Bol. Soc. Mat. Mex. (2025) 31:8
https://doi.org/10.1007/s40590-024-00689-2

Sociedad
Matemática
Mexicana

ORIGINAL ARTICLE

On the essential norms of Toeplitz operators on abstract Hardy spaces built upon Banach function spaces

Oleksiy Karlovych[1] · **Eugene Shargorodsky**[2]

Received: 22 August 2024 / Accepted: 21 October 2024 / Published online: 8 November 2024

Abstract

Let X be a Banach function space over the unit circle such that the Riesz projection P is bounded on X and let $H[X]$ be the abstract Hardy space built upon X. We show that the essential norm of the Toeplitz operator $T(a) : H[X] \to H[X]$ coincides with $\|a\|_{L^\infty}$ for every $a \in C + H^\infty$ if and only if the essential norm of the backward shift operator $T(\mathbf{e}_{-1}) : H[X] \to H[X]$ is equal to one, where $\mathbf{e}_{-1}(z) = z^{-1}$. This result extends an observation by Böttcher, Krupnik, and Silbermann for the case of classical Hardy spaces.

Keywords Banach function space · Abstract Hardy space · Toeplitz operator · Essential norm

Mathematics Subject Classification 47B35 · 46E30

1 Introduction and the main result

For a Banach space $\mathcal{X}$, let $\mathcal{B}(\mathcal{X})$ denote the Banach algebra of bounded linear operators on $\mathcal{X}$ and let $\mathcal{K}(\mathcal{X})$ be the closed two-sided ideal of $\mathcal{B}(\mathcal{X})$ consisting of all compact linear operators on $\mathcal{X}$. The norm of an operator $A \in \mathcal{B}(\mathcal{X})$ is denoted by $\|A\|_{\mathcal{B}(\mathcal{X})}$. The essential norm of $A \in \mathcal{B}(\mathcal{X})$ is defined as follows:

$$\|A\|_{\mathcal{B}(\mathcal{X}),\mathrm{e}} := \inf\{\|A - K\|_{\mathcal{B}(\mathcal{X})} \ \ K \in \mathcal{K}(\mathcal{X})\}.$$

✉ Oleksiy Karlovych
oyk@fct.unl.pt

Eugene Shargorodsky
eugene.shargorodsky@kcl.ac.uk

[1] Centro de Matemática e Aplicações, Departamento de Matemática, Faculdade de Ciências e Tecnologia, Universidade Nova de Lisboa, Quinta da Torre, 2829–516 Caparica, Portugal

[2] Department of Mathematics, King's College London, Strand, London WC2R 2LS, UK

For a function $f \in L^1$ on the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ equipped with the Lebesgue measure m normalised so that $m(\mathbb{T}) = 1$, let

$$\widehat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(e^{i\theta}\right) e^{-in\theta}\, d\theta, \quad n \in \mathbb{Z}$$

be the Fourier coefficients of f. Let X be a Banach function space on the unit circle $\mathbb{T}$. We postpone the definition of this notion until Sect. 2.1. Here we only mention that the class of Banach function spaces is very reach, it includes all Lebesgue spaces L^p, $1 \le p \le \infty$, Orlicz spaces L^Φ (see, e.g., [1, Ch. 4, Section 8]), and Lorentz spaces $L^{p,q}$, $1 < p < \infty$, $1 \le q \le \infty$ (see, e.g., [1, Ch. 4, Section 4]). Moreover, all mentioned above spaces are rearrangement-invariant (see Sect. 2.2 for their definition).

Let

$$H[X] := \{g \in X \;\; \widehat{g}(n) = 0 \quad \text{for all} \quad n < 0\}$$

denote the abstract Hardy space built upon the space X. In the case $X = L^p$, where $1 \le p \le \infty$, we will use the standard notation $H^p := H[L^p]$. Consider the operators S and P, defined for a function $f \in L^1$ and at a.e. point $t \in \mathbb{T}$ by

$$(Sf)(t) := \frac{1}{\pi i}\, \text{p.v.} \int_{\mathbb{T}} \frac{f(\tau)}{\tau - t}\, d\tau, \quad (Pf)(t) := \frac{1}{2}(f(t) + (Sf)(t)),$$

respectively, where the integral is understood in the Cauchy principal value sense. The operator S is called the Cauchy singular integral operator and the operator P is called the Riesz projection. Assume that the Riesz projection is bounded on X. For $a \in L^\infty$, the Toeplitz operator with symbol a is defined by

$$T(a)f = P(af), \quad f \in H[X].$$

It is clear that $T(a) \in \mathcal{B}(H[X])$ and

$$\|T(a)\|_{\mathcal{B}(H[X]),\mathrm{e}} \le \|T(a)\|_{\mathcal{B}(H[X])} \le \|P\|_{\mathcal{B}(X)} \|a\|_{L^\infty}. \tag{1.1}$$

Let C denote the Banach space of all complex-valued continuous functions on $\mathbb{T}$ with the supremum norm and let

$$C + H^\infty := \{f \in L^\infty \;\; f = g + h,\; g \in C,\; h \in H^\infty\}.$$

In 1967, Sarason observed that $C + H^\infty$ is a closed subalgebra of L^∞ (see, e.g., [4, Ch. IX, Theorem 2.2] for the proof of this fact).

Let $1 < p < \infty$ and $a \in L^\infty$. It follows from [3, Theorem 2.30] and (1.1) that

$$\|a\|_{L^\infty} \le \|T(a)\|_{\mathcal{B}(H^p),\mathrm{e}} \le \|T(a)\|_{\mathcal{B}(H^p)} \le \|P\|_{\mathcal{B}(L^p)} \|a\|_{L^\infty}.$$

Gohberg and Krupnik [5, Theorem 6] proved that $\|P\|_{\mathcal{B}(L^p),\mathrm{e}} \ge 1/\sin(\pi/p)$ and conjectured that $\|P\|_{\mathcal{B}(L^p)} = 1/\sin(\pi/p)$. This conjecture was confirmed by Hollenbeck and Verbitsky [6]. Thus

$$\|a\|_{L^\infty} \le \|T(a)\|_{\mathcal{B}(H^p),\mathrm{e}} \le 1/\sin(\pi/p)\|a\|_{L^\infty}, \quad a \in L^\infty. \tag{1.2}$$

Böttcher et al. [2, Section 7.6] asked whether the essential norm of Toeplitz operators $T(a)$ with $a \in C$ acting on the Hardy spaces H^p is independent of $p \in (1, \infty)$. The second author answered this question in the negative [12]. More precisely, it was shown that

$$\|T(a)\|_{\mathcal{B}(H^p),\mathrm{e}} = \|a\|_{L^\infty} \quad \text{for all} \quad a \in C \quad \text{if and only if} \quad p = 2. \tag{1.3}$$

Nevertheless, the following estimates for $\|T(a)\|_{\mathcal{B}(H^p),\mathrm{e}}$ were obtained for $1 < p < \infty$ and $a \in C + H^\infty$:

$$\|a\|_{L^\infty} \le \|T(a)\|_{\mathcal{B}(H^p),\mathrm{e}} \le \min\left\{2^{|1-2/p|}, 1/\sin(\pi/p)\right\} \|a\|_{L^\infty}$$

(see (1.2) and [12, Theorem 4.1]).

We will use the following notation:

$$\mathbf{e}_m(z) := z^m, \quad z \in \mathbb{C}, \quad m \in \mathbb{Z}.$$

The following result extends (1.3) to the class of rearrangement-invariant Banach function spaces.

Theorem 1.1 *Let X be a rearrangement-invariant Banach function space such that the Riesz projection P is bounded on X. Then the following statements are equivalent:*

(a) the equality

$$\|T(a)\|_{\mathcal{B}(H[X]),\mathrm{e}} = \|a\|_{L^\infty} \tag{1.4}$$

holds for every Toeplitz operator $T(a) : H[X] \to H[X]$ with $a \in L^\infty$;

(b) equality (1.4) holds for every Toeplitz operator $T(a) : H[X] \to H[X]$ with $a \in C + H^\infty$;

(c) $\|T(\mathbf{e}_{-1})\|_{\mathcal{B}(H[X]),\mathrm{e}} = 1$;

(d) $\|T(\mathbf{e}_{-1})\|_{\mathcal{B}(H[X])} = 1$;

(e) $\|P\|_{\mathcal{B}(X)} = 1$;

(f) $X = L^2$ and there exists $C \in (0, \infty)$ such that

$$\|g\|_X = C\|g\|_{L^2} \quad \textit{for all} \quad g \in X.$$

The implication (f) $\Longrightarrow$ (a) follows from inequalities (1.2), which become equalities for $p = 2$. The implications (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) are trivial. The equivalences (d) $\Longleftrightarrow$ (e) $\Longleftrightarrow$ (f) were proved in [10, Theorems 1.1–1.2] for arbitrary (not necessarily rearrangement-invariant) Banach function spaces X. The equality

$$\|T(\mathbf{e}_{-1})\|_{\mathcal{B}(H[X])} = \|T(\mathbf{e}_{-1})\|_{\mathcal{B}(H[X]),\mathrm{e}}$$

was proved in [8, Theorem 1.2] for rearrangement-invariant Banach function spaces X, which gives the equivalence (c) $\Longleftrightarrow$ (d) and completes the proof of Theorem 1.1.

Böttcher et al. [2, p. 472] provided an argument allowing to show directly that (c) $\Longrightarrow$ (b) in the case of classical Hardy spaces H^p, $1 < p < \infty$. The aim of this paper is to show that their reasoning can be extended to the case of arbitrary Banach function spaces (not necessarily rearrangement-invariant) on which the Riesz projection P is bounded. Our main result is the following.

Theorem 1.2 (Main result) *Let X be a Banach function space on which the Riesz projection is bounded. Then the following statements are equivalent:*

(i) the equality

$$\|T(\mathbf{e}_{-1})\|_{\mathcal{B}(H[X]),\mathrm{e}} = 1$$

holds for the backward shift operator $T(\mathbf{e}_{-1}) : H[X] \to H[X]$;

(ii) the equality

$$\|T(a)\|_{\mathcal{B}(H[X]),\mathrm{e}} = \|a\|_{L^\infty}$$

holds for every Toeplitz operator $T(a) : H[X] \to H[X]$ with $a \in C + H^\infty$.

The paper is organised as follows. In Sect. 2, we recall the definition of the class of Banach function spaces and of its distinguished subclasss of rearrangement-invariant Banach function spaces. In Sect. 3, we prove that the Toeplitz operators $T(\mathbf{e}_{-n}h)$ with $n \in \mathbb{Z}_+ := \{0, 1, 2, \dots\}$ and $h \in H^\infty$ are bounded on $H[X]$. Further, we show that $T(\mathbf{e}_{-1})T(\mathbf{e}_{-n}h) = T(\mathbf{e}_{-n-1}h)$ for $n \in \mathbb{Z}_+$ and $h \in H^\infty$ on the space $H[X]$. Although our main results have been obtained under the assumption that P is bounded on X, we do not make this assumption in Sect. 3 as we believe that this more general case is of an independent interest. Using the results of Sect. 3, we prove Theorem 1.2 in Sect. 4.

2 Preliminaries

2.1 Banach function spaces

Let $\mathcal{M}$ be the set of all measurable complex-valued functions on $\mathbb{T}$ equipped with the normalized Lebesgue measure m and let $\mathcal{M}^+$ be the subset of functions in $\mathcal{M}$ whose values lie in $[0, \infty]$. Following [1, Ch. 1, Definition 1.1], a mapping $\rho : \mathcal{M}^+ \to [0, \infty]$ is called a Banach function norm if, for all functions $f, g, f_n \in \mathcal{M}^+$ with $n \in \mathbb{N}$, and for all constants $a \geq 0$, the following properties hold:

(A1) $\rho(f) = 0 \Leftrightarrow f = 0$ a.e., $\rho(af) = a\rho(f)$, $\rho(f + g) \leq \rho(f) + \rho(g)$,

(A2) $0 \leq g \leq f$ a.e. $\Rightarrow$ $\rho(g) \leq \rho(f)$ (the lattice property),

(A3) $0 \leq f_n \uparrow f$ a.e. $\Rightarrow$ $\rho(f_n) \uparrow \rho(f)$ (the Fatou property),

(A4) $\rho(1) < \infty$,

(A5) $\displaystyle\int_{\mathbb{T}} f(t)\, dm(t) \leq C\rho(f)$

with a constant $C \in (0, \infty)$ that is independent of f. When functions differ only on a set of measure zero are identified, the set X of all functions $f \in \mathcal{M}$ for which $\rho(|f|) < \infty$ is called a Banach function space. For each $f \in X$, the norm of f is defined by $\|f\|_X := \rho(|f|)$. The set X equipped with the natural linear space operations and this norm becomes a Banach space (see [1, Ch. 1, Theorems 1.4 and 1.6]).

2.2 Rearrangement-invariant Banach function spaces

Let $\mathcal{M}_0$ (resp. $\mathcal{M}_0^+$) denote the set of all a.e. finite functions in $\mathcal{M}$ (resp. in $\mathcal{M}^+$). Following [1, Chap. 2, Definitions 1.1 and 1.2], the distribution function m_f of a function $f \in \mathcal{M}_0$ is given by

$$m_f(\lambda) := m\{t \in \mathbb{T} \ |f(t)| > \lambda\}, \quad \lambda \geq 0.$$

Two functions $f, g \in \mathcal{M}_0$ are said to be equimeasurable if $m_f(\lambda) = m_g(\lambda)$ for all $\lambda \geq 0$. A Banach function norm $\rho : \mathcal{M} \to [0, \infty]$ is said to be rearrangement-invariant if $\rho(f) = \rho(g)$ for every pair of equimeasurable functions $f, g \in \mathcal{M}_0^+$. In that case, the Banach function space X generated by ρ is said to be a rearrangement-invariant Banach function space (see [1, Ch. 2, Definition 4.1]).

3 Auxiliary results

3.1 Operator P_n

For $n \in \mathbb{N}$ and $f \in L^1$, put

$$P_n f := \sum_{k=0}^{n-1} \widehat{f}(k)\mathbf{e}_k \in H^1.$$

Lemma 3.1 *For every $n \in \mathbb{N}$, the operator $P_n : L^1 \to H^\infty$ is bounded and*

$$\|P_n\|_{\mathcal{B}(L^1, H^\infty)} \leq n.$$

Proof For every $f \in L^1$, one has

$$\begin{aligned}\|P_n f\|_{L^\infty} &= \left\| \sum_{k=0}^{n-1} \widehat{f}(k)\mathbf{e}_k \right\|_{L^\infty} \leq \sum_{k=0}^{n-1} \left|\widehat{f}(k)\right| \|\mathbf{e}_k\|_{L^\infty} \\ &= \sum_{k=0}^{n-1} \left|\widehat{f}(k)\right| \leq \sum_{k=0}^{n-1} \|f\|_{L^1} = n\|f\|_{L^1}.\end{aligned}$$

So, $P_n \in \mathcal{B}(L^1, H^\infty)$ and $\|P_n\|_{\mathcal{B}(L^1, H^\infty)} \leq n$. □

Corollary 3.2 *Let X be a Banach function space. For every $n \in \mathbb{N}$, the operator $P_n : X \to H[X]$ is bounded.*

Proof Axioms (A4) and (A5) imply the existence of a constant $C > 0$ such that

$$\|P_n\|_{\mathcal{B}(X,H[X])} \le C\|P_n\|_{\mathcal{B}(L^1,H^\infty)} \le Cn,$$

which completes the proof. ☐

3.2 Boundendess of a special Toeplitz operator

We will need the following auxiliary lemma

Lemma 3.3 [7, Lemma 3.1] *Let $f \in L^1$. Suppose there exists $g \in H^1$ such that $\widehat{f}(n) = \widehat{g}(n)$ for all $n \ge 0$. Then $Pf = g$.*

As a consequence of the results of the previous subsection and Lemma 3.3, we will show that special Toeplitz operators with symbols of the form $\mathbf{e}_{-n}h$, where $n \in \mathbb{Z}_+$ and $h \in H^\infty$, are bounded on abstract Hardy spaces $H[X]$ built upon Banach function spaces X even without the assumption that the Riesz projection P is bounded on X.

Lemma 3.4 *Let X be a Banach function space. If $n \in \mathbb{Z}_+$ and $h \in H^\infty$, then the Toeplitz operator $T(\mathbf{e}_{-n}h) : H[X] \to H[X]$ is bounded.*

Proof Let $f \in H[X] \subset H^1$. Since $h \in H^\infty$, it follows from [11, Section 3.3.1, properties (a), (g)] that $hf \in H^1$. In the case $n = 0$, Lemma 3.3 implies that

$$T(h)f = P(hf) = hf \tag{3.1}$$

and

$$\|T(h)f\|_{H[X]} = \|hf\|_{H[X]} \le \|h\|_{L^\infty}\|f\|_{H[X]},$$

whence

$$\|T(h)\|_{\mathcal{B}(H[X])} \le \|h\|_{L^\infty}. \tag{3.2}$$

If $n \in \mathbb{N}$, then

$$T(\mathbf{e}_{-n}h)f = P(\mathbf{e}_{-n}hf) = P(\mathbf{e}_{-n}P_n(hf)) + P(\mathbf{e}_{-n}(I - P_n)(hf)).$$

It follows from the definition of P_n that

$$\begin{aligned} &(\mathbf{e}_{-n}P_n(hf))\widehat{\ }(m) = 0, \quad m \in \mathbb{Z}_+; \\ &(\mathbf{e}_{-n}(I - P_n)(hf))\widehat{\ }(m) = 0, \quad m \in \mathbb{Z} \setminus \mathbb{Z}_+. \end{aligned} \tag{3.3}$$

Hence

$$P(\mathbf{e}_{-n}P_n(hf)) = 0, \quad P(\mathbf{e}_{-n}(I - P_n)(hf)) = \mathbf{e}_{-n}(I - P_n)(hf)$$

(see Lemma 3.3). So,

$$T(\mathbf{e}_{-n}h)f = P(\mathbf{e}_{-n}hf) = \mathbf{e}_{-n}(I - P_n)(hf). \tag{3.4}$$

Hence, taking into account Corollary 3.2, we obtain

$$\begin{aligned}\|T(\mathbf{e}_{-n}h)f\|_{H[X]} &= \|T(\mathbf{e}_{-n}h)f\|_X = \|\mathbf{e}_{-n}(I - P_n)(hf)\|_X \\ &= \|(I - P_n)(hf)\|_X \le \left(1 + \|P_n\|_{\mathcal{B}(X,H[X])}\right) \|hf\|_X \\ &= \left(1 + \|P_n\|_{\mathcal{B}(X,H[X])}\right) \|h\|_{L^\infty} \|f\|_{H[X]}.\end{aligned}$$

So, $T(\mathbf{e}_{-n}h) \in \mathcal{B}(H[X])$ and

$$\|T(\mathbf{e}_{-n}h)\|_{\mathcal{B}(H[X])} \le \left(1 + \|P_n\|_{\mathcal{B}(X,H[X])}\right) \|h\|_{L^\infty},$$

which completes the proof. □

The above lemma can be complemented by the following (cf. [3, Proposition 2.14]).

Lemma 3.5 *Let X be a Banach function space. If $n \in \mathbb{Z}_+$ and $h \in H^\infty$, then*

$$T(\mathbf{e}_{-1})T(\mathbf{e}_{-n}h) = T(\mathbf{e}_{-n-1}h)$$

on the space $H[X]$.

Proof It follows from Lemma 3.4 that the Toeplitz operators $T(\mathbf{e}_{-1})$, $T(\mathbf{e}_{-n}h)$ and $T(\mathbf{e}_{-n-1}h)$ are bounded on the space $H[X]$. Let $f \in H[X]$. If $n = 0$, then it follows from (3.1) that

$$T(\mathbf{e}_{-1})T(h)f = T(\mathbf{e}_{-1})(hf) = P(\mathbf{e}_{-1}hf) = T(\mathbf{e}_{-1}h)f.$$

If $n \in \mathbb{N}$, then (3.4) and (3.3) imply that

$$\begin{aligned}T(\mathbf{e}_{-1})T(\mathbf{e}_{-n}h)f &= \mathbf{e}_{-1}(I - P_1)(T(\mathbf{e}_{-n}h)f) \\ &= \mathbf{e}_{-1}\left[T(\mathbf{e}_{-n}h)f - (T(\mathbf{e}_{-n}h)f)\widehat{\ }(0)\right] \\ &= \mathbf{e}_{-1}\left[\mathbf{e}_{-n}(I - P_n)(hf) - (\mathbf{e}_{-n}(I - P_n)(hf))\widehat{\ }(0)\right] \\ &= \mathbf{e}_{-n-1}(I - P_n)(hf) - \mathbf{e}_{-1}(\mathbf{e}_{-n}hf)\widehat{\ }(0) \\ &= \mathbf{e}_{-n-1}(I - P_{n+1})(hf) \\ &\quad + \mathbf{e}_{-n-1}(P_{n+1} - P_n)(hf) - \mathbf{e}_{-1}(\mathbf{e}_{-n}hf)\widehat{\ }(0) \\ &= T(\mathbf{e}_{-n-1}h)f + \mathbf{e}_{-n-1}\widehat{hf}(n)\mathbf{e}_n - \mathbf{e}_{-1}\widehat{hf}(n) \\ &= T(\mathbf{e}_{-n-1}h)f,\end{aligned}$$

which completes the proof. □

4 Proof of the main result

4.1 Extending an observation by Böttcher, Krupnik, and Silbermann

We start with the following auxiliary result, containing the essence of the argument in [2, p. 472], in which we do not assume the boundedness of the Riesz projection on a Banach function space X.

Lemma 4.1 *Let X be a Banach function space. If $\|T(\mathbf{e}_{-1})\|_{\mathcal{B}(H[X]),\mathrm{e}} = 1$, then $\|T(a)\|_{\mathcal{B}(H[X]),\mathrm{e}} \le \|a\|_{L^\infty}$ for every Toeplitz operator $T(a) : H[X] \to H[X]$ with $a \in \{\mathbf{e}_{-n}h : n \in \mathbb{N}, h \in H^\infty\}$.*

Proof Let $a = \mathbf{e}_{-n}h$ for some $n \in \mathbb{N}$ and $h \in H^\infty$. By Lemma 3.5 and (3.2), we have

$$\begin{aligned}\|T(a)\|_{\mathcal{B}(H[X]),\mathrm{e}} &= \|T(\mathbf{e}_{-n}h)\|_{\mathcal{B}(H[X]),\mathrm{e}} = \|(T(\mathbf{e}_{-1}))^n T(h)\|_{\mathcal{B}(H[X]),\mathrm{e}}\\ &\le \|T(\mathbf{e}_{-1})\|^n_{\mathcal{B}(H[X]),\mathrm{e}} \|T(h)\|_{\mathcal{B}(H[X]),\mathrm{e}} = \|T(h)\|_{\mathcal{B}(H[X]),\mathrm{e}}\\ &\le \|T(h)\|_{\mathcal{B}(H[X])} \le \|h\|_{L^\infty} = \|a\|_{L^\infty},\end{aligned}$$

which completes the proof. □

4.2 Proof of Theorem 1.2

It is clear that (ii) implies (i). Suppose (i) holds and $a \in C + H^\infty$. Since the set $G := \{\mathbf{e}_{-n}h : n \in \mathbb{N}, h \in H^\infty\}$ is dense in $C + H^\infty$ (see, e.g., [4, Ch. IX, Theorem 2.2]), there is a sequence $\{a_m\}$ of elements of G such that $\|a - a_m\|_{L^\infty} \to 0$ as $m \to \infty$. By Lemma 4.1, $\|T(a_m)\|_{\mathcal{B}(H[X]),\mathrm{e}} \le \|a_m\|_{L^\infty}$ for all $m \in \mathbb{N}$. On the other hand, it follows from [9, Theorem 5.2] that $\|a_m\|_{L^\infty} \le \|T(a_m)\|_{\mathcal{B}(H[X]),\mathrm{e}}$ for all $m \in \mathbb{N}$. Thus $\|T(a_m)\|_{\mathcal{B}(H[X]),\mathrm{e}} = \|a_m\|_{L^\infty}$ for all $m \in \mathbb{N}$. Since

$$\left|\|T(a)\|_{\mathcal{B}(H[X]),\mathrm{e}} - \|T(a_m)\|_{\mathcal{B}(H[X]),\mathrm{e}}\right| \le \|T(a - a_m)\|_{\mathcal{B}(H[X]),\mathrm{e}} \le \|P\|_{\mathcal{B}(X)} \|a - a_m\|_{L^\infty}$$

and $|\,\|a\|_{L^\infty} - \|a_m\|_{L^\infty}| \le \|a - a_m\|_{L^\infty}$ for all $m \in \mathbb{N}$ and $\|a - a_m\|_{L^\infty} \to 0$ as $m \to \infty$, we get

$$\|T(a)\|_{\mathcal{B}(H[X]),\mathrm{e}} = \lim_{n\to\infty} \|T(a_m)\|_{\mathcal{B}(H[X]),\mathrm{e}} = \lim_{m\to\infty} \|a_m\|_{L^\infty} = \|a\|_{L^\infty},$$

which completes the proof of (ii). □

Acknowledgements This work is funded by national funds through the FCT - Fundação para a Ciência e a Tecnologia, I.P., under the scope of the projects UIDB/00297/2020 (https://doi.org/10.54499/UIDB/00297/2020) and UIDP/ 00297/2020 (https://doi.org/10.54499/UIDP/00297/2020) (Center for Mathematics and Applications).

Funding Open access funding provided by FCT|FCCN (b-on).

References

1. Bennett, C., Sharpley, R.: Interpolation of Operators. Pure and Applied Mathematics, vol. 129. Academic Press Inc., Boston (1988)
2. Böttcher, A., Krupnik, N., Silbermann, B.: A general look at local principles with special emphasis on the norm computation aspect. Integral Equ. Oper. Theory **11**(4), 455–479 (1988)
3. Böttcher, A., Silbermann, B.: Analysis of Toeplitz Operators. Springer Monographs in Mathematics, 2nd edn. Springer, Berlin (2006)
4. Garnett, J.B.: Bounded Analytic Functions, Volume 236 of Graduate Texts in Mathematics, 1st edn. Springer, New York (2007)
5. Gohberg, I.C., Krupnik, N.J.: The spectrum of singular integral operators in L_p spaces. Studia Math. **31**, 347–362 (1968)
6. Hollenbeck, B., Verbitsky, I.E.: Best constants for the Riesz projection. J. Funct. Anal. **175**(2), 370–392 (2000)
7. Karlovich, A.: Noncompactness of Toeplitz operators between abstract Hardy spaces. Adv. Oper. Theory **6**(2), 29 (2021)
8. Karlovych, O., Shargorodsky, E.: On the essential norms of singular integral operators with constant coefficients and of the backward shift. Proc. Am. Math. Soc. Ser. B **9**, 60–70 (2022)
9. Karlovych, O., Shargorodsky, E.: The Coburn lemma and the Hartman-Wintner-Simonenko theorem for Toeplitz operators on abstract Hardy spaces. Integral Equ. Oper. Theory **95**(1), 6 (2023)
10. Karlovych, O., Shargorodsky, E.: When are the norms of the Riesz projection and the backward shift operator equal to one? J. Funct. Anal. **285**(12), 110158 (2023)
11. Nikolski, N.: Hardy Spaces, Volume 179 of Cambridge Studies in Advanced Mathematics, French Cambridge University Press, Cambridge (2019)
12. Shargorodsky, E.: On the essential norms of Toeplitz operators with continuous symbols. J. Funct. Anal. **280**(2), 108835 (2021)