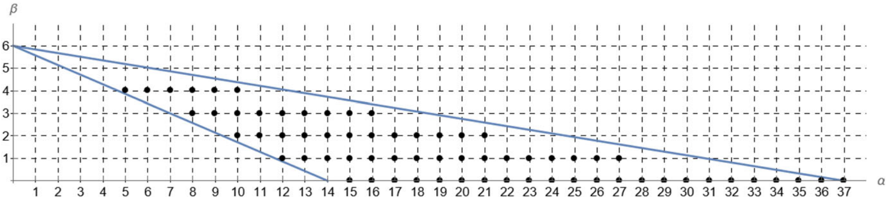

Fig. 1 Set *A*

The induced parameterizations are, respectively,

$$x = t^3,\ y = c_1t^7 + c_3t^8$$

and

$$\begin{aligned} x = t^6,\ y = c_1t^{14} + c_2t^{15} + c_3t^{16} + c_4t^{17} + c_5t^{19} + c_6t^{22} + c_7t^{23} \\ +c_8t^{25} + c_9t^{31} + c_{10}t^{37}. \end{aligned}$$

Let A be the set of black points in Fig. 1. We apply the Algorithm 1 to obtain

$$f_1(x, y) = y^3 - c_1^3x^7 - 3c_1c_3x^5y - c_3^3x^8$$

and

$$f_2(x, y) = f_1^2(x, y) + \sum_{(\alpha,\beta)\in A} d_{\alpha,\beta}x^\alpha y^\beta,$$

with $d_{\alpha,\beta}$ non-zero elements of $\mathbb{C}[c_i : i \in \{1, \dots, 10\}]$.

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