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Well-posedness and asymptotic analysis of a class of 2D and 3D third-grade fluids in bounded and unbounded domains

Kush Kinra and Fernanda Cipriano

Abstract. This work focuses on the study of non-autonomous incompressible third-grade fluid equations in two- and three-dimensional open and connected domains Ω, which may be either bounded or unbounded. Our main goals are to investigate the well-posedness of the equations and to analyze the asymptotic behavior of their solutions. First, we establish the existence of at least one weak solution for the system with the Dirichlet boundary condition. Next, we show that every weak solution satisfies the energy equality, ensuring continuity over time. This, in turn, helps to establish the uniqueness of weak solutions and guarantees that the underlying system can be associated with a continuous cocycle Ψ. Second, on bounded domains, we prove the existence of a unique global pullback attractor (GPA) for the underlying system using the compact Sobolev embedding $\mathbb{H}^1(\Omega) \hookrightarrow\hookrightarrow \mathbb{L}^2(\Omega)$. Lastly, we demonstrate the existence of a unique GPA by considering the underlying system on unbounded Poincaré domains. Since the compact Sobolev embedding $\mathbb{H}^1(\Omega) \hookrightarrow\hookrightarrow \mathbb{L}^2(\Omega)$ is no longer available in this setting, we employ the uniform-tail estimates approach to obtain the existence of a unique GPA on unbounded domains. This allows us to show the pullback asymptotic compactness of Ψ. We notice that the pullback asymptotic compactness of Ψ in unbounded domains cannot be obtained using the energy equality method due to the presence of strong nonlinear terms in the equations, which substantially complicates the analysis, making the study of unbounded domains more challenging. The main novelty of this work, compared to the existing literature, is that our results yield solutions that satisfy energy equality and are continuous in time. Moreover, to the best of the authors' knowledge, this is the first work that considers 2D as well as 3D non-autonomous incompressible third-grade fluids and establishes the existence of a unique GPA in bounded as well as unbounded domains.

1. Introduction

The well-posedness and existence of a unique global pullback attractor (GPA) for a class of incompressible non-Newtonian fluids occupying a two- or three-dimensional domain, which may be bounded or unbounded, under the Dirichlet boundary condition are the key concerns of this work. It is worth mentioning that while most studies in the literature have been devoted to Newtonian fluids governed by the Navier–Stokes equations, a wide range of real, industrial and biological flows (see [20,24,48] and their references) do not obey Newton's law of viscosity and, therefore, cannot be accurately modeled by these equations. These fluids exhibit complex rheological properties, such as shear-thinning, shear-thickening, and viscoelastic behavior, which conventional

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Newtonian models fail to capture. Consequently, more advanced mathematical frameworks and fluid models are necessary to properly describe their dynamics and predict their behavior in practical applications. Non-Newtonian viscoelastic fluids of differential type have received a lot of attention lately; see for instance [15]. However, in order to comprehend and explain the properties of various nanofluids, a number of simulation studies have been conducted using third-grade fluid models (see for instance [42,43] and the references therein). Nanofluids are engineered colloidal suspensions of nanoparticles in a base fluid, such as water, ethylene glycol, or oil, and they exhibit enhanced thermal conductivity in comparison with the base fluid, which shows great promise for use in technology and microelectronics. Thus, in order to comprehend the behaviors of these fluids, it is crucial to perform a mathematical analysis of third-grade fluid equations.

1.1. The underlying system

Let us now briefly recall how to obtain the fluid equations for non-Newtonian fluids of differential type, for more details about Kinematics of such fluids we refer to [15]. Denote the velocity field of the fluid by q and introduce the Rivlin–Ericksen kinematic tensors $\mathrm{A}_n, n \geq 1$, (see [44]), defined by

$$
\begin{aligned}
&\mathrm{A}_1(\mathrm{q}) = \nabla \mathrm{q} + (\nabla \mathrm{q})^T; \\
&\mathrm{A}_n(\mathrm{q}) = \frac{\mathrm{D}}{\mathrm{D}t}\mathrm{A}_{n-1}(\mathrm{q}) + \mathrm{A}_{n-1}(\mathrm{q})(\nabla \mathrm{q}) + (\nabla \mathrm{q})^T \mathrm{A}_{n-1}(\mathrm{q}), \; n = 2, 3, \cdots,
\end{aligned}
$$

where $\dfrac{\mathrm{D}}{\mathrm{D}t} \equiv \dfrac{\partial}{\partial t} + (\mathrm{q} \cdot \nabla)$ is material derivative.

The constitutive law of fluids of grade n reads $\mathbb{T} = -p\mathrm{I} + F(\mathrm{A}_1, \cdots, \mathrm{A}_n)$, where $\mathbb{T}$ is the Cauchy stress tensor, p is the pressure and F is an isotropic polynomial function of degree n, subject to the usual requirement of material frame indifference, see [15]. The constitutive law of third-grade fluids ($n = 3$) is given by the following equation

$$
\mathbb{T} = -p\mathrm{I} + \nu \mathrm{A}_1 + \alpha_1 \mathrm{A}_2 + \alpha_2 \mathrm{A}_1^2 + \beta_1 \mathrm{A}_3 + \beta_2(\mathrm{A}_1\mathrm{A}_2 + \mathrm{A}_2\mathrm{A}_1) + \beta_3 \mathrm{Tr}(\mathrm{A}_1^2)\mathrm{A}_1,
$$

where ν is the viscosity and $\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3$ are material moduli. We recall that the momentum equations, established by the Newton's second law, are given by

$$
\frac{\mathrm{D}\mathrm{q}}{\mathrm{D}t} = \frac{\partial \mathrm{q}}{\partial t} + (\mathrm{q} \cdot \nabla)\mathrm{q} = \mathrm{div}(\mathbb{T}).
$$

If $\beta_1 = \beta_2 = \beta_3 = 0$, the constitutive equations correspond to second grade fluids. It has been shown that the Clausius–Duhem inequality and the assumption that the Helmholtz free energy is a minimum in equilibrium require the viscosity and material moduli to satisfy

$$
\nu \geq 0, \quad \alpha_1 + \alpha_2 = 0, \quad \alpha_1 \geq 0. \tag{1.1}
$$

Although second grade fluids are mathematically more treatable, dealing with several non-Newtonian fluids, the rheologists have not confirmed these restrictions (1.1), thus give the conclusion that the fluids that have been tested are not fluids of second grade but are fluids that are characterized by a different constitutive structure, we refer to [24] and references therein for more details. Following [24], in order to allow the motion of the fluid to be compatible with thermodynamics, it should be imposed that

$$\nu \geq 0, \quad \alpha_1 \geq 0, \quad |\alpha_1 + \alpha_2| \leq \sqrt{24\nu\beta}, \quad \beta_1 = \beta_2 = 0, \beta_3 = \beta \geq 0. \tag{1.2}$$

Consequently, the velocity field $\mathfrak{q}$ satisfies the incompressible third-grade fluid equations

$$\begin{cases} \partial_t(z(\mathfrak{q})) - \nu\Delta\mathfrak{q} + (\mathfrak{q}\cdot\nabla)z(\mathfrak{q}) + \displaystyle\sum_{j=1}^{d} z(\mathfrak{q})^j \nabla v^j - (\alpha_1+\alpha_2)\mathrm{div}((\mathrm{A}(\mathfrak{q}))^2) \\ -\beta\mathrm{div}[\mathrm{Tr}(\mathrm{A}(\mathfrak{q})\mathrm{A}(\mathfrak{q})^T)\mathrm{A}(\mathfrak{q})] = -\nabla\mathbf{P} + \mathfrak{g}, \\ \mathrm{div}\ \mathfrak{q} = 0, \\ z(\mathfrak{q}) := \mathfrak{q} - \alpha_1\Delta\mathfrak{q}, \\ \mathrm{A}(\mathfrak{q}) := \nabla\mathfrak{q} + (\nabla\mathfrak{q})^T, \end{cases} \tag{1.3}$$

where the viscosity ν and the material moduli α_1, α_2, β verify (1.2), $\mathbf{P}$ denotes the pressure and $\mathfrak{g}$ denotes an external force. Notice that if $\alpha_1 = \alpha_2 = 0$ and β=0, we recover the Navier-Stokes equations (NSE). From a mathematical point of view, fluids of grade n constitute an hierarchy of fluids with increasing complexity and more nonlinear terms, then compared with Newtonian (grade 1) or second-grade fluids, third-grade fluids are more complex and require more involved analysis.

Our aim in this work is to study system (1.3) with a subset of physical conditions (1.2), namely

$$\nu > 0, \quad \alpha_1 = 0, \ \alpha_2 = \alpha, \quad |\alpha| < \sqrt{2\nu\beta}, \quad \beta_1 = \beta_2 = 0, \beta_3 = \beta > 0, \tag{1.4}$$

in the presence of time-dependent deterministic external force $\mathfrak{g}$ and the Dirichlet boundary condition, namely, the equations read

$$\begin{cases} \dfrac{\partial\mathfrak{q}}{\partial t} - \nu\Delta\mathfrak{q} + (\mathfrak{q}\cdot\nabla)\mathfrak{q} - \alpha\mathrm{div}((\mathrm{A}(\mathfrak{q}))^2) - \beta\mathrm{div}(|\mathrm{A}(\mathfrak{q})|^2\mathrm{A}(\mathfrak{q})) = \mathfrak{g} - \nabla\mathbf{P}; \\ \mathrm{div}\ \mathfrak{q} = 0. \end{cases} \tag{1.5}$$

1.2. Literature and scope of the article

First, we review some early research on the system (1.3) with $\alpha_1 > 0$. The authors of [1] demonstrated that the system (1.3) on bounded domains with the Dirichlet boundary condition has a local (in time) solution in $\mathbb{H}^3(\Omega)$ (see [47] as well). The global (in time) existence in $\mathbb{R}^d$, $d \in \{2, 3\}$ for $\mathbb{H}^2$-valued solution and uniqueness in $\mathbb{R}^2$ were later demonstrated by the authors in [6]. Keep in mind that the $\mathbb{H}^2$-valued solution's uniqueness result in 3D is still up for debate. Using a Navier-slip boundary condition, the authors of [7] examined system (1.3) and showed that a global (in time) solution for initial conditions exists in $\mathbb{H}^2(\mathbb{R}^d)$, $d \in \{2, 3\}$. They also demonstrated that uniqueness is true in 2D. We point out that it is challenging to obtain the existence of a solution with less regular initial data due to the complicated nonlinearities in (1.3), and that an extra constraint on the parameters $\alpha_1, \alpha_2, \beta$, and ν is required to provide any results. Indeed, the author in [41] showed that, under certain additional parameter restrictions, there is a global weak solution for (1.3) in $\mathbb{R}^d$, $d \in \{2, 3\}$. This allows the use of monotonicity techniques when the initial data only belong to $\mathbb{H}^1(\mathbb{R}^d)$ and $\alpha_1 > 0$ (see [8] for the stationary case). The author also proved the validity of the energy equality and a weak-strong uniqueness conclusion.

Now, let us focus on the literature related to system (1.5) (that is, (1.3) with $\alpha_1 = 0$). Firstly, we draw the reader's attention to the groundbreaking work [36], where the author introduced a new system for incompressible viscous fluids with the viscosity depending on the velocity gradient. This in fact has similar nonlinear terms as system (1.5). In [27], the authors proved the global well-posedness to system (1.5) in $\mathbb{R}^3$ with divergence-free initial data belonging to $\mathbb{L}^2(\mathbb{R}^3)$ using a monotonicity method and some extra restriction on the parameters ν, α and β. We refer readers to the works [31,41,56–58], etc., for the analysis of system (1.5).

There are two main aims of this article. The first objective of this article is to show the existence of a unique weak solution (in the sense of Definition 2.1) satisfying energy equality to system (1.5) with the Dirichlet boundary condition on any general domain $\Omega \subseteq \mathbb{R}^d$ ($d \in \{2, 3\}$). The other objective of this article is to investigate the large-time behavior, that is, the existence of a unique GPA for system (1.5) by making use of the abstract theory introduced in [11]. We remark that to the best of authors' knowledge, the large-time behavior of the solutions to system (1.5) has not been investigated in unbounded domains till the date. For bounded domains and torus, we refer readers to the works [12,16,17,19], etc. The theories and applications of global/pullback/exponential/trajectory attractors of deterministic dynamical systems can be referred to some well-known books by Temam [50], Robinson [45], Chepyzhov and Vishik [13] and many others.

Let us now state our main results whose proofs follow from Theorems 3.1, 5.7 and 6.11, respectively. Note that we impose some assumptions on $\mathfrak{g}$ to obtain the following results which are given in subsequent sections.

Theorem 1.1. *There exists a unique weak solution (in the sense of Definition 2.1) to system (1.5) satisfying the energy equality for any general domain $\Omega \subseteq \mathbb{R}^d$ (which may be bounded or unbounded).*

Theorem 1.2. *There exists a unique GPA for continuous cocycle associated with system (1.5) on bounded domains $\Omega \subset \mathbb{R}^d$.*

Theorem 1.3. *There exists a unique GPA for continuous cocycle associated with system (1.5) on unbounded Poincaré domains $\Omega \subset \mathbb{R}^d$ (see Hypothesis 6.1).*

Remark 1.1. We remark that a bounded domain is also a Poincaré domain. The reason to consider bounded and unbounded domains separately (see Theorems 1.2 and 1.3, respectively) is the following: We require external forcing $\mathfrak{g} \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R};\mathbb{L}^2(\Omega))$ to prove the existence of unique GPA (Theorem 1.3); however, in bounded domains, we are able to prove the existence of unique GPA (Theorem 1.2) with external forcing $\mathfrak{g} \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R};\mathbb{H}^{-1}(\Omega)) + \mathrm{L}^{\frac{4}{3}}_{\mathrm{loc}}(\mathbb{R};\mathbb{W}^{-1,\frac{4}{3}}(\Omega))$ instead of $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R};\mathbb{L}^2(\Omega))$.

1.3. Difficulties and approaches

The existence and uniqueness of weak solutions to system (1.5) on $\mathbb{R}^3$ were demonstrated by the authors in [27], which is insufficient for our objectives. One has to establish a continuous cocycle associated with system (1.5) (see Sect. 4) in order to analyze the global pullback attractors (see [11]). This requires that the solution to system (1.5) be unique and continuous in time. Therefore, we demonstrate that the weak solutions satisfy the energy equality (see Theorem 3.1), which provides the continuity of the solution in time. Next, we demonstrate that the weak solutions are unique. In this work, we use an approach given by [39] (see [23] also) based on monotonicity argument, known as Minty–Browder technique, to obtain the existence of weak solutions. Since the approach given in [39] does not require any compactness argument, we obtain the existence of weak solutions to system (1.5) in bounded as well as unbounded domains. We also mention that the energy equality satisfied by the solution is not immediate for the solution to system (1.5), therefore we provide an explicit proof to the energy equality. For this purpose, we first define a compact operator using the idea of the work [5, Subsection 2.3] (see Subsection 2.4). Finally, by enhancing the idea of the works [2,22,33] and using the eigenvalues and eigenfunctions of the above compact operator, we show that the solutions to system (1.5) satisfies the energy equality (see **Step II** of the proof of Theorem 3.1).

Second, we are interested in the large-time behavior of solutions, specifically whether system (1.5) has a unique GPA. Studying the long-time behavior of third-grade fluid equations on the whole domain $\mathbb{R}^d$ is a difficult problem, much like the case of NSE. However, such analyses may be performed on bounded or unbounded Poincaré domains (Hypothesis 6.1); for example, see [4,10,30,46,54], etc. In this study, we show that system (1.5) has a unique GPA by using the abstract theory put out in [11]. To apply the abstract theory, we first demonstrate that a continuous cocycle Ψ is generated by system (1.5). Following that, we must demonstrate that Ψ is pullback asymptotically compact and possesses a pullback absorbing set. Using the energy estimates, we demonstrate the existence of an absorbing set. Pullback asymptotic compactness of Ψ is obtained on bounded domains using the compact Sobolev embedding $\mathbb{H}^1(\Omega) \hookrightarrow\hookrightarrow \mathbb{L}^2(\Omega)$. Note that unlike the Navier–Stokes equations (see [4,11,46]), convective Brinkman-Forchheimer equations (see [32,33]), etc., we are not able to show that the solutions of third-grade fluids equations (1.5) satisfy the *weak continuity* property with respect to the initial data. Due to this reason, we are not able to use the energy equality method (due to Ball [3]) to obtain the pullback asymptotic compactness of Ψ on unbounded domains. Therefore, for unbounded domains, we explore the uniform-tail estimate method (due to Wang [53]) to prove the pullback asymptotic compactness of Ψ (Lemma 6.9).

Remark 1.2. As we have already mentioned that, in the context of attractor when Ω is a bounded domains, we have considered the forcing term in some distributional spaces (see Hypothesis 5.1). But for unbounded domain case, we need to restrict the forcing term to be in a regular space (see Hypotheses 6.2 and 6.3). If one proves the weak continuity of the solutions with respect to the initial data of the third-grade fluids equation, it will be possible to work with the same regularity as in bounded domain via energy equality method (see the works [11,32], etc., for Navier–Stokes equations and related models). Unfortunately, we are not able to prove the weak continuity of the solutions of third-grade fluids equations with respect to the initial data, so the method of energy equality is not applied.

Essentially unlike the parabolic or hyperbolic equations as considered in [9,37,53, 55], etc., the fluid equations like (1.5) contain the pressure term $\mathbf{P}$. When we derive these uniform tail-estimates, the pressure term $\mathbf{P}$ cannot be simply treated by the divergence theorem (does not vanish). However, by taking the divergence in (1.5) and using the incompressibility condition, we get the rigorous expression of the pressure term

$$\begin{aligned}\mathbf{P} &= (-\Delta)^{-1}\nabla\cdot[\nabla\cdot(\mathfrak{q}\otimes\mathfrak{q})] + \alpha(-\Delta)^{-1}\nabla\cdot\left[\nabla\cdot\left\{(\mathrm{A}(\mathfrak{q}))^2\right\}\right] \\ &\quad + \beta(-\Delta)^{-1}\nabla\cdot\left[\nabla\cdot\left\{|\mathrm{A}(\mathfrak{q})|^2\mathrm{A}(\mathfrak{q})\right\}\right] - (-\Delta)^{-1}\left[\nabla\cdot\mathfrak{g}\right].\end{aligned}$$

Then it is possible to derive uniform-tail estimates by estimating all nonlinear terms in system (1.5) and pressure $\mathbf{P}$ carefully, see Lemma 6.8. As a result of uniform-tail estimates, the pullback asymptotic compactness of Ψ on unbounded domains follows (see Lemma 6.9).

1.4. Organization of the article

The organization of this article is as follows: In the next section, we provide necessary function spaces, some important inequalities, linear and nonlinear operators with their properties for abstract formulation to system (1.5) and further analysis, abstract formulation to system (1.5) and a compact operator for demonstrating the energy equality satisfied by the solution to system (1.5). In Sect. 3, we prove our first main result, that is, the existence and uniqueness of weak solutions to system (1.5) which satisfy energy equality, Theorem 3.1. In Sect. 4, we show that system (1.5) generates a continuous cocycle/non-autonomous dynamical system Ψ. We consider $\Omega\subset\mathbb{R}^d$, $d\in\{2,3\}$ as a bounded domain in Section 5 and prove the existence of a unique GPA for Ψ using compact Sobolev embedding, Theorem 5.7. In final section, we consider $\Omega\subset\mathbb{R}^d$ as an unbounded Poincaré domain (see Hypothesis 6.1). First, we provide the uniform energy estimates satisfied by the solutions to system (1.5) (see Lemma

6.5) which helps us to prove the existence of pullback absorbing set for Ψ. Next, we show that the solutions to system (1.5) satisfy the uniform-tail estimates, see Lemma 6.8. Then, making use of the continuity property of solution with respect to initial data (Lemma 4.1) and the uniform-tail estimates (Lemma 6.8), we prove that Ψ is pullback asymptotically compact (see Lemma 6.9) which helps us to prove the existence of a unique GPA for Ψ, see Theorem 6.11.

2. Mathematical formulation and preliminary results

The goal of this work is to study the well-posedness and large-time behavior of the solutions of a class of third-grade fluids. For $d \in \{2, 3\}$, let the fluid fill open connected domain $\Omega \subseteq \mathbb{R}^d$ (which may be bounded or unbounded) with C^2-boundary $\partial\Omega$ and, for any given $\tau \in \mathbb{R}$, its dynamics is governed by the following equations

$$\begin{cases} \dfrac{\partial \mathfrak{q}}{\partial t} = \mathfrak{g} - \nabla \mathbf{P} + \nu \Delta \mathfrak{q} - (\mathfrak{q} \cdot \nabla)\mathfrak{q} + \alpha \mathrm{div}((\mathrm{A}(\mathfrak{q}))^2) & \\ \qquad + \beta \mathrm{div}(|\mathrm{A}(\mathfrak{q})|^2 \mathrm{A}(\mathfrak{q})) & \text{in } \Omega \times (\tau, \infty), \\ \mathrm{div}\ \mathfrak{q} = 0 & \text{in } \Omega \times [\tau, \infty), \\ \mathfrak{q} = \mathbf{0} & \text{on } \partial\Omega \times [\tau, \infty), \\ \mathfrak{q}(x, \tau) = \mathfrak{q}_\tau(x) & \text{in } \Omega, \end{cases} \tag{2.1}$$

where $\mathfrak{q}$ is the velocity of the fluid, $\mathbf{P}$ is the pressure, and $\mathfrak{g}$ corresponds to the external force. The operator A is defined by $\mathrm{A}(\mathfrak{q}) := \nabla \mathfrak{q} + (\nabla \mathfrak{q})^T$. In addition, ν denotes the viscosity of the fluid and α, β are material moduli. Note that for $\Omega = \mathbb{R}^d$, one needs to replace boundary condition in $(2.1)_3$ with $|\mathfrak{q}| \to 0$ as $|x| \to \infty$ (decay at infinity).

2.1. Notations and the functional setting

Let $T > 0$, for a Banach space E, we define

$$(E)^k := \{(f_1, \cdots, f_k) : f_l \in E, \quad l = 1, \cdots, k\} \text{ for positive integer } k.$$

The unknowns in system (2.1) are the velocity and the scalar pressure fields, $\mathfrak{g}$ is a given external forcing (we are considering time-dependent):

$$\begin{aligned} &\mathfrak{q} : \Omega \times [\tau, \infty) \to \mathbb{R}^d, \quad \mathbf{P} : \Omega \times [\tau, \infty) \to \mathbb{R}, \quad \mathfrak{g} : \Omega \times [\tau, \infty) \to \mathbb{R}^d \\ &(x, t) \mapsto \{q_i(x, t)\}_{i=1}^d, \quad (x, t) \mapsto \mathbf{P}(x, t), \quad (x, t) \mapsto \{g_i(x, t)\}_{i=1}^d. \end{aligned}$$

Let $m \in \mathbb{N}^* := \mathbb{N} \cup \{\infty\}$ and $1 \leq p < \infty$, we denote by $\mathbb{W}^{m,p}(\Omega)$ (resp. $\mathrm{W}^{m,p}(\Omega)$) the standard Sobolev space of matrix/vector-valued (resp. scalar-valued) functions whose weak derivative up to order m belong to the Lebesgue space $\mathbb{L}^p(\Omega)$ (resp. $\mathrm{L}^p(\Omega)$) and set $\mathbb{H}^m(\Omega) = \mathbb{W}^{m,2}(\Omega)$ and $\mathbb{H}^0(\Omega) = \mathbb{L}^2(\Omega)$.

Let us introduce the following spaces:

$$\begin{aligned}
\mathcal{V} &:= \{\mathfrak{q} \in (\mathrm{C}_c^\infty(\Omega))^d : \operatorname{div}(\mathfrak{q}) = 0\},\\
\mathbb{H} &:= \text{The closure of } \mathcal{V} \text{ in } \mathbb{L}^2(\Omega)\\
&= \{\mathfrak{q} \in \mathbb{L}^2(\Omega) : \operatorname{div}(\mathfrak{q}) = 0 \text{ in } \Omega \text{ and } \mathfrak{q}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega\},\\
\mathbb{V} &:= \text{The closure of } \mathcal{V} \text{ in } \mathbb{H}^1(\Omega) = \{\mathfrak{q} \in \mathbb{H}_0^1(\Omega) : \operatorname{div}(\mathfrak{q}) = 0 \text{ in } \Omega\},\\
\mathbb{V}_s &:= \text{the closure of } \mathcal{V} \text{ in the Sobolev space } \mathbb{H}^s(\Omega), \quad \text{for } s > 1.
\end{aligned}$$

Now, we recall the Leray–Helmholtz projector $\mathbb{P} : \mathbb{L}^2(\Omega) \to \mathbb{H}$, which is a linear bounded operator characterized by the following $\mathbb{L}^2$-orthogonal decomposition $\mathfrak{q} = \mathbb{P}\mathfrak{q} + \nabla\varphi,\ \varphi \in \mathrm{H}^1(\Omega)$, (cf. [49]).

Next, let us introduce the scalar product between two matrices $A : B = \mathrm{Tr}(AB^T)$ and denote $|A|^2 := A : A$. The divergence of a matrix $A \in \mathcal{M}_{d\times d}(E)$ is given by $(\operatorname{div}(A)_i)_{i=1}^{i=d} = (\sum_{j=1}^{d} \partial_j a_{ij})_{i=1}^{i=d}$. The space $\mathbb{H}$ is endowed with the $\mathbb{L}^2$-inner product $(\cdot,\cdot)$ and the associated norm $\|\cdot\|_2$. We recall that

$$(\mathfrak{p},\mathfrak{q}) = \sum_{i=1}^{d}\int_\Omega p_i q_i \mathrm{d}x, \quad \text{for all } \mathfrak{p},\mathfrak{q} \in \mathbb{L}^2(\Omega)$$

and

$$(A, B) = \int_\Omega A : B \mathrm{d}x; \quad \text{for all } A, B \in \mathcal{M}_{d\times d}(\mathrm{L}^2(\Omega)).$$

On the functional space $\mathbb{V}$, we will consider the following inner product

$$(\mathfrak{p},\mathfrak{q})_{\mathbb{V}} := (\mathfrak{p},\mathfrak{q}) + (\nabla\mathfrak{p},\nabla\mathfrak{q}),$$

and denote by $\|\cdot\|_{\mathbb{V}}$ the corresponding norm. The usual norms on the classical Lebesgue and Sobolev spaces $\mathrm{L}^p(\Omega)$ (resp. $\mathbb{L}^p(\Omega)$) and $\mathrm{W}^{m,p}(\Omega)$ (resp. $\mathbb{W}^{m,p}(\Omega)$) will be denoted by $\|\cdot\|_p$ and $\|\cdot\|_{\mathrm{W}^{m,p}}$ (resp. $\|\cdot\|_{\mathbb{W}^{m,p}}$), respectively. In addition, given a Banach space E, we will denote by E' its dual.

Let us introduce the following Banach space $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$

$$\mathbb{X} = \mathbb{W}_0^{1,4}(\Omega) \cap \mathbb{V}, \quad \text{with} \quad \|\cdot\|_{\mathbb{X}} := \|\cdot\|_{\mathbb{W}^{1,4}} + \|\cdot\|_{\mathbb{V}}.$$

Indeed, we recall that $\mathbb{W}_0^{1,4}(\Omega)$ endowed with $\|\cdot\|_{\mathbb{W}^{1,4}}$-norm is a Banach space, where

$$\|\mathfrak{u}\|_{\mathbb{W}^{1,4}}^4 = \int_\Omega |\mathfrak{u}(x)|^4 \mathrm{d}x + \int_\Omega |\nabla\mathfrak{u}(x)|^4 \mathrm{d}x.$$

We denote by $\langle\cdot,\cdot\rangle$ the duality pairing between $\mathbb{X}'$ and $\mathbb{X}$.

Let us recall an embedding result from [34, Subsection 2.4].

Lemma 2.1. *For $p > d$, there exists a constant $C = C(p) > 0$ such that for all $\mathfrak{u} \in \mathbb{W}_0^{1,p}(\Omega)$*

$$\|\mathfrak{u}\|_\infty \le C_{S,3}\|\nabla\mathfrak{u}\|_p.$$

Remark 2.1. Let us provide some inequalities which will be useful in the sequel.

(i) The well-known Gagliardo–Nirenberg inequality [40, Theorem 1] gives (for $d = 2, 3$)

$$\|\mathfrak{q}\|_4 \le C\|\mathfrak{q}\|_2^{\frac{4}{4+d}}\|\nabla\mathfrak{q}\|_4^{\frac{d}{4+d}},\ \mathfrak{q} \in \mathbb{L}^2(\Omega) \cap \mathbb{W}_0^{1,4}(\Omega). \tag{2.2}$$

(ii) A Korn-type inequality (see [18, Theorem 2 (ii)] and [35, Theorem 4, p.90]) gives that there exists a constant $C_K > 0$ such that

$$\|\nabla\mathfrak{u}\|_4 \le C_K\|\mathrm{A}(\mathfrak{u})\|_4, \quad \text{for all } \ \mathfrak{u} \in \mathbb{W}_0^{1,4}(\Omega). \tag{2.3}$$

For the sake of simplicity, we do not distinguish between scalar, vector or matrix-valued notations when it is clear from the context. In particular, $\|\cdot\|_E$ should be understood as follows

- $\|f\|_E^2 = \|f_1\|_E^2 + \cdots + \|f_d\|_E^2$ for any $f = (f_1, \cdots, f_d) \in (E)^d$.
- $\|f\|_E^2 = \sum_{i,j=1}^{d}\|f_{ij}\|_E^2$ for any $f \in \mathcal{M}_{d\times d}(E)$.

Throughout the article, we denote by C generic constant, which may vary from line to line.

2.2. Linear and nonlinear operators

Let us define linear operator by

$$\mathcal{A}\mathfrak{q} := -\mathbb{P}\Delta\mathfrak{q},\ \mathfrak{q} \in \mathrm{D}(\mathcal{A}) := \mathbb{V} \cap \mathbb{H}^2(\Omega).$$

Remember that the operator $\mathcal{A}$ is a non-negative, self-adjoint operator in $\mathbb{H}$ and

$$\langle\mathcal{A}\mathfrak{q}, \mathfrak{q}\rangle = \|\mathfrak{q}\|_{\mathbb{V}}^2, \quad \text{for all } \ \mathfrak{q} \in \mathbb{V}, \quad \text{so that } \ \|\mathcal{A}\mathfrak{q}\|_{\mathbb{V}'} \le \|\mathfrak{q}\|_{\mathbb{V}}. \tag{2.4}$$

Next, we define the trilinear form $b(\cdot,\cdot,\cdot) : \mathbb{V} \times \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ by

$$b(\mathfrak{p}, \mathfrak{q}, \mathfrak{u}) = \int_\Omega (\mathfrak{p}(x)\cdot\nabla)\mathfrak{q}(x)\cdot\mathfrak{u}(x)\mathrm{d}x = \sum_{i,j=1}^{d}\int_\Omega p_i(x)\frac{\partial q_j(x)}{\partial x_i}u_j(x)\mathrm{d}x.$$

If $\mathfrak{p}, \mathfrak{q}$ are such that the linear map $b(\mathfrak{p}, \mathfrak{q}, \cdot)$ is continuous on $\mathbb{V}$, the corresponding element of $\mathbb{V}'$ is denoted by $\mathcal{B}(\mathfrak{p}, \mathfrak{q})$. We represent $\mathcal{B}(\mathfrak{q}) = \mathcal{B}(\mathfrak{q}, \mathfrak{q}) = \mathbb{P}(\mathfrak{q}\cdot\nabla)\mathfrak{q}$. Using an integration by parts, it is immediate that

$$\begin{cases} b(\mathfrak{p}, \mathfrak{q}, \mathfrak{q}) = 0, \quad \text{for all } \ \mathfrak{p}, \mathfrak{q} \in \mathbb{V}, \\ b(\mathfrak{p}, \mathfrak{q}, \mathfrak{u}) = -b(\mathfrak{p}, \mathfrak{u}, \mathfrak{q}), \quad \text{for all } \ \mathfrak{p}, \mathfrak{q}, \mathfrak{u} \in \mathbb{V}. \end{cases} \tag{2.5}$$

For $\mathfrak{p}, \mathfrak{q} \in \mathbb{V}$, we have

$$\begin{aligned}\langle \mathcal{B}(\mathfrak{p}) - \mathcal{B}(\mathfrak{q}), \mathfrak{u}\rangle &= b(\mathfrak{p}, \mathfrak{p}-\mathfrak{q}, \mathfrak{u}) + b(\mathfrak{p}-\mathfrak{q}, \mathfrak{q}, \mathfrak{u}) \\ &\leq \big[\|\mathfrak{p}\|_4\|\nabla(\mathfrak{p}-\mathfrak{q})\|_2 + \|\mathfrak{p}-\mathfrak{q}\|_4\|\nabla\mathfrak{q}\|_2\big]\|\mathfrak{u}\|_4, \end{aligned} \tag{2.6}$$

for all $\mathfrak{p}, \mathfrak{q}, \mathfrak{u} \in \mathbb{V}$. Thus the operator $\mathcal{B}(\cdot) : \mathbb{V} \to \mathbb{V}'$ is locally Lipschitz.

Let us now define the operator $\mathcal{J}(\mathfrak{q}) := -\mathbb{P}\mathrm{div}(\mathrm{A}(\mathfrak{q})\mathrm{A}(\mathfrak{q}))$. Note that for $\mathfrak{q} \in \mathbb{X}$, we have

$$\|\mathcal{J}(\mathfrak{q})\|_{\mathbb{X}'} \leq C\|\mathfrak{q}\|_{\mathbb{X}}^2,$$

and hence the map $\mathcal{J}(\cdot) : \mathbb{X} \to \mathbb{X}'$. For $\mathfrak{p}, \mathfrak{q} \in \mathbb{X}$, we have

$$\begin{aligned}\langle \mathcal{J}(\mathfrak{p}) - \mathcal{J}(\mathfrak{q}), \mathfrak{u}\rangle &= \frac{1}{2}\int_\Omega [\mathrm{A}(\mathfrak{p}-\mathfrak{q})\mathrm{A}(\mathfrak{p}) + \mathrm{A}(\mathfrak{q})\mathrm{A}(\mathfrak{p}-\mathfrak{q})] : \mathrm{A}(\mathfrak{u})\mathrm{d}x \\ &\leq C\big[\|\mathrm{A}(\mathfrak{p})\|_4 + \|\mathrm{A}(\mathfrak{q})\|_4\big]\|\mathrm{A}(\mathfrak{p}-\mathfrak{q})\|_4\|\mathrm{A}(\mathfrak{u})\|_2, \end{aligned} \tag{2.7}$$

for all $\mathfrak{p}, \mathfrak{q}, \mathfrak{u} \in \mathbb{X}$. Thus the operator $\mathcal{J}(\cdot) : \mathbb{X} \to \mathbb{X}'$ is locally Lipschitz.

Finally, we define the operator $\mathcal{K}(\mathfrak{q}) := -\mathbb{P}\mathrm{div}(|\mathrm{A}(\mathfrak{q})|^2\mathrm{A}(\mathfrak{q}))$. It is immediate that

$$\langle \mathcal{K}(\mathfrak{q}), \mathfrak{q}\rangle = \frac{1}{2}\|\mathrm{A}(\mathfrak{q})\|_4^4.$$

Note that for $\mathfrak{q} \in \mathbb{X}$, we have

$$\|\mathcal{K}(\mathfrak{q})\|_{\mathbb{X}'} \leq C\|\mathfrak{q}\|_{\mathbb{X}}^3,$$

and hence the map $\mathcal{K}(\cdot) : \mathbb{X} \to \mathbb{X}'$. For $\mathfrak{p}, \mathfrak{q} \in \mathbb{X}$, we have

$$\begin{aligned}\langle \mathcal{K}(\mathfrak{p}) - \mathcal{K}(\mathfrak{q}), \mathfrak{u}\rangle &= \frac{1}{2}\int_\Omega [\mathrm{A}(\mathfrak{p}-\mathfrak{q}) : \mathrm{A}(\mathfrak{p}) + \mathrm{A}(\mathfrak{q}) : \mathrm{A}(\mathfrak{p}-\mathfrak{q})]\mathrm{A}(\mathfrak{p}) : \mathrm{A}(\mathfrak{u})\mathrm{d}x \\ &\quad + \frac{1}{2}\int_\Omega |\mathrm{A}(\mathfrak{q})|^2\mathrm{A}(\mathfrak{p}-\mathfrak{q}) : \mathrm{A}(\mathfrak{u})\mathrm{d}x \\ &\leq C\big[\|\mathrm{A}(\mathfrak{p})\|_4^2 + \|\mathrm{A}(\mathfrak{q})\|_4^2\big]\|\mathrm{A}(\mathfrak{p}-\mathfrak{q})\|_4\|\mathrm{A}(\mathfrak{u})\|_4, \end{aligned} \tag{2.8}$$

for all $\mathfrak{p}, \mathfrak{q}, \mathfrak{u} \in \mathbb{X}$. Thus the operator $\mathcal{K}(\cdot) : \mathbb{X} \to \mathbb{X}'$ is locally Lipschitz.

Lemma 2.2. *For* $\varepsilon_0 := 1 - \sqrt{\frac{\alpha^2}{2\beta\nu}} \in (0, 1)$*, the operator* $\mathscr{G} : \mathbb{X} \to \mathbb{X}'$ *given by*

$$\mathscr{G}(\cdot) := \nu\mathcal{A}\cdot + \mathcal{B}(\cdot) + \alpha\mathcal{J}(\cdot) + \beta\mathcal{K}(\cdot) \tag{2.9}$$

satisfies

$$\begin{aligned}&\langle \mathscr{G}(\mathfrak{q}_1) - \mathscr{G}(\mathfrak{q}_2), \mathfrak{q}_1 - \mathfrak{q}_2\rangle + \frac{(C_{S,3}C_K)^2}{\nu\varepsilon_0}\|\mathrm{A}(\mathfrak{q}_2)\|_4^2\|\mathfrak{q}_1 - \mathfrak{q}_2\|_2^2 \\ &\geq \frac{\nu\varepsilon_0}{4}\|\mathrm{A}(\mathfrak{q}_1 - \mathfrak{q}_2)\|_2^2 + \frac{\beta\varepsilon_0}{2}\int_\Omega |\mathrm{A}(\mathfrak{q}_1(x) - \mathfrak{q}_2(x))|^2(|\mathrm{A}(\mathfrak{q}_1(x))|^2 + |\mathrm{A}(\mathfrak{q}_2(x))|^2)\mathrm{d}x \\ &\geq \frac{\nu\varepsilon_0}{4}\|\mathrm{A}(\mathfrak{q}_1 - \mathfrak{q}_2)\|_2^2 + \frac{\beta\varepsilon_0}{4}\|\mathrm{A}(\mathfrak{q}_1 - \mathfrak{q}_2)\|_4^4 \geq 0, \end{aligned} \tag{2.10}$$

for any $\mathfrak{q}_1, \mathfrak{q}_2 \in \mathbb{X}$.

Proof. Firstly, note that $\varepsilon_0 = 1 - \sqrt{\frac{\alpha^2}{2\beta\nu}}$ implies $\frac{\alpha^2}{4\nu(1-\varepsilon_0)} = \frac{\beta(1-\varepsilon_0)}{2}$. We know that

$$\langle \nu\mathcal{A}\mathfrak{q}_1 - \nu\mathcal{A}\mathfrak{q}_2, \mathfrak{q}_1 - \mathfrak{q}_2\rangle = \frac{\nu}{2}\|\mathrm{A}(\mathfrak{q}_1 - \mathfrak{q}_2)\|_2^2. \tag{2.11}$$

Now, using (2.5), Hölder's inequality, Lemma 2.1, (2.3) and Young's inequality, we estimate

$$\begin{aligned}
|\langle \mathcal{B}(\mathfrak{q}_1) - \mathcal{B}(\mathfrak{q}_2), \mathfrak{q}_1 - \mathfrak{q}_2\rangle| &\leq |b(\mathfrak{q}_1 - \mathfrak{q}_2, \mathfrak{q}_1 - \mathfrak{q}_2, \mathfrak{q}_2| + |b(\mathfrak{q}_2, \mathfrak{q}_1 - \mathfrak{q}_2, \mathfrak{q}_1 - \mathfrak{q}_2)| \\
&\leq 2\|\mathfrak{q}_1 - \mathfrak{q}_2\|_2\|\nabla(\mathfrak{q}_1 - \mathfrak{q}_2)\|_2\|\mathfrak{q}_2\|_\infty \\
&\leq C_{S,3}C_K\|\mathfrak{q}_1 - \mathfrak{q}_2\|_2\|\mathrm{A}(\mathfrak{q}_1 - \mathfrak{q}_2)\|_2\|\mathrm{A}(\mathfrak{q}_2)\|_4 \\
&\leq \frac{\nu\varepsilon_0}{4}\|\mathrm{A}(\mathfrak{q}_1 - \mathfrak{q}_2)\|_2^2 + \frac{(C_{S,3}C_K)^2}{\nu\varepsilon_0}\|\mathrm{A}(\mathfrak{q}_2)\|_4^2\|\mathfrak{q}_1 - \mathfrak{q}_2\|_2^2,
\end{aligned}$$

which implies

$$\langle \mathcal{B}(\mathfrak{q}_1) - \mathcal{B}(\mathfrak{q}_2), \mathfrak{q}_1 - \mathfrak{q}_2\rangle \geq -\frac{\nu\varepsilon_0}{4}\|\mathrm{A}(\mathfrak{q}_1 - \mathfrak{q}_2)\|_2^2 - \frac{(C_{S,3}C_K)^2}{\nu\varepsilon_0}\|\mathrm{A}(\mathfrak{q}_2)\|_4^2\|\mathfrak{q}_1 - \mathfrak{q}_2\|_2^2. \tag{2.12}$$

From [27, Equation (2.21)], we have

$$\begin{aligned}
|\alpha\langle \mathcal{J}(\mathfrak{q}_1) - \mathcal{J}(\mathfrak{q}_2), \mathfrak{q}_1 - \mathfrak{q}_2\rangle| \leq\ & \frac{\nu(1-\varepsilon_0)}{2}\|\mathrm{A}(\mathfrak{q}_1 - \mathfrak{q}_2)\|_2^2 \\
&+ \frac{\alpha^2}{4\nu(1-\varepsilon_0)}\int_\Omega |\mathrm{A}(\mathfrak{q}_1 - \mathfrak{q}_2)|^2(|\mathrm{A}(\mathfrak{q}_1)|^2 + |\mathrm{A}(\mathfrak{q}_2)|^2)\mathrm{d}x \\
\leq\ & \frac{\nu(1-\varepsilon_0)}{2}\|\mathrm{A}(\mathfrak{q}_1 - \mathfrak{q}_2)\|_2^2 \\
&+ \frac{\beta(1-\varepsilon_0)}{2}\int_\Omega |\mathrm{A}(\mathfrak{q}_1 - \mathfrak{q}_2)|^2(|\mathrm{A}(\mathfrak{q}_1)|^2 + |\mathrm{A}(\mathfrak{q}_2)|^2)\mathrm{d}x,
\end{aligned}$$

which implies

$$\begin{aligned}
\alpha\langle \mathcal{J}(\mathfrak{q}_1) - \mathcal{J}(\mathfrak{q}_2), \mathfrak{q}_1 - \mathfrak{q}_2\rangle \geq -\frac{\nu(1-\varepsilon_0)}{2}\|\mathrm{A}(\mathfrak{q}_1 - \mathfrak{q}_2)\|_2^2 - \frac{\beta(1-\varepsilon_0)}{2} \\
\int_\Omega |\mathrm{A}(\mathfrak{q}_1 - \mathfrak{q}_2)|^2(|\mathrm{A}(\mathfrak{q}_1)|^2 + |\mathrm{A}(\mathfrak{q}_2)|^2)\mathrm{d}x.
\end{aligned}$$

Now, from [27, Equation (2.13)], we have

$$\begin{aligned}
&\beta\langle \mathcal{K}(\mathfrak{q}_1) - \mathcal{K}(\mathfrak{q}_2), \mathfrak{q}_1 - \mathfrak{q}_2\rangle \\
&= \frac{\beta}{2}\int_\Omega (|\mathrm{A}(\mathfrak{q}_1)|^2 - |\mathrm{A}(\mathfrak{q}_2)|^2)^2\mathrm{d}x + \frac{\beta}{2}\int_\Omega |\mathrm{A}(\mathfrak{q}_1 - \mathfrak{q}_2)|^2(|\mathrm{A}(\mathfrak{q}_1)|^2 + |\mathrm{A}(\mathfrak{q}_2)|^2)\mathrm{d}x.
\end{aligned}$$

In view of (2.11) and (2.12), we obtain (2.10). This completes the proof. □

Lemma 2.3. *The operator* $\mathscr{G} : \mathbb{X} \to \mathbb{X}'$ *is demicontinuous.*

Proof. Let us take a sequence $\mathfrak{q}_n \to \mathfrak{q}$ in $\mathbb{X}$, that is, $\|\mathfrak{q}_n - \mathfrak{q}\|_{\mathbb{X}} = \|\mathfrak{q}_n - \mathfrak{q}\|_{\mathbb{V}} + \|\mathfrak{q}_n - \mathfrak{q}\|_{\mathbb{W}^{1,4}} \to 0$ as $n \to \infty$. For any $\boldsymbol{\psi} \in \mathbb{X}$, we consider

$$\langle \mathscr{G}(\mathfrak{q}_n) - \mathscr{G}(\mathfrak{q}), \boldsymbol{\psi}\rangle = \nu\langle \mathcal{A}\mathfrak{q}_n - \mathcal{A}\mathfrak{q}, \boldsymbol{\psi}\rangle + \langle \mathcal{B}(\mathfrak{q}_n) - \mathcal{B}(\mathfrak{q}), \boldsymbol{\psi}\rangle + \alpha\langle \mathcal{J}(\mathfrak{q}_n) - \mathcal{J}(\mathfrak{q}), \boldsymbol{\psi}\rangle \\ + \beta\langle \mathcal{K}(\mathfrak{q}_n) - \mathcal{K}(\mathfrak{q}), \boldsymbol{\psi}\rangle. \tag{2.13}$$

Next, we take $\langle \mathcal{A}\mathfrak{q}_n - \mathcal{A}\mathfrak{q}, \boldsymbol{\psi}\rangle$ from (2.13) and estimate as follows:

$$|\langle \mathcal{A}\mathfrak{q}_n - \mathcal{A}\mathfrak{q}, \boldsymbol{\psi}\rangle| = |(\nabla(\mathfrak{q}_n - \mathfrak{q}), \nabla\boldsymbol{\psi})| \leq \|\mathfrak{q}_n - \mathfrak{q}\|_{\mathbb{V}}\|\boldsymbol{\psi}\|_{\mathbb{V}} \to 0, \quad \text{as } n \to \infty, \tag{2.14}$$

since $\mathfrak{q}_n \to \mathfrak{q}$ in $\mathbb{V}$. We estimate the term $\langle \mathcal{B}(\mathfrak{q}_n) - \mathcal{B}(\mathfrak{q}), \boldsymbol{\psi}\rangle$ from (2.13) using (2.5), Hölder's inequality and Lemma 2.1 as

$$\begin{aligned} &|\langle \mathcal{B}(\mathfrak{q}_n) - \mathcal{B}(\mathfrak{q}), \boldsymbol{\psi}\rangle| \\ &\quad \leq |b(\mathfrak{q}_n - \mathfrak{q}, \boldsymbol{\psi}, \mathfrak{q}_n - \mathfrak{q})| + |b(\mathfrak{q}_n - \mathfrak{q}, \boldsymbol{\psi}, \mathfrak{q})| + |b(\mathfrak{q}, \boldsymbol{\psi}, \mathfrak{q}_n - \mathfrak{q})| \\ &\quad \leq \|\mathfrak{q}_n - \mathfrak{q}\|_2\|\mathfrak{q}_n - \mathfrak{q}\|_\infty\|\boldsymbol{\psi}\|_{\mathbb{V}} + \|\mathfrak{q}_n - \mathfrak{q}\|_2\|\mathfrak{q}\|_\infty\|\boldsymbol{\psi}\|_{\mathbb{V}} + \|\mathfrak{q}_n - \mathfrak{q}\|_2\|\mathfrak{q}\|_\infty\|\boldsymbol{\psi}\|_{\mathbb{V}} \\ &\quad \leq C\|\mathfrak{q}_n - \mathfrak{q}\|_2\|\mathfrak{q}_n - \mathfrak{q}\|_{\mathbb{W}^{1,4}}\|\boldsymbol{\psi}\|_{\mathbb{V}} + \|\mathfrak{q}_n - \mathfrak{q}\|_2\|\mathfrak{q}\|_{\mathbb{W}^{1,4}}\|\boldsymbol{\psi}\|_{\mathbb{V}} \\ &\qquad + \|\mathfrak{q}_n - \mathfrak{q}\|_2\|\mathfrak{q}\|_{\mathbb{W}^{1,4}}\|\boldsymbol{\psi}\|_{\mathbb{V}} \\ &\quad \to 0, \quad \text{as } n \to \infty, \end{aligned} \tag{2.15}$$

since $\mathfrak{q}_n \to \mathfrak{q}$ in $\mathbb{X}$. We estimate the term $\alpha\langle \mathcal{J}(\mathfrak{q}_n) - \mathcal{J}(\mathfrak{q}), \boldsymbol{\psi}\rangle$ from (2.13) using Hölder's inequality as

$$\begin{aligned} |\alpha\langle \mathcal{J}(\mathfrak{q}_n) - \mathcal{J}(\mathfrak{q}), \boldsymbol{\psi}\rangle| &= |\alpha||\langle \mathrm{A}^2(\mathfrak{q}_n) - \mathrm{A}^2(\mathfrak{q}), \nabla\boldsymbol{\psi}\rangle| \\ &\leq |\alpha||\langle \mathrm{A}^2(\mathfrak{q}_n - \mathfrak{q}), \nabla\boldsymbol{\psi}\rangle| + |\alpha||\langle \mathrm{A}(\mathfrak{q}_n - \mathfrak{q})\mathrm{A}(\mathfrak{q}), \nabla\boldsymbol{\psi}\rangle| \\ &\quad + |\alpha||\langle \mathrm{A}(\mathfrak{q})\mathrm{A}(\mathfrak{q}_n - \mathfrak{q}), \nabla\boldsymbol{\psi}\rangle| \\ &\leq |\alpha|\|\mathfrak{q}_n - \mathfrak{q}\|^2_{\mathbb{W}^{1,4}}\|\boldsymbol{\psi}\|_{\mathbb{V}} + |\alpha|\|\mathfrak{q}_n - \mathfrak{q}\|_{\mathbb{W}^{1,4}}\|\mathfrak{q}\|_{\mathbb{W}^{1,4}}\|\boldsymbol{\psi}\|_{\mathbb{V}} \\ &\quad + |\alpha|\|\mathfrak{q}_n - \mathfrak{q}\|_{\mathbb{W}^{1,4}}\|\mathfrak{q}\|_{\mathbb{W}^{1,4}}\|\boldsymbol{\psi}\|_{\mathbb{V}} \to 0, \quad \text{as } n \to \infty, \end{aligned} \tag{2.16}$$

since $\mathfrak{q}_n \to \mathfrak{q}$ in $\mathbb{W}^{1,4}(\Omega)$. We estimate the term $\langle \mathcal{K}(\mathfrak{q}_n) - \mathcal{K}(\mathfrak{q}), \boldsymbol{\psi}\rangle$ from (2.13) using Hölder's inequality as

$$\begin{aligned} &|\beta\langle \mathcal{K}(\mathfrak{q}_n) - \mathcal{K}(\mathfrak{q}), \boldsymbol{\psi}\rangle| \\ &= \beta|\langle |\mathrm{A}(\mathfrak{q}_n)|^2\mathrm{A}(\mathfrak{q}_n) - |\mathrm{A}(\mathfrak{q})|^2\mathrm{A}(\mathfrak{q}), \nabla\boldsymbol{\psi}\rangle| \\ &\leq \beta|\langle |\mathrm{A}(\mathfrak{q}_n - \mathfrak{q})|^2\mathrm{A}(\mathfrak{q}_n - \mathfrak{q}), \nabla\boldsymbol{\psi}\rangle| + \beta|\langle |\mathrm{A}(\mathfrak{q}_n - \mathfrak{q})|^2\mathrm{A}(\mathfrak{q}), \nabla\boldsymbol{\psi}\rangle| \\ &\quad + 2\beta|\langle [\mathrm{A}(\mathfrak{q}_n - \mathfrak{q}) : \mathrm{A}(\mathfrak{q})]\mathrm{A}(\mathfrak{q}_n - \mathfrak{q}), \nabla\boldsymbol{\psi}\rangle| + 2\beta|\langle [\mathrm{A}(\mathfrak{q}_n - \mathfrak{q}) : \mathrm{A}(\mathfrak{q})]\mathrm{A}(\mathfrak{q}), \nabla\boldsymbol{\psi}\rangle| \\ &\quad + \beta|\langle |\mathrm{A}(\mathfrak{q})|^2\mathrm{A}(\mathfrak{q}_n - \mathfrak{q}), \nabla\boldsymbol{\psi}\rangle| \\ &\leq \beta\|\mathfrak{q}_n - \mathfrak{q}\|^3_{\mathbb{W}^{1,4}}\|\boldsymbol{\psi}\|_{\mathbb{W}^{1,4}} + 3\beta\|\mathfrak{q}_n - \mathfrak{q}\|^2_{\mathbb{W}^{1,4}}\|\mathfrak{q}\|_{\mathbb{W}^{1,4}}\|\boldsymbol{\psi}\|_{\mathbb{W}^{1,4}} \\ &\quad + 3\beta\|\mathfrak{q}_n - \mathfrak{q}\|_{\mathbb{W}^{1,4}}\|\mathfrak{q}\|^2_{\mathbb{W}^{1,4}}\|\boldsymbol{\psi}\|_{\mathbb{W}^{1,4}} \end{aligned}$$

$$\to 0, \quad \text{as} \ \ n \to \infty, \tag{2.17}$$

since $\mathfrak{q}_n \to \mathfrak{q}$ in $\mathbb{W}^{1,4}(\Omega)$.

From the above convergences, it is immediate that $\langle \mathscr{G}(\mathfrak{q}_n) - \mathscr{G}(\mathfrak{q}), \boldsymbol{\psi} \rangle \to 0$, for all $\boldsymbol{\psi} \in \mathbb{X}$. Hence the operator $\mathscr{G} : \mathbb{X} \to \mathbb{X}'$ is demicontinuous, which implies that the operator $\mathscr{G}(\cdot)$ is hemicontinuous also. □

2.3. Abstract formulation and weak solution

Let us write the abstract formulation to system (2.1) by taking the projection $\mathbb{P}$ as:

$$\begin{cases} \dfrac{\mathrm{d}\mathfrak{q}(t)}{\mathrm{d}t} + \nu \mathcal{A}\mathfrak{q}(t) + \mathcal{B}(\mathfrak{q}(t)) + \alpha \mathcal{J}(\mathfrak{q}(t)) + \beta \mathcal{K}(\mathfrak{q}(t)) = \mathbb{P}\mathfrak{g}(t), \ \ t \in (\tau, \infty), \\ \mathfrak{q}(\tau) = \mathfrak{q}_\tau \in \mathbb{H}, \end{cases} \tag{2.18}$$

where $\mathfrak{g} \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}; \mathbb{H}^{-1}(\Omega)) + \mathrm{L}^{\frac{4}{3}}_{\mathrm{loc}}(\mathbb{R}; \mathbb{W}^{-1,\frac{4}{3}}(\Omega))$, that is, we can write $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$ such that $\mathfrak{g}_1 \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}; \mathbb{H}^{-1}(\Omega))$ and $\mathfrak{g}_2 \in \mathrm{L}^{\frac{4}{3}}_{\mathrm{loc}}(\mathbb{R}; \mathbb{W}^{-1,\frac{4}{3}}(\Omega))$. Let us now provide the notion of weak solution to system (2.18).

Definition 2.1. A function $\mathfrak{q}(\cdot) := \mathfrak{q}(\cdot; \tau, \mathfrak{q}_\tau)$ is called a *weak solution* to system (2.1) on time interval $[\tau, \infty)$, if

$$\mathfrak{q} \in \mathrm{C}_w([\tau, \infty); \mathbb{H}) \cap \mathrm{L}^2_{\mathrm{loc}}(\tau, \infty; \mathbb{V}) \cap \mathrm{L}^4_{\mathrm{loc}}(\tau, \infty; \mathbb{W}^{1,4}_0(\Omega)),$$

$$\frac{\mathrm{d}\mathfrak{q}}{\mathrm{d}t} \in \mathrm{L}^2_{\mathrm{loc}}(\tau, \infty; \mathbb{V}') + \mathrm{L}^{\frac{4}{3}}_{\mathrm{loc}}(\tau, \infty; \mathbb{W}^{-1,\frac{4}{3}}(\Omega)),$$

[1] and it satisfies

(i) for any $\psi \in \mathbb{X}$,

$$\left\langle \frac{\mathrm{d}\mathfrak{q}(t)}{\mathrm{d}t}, \psi \right\rangle = - \left\langle \nu \mathcal{A}\mathfrak{q}(t) + \mathcal{B}(\mathfrak{q}(t)) + \alpha \mathcal{J}(\mathfrak{q}(t)) + \beta \mathcal{K}(\mathfrak{q}(t)) - \mathfrak{g}, \psi \right\rangle, \tag{2.19}$$

for a.e. $t \in [\tau, \infty)$;

(ii) the initial data:

$$\mathfrak{q}(\tau) = \mathfrak{q}_\tau \ \ \text{in} \ \ \mathbb{H}.$$

2.4. A compact operator

We follow [5, Subsection 2.3] for the contents of this section. Consider the natural embedding $j : \mathbb{V} \hookrightarrow \mathbb{H}$ and its adjoint $j^* : \mathbb{H} \hookrightarrow \mathbb{V}$. Since the range of j is dense in $\mathbb{H}$, the map j^* is one-to-one. Let us define

[1] $\mathrm{C}_w([\tau, \infty); \mathbb{H})$ denotes the space of functions $\mathfrak{q} : [\tau, \infty) \to \mathbb{H}$, which are weakly continuous.

$$\begin{aligned} \mathrm{D}(\mathbb{A}) &:= j^*(\mathbb{H}) \subset \mathbb{V}, \\ \mathbb{A}\mathfrak{q} &:= (j^*)^{-1}\mathfrak{q}, \quad \mathfrak{q} \in \mathrm{D}(\mathbb{A}). \end{aligned} \tag{2.20}$$

Note that for all $\mathfrak{q} \in \mathrm{D}(\mathbb{A})$ and $\mathfrak{q} \in \mathbb{V}$

$$(\mathbb{A}\mathfrak{p}, \mathfrak{q})_{\mathbb{H}} = (\mathfrak{p}, \mathfrak{q})_{\mathbb{V}}.$$

Let us assume that $s > 2$. It is clear that $\mathbb{V}_s$ is dense in $\mathbb{V}$ and the embedding $j_s : \mathbb{V}_s \hookrightarrow \mathbb{V}$ is continuous. Then, there exists a Hilbert space $\mathbb{U}$ (cf. [29], [5, Lemma C.1]) such that $\mathbb{U} \subset \mathbb{V}_s$, $\mathbb{U}$ is dense in $\mathbb{V}_s$ and

$$\text{the natural embedding } \iota_s : \mathbb{U} \hookrightarrow \mathbb{V}_s \text{ is compact.}$$

It implies that

$$\mathbb{U} \underset{\iota_s}{\hookrightarrow} \mathbb{V}_s \underset{j_s}{\hookrightarrow} \mathbb{V} \underset{j}{\hookrightarrow} \mathbb{H} \cong \mathbb{H}' \underset{j'}{\hookrightarrow} \mathbb{V}' \underset{j_s'}{\hookrightarrow} \mathbb{V}_s' \underset{\iota_s'}{\hookrightarrow} \mathbb{U}'.$$

Consider the composition

$$\iota := j \circ j_s \circ \iota_s : \mathbb{U} \to \mathbb{H}$$

and its adjoint

$$\iota^* := (j \circ j_s \circ \iota_s)^* = \iota_s^* \circ j_s^* \circ j^* : \mathbb{H} \to \mathbb{U}.$$

We have that ι is compact and since its range is dense in $\mathbb{H}$, $\iota^* : \mathbb{H} \to \mathbb{U}$ is one-one. Let us define

$$\begin{aligned} \mathrm{D}(\mathfrak{L}) &:= \iota^*(\mathbb{H}) \subset \mathbb{U}, \\ \mathfrak{L}\mathfrak{q} &:= (\iota^*)^{-1}\mathfrak{q}, \quad \mathfrak{q} \in \mathrm{D}(\mathfrak{L}). \end{aligned} \tag{2.21}$$

Also we have that $\mathfrak{L} : \mathrm{D}(\mathfrak{L}) \to \mathbb{H}$ is onto, $\mathrm{D}(\mathfrak{L})$ is dense in $\mathbb{H}$ and

$$(\mathfrak{L}\mathfrak{q}, \mathfrak{u})_{\mathbb{H}} = (\mathfrak{q}, \mathfrak{u})_{\mathbb{U}}, \quad \mathfrak{q} \in \mathrm{D}(\mathfrak{L}),\ \mathfrak{u} \in \mathbb{U}.$$

Furthermore, for $\mathfrak{q} \in \mathrm{D}(\mathfrak{L})$,

$$\mathfrak{L}\mathfrak{q} = (\iota^*)^{-1}\mathfrak{q} = (j^*)^{-1} \circ (j_s^*)^{-1} \circ (\iota_s^*)^{-1}\mathfrak{q} = \mathbb{A} \circ (j_s^*)^{-1} \circ (\iota_s^*)^{-1}\mathfrak{q},$$

where $\mathbb{A}$ is defined in (2.20). Since the operator $\mathfrak{L}$ is self-adjoint and $\mathfrak{L}^{-1}$ is compact, there exists an orthonormal basis $\{\boldsymbol{e}_i\}_{i\in\mathbb{N}}$ of $\mathbb{H}$ such that

$$\mathfrak{L}\boldsymbol{e}_i = \mu_i \boldsymbol{e}_i, \quad i \in \mathbb{N}, \tag{2.22}$$

that is, $\{\boldsymbol{e}_i\}$ are the eigenfunctions and $\{\mu_i\}$ are the corresponding eigenvalues of operator $\mathfrak{L}$. Note that $\boldsymbol{e}_i \in \mathbb{U},\ i \in \mathbb{N}$, because $\mathrm{D}(\mathfrak{L}) \subset \mathbb{U}$.

Let us fix $m \in \mathbb{N}$ and let P_m be the operator from $\mathbb{U}'$ to $\mathrm{span}\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_m\} =: \mathbb{H}_m$ defined by

$$\mathrm{P}_m \mathfrak{q}^* := \sum_{i=1}^{m} \langle \mathfrak{q}^*, \boldsymbol{e}_i \rangle_{\mathbb{U}' \times \mathbb{U}} \boldsymbol{e}_i, \qquad \mathfrak{q}^* \in \mathbb{U}'. \tag{2.23}$$

We will consider the restriction of operator P_m to the space $\mathbb{H}$ denoted still by the same. In particular, we have $\mathbb{H} \hookrightarrow \mathbb{U}'$, that is, every element $\mathfrak{q} \in \mathbb{H}$ induces a functional $\mathfrak{q}^* \in \mathbb{U}'$ by

$$\langle \mathfrak{q}^*, \mathfrak{p} \rangle_{\mathbb{U}' \times \mathbb{U}} := (\mathfrak{q}, \mathfrak{p}), \quad \mathfrak{p} \in \mathbb{U}. \tag{2.24}$$

Thus the restriction of P_m to $\mathbb{H}$ is given by

$$\mathrm{P}_m \mathfrak{q} := \sum_{i=1}^{m} (\mathfrak{q}, \boldsymbol{e}_i) \boldsymbol{e}_i, \qquad \mathfrak{q} \in \mathbb{H}. \tag{2.25}$$

Hence, in particular, P_m is the orthogonal projection from $\mathbb{H}$ onto $\mathbb{H}_m$.

3. Existence of global weak solutions satisfying energy equality and uniqueness

In this section, we establish the existence of unique global weak solution (in the sense of Definition 2.1) to system (2.1) which satisfies energy equality. Due to (1.4), we can define

$$\varepsilon_0 := 1 - \frac{\alpha^2}{2\nu\beta}. \tag{3.1}$$

Next theorem is the main result of this section which provides the existence and uniqueness of global weak solutions to system (2.1) satisfying energy equality.

Theorem 3.1. *Let* (1.4) *hold. For* $\mathfrak{q}_\tau \in \mathbb{H}$ *and* $\mathfrak{g} \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}; \mathbb{H}^{-1}(\Omega)) + \mathrm{L}^{\frac{4}{3}}_{\mathrm{loc}}(\mathbb{R}; \mathbb{W}^{-1,\frac{4}{3}}(\Omega))$ *(that is, we can write* $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$ *such that* $\mathfrak{g}_1 \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}; \mathbb{H}^{-1}(\Omega))$ *and* $\mathfrak{g}_2 \in \mathrm{L}^{\frac{4}{3}}_{\mathrm{loc}}(\mathbb{R}; \mathbb{W}^{-1,\frac{4}{3}}(\Omega))$*), there exists a unique weak solution* $\mathfrak{q}(\cdot) := \mathfrak{q}(\cdot; \tau, \mathfrak{q}_\tau)$ *to system* (2.1) *in the sense of Definition 2.1 which satisfies the following energy equality:*

$$\begin{aligned} \|\mathfrak{q}(t)\|_2^2 + \nu \int_\tau^t \|\mathrm{A}(\mathfrak{q}(s))\|_2^2 \mathrm{d}s + \beta \int_\tau^t \|\mathrm{A}(\mathfrak{q}(s))\|_4^4 \mathrm{d}s + 2\alpha \int_\tau^t \langle \mathcal{J}(\mathfrak{q}(s)), \mathfrak{q}(s) \rangle \mathrm{d}s \\ = \|\mathfrak{q}_\tau\|_2^2 + 2 \int_\tau^t \langle \mathfrak{g}(s), \mathfrak{q}(s) \rangle \mathrm{d}s, \end{aligned} \tag{3.2}$$

for all $t \in [\tau, \infty)$*. Consequently,* $\mathfrak{q} \in \mathrm{C}([\tau, \infty); \mathbb{H})$ *(cf. [25]).*

Proof of Theorem 3.1. Let us fix $T > 0$. Note that it is enough to prove the result on the interval $[\tau, \tau + T]$. The proof is divided into the following four steps.

Step I. *Finite-dimensional approximation and weak limits.* Let us consider the following approximate equation for system (2.18) on the finite-dimensional space $\mathbb{H}_n$ (see Sect. 2.4):

$$\begin{cases} \dfrac{\mathrm{d}\mathfrak{q}^n(t)}{\mathrm{d}t} = -\nu\mathcal{A}_n\mathfrak{q}^n(t) - \mathcal{B}_n(\mathfrak{q}^n(t)) - \alpha\mathcal{J}_n(\mathfrak{q}^n(t)) - \beta\mathcal{K}_n(\mathfrak{q}^n(t)) + \mathfrak{g}_n(t), \\ \mathfrak{q}^n(\tau) = \mathfrak{q}_{\tau n}, \end{cases} \tag{3.3}$$

where $\mathcal{A}_n\mathfrak{q}^n = \mathrm{P}_n\mathcal{A}\mathfrak{q}^n$, $\mathcal{B}_n(\mathfrak{q}^n) = \mathrm{P}_n\mathcal{B}(\mathfrak{q}^n)$, $\mathcal{J}_n(\mathfrak{q}^n) = \mathrm{P}_n\mathcal{J}(\mathfrak{q}^n)$, $\mathcal{K}_n(\mathfrak{q}^n) = \mathrm{P}_n\mathcal{K}(\mathfrak{q}^n)$, $\mathfrak{g}_n = \mathrm{P}_n[\mathbb{P}\mathfrak{g}]$ and $\mathfrak{q}_{\tau n} = \mathrm{P}_n[\mathfrak{q}_\tau]$. Since $\mathcal{A}_n \cdot + \mathcal{B}_n(\cdot) + \alpha\mathcal{J}_n(\cdot) + \beta\mathcal{K}_n(\cdot)$ is locally Lipschitz (see (2.4), (2.6), (2.7) and (2.8)), system (3.3) has a unique local solution $\mathfrak{q}^n \in \mathrm{C}([\tau, T^*]; \mathbb{H}_n)$, for some $\tau < T^* < \tau + T$. The following a priori estimates show that the time T^* can be extended to time $\tau + T$. Taking the inner product with $\mathfrak{q}^n(\cdot)$ to the first equation of (3.3), we obtain

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathfrak{q}^n(t)\|_2^2 = &-\frac{\nu}{2}\|\mathrm{A}(\mathfrak{q}^n(t))\|_2^2 - \frac{\beta}{2}\|\mathrm{A}(\mathfrak{q}^n(t))\|_4^4 \\ &+ \frac{\alpha}{2}\int_\Omega (\mathrm{A}(\mathfrak{q}^n(t)))^2 : \mathrm{A}(\mathfrak{q}^n(t))\mathrm{d}x + \langle \mathfrak{g}(t), \mathfrak{q}^n(t)\rangle. \end{aligned} \tag{3.4}$$

Next, using the Hölder's inequality, (1.4) and Young's inequality, we estimate the terms of the right hand side of (3.4) as

$$\begin{aligned} \left|\frac{\alpha}{2}\int_\Omega (\mathrm{A}(\mathfrak{q}^n))^2 : \mathrm{A}(\mathfrak{q}^n)\mathrm{d}x\right| &\leq \frac{|\alpha|}{2}\int_\Omega \left|\mathrm{A}(\mathfrak{q}^n)\right|^2\left|\mathrm{A}(\mathfrak{q}^n)\right|\mathrm{d}x \leq \frac{|\alpha|}{2}\|\mathrm{A}(\mathfrak{q}^n)\|_4^2\|\mathrm{A}(\mathfrak{q}^n)\|_2 \\ &\leq \frac{\nu(1-\varepsilon_0)}{4}\|\mathrm{A}(\mathfrak{q}^n)\|_2^2 + \frac{\alpha^2}{4\nu(1-\varepsilon_0)}\|\mathrm{A}(\mathfrak{q}^n)\|_4^4 \\ &\leq \frac{\nu(1-\varepsilon_0)}{4}\|\mathrm{A}(\mathfrak{q}^n)\|_2^2 + \frac{\beta(1-\varepsilon_0)}{2}\|\mathrm{A}(\mathfrak{q}^n)\|_4^4 \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \left|\langle\mathfrak{g}, \mathfrak{q}^n\rangle\right| &= \left|\langle\mathfrak{g}_1 + \mathfrak{g}_2, \mathfrak{q}^n\rangle\right| \\ &\leq \|\mathfrak{g}_1\|_{\mathbb{H}^{-1}}\|\mathfrak{q}^n\|_{\mathbb{V}} + \|\mathfrak{g}_2\|_{\mathbb{W}^{-1,\frac{4}{3}}}\|\mathfrak{q}^n\|_{\mathbb{W}^{1,4}} \\ &\leq C\|\mathfrak{g}_1\|_{\mathbb{H}^{-1}}^2 + \frac{\nu\varepsilon_0}{4}\|\mathfrak{q}^n\|_{\mathbb{V}}^2 + C\|\mathfrak{g}_2\|_{\mathbb{W}^{-1,\frac{4}{3}}}(\|\mathfrak{q}^n\|_4 + \|\nabla\mathfrak{q}^n\|_4) \\ &\leq C\|\mathfrak{g}_1\|_{\mathbb{H}^{-1}}^2 + \frac{\nu\varepsilon_0}{4}\|\mathfrak{q}^n\|_2^2 + \frac{\nu\varepsilon_0}{4}\|\nabla\mathfrak{q}^n\|_2^2 + C\|\mathfrak{g}_2\|_{\mathbb{W}^{-1,\frac{4}{3}}} \\ &\quad(\|\mathrm{A}(\mathfrak{q}^n)\|_4^{\frac{d}{4+d}}\|\mathfrak{q}^n\|_2^{\frac{4}{4+d}} + \|\mathrm{A}(\mathfrak{q}^n)\|_4) \\ &\leq C\|\mathfrak{g}_1\|_{\mathbb{H}^{-1}}^2 + \frac{\nu\varepsilon_0}{4}\|\mathfrak{q}^n\|_2^2 + \frac{\nu\varepsilon_0}{8}\|\mathrm{A}(\mathfrak{q}^n)\|_2^2 + C\|\mathfrak{g}_2\|_{\mathbb{W}^{-1,\frac{4}{3}}}^{\frac{4}{3}} + \frac{\beta\varepsilon_0}{4}\|\mathrm{A}(\mathfrak{q}^n)\|_4^4 \\ &\quad + C\|\mathfrak{g}_2\|_{\mathbb{W}^{-1,\frac{4}{3}}}^{\frac{4(4+d)}{16+3d}}\|\mathfrak{q}^n\|_2^{\frac{16}{16+3d}} \\ &\leq C\|\mathfrak{g}_1\|_{\mathbb{H}^{-1}}^2 + \frac{\nu\varepsilon_0}{4}\|\mathfrak{q}^n\|_2^2 + \frac{\nu\varepsilon_0}{8}\|\mathrm{A}(\mathfrak{q}^n)\|_2^2 + C\|\mathfrak{g}_2\|_{\mathbb{W}^{-1,\frac{4}{3}}}^{\frac{4}{3}} + \frac{\beta\varepsilon_0}{4}\|\mathrm{A}(\mathfrak{q}^n)\|_4^4 \end{aligned}$$

$$
\begin{aligned}
&\quad + C(\|\mathfrak{g}_2\|_{\mathbb{W}^{-1,\frac{4}{3}}}^{\frac{4}{3}} + 1)(\|\mathfrak{q}^n\|_2^2 + 1)\\
&\leq \frac{\nu\varepsilon_0}{8}\|\mathrm{A}(\mathfrak{q}^n)\|_2^2 + \frac{\beta\varepsilon_0}{4}\|\mathrm{A}(\mathfrak{q}^n)\|_4^4 + C(\|\mathfrak{g}_1\|_{\mathbb{H}^{-1}}^2 + \|\mathfrak{g}_2\|_{\mathbb{W}^{-1,\frac{4}{3}}}^{\frac{4}{3}} + 1)\\
&\quad + C(\|\mathfrak{g}_2\|_{\mathbb{W}^{-1,\frac{4}{3}}}^{\frac{4}{3}} + 1)\|\mathfrak{q}^n\|_2^2. \qquad (3.6)
\end{aligned}
$$

Combining (3.4)–(3.6), we deduce

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\mathfrak{q}^n(t)\|_2^2 + \frac{\nu}{2}\left(1 + \frac{\varepsilon_0}{2}\right)\|\mathrm{A}(\mathfrak{q}^n(t))\|_2^2 + \frac{\beta\varepsilon_0}{2}\|\mathrm{A}(\mathfrak{q}^n(t))\|_4^4\\
&\leq C(\|\mathfrak{g}_1(t)\|_{\mathbb{H}^{-1}}^2 + \|\mathfrak{g}_2(t)\|_{\mathbb{W}^{-1,\frac{4}{3}}}^{\frac{4}{3}} + 1) + C(\|\mathfrak{g}_2(t)\|_{\mathbb{W}^{-1,\frac{4}{3}}}^{\frac{4}{3}} + 1)\|\mathfrak{q}^n(t)\|_2^2, \qquad (3.7)
\end{aligned}
$$

which gives for all $t \in [\tau, \tau + T]$

$$
\begin{aligned}
&\|\mathfrak{q}^n(t)\|_2^2 + \frac{\nu}{2}\left(1 + \frac{\varepsilon_0}{2}\right)\int_\tau^t \|\mathrm{A}(\mathfrak{q}^n(s))\|_2^2\mathrm{d}s + \frac{\beta\varepsilon_0}{2}\int_\tau^t \|\mathrm{A}(\mathfrak{q}^n(s))\|_4^4\mathrm{d}s\\
&\leq \|\mathfrak{q}^n(\tau)\|_2^2 + C\int_\tau^t \Big[\|\mathfrak{g}_1(s)\|_{\mathbb{H}^{-1}}^2 + \|\mathfrak{g}_2(s)\|_{\mathbb{W}^{-1,\frac{4}{3}}}^{\frac{4}{3}} + 1\\
&\quad + (\|\mathfrak{g}_2(s)\|_{\mathbb{W}^{-1,\frac{4}{3}}}^{\frac{4}{3}} + 1)\|\mathfrak{q}^n(s)\|_2^2\Big]\mathrm{d}s. \qquad (3.8)
\end{aligned}
$$

Hence, an application of Gronwall's inequality and the fact that $\|\mathfrak{q}^n(\tau)\|_2 \leq \|\mathfrak{q}(\tau)\|_2$, $\mathfrak{g}_1 \in \mathrm{L}^2(\tau, \tau + T; \mathbb{H}^{-1}(\Omega))$ and $\mathfrak{g}_2 \in \mathrm{L}^{\frac{4}{3}}(\tau, \tau + T; \mathbb{W}^{-1,\frac{4}{3}}(\Omega))$, we have from (3.8) that

$$
\begin{aligned}
&\{\mathfrak{q}^n\}_{n\in\mathbb{N}} \text{ is a bounded sequence in } \mathrm{L}^\infty(\tau, \tau + T; \mathbb{H}) \cap \mathrm{L}^2(\tau, \tau + T; \mathbb{V})\\
&\qquad \cap \mathrm{L}^4(\tau, \tau + T; \mathbb{W}^{1,4}(\Omega)). \qquad (3.9)
\end{aligned}
$$

For any arbitrary element $\boldsymbol{\psi} \in \mathrm{L}^2(\tau, \tau + T; \mathbb{V}) \cap \mathrm{L}^4(\tau, \tau + T; \mathbb{W}^{1,4}(\Omega))$, using Hölder's inequality and Lemma 2.1, we have from (3.3) that

$$
\begin{aligned}
&\left|\int_\tau^{\tau+T}\left\langle\frac{\mathrm{d}\mathfrak{q}^n(t)}{\mathrm{d}t}, \boldsymbol{\psi}(t)\right\rangle\mathrm{d}t\right|\\
&\leq \int_\tau^{\tau+T}\Big[\nu\left|(\nabla\mathfrak{q}^n(t), \mathrm{P}_n\nabla\boldsymbol{\psi}(t))\right| + \left|b(\mathfrak{q}^n(t), \mathrm{P}_n\boldsymbol{\psi}(t), \mathfrak{q}^n(t))\right| + |\langle\mathfrak{g}, \mathrm{P}_n\boldsymbol{\psi}(t)\rangle|\Big]\mathrm{d}t\\
&\quad + |\alpha|\int_\tau^{\tau+T}\int_\Omega |\mathrm{A}(\mathfrak{q}^n(t))|^2|\mathrm{P}_n\nabla\boldsymbol{\psi}(t)|\mathrm{d}x\mathrm{d}t + \beta\int_0^T\int_\Omega |\mathrm{A}(\mathfrak{q}^n(t))|^3|\mathrm{P}_n\nabla\boldsymbol{\psi}(t)|\mathrm{d}x\mathrm{d}t\\
&\leq C\Big[\|\mathfrak{q}^n\|_{\mathrm{L}^2(\tau,\tau+T;\mathbb{V})} + \|\mathfrak{q}^n\|_{\mathrm{L}^4(\tau,\tau+T;\mathbb{W}^{1,4})} + \|\mathfrak{g}\|_{\mathrm{L}^2(\tau,\tau+T;\mathbb{H}^{-1})+\mathrm{L}^{\frac{4}{3}}(\tau,\tau+T;\mathbb{W}^{-1,\frac{4}{3}})}\Big]\\
&\quad \times \Big[\|\boldsymbol{\psi}\|_{\mathrm{L}^2(\tau,\tau+T;\mathbb{V})\cap\mathrm{L}^4(\tau,\tau+T;\mathbb{W}^{1,4})}\Big],
\end{aligned}
$$

which implies that the sequences

$$
\{\tfrac{\mathrm{d}\mathfrak{q}^n}{\mathrm{d}t}\}_{n\in\mathbb{N}} \text{ and } \{\mathscr{G}(\mathfrak{q}^n)\}_{n\in\mathbb{N}} \text{ are bounded in } \mathrm{L}^2(\tau, \tau + T; \mathbb{V}') + \mathrm{L}^{\frac{4}{3}}(\tau, \tau + T; \mathbb{W}^{-1,\frac{4}{3}}(\Omega)), \qquad (3.10)
$$

where $\mathscr{G}(\cdot)$ given by (2.9). Using (3.9), (3.10) and the *Banach–Alaoglu theorem*, we infer the existence of an element $\mathfrak{q} \in \mathrm{L}^{\infty}(\tau, \tau+T; \mathbb{H}) \cap \mathrm{L}^{2}(\tau, \tau+T; \mathbb{V}) \cap \mathrm{L}^{4}(\tau, \tau+T; \mathbb{W}^{1,4}(\Omega))$ with $\frac{\mathrm{d}\mathfrak{q}}{\mathrm{d}t} \in \mathrm{L}^{2}(\tau, \tau+T; \mathbb{V}') + \mathrm{L}^{\frac{4}{3}}(\tau, \tau+T; \mathbb{W}^{-1,\frac{4}{3}}(\Omega))$ and an element $\mathscr{G}_0 \in \mathrm{L}^{2}(\tau, \tau+T; \mathbb{V}') + \mathrm{L}^{\frac{4}{3}}(\tau, \tau+T; \mathbb{W}^{-1,\frac{4}{3}}(\Omega))$ such that

$$\mathfrak{q}^n \xrightharpoonup{w^*} \mathfrak{q} \quad \text{in} \quad \mathrm{L}^{\infty}(\tau, \tau+T; \mathbb{H}), \tag{3.11}$$

$$\mathfrak{q}^n \xrightharpoonup{w} \mathfrak{q} \quad \text{in} \quad \mathrm{L}^{2}(\tau, \tau+T; \mathbb{V}), \tag{3.12}$$

$$\mathfrak{q}^n \xrightharpoonup{w} \mathfrak{q} \quad \text{in} \quad \mathrm{L}^{4}(\tau, \tau+T; \mathbb{W}^{1,4}(\Omega)), \tag{3.13}$$

$$\frac{\mathrm{d}\mathfrak{q}^n}{\mathrm{d}t} \xrightharpoonup{w} \frac{\mathrm{d}\mathfrak{q}}{\mathrm{d}t} \quad \text{in} \quad \mathrm{L}^{2}(\tau, \tau+T; \mathbb{V}') + \mathrm{L}^{\frac{4}{3}}(\tau, \tau+T; \mathbb{W}^{-1,\frac{4}{3}}(\Omega)), \tag{3.14}$$

$$\mathscr{G}(\mathfrak{q}^n) \xrightharpoonup{w} \mathscr{G}_0 \quad \text{in} \quad \mathrm{L}^{2}(\tau, \tau+T; \mathbb{V}') + \mathrm{L}^{\frac{4}{3}}(\tau, \tau+T; \mathbb{W}^{-1,\frac{4}{3}}(\Omega)), \tag{3.15}$$

along a subsequence (still denoted by the same symbol). Note that $\mathfrak{g}_n \to \mathbb{P}\mathfrak{g}$ in $\mathrm{L}^{2}(\tau, \tau+T; \mathbb{H}^{-1}(\Omega)) + \mathrm{L}^{\frac{4}{3}}(\tau, \tau+T; \mathbb{W}^{-1,\frac{4}{3}}(\Omega))$. Therefore, on passing to limit as $n \to \infty$ in (3.3), the limit $\mathfrak{q}(\cdot)$ satisfies:

$$\frac{\mathrm{d}\mathfrak{q}}{\mathrm{d}t} = -\mathscr{G}_0 + \mathbb{P}\mathfrak{g}, \tag{3.16}$$

in the sense of distribution. Note that the embedding of $\mathbb{H} \subset \mathbb{V}'$ is continuous and $\mathfrak{q} \in \mathrm{L}^{\infty}(\tau, \tau+T; \mathbb{H})$ implies $\mathfrak{q} \in \mathrm{L}^{\infty}(\tau, \tau+T; \mathbb{V}' + \mathbb{W}^{-1,\frac{4}{3}}(\Omega))$. Thus, we get $\mathfrak{q}, \frac{\mathrm{d}\mathfrak{q}}{\mathrm{d}t} \in \mathrm{L}^{\frac{4}{3}}(\tau, \tau+T; \mathbb{V}' + \mathbb{W}^{-1,\frac{4}{3}}(\Omega))$ and then invoking [21, Theorem 2, Section 5.9.2], it is immediate that $\mathfrak{q} \in \mathrm{C}([\tau, \tau+T]; \mathbb{V}' + \mathbb{W}^{-1,\frac{4}{3}}(\Omega))$. Since $\mathbb{H}$ is reflexive, using [14, Proposition 1.7.1], we obtain $\mathfrak{q} \in \mathrm{C}_w([\tau, \tau+T]; \mathbb{H})$ and the map $t \mapsto \|\mathfrak{q}(t)\|_2$ is bounded. Thus the condition (ii) in the Definition (2.1) makes sense.

Step II. *Energy equality satisfied by* $\mathfrak{q}(\cdot)$. Let us first note that, from [13, Chapter II, Theorem 1.8], $\mathfrak{q} \in \mathrm{L}^{2}(\tau, \tau+T; \mathbb{V}) \cap \mathrm{L}^{4}(\tau, \tau+T; \mathbb{W}^{1,4}(\Omega))$ and $\frac{\mathrm{d}\mathfrak{q}}{\mathrm{d}t} \in \mathrm{L}^{2}(\tau, \tau+T; \mathbb{V}') + \mathrm{L}^{\frac{4}{3}}(\tau, \tau+T; \mathbb{W}^{-1,\frac{4}{3}}(\Omega))$ imply $\mathfrak{q} \in \mathrm{C}([\tau, \tau+T]; \mathbb{H})$, the real-valued function $t \mapsto \|\mathfrak{q}(t)\|_2^2$ is absolutely continuous and the following equality is satisfied:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathfrak{q}(t)\|_2^2 = 2\left\langle \frac{\mathrm{d}\mathfrak{q}(t)}{\mathrm{d}t}, \mathfrak{q}(t) \right\rangle, \qquad \text{for a.e. } t \in [\tau, \tau+T]. \tag{3.17}$$

The proof of Theorem 1.8 in [13, Chapter II] requires a regularization technique which is not explicitly provided in [13]. Therefore we are not using Theorem 1.8 from [13, Chapter II] directly. We provide a sketch of proof of the energy equality (3.17) explicitly. We also mention that the similar process has been adopted to prove the energy equality for the solution of convective Brinkman-Forchheimer equations in periodic domains, bounded domains and unbounded domains in [2,22,26,33], respectively. Since we are working on unbounded domains, we discuss the sketch of the idea used in [33], that is, we use the eigenfunctions of operator $\mathfrak{L}$ (cf. (2.21) and (2.22)) to obtain a sequence which approximates $\mathfrak{q}(\cdot)$. Set

$$\mathfrak{q}_m(t) := \mathrm{P}_{1/m}\mathfrak{q}(t) = \sum_{\mu_j < m^2} e^{-\mu_j/m} \langle \mathfrak{q}(t), \boldsymbol{e}_j \rangle_{\mathbb{U}' \times \mathbb{U}} \boldsymbol{e}_j. \tag{3.18}$$

Since, for $s > 2$, $\{\boldsymbol{e}_j\}_{j\in\mathbb{N}} \subset \mathrm{D}(\mathcal{L}) \subset \mathbb{V}_s \subset \mathbb{V}\cap\mathbb{W}^{1,4}(\Omega)$, one can obtain (cf. [2,22,33])

$$\|\mathfrak{q}_m - \mathfrak{q}\|_{\mathrm{L}^2(\tau,\tau+T;\mathbb{V})\cap\mathrm{L}^4(\tau,\tau+T;\mathbb{W}^{1,4}(\Omega))} \to 0 \text{ as } m\to\infty. \tag{3.19}$$

Now, we define $\mathcal{V}_T := \{\boldsymbol{\phi}\in \mathrm{C}_0^\infty(\Omega\times[\tau,\tau+T)) : \nabla\cdot\boldsymbol{\phi}(x,t)=0\}$. Observe that, for each $\boldsymbol{\phi}\in\mathcal{V}_T$, $\boldsymbol{\phi}(\cdot,\tau+T)=0$ and $\mathcal{V}_T$ is dense in $\mathrm{L}^p(\tau,\tau+T;\mathbb{H}^1(\Omega)\cap\mathbb{W}^{1,4}(\Omega))$ (cf. [25, Lemmas 2.5, 2.6]). For some $t>\tau$, $\mathfrak{q}\in\mathrm{L}^p(\tau,\tau+T;\mathrm{X})$, $1\leq p<\infty$ and $\tau<h+\tau<\tau+T-t$ and $h<t$, the mollifier $\mathfrak{q}_h$ (in the sense of Friederichs) of $\mathfrak{q}$ is defined by

$$\mathfrak{q}_h(s) := (\mathfrak{q}*j_h)(s) = \int_\tau^{\tau+T} 1_{[\tau,t]}j_h(s-\zeta)\mathfrak{q}(\zeta)\mathrm{d}\zeta,$$

where $j_h(\cdot)$ is an infinite times differentiable function having support in $(-h,h)$, which is even and positive, such that $\int_{-\infty}^{+\infty} j_h(\zeta)\mathrm{d}\zeta = 1$. In view of [25, Lemma 2.5], we have that for $\mathfrak{q}\in\mathrm{L}^p(\tau,\tau+T;\mathrm{X})$ with $1\leq p<\infty$, $\mathfrak{q}_h\in\mathrm{C}^k([\tau,\tau+T];\mathrm{X})$ for all $k\geq 0$ and

$$\lim_{h\to 0}\|\mathfrak{q}_h-\mathfrak{q}\|_{\mathrm{L}^p(\tau,\tau+T;\mathrm{X})}=0. \tag{3.20}$$

Furthermore, if $\{\mathfrak{q}_m\}_{m\in\mathbb{N}}\subset\mathrm{L}^p(\tau,\tau+T;\mathrm{X})$ converges to $\mathfrak{q}$ in the norm of $\mathrm{L}^p(\tau,\tau+T;\mathrm{X})$, then

$$\lim_{m\to\infty}\|(\mathfrak{q}_m)_h-\mathfrak{q}_h\|_{\mathrm{L}^p(\tau,\tau+T;\mathrm{X})}=0. \tag{3.21}$$

In particular, form (3.21) and (3.19), we have

$$\lim_{m\to\infty}\|(\mathfrak{q}_m)_h-\mathfrak{q}_h\|_{\mathrm{L}^2(\tau,\tau+T;\mathbb{V})\cap\mathrm{L}^4(\tau,\tau+T;\mathbb{W}^{1,4})}=0. \tag{3.22}$$

We write the weak solution of (2.18) as

$$\int_\tau^t\left\langle\frac{\mathrm{d}\mathfrak{q}}{\mathrm{d}t}+\mathscr{G}_0-\mathbb{P}\mathfrak{g},\boldsymbol{\phi}\right\rangle\mathrm{d}\zeta=0, \tag{3.23}$$

for all $t<\tau+T$ and $\boldsymbol{\phi}\in\mathcal{V}_T$. Choosing $\boldsymbol{\phi}=(\mathfrak{q}_m)_h =: \mathfrak{q}_{m,h}$ in (3.23), where $(\cdot)_h$ is the mollification operator discussed above, for $\tau\leq t<\tau+T$, we get

$$\int_\tau^t\left\langle\frac{\mathrm{d}\mathfrak{q}}{\mathrm{d}t}+\mathscr{G}_0-\mathbb{P}\mathfrak{g},\mathfrak{q}_{m,h}\right\rangle\mathrm{d}\zeta=0. \tag{3.24}$$

Using (3.21), we obtain

$$\left|\int_\tau^t \left\langle \frac{\mathrm{d}\mathfrak{q}}{\mathrm{d}t} + \mathscr{G}_0 - \mathbb{P}\mathfrak{g}, \mathfrak{q}_{m,h} - \mathfrak{q}_h \right\rangle \mathrm{d}\zeta \right| \leq \left\| \frac{\mathrm{d}\mathfrak{q}}{\mathrm{d}t} + \mathscr{G}_0 - \mathbb{P}\mathfrak{g} \right\|_{\mathrm{L}^2(\tau,\tau+T;\mathbb{V}')+\mathrm{L}^{\frac{4}{3}}(\tau,\tau+T;\mathbb{W}^{-1,\frac{4}{3}})} \\ \|\mathfrak{q}_{m,h} - \mathfrak{q}_h\|_{\mathrm{L}^2(\tau,\tau+T;\mathbb{V})\cap\mathrm{L}^4(\tau,\tau+T;\mathbb{W}^{1,4})} \to 0.$$

Hence, passing limit $m \to \infty$ in (3.24), we obtain

$$\int_\tau^t \left\langle \frac{\mathrm{d}\mathfrak{q}}{\mathrm{d}t} + \mathscr{G}_0 - \mathbb{P}\mathfrak{g}, \mathfrak{q}_h \right\rangle \mathrm{d}\zeta = 0. \tag{3.25}$$

Using (3.20) and similar arguments as above, we find the following convergence also:

$$\lim_{h\to 0} \int_\tau^t \langle \mathscr{G}_0 - \mathbb{P}\mathfrak{g}, \mathfrak{q}_h \rangle \, \mathrm{d}\zeta = \int_\tau^t \langle \mathscr{G}_0 - \mathbb{P}\mathfrak{g}, \mathfrak{q} \rangle \, \mathrm{d}\zeta. \tag{3.26}$$

An integration by parts results to

$$\begin{aligned}\int_\tau^t \left\langle \frac{\mathrm{d}\mathfrak{q}}{\mathrm{d}t}, \mathfrak{q}_h \right\rangle \mathrm{d}\zeta &= -\int_\tau^t \left\langle \mathfrak{q}, \frac{\mathrm{d}\mathfrak{q}_h}{\mathrm{d}t} \right\rangle \mathrm{d}\zeta - (\mathfrak{q}(\tau), \mathfrak{q}_h(\tau)) + (\mathfrak{q}(t), \mathfrak{q}_h(t)) \\ &= -\int_\tau^t \int_\tau^t \frac{\mathrm{d}\, j_h(\zeta - s)}{\mathrm{d}t} (\mathfrak{q}(\zeta), \mathfrak{q}(s))\, \mathrm{d}s\mathrm{d}\zeta \\ &\quad - (\mathfrak{q}(\tau), \mathfrak{q}_h(\tau)) + (\mathfrak{q}(t), \mathfrak{q}_h(t)) = -(\mathfrak{q}(\tau), \mathfrak{q}_h(\tau)) + (\mathfrak{q}(t), \mathfrak{q}_h(t)) \\ &\to -\frac{1}{2}(\mathfrak{q}(\tau), \mathfrak{q}(\tau)) + \frac{1}{2}(\mathfrak{q}(t), \mathfrak{q}(t)) = -\frac{1}{2}\|\mathfrak{q}(\tau)\|_2^2 + \frac{1}{2}\|\mathfrak{q}(t)\|_2^2,\end{aligned} \tag{3.27}$$

as $h \to 0$, where we have used the property of mollifiers and the fact that the kernel $j_h(\cdot)$ in the definition of mollifier is even in $(-h, h)$. From (3.25)–(3.27), we infer that $\mathfrak{q}(\cdot)$ satisfies the energy equality, that is,

$$\|\mathfrak{q}(t)\|_2^2 + 2\int_\tau^t \langle \mathscr{G}_0(s) - \mathbb{P}\mathfrak{g}(s), \mathfrak{q}(s) \rangle \, \mathrm{d}s = \|\mathfrak{q}(\tau)\|_2^2.$$

Hence, an application of Fundamental Theorem of Calculus gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathfrak{q}(t)\|_2^2 = 2\langle \mathscr{G}_0(t) - \mathbb{P}\mathfrak{g}(t), \mathfrak{q}(t) \rangle = 2\left\langle \frac{\mathrm{d}\mathfrak{q}(t)}{\mathrm{d}t}, \mathfrak{q}(t) \right\rangle, \quad \text{for a.e. } t \in [\tau, \tau + T]. \tag{3.28}$$

Recalling that every weak solution of (3.16) is $\mathbb{H}$-weakly continuous in time, all weak solutions satisfy the energy equality (3.28) and so, all weak solutions of (3.16) belong to $\mathrm{C}([\tau, \tau + T]; \mathbb{H})$ (cf. [14, Proposition 1.7.1] and see [25,26] also).

Step III. *Minty–Browder technique:* From (3.28), for a measurable function $\eta(t) \geq 0$, we have the following equality:

$$e^{-2\eta(t)}\|\mathfrak{q}(t)\|_2^2 + 2\int_\tau^t e^{-2\eta(s)} \left\langle \mathscr{G}_0(s) - \mathbb{P}\mathfrak{g}(s) + \eta'(s)\mathfrak{q}(s), \mathfrak{q}(s) \right\rangle \mathrm{d}s = \|\mathfrak{q}(\tau)\|_2^2, \tag{3.29}$$

for all $t \in [\tau, \tau + T]$. Similar to (3.29), for system (3.3), we obtain the following energy equality:

$$e^{-2\eta(t)}\|\mathfrak{q}^n(t)\|_2^2 + 2\int_\tau^t e^{-2\eta(s)}\left\langle \mathscr{G}(\mathfrak{q}^n(s)) - \mathfrak{g}_n(s) + \eta'(s)\mathfrak{q}^n(s), \mathfrak{q}^n(s)\right\rangle \mathrm{d}s = \|\mathfrak{q}^n(\tau)\|_2^2, \tag{3.30}$$

for all $t \in [\tau, \tau + T]$. Remember that $\mathfrak{q}^n(\tau) = \mathrm{P}_n\mathfrak{q}(\tau)$, and hence the initial value $\mathfrak{q}^n(\tau)$ converges strongly in $\mathbb{H}$, that is, we have

$$\lim_{n\to\infty} \|\mathfrak{q}^n(\tau) - \mathfrak{q}(\tau)\|_2 = 0. \tag{3.31}$$

For any $\boldsymbol{\psi} \in \mathrm{L}^\infty(\tau, \tau + T; \mathbb{H}_m)$ with $m < n$, we define

$$\eta(t) = \frac{(C_S C_K)^2}{\nu\varepsilon_0}\int_\tau^t \|\mathrm{A}(\boldsymbol{\psi}(s))\|_4^2 \mathrm{d}s, \text{ for all } t \in [\tau, \tau + T].$$

Using (2.10), we obtain

$$\int_\tau^{\tau+T} e^{-2\eta(t)}\{\left\langle \mathscr{G}(\boldsymbol{\psi}(t)) - \mathscr{G}(\mathfrak{q}^n(t)), \boldsymbol{\psi}(t) - \mathfrak{q}^n(t)\right\rangle + \eta'(t)(\boldsymbol{\psi}(t) - \mathfrak{q}^n(t), \boldsymbol{\psi}(t) - \mathfrak{q}^n(t))\}\mathrm{d}t \geq 0. \tag{3.32}$$

Making use of (3.30) in (3.32), we obtain

$$\begin{aligned}
&\int_\tau^{\tau+T} e^{-2\eta(t)}\left\langle \mathscr{G}(\boldsymbol{\psi}(t)) + \eta'(t)\boldsymbol{\psi}(t), \boldsymbol{\psi}(t) - \mathfrak{q}^n(t)\right\rangle \mathrm{d}t \\
&\quad\geq \int_\tau^{\tau+T} e^{-2\eta(t)}\left\langle \mathscr{G}(\mathfrak{q}^n(t)) + \eta'(t)\mathfrak{q}^n(t), \boldsymbol{\psi}(t) - \mathfrak{q}^n(t)\right\rangle \mathrm{d}t \\
&\quad= \int_\tau^{\tau+T} e^{-2\eta(t)}\left\langle \mathscr{G}(\mathfrak{q}^n(t)) + \eta'(t)\mathfrak{q}^n(t), \boldsymbol{\psi}(t)\right\rangle \mathrm{d}t \\
&\qquad+ \frac{1}{2}\left[e^{-2\eta(T)}\|\mathfrak{q}^n(\tau + T)\|_2^2 - \|\mathfrak{q}^n(\tau)\|_2^2\right] \\
&\qquad- \int_\tau^{\tau+T} e^{-2\eta(t)}\left\langle \mathfrak{g}_n(t), \mathfrak{q}^n(t)\right\rangle \mathrm{d}t.
\end{aligned} \tag{3.33}$$

Taking limit infimum on both sides of (3.33), we deduce

$$\begin{aligned}
&\int_\tau^{\tau+T} e^{-2\eta(t)}\left\langle \mathscr{G}(\boldsymbol{\psi}(t)) + \eta'(t)\boldsymbol{\psi}(t), \boldsymbol{\psi}(t) - \mathfrak{q}(t)\right\rangle \mathrm{d}t \\
&\quad\geq \int_\tau^{\tau+T} e^{-2\eta(t)}\left\langle \mathscr{G}_0(t) + \eta'(t)\mathfrak{q}(t), \boldsymbol{\psi}(t)\right\rangle \mathrm{d}t \\
&\qquad+ \frac{1}{2}\liminf_{n\to\infty}\left[e^{-2\eta(T)}\|\mathfrak{q}^n(\tau + T)\|_2^2 - \|\mathfrak{q}^n(\tau)\|_2^2\right] \\
&\qquad- \int_\tau^{\tau+T} e^{-2\eta(t)}\left\langle \mathfrak{g}(t), \mathfrak{q}(t)\right\rangle \mathrm{d}t
\end{aligned}$$

$$\geq \int_{\tau}^{\tau+T} e^{-2\eta(t)} \left\langle \mathscr{G}_0(t) + \eta'(t)\mathfrak{q}(t), \boldsymbol{\psi}(t)\right\rangle \mathrm{d}t$$
$$+ \frac{1}{2}\left[e^{-2\eta(T)}\|\mathfrak{q}(\tau+T)\|_2^2 - \|\mathfrak{q}(\tau)\|_2^2\right] - \int_{\tau}^{\tau+T} e^{-2\eta(t)} \left\langle \mathfrak{g}(t), \mathfrak{q}(t)\right\rangle \mathrm{d}t, \quad (3.34)$$

where we have used the weak lower semicontinuity property of the $\mathbb{H}$-norm and the strong convergence of the initial data (3.31) in the final inequality. Now, using the equality (3.29) in (3.34), we further have

$$\int_{\tau}^{\tau+T} e^{-2\eta(t)} \left\langle \mathscr{G}(\boldsymbol{\psi}(t)) + \eta'(t)\boldsymbol{\psi}(t), \boldsymbol{\psi}(t) - \mathfrak{q}(t)\right\rangle \mathrm{d}t$$
$$\geq \int_{\tau}^{\tau+T} e^{-2\eta(t)} \left\langle \mathscr{G}_0(t) + \eta'(t)\mathfrak{q}(t), \boldsymbol{\psi}(t)\right\rangle \mathrm{d}t$$
$$- \int_{\tau}^{\tau+T} e^{-2\eta(t)} \left\langle \mathscr{G}_0(t) + \eta'(t)\mathfrak{q}(t), \mathfrak{q}(t)\right\rangle \mathrm{d}t$$
$$\geq \int_{\tau}^{\tau+T} e^{-2\eta(t)} \left\langle \mathscr{G}_0(t) + \eta'(t)\mathfrak{q}(t), \boldsymbol{\psi}(t) - \mathfrak{q}(t)\right\rangle \mathrm{d}t. \quad (3.35)$$

Note that the estimate (3.35) holds true for any $\boldsymbol{\psi} \in \mathrm{L}^{\infty}(\tau, \tau+; \mathbb{H}_m)$, $m \in \mathbb{N}$, since the inequality given in (3.35) is independent of both m and n. Using a density argument, one can show that the inequality (3.35) remains true for any $\boldsymbol{\psi} \in \mathrm{L}^{\infty}(\tau, \tau + T; \mathbb{H}) \cap \mathrm{L}^2(\tau, \tau + T; \mathbb{V}) \cap \mathrm{L}^4(\tau, \tau + T; \mathbb{W}^{1,4}(\Omega))$. In fact, for any $\boldsymbol{\psi} \in \mathrm{L}^{\infty}(\tau, \tau + T; \mathbb{H}) \cap \mathrm{L}^2(\tau, \tau+T; \mathbb{V}) \cap \mathrm{L}^4(\tau, \tau+T; \mathbb{W}^{1,4}(\Omega))$, there exists a strongly convergent subsequence $\boldsymbol{\psi}_m \in \mathrm{L}^{\infty}(\tau, \tau + T; \mathbb{H}) \cap \mathrm{L}^2(\tau, \tau + T; \mathbb{V}) \cap \mathrm{L}^4(\tau, \tau + T; \mathbb{W}^{1,4}(\Omega))$, that satisfies the inequality (3.35).

Taking $\boldsymbol{\psi} = \mathfrak{q} + r\mathfrak{u}$, $r > 0$, where $\mathfrak{u} \in \mathrm{L}^{\infty}(\tau, \tau + T; \mathbb{H}) \cap \mathrm{L}^2(\tau, \tau + T; \mathbb{V}) \cap \mathrm{L}^4(\tau, \tau + T; \mathbb{W}^{1,4}(\Omega))$, and substituting for $\boldsymbol{\psi}$ in (3.35), we get

$$\int_{\tau}^{\tau+T} e^{-2\eta(t)} \left\langle \mathscr{G}(\mathfrak{q}(t) + r\mathfrak{u}(t)) - \mathscr{G}_0(t) + r\eta(t)\mathfrak{u}(t), r\mathfrak{u}(t)\right\rangle \mathrm{d}t \geq 0. \quad (3.36)$$

Dividing the inequality (3.36) by r, using the hemicontinuity property of the operator $\mathscr{G}(\cdot)$ (see Lemma 2.3), and then passing $r \to 0$, we find

$$\int_{\tau}^{\tau+T} e^{-2\eta(t)} \left\langle \mathscr{G}(\mathfrak{q}(t)) - \mathscr{G}_0(t), \mathfrak{u}(t)\right\rangle \mathrm{d}t \geq 0, \quad (3.37)$$

for any $\mathfrak{u} \in \mathrm{L}^{\infty}(\tau, \tau+T; \mathbb{H}) \cap \mathrm{L}^2(\tau, \tau+T; \mathbb{V}) \cap \mathrm{L}^4(\tau, \tau+T; \mathbb{W}^{1,4}(\Omega))$. Therefore, from (3.37), we deduce that $\mathscr{G}(\mathfrak{q}(\cdot)) = \mathscr{G}_0(\cdot)$. In addition, $\mathfrak{q}(\cdot)$ satisfies the energy equality (3.2) for all $t \in [\tau, \tau + T]$ (see (3.28)).

Step IV. *Uniqueness:* Define $\mathfrak{F} = \mathfrak{q}_1 - \mathfrak{q}_2$, where $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are two weak solutions to system (2.18) in the sense of Definition 2.1. Then $\mathfrak{F} \in \mathrm{C}([\tau, \tau + T]; \mathbb{H}) \cap \mathrm{L}^2(\tau, \tau + T; \mathbb{V}) \cap \mathrm{L}^4(\tau, \tau + T; \mathbb{W}^{1,4}(\Omega))$ and satisfies

$$\begin{cases} \dfrac{\mathrm{d}\mathfrak{F}(t)}{\mathrm{d}t} = -\left[\mathscr{G}(\mathfrak{q}_1(t)) - \mathscr{G}(\mathfrak{q}_2(t))\right], \\ \mathfrak{F}(\tau) = \mathbf{0}, \end{cases} \quad (3.38)$$

in the weak sense. Therefore, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathfrak{F}(t)\|_2^2 &= -\langle \mathscr{G}(\mathfrak{q}_1(t)) - \mathscr{G}(\mathfrak{q}_2(t)), \mathfrak{q}_1(t) - \mathfrak{q}_2(t)\rangle \\ &\leq \frac{(C_S C_K)^2}{\nu\varepsilon_0}\|\mathcal{A}(\mathfrak{q}_2)(t)\|_4^2\|\mathfrak{F}(t)\|_2^2, \end{aligned} \tag{3.39}$$

for a.e. $t \in [\tau, \tau + T]$, where we have used (2.10). An application of the variation of constants formula and the fact that $\mathfrak{F}(\tau) = \mathbf{0}$ give $\mathfrak{q}_1(t) = \mathfrak{q}_2(t)$, for all $t \in [\tau, \tau + T]$ in $\mathbb{H}$, which proves the uniqueness.

□

4. Non-autonomous dynamical system

In this section, we demonstrate the existence of global pullback attractor for non-autonomous third-grade fluid equations on bounded as well as unbounded Poincaré domains. In order to define the non-autonomous dynamical system associated with (2.18), we consider a family of shift operators $\{\vartheta_t\}_{t\in\mathbb{R}}$, for each $t \in \mathbb{R}$, given by

$$\vartheta_t\tau = \tau + t, \quad \text{for all } \tau \in \mathbb{R}. \tag{4.1}$$

and define a map $\Psi : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{H} \to \mathbb{H}$ by

$$\Psi(t, \tau, \mathfrak{q}_\tau) := \mathfrak{q}(t + \tau; \tau, \mathfrak{q}_\tau). \tag{4.2}$$

Since, $\mathfrak{q}(\cdot; \tau, \mathfrak{q}_\tau)$ is the unique solution to system (2.18), it implies that

$$\Psi(t + s, \tau, \mathfrak{q}_\tau) = \Psi(t, s + \tau, \Psi(s, \tau, \mathfrak{q}_\tau)), \quad \tau \in \mathbb{R}, \ t, s \geq 0 \text{ and } \mathfrak{q}_\tau \in \mathbb{H}. \tag{4.3}$$

4.1. Continuous cocycle

In the next lemma, we show that the operator $\Psi(t, \tau, \cdot) : \mathbb{H} \to \mathbb{H}$, for $t \geq 0$ and $\tau \in \mathbb{R}$, is continuous.

Lemma 4.1. *Let* (1.4) *hold. Let $T > 0$ be fixed, $\mathfrak{q}_\tau^n \to \mathfrak{q}_\tau$ in $\mathbb{H}$ and $\mathfrak{g}_n \to \mathfrak{g}$ in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}; \mathbb{H}^{-1}(\Omega)) + \mathrm{L}^{\frac{4}{3}}_{\mathrm{loc}}(\mathbb{R}; \mathbb{W}^{-1,\frac{4}{3}}(\Omega))$. Let us denote by $\mathfrak{q}(\cdot) := \mathfrak{q}(\cdot; \tau, \mathfrak{q}_\tau)$ for the solution to system* (2.18) *and by $\mathfrak{q}_n(\cdot) := \mathfrak{q}(\cdot; \tau, \mathfrak{q}_\tau^n)$ for the solution to system* (2.18) *with $\mathfrak{q}_\tau$, $\mathfrak{g}$ being replaced by $\mathfrak{q}_\tau^n$, $\mathfrak{g}_n$. Then*

$$\mathfrak{q}_n(\cdot) \to \mathfrak{q}(\cdot) \ \textit{in} \ \mathrm{C}([\tau, \tau + T]; \mathbb{H}) \cap \mathrm{L}^2(\tau, \tau + T; \mathbb{V}) \cap \mathrm{L}^4(\tau, \tau + T; \mathbb{W}^{1,4}(\Omega)).$$

Proof. Let us introduce $\mathfrak{u}_n(\cdot) := \mathfrak{q}_n(\cdot) - \mathfrak{q}(\cdot)$ and $\hat{\mathfrak{g}}_n(\cdot) := \mathfrak{g}_n(\cdot) - \mathfrak{g}(\cdot)$. It is easy to see that $\mathfrak{u}_n(\cdot)$ solves the following initial value problem:

$$\begin{cases} \dfrac{\mathrm{d}\mathfrak{u}_n}{\mathrm{d}t} = -[\mathscr{G}(\mathfrak{q}_n) - \mathscr{G}(\mathfrak{q})] + \mathbb{P}\hat{\mathfrak{g}}_n, \\ \mathfrak{u}_n(\tau) = \mathfrak{q}_\tau^n - \mathfrak{q}_\tau, \end{cases} \tag{4.4}$$

in $\mathbb{V}'+\mathbb{W}^{-1,\frac{4}{3}}(\Omega)$. Taking the inner product with $\mathfrak{u}_n(\cdot)$ to the first equation of (4.4) and using (2.10), we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathfrak{u}_n(t)\|_2^2=-\frac{\nu\varepsilon_0}{4}\|\mathrm{A}(\mathfrak{u}_n(t))\|_2^2-\frac{\beta\varepsilon_0}{4}\|\mathrm{A}(\mathfrak{u}_n(t))\|_4^4 \\ +\frac{(C_{S,3}C_K)^2}{\nu\varepsilon_0}\|\mathrm{A}(\mathfrak{q}(t))\|_4^2\|\mathfrak{u}_n(t)\|_2^2+\big\langle\hat{\mathfrak{g}}_n(t),\mathfrak{u}_n(t)\big\rangle, \tag{4.5}$$

for a.e. $t\in[\tau,\tau+T]$. Similar to (3.6), we find for $\mathfrak{g}_n=\mathfrak{g}_{1,n}+\mathfrak{g}_{2,n}$, $\mathfrak{g}=\mathfrak{g}_1+\mathfrak{g}_2$ and $\hat{\mathfrak{g}}_n=\hat{\mathfrak{g}}_{1,n}+\hat{\mathfrak{g}}_{2,n}$

$$\big|\langle\hat{\mathfrak{g}}_n,\mathfrak{u}_n\rangle\big|\leq\frac{\nu\varepsilon_0}{8}\|\mathrm{A}(\mathfrak{u}_n)\|_2^2+\frac{\beta\varepsilon_0}{8}\|\mathrm{A}(\mathfrak{u}_n)\|_4^4+C\Bigg[\|\hat{\mathfrak{g}}_{1,n}\|_{\mathbb{H}^{-1}}^2+\|\hat{\mathfrak{g}}_{2,n}\|_{\mathbb{W}^{-1,\frac{4}{3}}}^{\frac{4}{3}} \\ +\|\mathfrak{g}_{2,n}\|_{\mathbb{W}^{-1,\frac{4}{3}}}^{\frac{4(4+d)}{16+3d}}\Bigg]+C\|\mathfrak{g}_{2,n}\|_{\mathbb{W}^{-1,\frac{4}{3}}}^{\frac{4(4+d)}{16+3d}}\|\mathfrak{u}_n\|_2^2. \tag{4.6}$$

Combining (4.5)–(4.6), we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathfrak{u}_n(t)\|_{\mathbb{H}}^2+\frac{\nu\varepsilon_0}{4}\|\mathrm{A}(\mathfrak{u}_n(t))\|_2^2+\frac{\beta\varepsilon_0}{4}\|\mathrm{A}(\mathfrak{u}_n(t))\|_4^4 \\ \leq C\Bigg[\|\hat{\mathfrak{g}}_{1,n}(t)\|_{\mathbb{H}^{-1}}^2+\|\hat{\mathfrak{g}}_{2,n}(t)\|_{\mathbb{W}^{-1,\frac{4}{3}}}^{\frac{4}{3}}+\|\mathfrak{g}_{2,n}(t)\|_{\mathbb{W}^{-1,\frac{4}{3}}}^{\frac{4(4+d)}{16+3d}}\Bigg] \\ +\Bigg[\frac{(C_{S,3}C_K)^2}{\nu\varepsilon_0}\|\mathrm{A}(\mathfrak{q}(t))\|_4^2+C\|\mathfrak{g}_{2,n}(t)\|_{\mathbb{W}^{-1,\frac{4}{3}}}^{\frac{4(4+d)}{16+3d}}\Bigg]\|\mathfrak{u}_n(t)\|_2^2, \tag{4.7}$$

for a.e. $t\in[\tau,\tau+T]$. Then by the classical Gronwall lemma, we arrive at

$$\|\mathfrak{u}_n(t)\|_{\mathbb{H}}^2+\frac{\nu\varepsilon_0}{4}\int_\tau^t\|\mathrm{A}(\mathfrak{u}_n(s))\|_2^2\mathrm{d}s+\frac{\beta\varepsilon_0}{4}\int_\tau^t\|\mathrm{A}(\mathfrak{u}_n(s))\|_4^4\mathrm{d}s \\ \leq\Bigg[\|\mathfrak{u}_n(\tau)\|_{\mathbb{H}}^2+C\int_\tau^{\tau+T}\Bigg[\|\hat{\mathfrak{g}}_{1,n}(s)\|_{\mathbb{H}^{-1}}^2+\|\hat{\mathfrak{g}}_{2,n}(s)\|_{\mathbb{W}^{-1,\frac{4}{3}}}^{\frac{4}{3}}+\|\mathfrak{g}_{2,n}(s)\|_{\mathbb{W}^{-1,\frac{4}{3}}}^{\frac{4(4+d)}{16+3d}}\Bigg]\mathrm{d}s\Bigg] \\ \times\exp\left\{\frac{(C_{S,3}C_K)^2}{\nu\varepsilon_0}\int_\tau^{\tau+T}\|\mathrm{A}(\mathfrak{q}(s))\|_4^2\mathrm{d}s+C\int_\tau^{\tau+T}\|\mathfrak{g}_{2,n}(s)\|_{\mathbb{W}^{-1,\frac{4}{3}}}^{\frac{4(4+d)}{16+3d}}\mathrm{d}s\right\}, \tag{4.8}$$

for all $t\in[\tau,\tau+T]$. Since $\mathfrak{q}_\tau^n\to\mathfrak{q}_\tau$ in $\mathbb{H}$, $\mathfrak{g}_n\to\mathfrak{g}$ in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R};\mathbb{H}^{-1}(\Omega))+\mathrm{L}^{\frac{4}{3}}_{\mathrm{loc}}(\mathbb{R};\mathbb{W}^{-1,\frac{4}{3}}(\Omega))$ and $\mathrm{A}(\mathfrak{q})\in\mathrm{L}^4(\tau,\tau+T;\mathbb{L}^4(\Omega))$, by (4.8), we infer that $\mathfrak{q}_n(\cdot)\to\mathfrak{q}(\cdot)$ in

$$\mathrm{C}([\tau,\tau+T];\mathbb{H})\cap\mathrm{L}^2(\tau,\tau+T;\mathbb{V})\cap\mathrm{L}^4(\tau,\tau+T;\mathbb{W}^{1,4}(\Omega))$$

as $n\to\infty$. This completes the proof. □

Lemma 4.1 shows that for all $\tau\in\mathbb{R}$, $t\geq0$ the mapping $\Psi(t,\tau,\cdot)$ defined in (4.2) is continuous from $\mathbb{H}$ into itself. Therefore, the mapping Ψ given by (4.2) is continuous ϑ-cocycle on $\mathbb{H}$ ([11]).

4.2. A class of families of nonempty subsets of $\mathbb{H}$

Let $\mathcal{R}$ be the set of all functions $r : \mathbb{R} \to (0, +\infty)$ such that

$$\lim_{t\to-\infty} e^{ct}[r(t)]^2 = 0, \text{ for all } c > 0, \tag{4.9}$$

and denote by $\mathcal{D}$ the class of all families $\hat{D} = \{D(t) \ : \ t \in \mathbb{R}\} \subset \mathcal{P}(\mathbb{H})$ such that $D(t) \subset \bar{B}(0, r_{\hat{D}}(t))$, for some $r_{\hat{D}} \in \mathcal{R}$, where $\mathcal{P}(\mathbb{H})$ denotes the family of all nonempty subsets of $\mathbb{H}$ and $\bar{B}(0, r_{\hat{D}}(t))$ denotes the closed ball in $\mathbb{H}$ centered at zero with radius $r_{\hat{D}}(t)$.

5. Global pullback attractor: Bounded domains

In this section, we establish the existence of a unique global pullback attractor on bounded domains. It is well known that for a bounded domain Ω, the operator $\mathcal{A}$ is invertible and its inverse $\mathcal{A}^{-1}$ is bounded, self-adjoint and compact in $\mathbb{H}$. Hence the spectrum of $\mathcal{A}$ consists of an infinite sequence $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots$, with $\lambda_k \to \infty$ as $k \to \infty$ of eigenvalues, and there exists an orthogonal basis $\{w_k\}_{k=1}^{\infty}$ of $\mathbb{H}$ consisting of eigenfunctions of $\mathcal{A}$ such that $\mathcal{A}w_k = \lambda_k w_k$, for all $k \in \mathbb{N}$. Since $\mathfrak{q} = \sum_{j=1}^{\infty}(\mathfrak{q}, w_j)w_j$, we have $\mathcal{A}\mathfrak{q} = \sum_{j=1}^{\infty} \lambda_j(\mathfrak{q}, w_j)w_j$. Thus, it is immediate that

$$\|\nabla\mathfrak{q}\|_2^2 = \langle\mathcal{A}\mathfrak{q}, \mathfrak{q}\rangle = \sum_{j=1}^{\infty} \lambda_j |(\mathfrak{q}, w_j)|^2 \geq \lambda_1 \sum_{j=1}^{\infty} |(\mathfrak{q}, w_j)|^2 = \lambda_1\|\mathfrak{q}\|_2^2, \tag{5.1}$$

for all $\mathfrak{q} \in \mathbb{V}$, which is known as *Poincaré inequality*. Throughout this subsection, we assume that external forcing term $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$ satisfies following hypothesis:

Hypothesis 5.1. Let $\mathfrak{g} \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}; \mathbb{H}^{-1}(\Omega)) + \mathrm{L}^{\frac{4}{3}}_{\mathrm{loc}}(\mathbb{R}; \mathbb{W}^{-1,\frac{4}{3}}(\Omega))$ and satisfy the following condition:

$$\lim_{t\to-\infty} e^{ct} \int_{-\infty}^{0} e^{\nu^*\lambda_1\xi} \left[\|\mathfrak{g}_1(\xi + t)\|^2_{\mathbb{H}^{-1}} + \|\mathfrak{g}_2(\xi + t)\|^{\frac{4(4+d)}{8+3d}}_{\mathbb{W}^{-1,\frac{4}{3}}}\right] \mathrm{d}\xi < +\infty, \text{ for all } c > 0, \tag{5.2}$$

where $\mathfrak{g}_1 \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}; \mathbb{H}^{-1}(\Omega))$ and $\mathfrak{g}_2 \in \mathrm{L}^{\frac{4}{3}}_{\mathrm{loc}}(\mathbb{R}; \mathbb{W}^{-1,\frac{4}{3}}(\Omega))$ are such that $\mathfrak{g}$ can be uniquely written as $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$. Also, here

$$0 < \nu^* = \begin{cases} 2\nu\left(1 - \frac{\varepsilon_0}{4}\right), & for \ \ d = 2; \\ \nu\left(1 + \frac{\varepsilon_0}{2}\right), & for \ \ d = 3; \end{cases} \tag{5.3}$$

is constant related to the parameters defined in (3.1) and $\lambda_1 > 0$ is a constant as in (5.1).

A direct consequence of above Hypothesis 5.1 is as follows:

Proposition 5.2 (Proposition 4.2, [32]). *Assume that Hypothesis 5.1 holds. Then*

$$\int_{-\infty}^{\tau} e^{\nu^*\lambda_1\xi}\left[\|\mathfrak{g}_1(\xi)\|_{\mathbb{H}^{-1}}^2+\|\mathfrak{g}_2(\xi)\|_{\mathbb{W}^{-1,\frac{4}{3}}}^{\frac{4(4+d)}{8+3d}}\right]\mathrm{d}\xi<\infty, \quad \textit{for all } \tau\in\mathbb{R}. \tag{5.4}$$

Remark 5.1. The condition (5.2) does not need $\mathfrak{g}$ to be bounded in $\mathbb{H}^{-1}(\Omega)+\mathbb{W}^{-1,\frac{4}{3}}(\Omega)$ at $\pm\infty$ (cf. [52]). For instance, for any $p\geq 0$ and $\mathfrak{g}^*\in\mathbb{H}^{-1}(\Omega)+\mathbb{W}^{-1,\frac{4}{3}}(\Omega)$, the function $\mathfrak{g}(\cdot,t)=t^p\mathfrak{g}^*$ satisfies both (5.4) and (5.2).

Lemma 5.3. *Assume that* (1.4) *and Hypothesis 5.1 are satisfied. Then, for every $\tau\in\mathbb{R}$ and $\hat{K}=\{K(\tau):\tau\in\mathbb{R}\}\in\mathfrak{D}$, and for all $s\in[\tau-1,\tau]$, there exists $\mathscr{T}=\mathscr{T}(\tau,\hat{K})>0$ and a constant $C_{bd}>0$ such that for all $t\geq\mathscr{T}$,*

$$\|\mathfrak{q}(s;\tau-t,\mathfrak{q}_*)\|_2^2\leq C_{bd}\int_{-\infty}^{\tau} e^{-\nu^*\lambda_1(\tau-\xi)}\left[\|\mathfrak{g}_1(\xi)\|_{\mathbb{H}^{-1}}^2+\|\mathfrak{g}_2(\xi)\|_{\mathbb{W}^{-1,\frac{4}{3}}}^{\frac{4(4+d)}{8+3d}}+1\right]\mathrm{d}\xi, \tag{5.5}$$

where $\mathfrak{q}_*\in K(\tau-t)$.

Proof. Let us take $\mathfrak{q}(\cdot)=\mathfrak{q}(\cdot;\tau-t,\mathfrak{q}_*)$. By taking the inner product with $\mathfrak{q}(\cdot)$ to the first equation to system (2.18) and keeping in mind that $\langle\mathcal{B}(\mathfrak{q}),\mathfrak{q}\rangle=0$, we get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\|\mathfrak{q}(s)\|_2^2&=-\frac{\nu}{2}\|\mathrm{A}(\mathfrak{q}(s))\|_2^2-\frac{\beta}{2}\|\mathrm{A}(\mathfrak{q}(s))\|_4^4\\&\quad+\frac{\alpha}{2}\int_\Omega(\mathrm{A}(\mathfrak{q}(s)))^2:\mathrm{A}(\mathfrak{q}(s))\mathrm{d}x+\langle\mathfrak{g}(s),\mathfrak{q}(s)\rangle,\end{aligned} \tag{5.6}$$

for a.e. $s\in[\tau-t,\tau-t+T]$ with $T>0$. Using the Hölder's inequality, (1.4) and Young's inequality, we obtain (see (3.5))

$$\begin{aligned}\left|\frac{\alpha}{2}\int_\Omega(\mathrm{A}(\mathfrak{q}))^2:\mathrm{A}(\mathfrak{q})\mathrm{d}x\right|&=\left|\frac{\alpha}{2}\int_\Omega\mathrm{Tr}[(\mathrm{A}(\mathfrak{q}))^3]\mathrm{d}x\right|\\&\leq\begin{cases}0, & d=2;\\ \frac{\nu(1-\varepsilon_0)}{4}\|\mathrm{A}(\mathfrak{q})\|_2^2+\frac{\beta(1-\varepsilon_0)}{2}\|\mathrm{A}(\mathfrak{q})\|_4^4, & d=3.\end{cases}\end{aligned} \tag{5.7}$$

Using the Cauchy–Schwarz, (5.1), (2.2), (2.3) and Young's inequalities, we obtain

$$\begin{aligned}|\langle\mathfrak{g},\mathfrak{q}\rangle|&\leq\|\mathfrak{g}_1\|_{\mathbb{V}'}\|\mathfrak{q}\|_{\mathbb{V}}+\|\mathfrak{g}_2\|_{\mathbb{W}^{-1,\frac{4}{3}}}\|\mathfrak{q}\|_{\mathbb{W}^{1,4}}\\&\leq C\|\mathfrak{g}_1\|_{\mathbb{H}^{-1}}\|\mathrm{A}(\mathfrak{q})\|_2+\|\mathfrak{g}_2\|_{\mathbb{W}^{-1,\frac{4}{3}}}\|\mathfrak{q}\|_4+\|\mathfrak{g}_2\|_{\mathbb{W}^{-1,\frac{4}{3}}}\|\nabla\mathfrak{q}\|_4\\&\leq C\|\mathfrak{g}_1\|_{\mathbb{H}^{-1}}\|\mathrm{A}(\mathfrak{q})\|_2+C\|\mathfrak{g}_2\|_{\mathbb{W}^{-1,\frac{4}{3}}}\|\nabla\mathfrak{q}\|_4^{\frac{d}{4+d}}\|\mathfrak{q}\|_2^{\frac{4}{4+d}}+\|\mathfrak{g}_2\|_{\mathbb{W}^{-1,\frac{4}{3}}}\|\nabla\mathfrak{q}\|_4\\&\leq C\|\mathfrak{g}_1\|_{\mathbb{H}^{-1}}\|\mathrm{A}(\mathfrak{q})\|_2+C\|\mathfrak{g}_2\|_{\mathbb{W}^{-1,\frac{4}{3}}}\|\mathrm{A}(\mathfrak{q})\|_4^{\frac{d}{4+d}}\|\mathrm{A}(\mathfrak{q})\|_2^{\frac{4}{4+d}}\\&\quad+C\|\mathfrak{g}_2\|_{\mathbb{W}^{-1,\frac{4}{3}}}\|\mathrm{A}(\mathfrak{q})\|_4\\&\leq\frac{\nu\varepsilon_0}{8}\|\mathrm{A}(\mathfrak{q})\|_2^2+\frac{\beta\varepsilon_0}{4}\|\mathrm{A}(\mathfrak{q})\|_4^4+C\left[\|\mathfrak{g}_1\|_{\mathbb{H}^{-1}}^2+\|\mathfrak{g}_2\|_{\mathbb{W}^{-1,\frac{4}{3}}}^{\frac{4}{3}}+\|\mathfrak{g}_2\|_{\mathbb{W}^{-1,\frac{4}{3}}}^{\frac{4(4+d)}{8+3d}}\right]\end{aligned}$$

$$\leq \frac{\nu\varepsilon_0}{8}\|\mathrm{A}(\mathfrak{q})\|_2^2 + \frac{\beta\varepsilon_0}{4}\|\mathrm{A}(\mathfrak{q})\|_4^4 + C\left[\|\mathfrak{g}_1\|_{\mathbb{H}^{-1}}^2 + \|\mathfrak{g}_2\|_{\mathbb{W}^{-1,\frac{4}{3}}}^{\frac{4(4+d)}{8+3d}} + 1\right]. \quad (5.8)$$

Combining (5.6)–(5.8), we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}s}\|\mathfrak{q}(s)\|_2^2 + \frac{\nu^*}{2}\|\mathrm{A}(\mathfrak{q}(s))\|_2^2 + \frac{\beta\varepsilon_0}{2}\|\mathrm{A}(\mathfrak{q}(s))\|_4^4 \\ &\leq C\left[\|\mathfrak{g}_1(s)\|_{\mathbb{H}^{-1}}^2 + \|\mathfrak{g}_2(s)\|_{\mathbb{W}^{-1,\frac{4}{3}}}^{\frac{4(4+d)}{8+3d}} + 1\right],\end{aligned}$$

for a.e. $s \in [\tau - t, \tau - t + T]$ with $T > 0$, which due to Poincaré inequality (5.1) implies

$$\frac{\mathrm{d}}{\mathrm{d}s}\|\mathfrak{q}(s)\|_2^2 + \nu^*\lambda_1\|\mathfrak{q}(s)\|_2^2 \leq C\left[\|\mathfrak{g}_1(s)\|_{\mathbb{H}^{-1}}^2 + \|\mathfrak{g}_2(s)\|_{\mathbb{W}^{-1,\frac{4}{3}}}^{\frac{4(4+d)}{8+3d}} + 1\right], \quad (5.9)$$

for a.e. $s \in [\tau - t, \tau - t + T]$ with $T > 0$, where $\nu^* > 0$ is a constant defined in (5.3). Now, an application of the variation of constants formula gives

$$\begin{aligned}e^{\nu^*\lambda_1 s}\|\mathfrak{q}(s;\tau - t, \mathfrak{q}_*)\|_2^2 &\leq e^{\nu^*\lambda_1(\tau-t)}\|\mathfrak{q}_*\|_2^2 + C\int_{\tau-t}^{\tau} e^{\nu^*\lambda_1\xi}\Big[\|\mathfrak{g}_1(\xi)\|_{\mathbb{H}^{-1}}^2 \\ &\quad + \|\mathfrak{g}_2(\xi)\|_{\mathbb{W}^{-1,\frac{4}{3}}}^{\frac{4(4+d)}{8+3d}} + 1\Big]\mathrm{d}\xi, \quad (5.10)\end{aligned}$$

for $t > 1$ and for all $s \in [\tau-1, \tau]$. Since $\mathfrak{q}_* \in K(\tau-t)$, we have $\|\mathfrak{q}_*\|_2^2 \leq [r_{\hat{K}}(\tau-t)]^2$. Therefore, there exists a time $\mathscr{T} = \mathscr{T}(\tau, \hat{K}) > 1$ such that for all $t \geq \mathscr{T}$,

$$e^{\nu^*\lambda_1(\tau-t)}\|\mathfrak{q}_*\|_2^2 \leq C\int_{\tau-t}^{\tau} e^{\nu^*\lambda_1\xi}\left[\|\mathfrak{g}_1(\xi)\|_{\mathbb{H}^{-1}}^2 + \|\mathfrak{g}_2(\xi)\|_{\mathbb{W}^{-1,\frac{4}{3}}}^{\frac{4(4+d)}{8+3d}} + 1\right]\mathrm{d}\xi. \quad (5.11)$$

Putting (5.11) in (5.10), we infer that there exists $\mathscr{T} = \mathscr{T}(\tau, \hat{K})$ such that for all $t \geq \mathscr{T}$,

$$\|\mathfrak{q}(s;\tau - t, \mathfrak{q}_*)\|_2^2 \leq 2Ce^{-\nu^*\lambda_1 s}\int_{\tau-t}^{\tau} e^{\nu^*\lambda_1\xi}\left[\|\mathfrak{g}_1(\xi)\|_{\mathbb{H}^{-1}}^2 + \|\mathfrak{g}_2(\xi)\|_{\mathbb{W}^{-1,\frac{4}{3}}}^{\frac{4(4+d)}{8+3d}} + 1\right]\mathrm{d}\xi,$$

for all $s \in [\tau - 1, \tau]$. This completes the proof. □

5.1. Pullback asymptotic compactness of Ψ

Now, we prove $\mathfrak{D}$-pullback asymptotic compactness using compact Sobolev embeddings on bounded domains.

Lemma 5.4. *Assume that* (1.4) *and Hypothesis* 5.1 *are satisfied. Then for every* $\tau \in \mathbb{R}$ *and* $t > \tau$*, the solution mapping* $\mathfrak{q}(t;\tau,\cdot) : \mathbb{H} \to \mathbb{H}$ *is compact, that is, for every bounded set* B *in* $\mathbb{H}$*, the image* $\mathfrak{q}(t;\tau,B)$ *is precompact in* $\mathbb{H}$.

Proof. Consider the solution $\mathfrak{q}(s;\tau,\cdot)$ of (2.18) for $s\in[\tau,\tau+T]$, where $T>0$. Assume that the sequence $\{\mathfrak{q}_{0m}\}_{m\in\mathbb{N}}\subset B$. We know that (see the proof of Theorem 3.1)

$$\begin{aligned}\{\mathfrak{q}(\cdot;\tau,\mathfrak{q}_{0m})\}_{m\in\mathbb{N}} &\text{ is bounded in } \mathrm{L}^{\infty}(\tau,\tau+T;\mathbb{H})\cap\mathrm{L}^2(\tau,\tau+T;\mathbb{V})\\ &\cap\mathrm{L}^4(\tau,\tau+T;\mathbb{W}^{1,4}(\Omega)),\end{aligned}$$

and

$$\left\{\frac{\mathrm{d}}{\mathrm{d}t}(\mathfrak{q}(\cdot;\tau,\mathfrak{q}_{0m}))\right\}_{m\in\mathbb{N}} \text{ is bounded in } \mathrm{L}^2(\tau,\tau+T;\mathbb{V}')+\mathrm{L}^{\frac43}(\tau,\tau+T;\mathbb{W}^{-1,\frac43}(\Omega)).$$

Since $\mathrm{L}^2(\tau,\tau+T;\mathbb{V}')+\mathrm{L}^{\frac43}(\tau,\tau+T;\mathbb{W}^{-1,\frac43})\subset\mathrm{L}^{\frac43}(\tau,\tau+T;\mathbb{V}'+\mathbb{W}^{-1,\frac43})$, the above sequence is bounded in $\mathrm{L}^{\frac43}(\tau,\tau+T;\mathbb{V}'+\mathbb{W}^{-1,\frac43}(\Omega))$. Note also that $\mathbb{V}\subset\mathbb{H}\subset\mathbb{V}'+\mathbb{W}^{-1,\frac43}$ and the embedding of $\mathbb{V}\subset\mathbb{H}$ is compact. By the *Aubin-Lions compactness lemma*, there exists a subsequence (keeping as it is) and $\mathfrak{p}\in\mathrm{L}^2(\tau,\tau+T;\mathbb{H})$ such that

$$\mathfrak{q}(\cdot;\tau,\mathfrak{q}_{0m})\to\mathfrak{p}(\cdot)\ \text{ strongly in }\ \mathrm{L}^2(\tau,\tau+T;\mathbb{H}). \tag{5.12}$$

Along a further subsequence (again not relabeling), we infer from (5.12) that

$$\mathfrak{q}(s;\tau,\mathfrak{q}_{0m})\to\mathfrak{p}(s) \text{ in } \mathbb{H}\ \text{ for almost all }\ s\in(\tau,\tau+T). \tag{5.13}$$

Since $\tau<t<\tau+T$, we obtain from (5.13) that there exists $s\in(\tau,t)$ such that (5.13) holds true for this particular s. Then by Lemma 4.1, we obtain

$$\mathfrak{q}(t;\tau,\mathfrak{q}_{0m})=\mathfrak{q}(t;s,\mathfrak{q}(s;\tau,\mathfrak{q}_{0m}))\to\mathfrak{q}(t;s,\mathfrak{p}(s))\quad\text{in}\quad\mathbb{H}.$$

This completes the proof. □

Interestingly, Lemma 5.4 helps us to prove the $\mathfrak{D}$-pullback asymptotic compactness of Ψ in $\mathbb{H}$ on bounded domains.

Corollary 5.5. *Assume that* (1.4) *and Hypothesis 5.1 are satisfied. Then for every* $\tau\in\mathbb{R}$, $\hat{K}=\{K(\tau):\tau\in\mathbb{R}\}\in\mathfrak{D}$ *and* $t_m\to\infty$, $\mathfrak{q}_{0m}\in K(\tau-t_m)$, *the sequence* $\mathfrak{q}(\tau;\tau-t_m,\mathfrak{q}_{0m})$ *of solutions to system* (2.18) *has a convergent subsequence in* $\mathbb{H}$.

Proof. From Lemma 5.3 with $s=\tau-1$, we have that there exists $\mathscr{T}=\mathscr{T}(\tau,\hat{K})>0$ such that for all $t\geq\mathscr{T}$ and $\mathfrak{q}_*\in K(\tau-t)$,

$$\mathfrak{q}(\tau-1;\tau-t,\mathfrak{q}_*)\in\mathbb{H}. \tag{5.14}$$

Since $t_m\to\infty$ and $\mathfrak{q}_{0m}\in K(\tau-t_m)$, from (5.14), we infer that there exists $M_1=M_1(\tau,\hat{K})\in\mathbb{N}$ such that the sequence

$$\{\mathfrak{q}(\tau-1;\tau-t_m,\mathfrak{q}_{0m})\}_{m\geq M_1}\subset\mathbb{H}, \tag{5.15}$$

is bounded. Hence, by (5.15) and Lemma 5.4, we conclude that the sequence

$$\{\mathfrak{q}(\tau;\tau-t_m,\mathfrak{q}_{0m})\}_{m\geq M_1}=\{\mathfrak{q}(\tau;\tau-1,\mathfrak{q}(\tau-1;\tau-t_m,\mathfrak{q}_{0m}))\}_{m\geq M_1}$$

has a convergent subsequence in $\mathbb{H}$, which completes the proof. □

5.2. Existence of a unique global $\mathcal{D}$-pullback attractor

In this subsection, we first give the result on the existence of a $\mathcal{D}$-pullback absorbing set in $\mathbb{H}$ for Ψ. Then, we prove the existence of a unique global $\mathcal{D}$-pullback attractor for Ψ.

Lemma 5.6. *Assume that* (1.4) *and Hypothesis* 5.1 *are satisfied. Then, there exists a closed* $\mathcal{D}$*-pullback absorbing set* $\mathcal{K}_{bd} = \{\mathcal{K}_{bd}(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}$ *for* Ψ.

Proof. Let us denote, for given $\tau \in \mathbb{R}$,

$$\mathcal{K}_{bd}(\tau) = \{\mathfrak{q} \in \mathbb{H} : \|\mathfrak{q}\|_2^2 \leq \mathcal{M}_{bd}(\tau)\},$$

where

$$\mathcal{M}_{bd}(\tau) = C_{bd} \int_{-\infty}^{\tau} e^{-\nu^*\lambda_1(\tau-\xi)} \left[\|\mathfrak{g}_1(\xi)\|_{\mathbb{H}^{-1}}^2 + \|\mathfrak{g}_2(\xi)\|_{\mathbb{W}^{-1,\frac{4}{3}}}^{\frac{4(4+d)}{8+3d}} + 1\right] \mathrm{d}\xi.$$

From Lemma 5.3, we have that for each $\tau \in \mathbb{R}$ and $\hat{K} \in \mathcal{D}$, there exists $\mathscr{T} = \mathscr{T}(\tau, \hat{K}) > 0$ such that for all $t \geq \mathscr{T}$,

$$\Psi(t, \tau - t, K(\tau - t)) = \mathfrak{q}(\tau; \tau - t, K(\tau - t)) \subseteq \mathcal{K}_{bd}(\tau). \tag{5.16}$$

Now, in order to complete the proof, we only need to prove that $\mathcal{K}_{bd} \in \mathcal{D}$, that is, for every $c > 0$ and $\tau \in \mathbb{R}$

$$\lim_{t\to-\infty} e^{ct}\mathcal{M}_{bd}(t) = 0.$$

For every given $c > 0$ and $\tau \in \mathbb{R}$,

$$\begin{aligned}\lim_{t\to-\infty} e^{ct}\mathcal{M}_{bd}(t) &= \lim_{t\to-\infty} e^{ct} C_{bd} \int_{-\infty}^{t} e^{-\nu^*\lambda_1(t-\xi)} \left[\|\mathfrak{g}_1(\xi)\|_{\mathbb{H}^{-1}}^2 + \|\mathfrak{g}_2(\xi)\|_{\mathbb{W}^{-1,\frac{4}{3}}}^{\frac{4(4+d)}{8+3d}} + 1\right] \mathrm{d}\xi \\ &= C_{bd} \lim_{t\to-\infty} e^{ct} \int_{-\infty}^{0} e^{\nu^*\lambda_1\xi} \left[\|\mathfrak{g}_1(\xi+t)\|_{\mathbb{H}^{-1}}^2 + \|\mathfrak{g}_2(\xi+t)\|_{\mathbb{W}^{-1,\frac{4}{3}}}^{\frac{4(4+d)}{8+3d}} + 1\right] \mathrm{d}\xi = 0,\end{aligned}$$

where we have used (5.2) in final equality. This completes the proof. □

Remark 5.2. We address an abstract result for pullback attractors in this remark that was developed in [11] (see to [11, Theorem 7 and Remark 14]). Let $\mathcal{D}$ be a class of nonempty subsets of $\mathbb{H}$. There is a minimal global $\mathcal{D}$-pullback attractor for Ψ if the ϑ-cocycle Ψ is continuous and $\mathcal{D}$-pullback asymptotically compact, and there exists $\mathcal{K} \in \mathcal{D}$ which is $\mathcal{D}$-pullback absorbing (see [11, Theorem 7]). Furthermore, this global $\mathcal{D}$-pullback attractor is unique if $\mathcal{K}(\tau)$ is closed for every $\tau \in \mathbb{R}$ and $\mathcal{D}$ is inclusion-closed (see [11, Remark 14]).

Finally, in view of the abstract result introduced in [11], we obtain the main result of this section.

Theorem 5.7. *Assume that* (1.4) *and Hypothesis 5.1 are satisfied. Then, there exists a unique global* $\mathcal{D}$*-pullback attractor for the continuous* ϑ*-cocycle* Ψ *associated with system* (2.18).

Proof. Corollary 5.5 indicates that Ψ is $\mathcal{D}$-pullback asymptotically compact. Furthermore, for all $\tau \in \mathbb{R}$, $\mathcal{K}_{bd}(\tau)$ is a closed $\mathcal{D}$-pullback absorbing set of Ψ in $\mathbb{H}$, as demonstrated by Lemma 5.6. The definition of $\mathcal{D}$ makes it evident that it is inclusion-closed. We therefore deduce that there is a unique global $\mathcal{D}$-pullback attractor for the continuous ϑ-cocycle Ψ associated with system (2.18) based on the abstract result obtained in [11] (see Remark 5.2).

6. Global pullback attractors: Unbounded Poincaré domains

We prove the existence of a unique global pullback attractor on unbounded Poincaré domains in this section. When we refer to an unbounded Poincaré domain, we mean one that satisfies the Poincaré inequality (see inequality (6.1)). $\Omega = \mathbb{R} \times (-L, L)$ and $\Omega = \mathbb{R}^2 \times (-L, L)$ with $L > 0$ are common examples of an unbounded Poincaré domain in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively; see [50, p.306] and [45, p. 117]. More precisely, we consider the following hypothesis on the domain Ω:

Hypothesis 6.1. Let Ω be an open, connected and unbounded subset of $\mathbb{R}^d$, the boundary of which is uniformly of class C^3 (see [28]). We assume that, there exists a positive constant λ such that the following Poincaré inequality is satisfied:

$$\lambda \int_{\Omega} |\psi(x)|^2 \mathrm{d}x \leq \int_{\Omega} |\nabla \psi(x)|^2 \mathrm{d}x, \quad \text{for all } \psi \in \mathbb{H}_0^1(\Omega). \tag{6.1}$$

In order to prove the result of this subsection, we need the following hypotheses on non-autonomous forcing term $\mathfrak{g}(\cdot, \cdot)$.

Hypothesis 6.2. Let $\mathfrak{g} \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}; \mathbb{L}^2(\Omega))$ and satisfies the following condition:

$$\lim_{t \to -\infty} e^{ct} \int_{-\infty}^{0} e^{\nu^* \lambda s} \|\mathfrak{g}(s+t)\|_2^2 \mathrm{d}s = 0, \tag{6.2}$$

where $\nu^* > 0$ is a constant given in (5.3) and $\lambda > 0$ is a constant given in Hypothesis 6.1.

Hypothesis 6.3. Let $\mathfrak{g} \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}; \mathbb{L}^2(\Omega))$ and there exists $\delta \in [0, \frac{\nu^* \lambda}{2})$ such that

$$\int_{-\infty}^{\tau} e^{\delta s} \|\mathfrak{g}(s)\|_2^2 \mathrm{d}s < \infty, \quad \text{for all } \tau \in \mathbb{R}. \tag{6.3}$$

Remark 6.1. It is worth mentioning here that we require Hypotheses 6.2 and 6.3 to obtain uniform-tail estimates satisfied by the solutions to system (2.1) (see Lemmas 6.6 and 6.8) which helps us to obtain $\mathcal{D}$-pullback asymptotic compactness of Ψ. Also, note that Hypothesis 5.1 is enough if one uses energy equality method given by [3] to show $\mathcal{D}$-pullback asymptotic compactness of Ψ (see the articles [11] and [32] for Navier–Stokes equations and convective Brinkman-Forchheimer equations, respectively). Unfortunately, for third-grade fluid equations (2.1), we are not able to use energy equality method.

A direct consequence of above Hypothesis 6.2 is as follows:

Proposition 6.4. (Proposition 4.2, [32]) *Assume that Hypothesis 6.2 holds. Then*

$$\int_{-\infty}^{\tau} e^{\nu^*\lambda s}\|\mathfrak{g}(s)\|_2^2 \mathrm{d}s < \infty, \quad \text{for all } \tau \in \mathbb{R}, \tag{6.4}$$

Moreover, (6.3) *implies that*

$$\lim_{k\to\infty}\int_{-\infty}^{\tau}\int_{\Omega\cap\{|x|\geq k\}} e^{\nu^*\lambda s}|\mathfrak{g}(x, s+\tau)|^2 \mathrm{d}x\mathrm{d}s = 0, \quad \textit{for all} \ \ \tau \in \mathbb{R}. \tag{6.5}$$

Lemma 6.5. *Assume that* (1.4)*, and Hypotheses* 6.1 *and* 6.2 *are satisfied. Then, for every* $\tau \in \mathbb{R}$ *and* $\hat{K} = \{K(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}$*, and for all* $s \in [\tau-1, \tau]$*, there exists* $\mathcal{T}_1 = \mathcal{T}_1(\tau, \hat{K}) > 0$ *and a constant* $C_{ubd} > 0$ *such that for all* $t \geq \mathcal{T}_1$*,*

$$\begin{aligned} &\|\mathfrak{q}(s; \tau-t, \mathfrak{q}_*)\|_2^2 + \int_{\tau-t}^{s} e^{\nu^*\lambda(s-\tau)}\|\mathrm{A}(\mathfrak{q}(s; \tau-t, \mathfrak{q}_*))\|_4^4 \mathrm{d}s \\ &\leq C_{ubd}\int_{-\infty}^{\tau} e^{-\nu^*\lambda(\tau-\xi)}\|\mathfrak{g}(\xi)\|_2^2 \mathrm{d}\xi, \end{aligned} \tag{6.6}$$

where $\mathfrak{q}_* \in K(\tau - t)$.

Proof. Let us take $\mathfrak{q}(\cdot) = \mathfrak{q}(\cdot; \tau-t, \mathfrak{q}_*)$. By taking the inner product with $\mathfrak{q}(\cdot)$ to the first equation to system (2.18) and keeping in mind that $\langle \mathcal{B}(\mathfrak{q}), \mathfrak{q}\rangle = 0$, we get

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\|\mathfrak{q}(s)\|_2^2 &= -\frac{\nu}{2}\|\mathrm{A}(\mathfrak{q}(s))\|_2^2 - \frac{\beta}{2}\|\mathrm{A}(\mathfrak{q}(s))\|_4^4 \\ &\quad + \frac{\alpha}{2}\int_\Omega (\mathrm{A}(\mathfrak{q}(s)))^2 : \mathrm{A}(\mathfrak{q}(s))\mathrm{d}x + (\mathfrak{g}(s), \mathfrak{q}(s)), \end{aligned} \tag{6.7}$$

for a.e. $s \in [\tau-t, \tau-t+T]$ with $T > 0$. Using the Hölder's inequality, (1.4) and Young's inequality (see (3.5)), we obtain

$$\begin{aligned} \left|\frac{\alpha}{2}\int_\Omega (\mathrm{A}(\mathfrak{q}))^2 : \mathrm{A}(\mathfrak{q})\mathrm{d}x\right| &= \left|\frac{\alpha}{2}\int_\Omega \mathrm{Tr}[(\mathrm{A}(\mathfrak{q}))^3]\mathrm{d}x\right| \\ &\leq \begin{cases} 0, & for\ d=2; \\ \frac{\nu(1-\varepsilon_0)}{4}\|\mathrm{A}(\mathfrak{q})\|_2^2 + \frac{\beta(1-\varepsilon_0)}{2}\|\mathrm{A}(\mathfrak{q})\|_4^4, & for\ d=3. \end{cases} \end{aligned} \tag{6.8}$$

Using the Cauchy–Schwarz, (6.1), (2.2), (2.3) and Young's inequalities, we obtain

$$|(\mathfrak{g}, \mathfrak{q})| \leq \|\mathfrak{g}\|_2\|\mathfrak{q}\|_2 \leq C\|\mathfrak{g}\|_2\|\mathrm{A}(\mathfrak{q})\|_2 \leq \frac{\nu\varepsilon_0}{8}\|\mathrm{A}(\mathfrak{q})\|_2^2 + C\|\mathfrak{g}\|_2^2. \tag{6.9}$$

Combining (6.7)–(6.9), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}s}\|\mathfrak{q}(s)\|_2^2 + \frac{\nu^*}{2}\|\mathrm{A}(\mathfrak{q}(s))\|_2^2 + \beta\varepsilon_0\|\mathrm{A}(\mathfrak{q}(s))\|_4^4 \leq C\|\mathfrak{g}(s)\|_2^2, \tag{6.10}$$

for a.e. $s \in [\tau - t, \tau - t + T]$ with $T > 0$, which due to Poincaré inequality (6.1) implies

$$\frac{\mathrm{d}}{\mathrm{d}s}\|\mathfrak{q}(s)\|_2^2 + \nu^*\lambda\|\mathfrak{q}(s)\|_2^2 + \beta\varepsilon_0\|\mathrm{A}(\mathfrak{q}(s))\|_4^4 \le C\|\mathfrak{g}(s)\|_2^2, \tag{6.11}$$

for a.e. $s \in [\tau - t, \tau - t + T]$ with $T > 0$, where $\nu^* > 0$ is a constant defined in Hypothesis 6.2. Now, an application of the variation of constants formula gives

$$\begin{aligned} &e^{\nu^*\lambda\xi}\|\mathfrak{q}(\xi; \tau - t, \mathfrak{q}_*)\|_2^2 + \beta\varepsilon_0\int_{\tau-t}^{\xi} e^{\nu^*\lambda s}\|\mathrm{A}(\mathfrak{q}(s; \tau - t, \mathfrak{q}_*))\|_4^4\mathrm{d}s \\ &\le e^{\nu^*\lambda(\tau-t)}\|\mathfrak{q}_*\|_2^2 + C\int_{\tau-t}^{\xi} e^{\nu^*\lambda s}\|\mathfrak{g}(s)\|_2^2\mathrm{d}s, \end{aligned} \tag{6.12}$$

for all $\tau \ge \xi > \tau - t$. In particular, for $t > 1$ and for all $s \in [\tau - 1, \tau]$, we have

$$\begin{aligned} &e^{\nu^*\lambda s}\|\mathfrak{q}(s; \tau - t, \mathfrak{q}_*)\|_2^2 + \beta\varepsilon_0\int_{\tau-t}^{s} e^{\nu^*\lambda\xi}\|\mathrm{A}(\mathfrak{q}(\xi; \tau - t, \mathfrak{q}_*))\|_4^4\mathrm{d}\xi \\ &\le e^{\nu^*\lambda(\tau-t)}\|\mathfrak{q}_*\|_2^2 + C\int_{-\infty}^{\tau} e^{\nu^*\lambda\xi}\|\mathfrak{g}(\xi)\|_2^2\mathrm{d}\xi. \end{aligned} \tag{6.13}$$

Since $\mathfrak{q}_* \in K(\tau - t)$, we have $\|\mathfrak{q}_*\|_2^2 \le [r_{\hat{K}}(\tau - t)]^2$. Therefore, there exists a time $\mathcal{T}_1 = \mathcal{T}_1(\tau, \hat{K}) > 1$ such that for all $t \ge \mathcal{T}_1$,

$$e^{-\nu^*\lambda(\tau-t)}\|\mathfrak{q}_*\|_2^2 \le C\int_{-\infty}^{\tau} e^{\nu^*\lambda\xi}\|\mathfrak{g}(\xi)\|_2^2\mathrm{d}\xi. \tag{6.14}$$

Putting (6.14) in (6.13), we infer that there exists $\mathcal{T}_1 = \mathcal{T}_1(\tau, \hat{K})$ such that for all $t \ge \mathcal{T}_1$,

$$\begin{aligned} &\|\mathfrak{q}(s; \tau - t, \mathfrak{q}_*)\|_2^2 + \beta\varepsilon_0\int_{\tau-t}^{s} e^{\nu^*\lambda(\xi-\tau)}\|\mathrm{A}(\mathfrak{q}(\xi; \tau - t, \mathfrak{q}_*))\|_4^4\mathrm{d}\xi \\ &\le 2Ce^{-\nu^*\lambda\tau}\int_{-\infty}^{\tau} e^{\nu^*\lambda\xi}\|\mathfrak{g}(\xi)\|_2^2\mathrm{d}\xi, \end{aligned}$$

for all $s \in [\tau - 1, \tau]$, which completes the proof. □

6.1. Pullback asymptotic compactness of Ψ

The following result is needed to obtain the uniform-tail estimate satisfied by the solutions to system (2.1) (see the proof of Lemma 6.8).

Lemma 6.6. *Assume that* (1.4)*, and Hypotheses* 6.1 *and* 6.3 *are satisfied. Then, for every* $\tau \in \mathbb{R}$ *and* $\hat{K} = \{K(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}$*, there exists* $\mathcal{T}_2 = \mathcal{T}_2(\tau, \hat{K}) > 0$ *and a constant* $C_1 > 0$ *such that for all* $t \ge \mathcal{T}_2$,

$$\int_{\tau-t}^{\tau} e^{\nu^*\lambda(s-\tau)}\|\mathfrak{q}(s; \tau - t, \mathfrak{q}_*)\|_2^4\mathrm{d}s \le C_1\left[\int_{-\infty}^{\tau} e^{\delta(s-\tau)}\|\mathfrak{g}(s)\|_2^2\mathrm{d}s\right]^2 < +\infty, \tag{6.15}$$

and there exists $\mathcal{T}_3 = \mathcal{T}_3(\tau, \hat{K}) \geq \mathcal{T}_2$ *and a constant* $C_2 > 0$ *such that for all* $t \geq \mathcal{T}_3$,

$$\begin{aligned}
&\int_{\tau-t}^{\tau} e^{\nu^*\lambda(s-\tau)} \|\mathrm{A}(\mathfrak{q}(s; \tau-t, \mathfrak{q}_*))\|_2^2 \mathrm{d}s \\
&\leq C_2 \int_{-\infty}^{\tau} e^{\nu^*\lambda(s-\tau)} \left[\|\mathfrak{g}(s)\|_2^2 + 1\right] \mathrm{d}s + C_2 \left[\int_{-\infty}^{\tau} e^{\delta(s-\tau)} \|\mathfrak{g}(s)\|_2^2 \mathrm{d}s\right]^2 < +\infty,
\end{aligned} \tag{6.16}$$

where $\mathfrak{q}_* \in K(\tau - t)$ *and* $\delta > 0$ *is the constant same as in Hypothesis 6.2.*

Proof. We know from (6.12) that

$$e^{\nu^*\lambda\xi} \|\mathfrak{q}(\xi; \tau-t, \mathfrak{q}_*)\|_2^2 \leq e^{\nu^*\lambda(\tau-t)} \|\mathfrak{q}_*\|_2^2 + C \int_{\tau-t}^{\xi} e^{\nu^*\lambda s} \|\mathfrak{g}(s)\|_2^2 \mathrm{d}s, \tag{6.17}$$

for all $\tau \geq \xi > \tau - t$, which leads to

$$\begin{aligned}
&\int_{\tau-t}^{\tau} e^{\nu^*\lambda\xi} \|\mathfrak{q}(\xi; \tau-t, \mathfrak{q}_*)\|_2^4 \mathrm{d}\xi \\
&\leq 2 \int_{\tau-t}^{\tau} \left\{ e^{-\nu^*\lambda\xi} e^{2\nu^*\lambda(\tau-t)} \|\mathfrak{q}_*\|_2^4 + e^{-\nu^*\lambda\xi} \left[C \int_{\tau-t}^{\xi} e^{\nu^*\lambda s} \|\mathfrak{g}(s)\|_2^2 \mathrm{d}s\right]^2 \right\} \mathrm{d}\xi \\
&\leq 2 \int_{\tau-t}^{\tau} \left\{ e^{\nu^*\lambda(\tau-t)} \|\mathfrak{q}_*\|_2^4 + \left[C \int_{\tau-t}^{\xi} e^{\frac{\nu^*\lambda}{2}s} \|\mathfrak{g}(s)\|_2^2 \mathrm{d}s\right]^2 \right\} \mathrm{d}\xi \\
&\leq 2 \int_{\tau-t}^{\tau} e^{(\nu^*\lambda-2\delta)\xi} \mathrm{d}\xi \left\{ e^{2\delta(\tau-t)} \|\mathfrak{q}_*\|_2^4 + \left[C \int_{-\infty}^{\tau} e^{\delta s} \|\mathfrak{g}(s)\|_2^2 \mathrm{d}s\right]^2 \right\},
\end{aligned} \tag{6.18}$$

where δ is same as Hypothesis 6.2. Since $\mathfrak{q}_* \in K(\tau-t)$, we have $\|\mathfrak{q}_*\|_2^2 \leq [r_{\hat{K}}(\tau-t)]^2$. Therefore, there exists a time $\mathcal{T}_2 = \mathcal{T}_2(\tau, \hat{K})$ such that for all $t \geq \mathcal{T}_2$,

$$e^{\delta(\tau-t)} \|\mathfrak{q}_*\|_2^2 \leq C \int_{-\infty}^{\tau} e^{\delta s} \|\mathfrak{g}(s)\|_2^2 \mathrm{d}s. \tag{6.19}$$

Putting (6.19) in (6.18), we infer that there exists $\mathcal{T}_2 = \mathcal{T}_2(\tau, \hat{K})$ such that for all $t \geq \mathcal{T}_2$,

$$\int_{\tau-t}^{\tau} e^{\nu^*\lambda s} \|\mathfrak{q}(s; \tau-t, \mathfrak{q}_*)\|_2^4 \mathrm{d}s \leq C \left[\int_{-\infty}^{\tau} e^{\delta s} \|\mathfrak{g}(s)\|_2^2 \mathrm{d}s\right]^2, \tag{6.20}$$

which is finite by Hypothesis 6.3.

Next, from (6.10), we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s} \|\mathfrak{q}(s)\|_2^2 + \nu^*\lambda \|\mathfrak{q}(s)\|_2^2 + \frac{\nu^*}{2} \|\mathrm{A}(\mathfrak{q}(s))\|_2^2 &\leq C \|\mathfrak{g}(s)\|_2^2 + \nu^*\lambda \|\mathfrak{q}(s)\|_2^2 \\
&\leq C\Big[\|\mathfrak{g}(s)\|_2^2 + \|\mathfrak{q}(s)\|_2^4 + 1\Big],
\end{aligned}$$

for a.e. $s \in [\tau - t, \tau - t + T]$ with $T > 0$. Again, an application of the variation of constants formula implies

$$\begin{aligned} &\frac{\nu^*}{2}\int_{\tau-t}^{\tau} e^{\nu^*\lambda s}\|\mathrm{A}(\mathfrak{q}(s;\tau-t,\mathfrak{q}_*))\|_2^2\mathrm{d}s \\ &\quad\leq e^{\nu^*\lambda(\tau-t)}\|\mathfrak{q}_*\|_2^2 + C\int_{\tau-t}^{\tau} e^{\nu^*\lambda s}\left[\|\mathfrak{g}(s)\|_2^2+1\right]\mathrm{d}s \\ &\quad\quad + \int_{\tau-t}^{\tau} e^{\nu^*\lambda s}\|\mathfrak{q}(s;\tau-t,\mathfrak{q}_*)\|_2^4\mathrm{d}s. \end{aligned} \tag{6.21}$$

Since $\mathfrak{q}_* \in K(\tau - t)$, we have $\|\mathfrak{q}_*\|_2^2 \leq [r_{\hat{K}}(\tau - t)]^2$. Therefore, there exists a time $\mathcal{T}_3 = \mathcal{T}_3(\tau, \hat{K}) \geq \mathcal{T}_2$ such that for all $t \geq \mathcal{T}_3$,

$$e^{\nu^*\lambda(\tau-t)}\|\mathfrak{q}_*\|_2^2 \leq C\int_{-\infty}^{\tau} e^{\nu^*\lambda s}\left[\|\mathfrak{g}(s)\|_2^2+1\right]\mathrm{d}s. \tag{6.22}$$

In view of (6.20), (6.21) and (6.22), we infer that there exists $\mathcal{T}_2 = \mathcal{T}_2(\tau, \hat{K})$ such that for all $t \geq \mathcal{T}_2$,

$$\begin{aligned} \frac{\nu^*}{2}\int_{\tau-t}^{\tau} e^{\nu^*\lambda s}\|\mathrm{A}(\mathfrak{q}(s;\tau-t,\mathfrak{q}_*))\|_2^2\mathrm{d}s &\leq C\int_{-\infty}^{\tau} e^{\nu^*\lambda s}\left[\|\mathfrak{g}(s)\|_2^2+1\right]\mathrm{d}s \\ &\quad + C\left[\int_{-\infty}^{\tau} e^{\delta s}\|\mathfrak{g}(s)\|_2^2\mathrm{d}s\right]^2, \end{aligned} \tag{6.23}$$

which is finite by Hypothesis 6.3. This completes the proof. □

The following result is used to obtain the pullback asymptotic compactness of Ψ.

Lemma 6.7. *Assume that* $\mathfrak{g} \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}; \mathbb{H}^{-1}(\Omega)) + \mathrm{L}^{\frac{4}{3}}_{\mathrm{loc}}(\mathbb{R}; \mathbb{W}^{-1,\frac{4}{3}}(\Omega))$ *and* $\{\mathfrak{q}_{0m}\}_{m\in\mathbb{N}}$ *be a bounded sequence in* $\mathbb{H}$*. Then, there exists* $\tilde{\mathfrak{q}} \in \mathrm{L}^\infty(\tau, \tau + T; \mathbb{H}) \cap \mathrm{L}^2(\tau, \tau + T; \mathbb{V}) \cap \mathrm{L}^4(\tau, \tau + T; \mathbb{W}^{1,4}(\Omega))$ *such that along a subsequence*

$$\mathfrak{q}(\cdot;\tau,\mathfrak{q}_{0m}) \to \tilde{\mathfrak{q}} \ \text{ in } \ \mathrm{L}^2(\tau, \tau+T; \mathbb{L}^2_{\mathrm{loc}}(\Omega)), \tag{6.24}$$

for every $T > 0$.

Proof. Let $\mathfrak{q}_m(\cdot) = \mathfrak{q}(\cdot; \tau, \mathfrak{q}_{0m})$. Since $\{\mathfrak{q}_{0m}\}_{m\in\mathbb{N}}$ is a bounded sequence in $\mathbb{H}$, the sequence

$$\begin{aligned}&\{\mathfrak{q}_m\}_{n\in\mathbb{N}} \text{ is bounded in } \mathrm{L}^{\infty}(\tau,\tau+T;\mathbb{H})\cap\mathrm{L}^2(\tau,\tau+T;\mathbb{V})\\&\quad\cap\mathrm{L}^4(\tau,\tau+T;\mathbb{W}^{1,4}(\Omega)).\end{aligned}\tag{6.25}$$

Hence, there exists a subsequence $\{\mathfrak{q}_{m'}\}_{m'\in\mathbb{N}}$ of $\{\mathfrak{q}_m\}_{m\in\mathbb{N}}$ and

$$\widetilde{\mathfrak{q}}\in\mathrm{L}^{\infty}(\tau,\tau+T;\mathbb{H})\cap\mathrm{L}^2(\tau,\tau+T;\mathbb{V})\cap\mathrm{L}^4(\tau,\tau+T;\mathbb{W}^{1,4}(\Omega)),$$

such that, as $m'\to\infty$, (by the Banach–Alaoglu theorem)

$$\begin{cases}\mathfrak{q}_{m'}\xrightarrow{w^*}\widetilde{\mathfrak{q}} \text{ in } \mathrm{L}^{\infty}(\tau,\tau+T;\mathbb{H});\\ \mathfrak{q}_{m'}\xrightarrow{w}\widetilde{\mathfrak{q}} \text{ in } \mathrm{L}^2(\tau,\tau+T;\mathbb{V})\cap\mathrm{L}^4(\tau,\tau+T;\mathbb{W}^{1,4}(\Omega)).\end{cases}\tag{6.26}$$

From Theorem 3.1, we have $\left\|\frac{\mathrm{d}\mathfrak{q}_m}{\mathrm{d}t}\right\|_{\mathrm{L}^{\frac{4}{3}}(\tau,\tau+T;\mathbb{V}'+\mathbb{W}^{-1,\frac{4}{3}})}\le C$, for some $C>0$ and all $m\in\mathbb{N}$. Therefore, by the Cauchy–Schwarz inequality, for all $\tau\le t\le t+a\le\tau+T$, $m\in\mathbb{N}$ and $\boldsymbol{\varphi}\in\mathbb{V}\cap\mathbb{W}^{1,4}(\Omega)$, we obtain

$$|(\mathfrak{q}_m(t+a)-\mathfrak{q}_m(t),\boldsymbol{\varphi})|\le\int_t^{t+a}\left|\left\langle\frac{\mathrm{d}\mathfrak{q}_m(s)}{\mathrm{d}t},\boldsymbol{\varphi}\right\rangle\right|\mathrm{d}s\le C(T)a^{\frac{1}{4}}\|\boldsymbol{\varphi}\|_{\mathbb{V}\cap\mathbb{W}^{1,4}}.\tag{6.27}$$

Since $\mathfrak{q}_m(t+a)-\mathfrak{q}_m(t)\in\mathbb{V}\cap\mathbb{W}^{1,4}(\Omega)$, for a.e. $t\in(\tau,\tau+T)$, choosing $\boldsymbol{\varphi}=\mathfrak{q}_m(t+a)-\mathfrak{q}_m(t)$ in (6.27), we obtain

$$\|\mathfrak{q}_m(t+a)-\mathfrak{q}_m(t)\|_2^2\le C(T)a^{\frac{1}{4}}\|\mathfrak{q}_m(t+a)-\mathfrak{q}_m(t)\|_{\mathbb{V}\cap\mathbb{W}^{1,4}}.\tag{6.28}$$

Integrating from τ to $\tau+T-a$, we further find

$$\begin{aligned}&\int_{\tau}^{\tau+T-a}\|\mathfrak{q}_m(t+a)-\mathfrak{q}_m(t)\|_2^2\mathrm{d}t\\&\quad\le C(T)a^{\frac{1}{4}}\int_{\tau}^{\tau+T-a}\|\mathfrak{q}_m(t+a)-\mathfrak{q}_m(t)\|_{\mathbb{V}\cap\mathbb{W}^{1,4}}\mathrm{d}t\\&\quad\le C(T)a^{\frac{1}{4}}\Bigg[(T-a)^{1/2}\left(\int_{\tau}^{\tau+T-a}\|\mathfrak{q}_m(t+a)-\mathfrak{q}_m(t)\|_{\mathbb{V}}^2\mathrm{d}t\right)^{1/2}\\&\qquad+(T-a)^{\frac{3}{4}}\left(\int_{\tau}^{\tau+T-a}\|\mathfrak{q}_m(t+a)-\mathfrak{q}_m(t)\|_{\mathbb{W}^{1,4}}^4\mathrm{d}t\right)^{\frac{1}{4}}\Bigg]\\&\quad\le\widetilde{C}(T)a^{\frac{1}{4}},\end{aligned}\tag{6.29}$$

where we have used the Hölder's inequality and (6.25). Also, $\widetilde{C}(T)$ is an another positive constant independent of m. Furthermore, we have

$$\lim_{a\to0}\sup_m\int_{\tau}^{\tau+T-a}\|\mathfrak{q}_m(t+a)-\mathfrak{q}_m(t)\|_2^2\mathrm{d}t=0.\tag{6.30}$$

Let us now consider a truncation function $\chi \in \mathrm{C}^1(\mathbb{R}^+)$ with $\chi(s) = 1$ for $s \in [0, 1]$ and $\chi(s) = 0$ for $s \in [4, \infty)$. For each $k > 0$, let us define

$$\mathfrak{q}_{m,k}(x) := \chi\left(\frac{|x|^2}{k^2}\right)\mathfrak{q}_m(x), \quad \text{for } x \in \Omega_{2k},$$

where $\Omega_k = \{x \in \Omega : |x| < k\}$. It can easily be seen from (6.30) that

$$\lim_{a\to 0}\sup_m \int_\tau^{\tau+T-a} \|\mathfrak{q}_{m,k}(t+a) - \mathfrak{q}_{m,k}(t)\|^2_{\mathbb{L}^2(\Omega_{2k})}\mathrm{d}t = 0, \tag{6.31}$$

for all $T, k > 0$. Moreover, from (6.25), for all $T, k > 0$, we infer

$$\{\mathfrak{q}_{m,k}\}_{m\in\mathbb{N}} \text{ is bounded in } \mathrm{L}^\infty(\tau, \tau+T; \mathbb{L}^2(\Omega_{2k})) \cap \mathrm{L}^2(\tau, \tau+T; \mathbb{H}^1_0(\Omega_{2k})). \tag{6.32}$$

Since the injection $\mathbb{H}^1_0(\Omega_{2k}) \subset \mathbb{L}^2(\Omega_{2k})$ is compact, we can apply [38, Theorem 16.3] (see [51, Theorem 13.3] also) to obtain

$$\{\mathfrak{q}_{m,k}\}_{m\in\mathbb{N}} \text{ is relatively compact in } \mathrm{L}^2(\tau, \tau+T; \mathbb{L}^2(\Omega_{2k})), \tag{6.33}$$

for all $T, k > 0$. From (6.33), we further infer

$$\{\mathfrak{q}_m\}_{m\in\mathbb{N}} \text{ is relatively compact in } \mathrm{L}^2(\tau, \tau+T; \mathbb{L}^2(\Omega_k)), \tag{6.34}$$

for all $T, k > 0$. Using the estimates (6.25) and (6.34), we can extract a subsequence of $\{\mathfrak{q}_m\}_{m\in\mathbb{N}}$ (not relabeling) such that $\mathfrak{q}_m \to \tilde{\mathfrak{q}}$ in $\mathrm{L}^2(\tau, \tau+T; \mathbb{L}^2(\Omega_k))$, for all $T, k > 0$. This completes the proof. □

Next, we show that the solution to system (2.1) satisfies the uniform-tail estimates. Let Θ be a smooth function such that $0 \leq \Theta(\xi) \leq 1$ for $\xi \in \mathbb{R}$ and

$$\Theta(\xi) = \begin{cases} 0, & \text{for } |\xi| \leq 1, \\ 1, & \text{for } |\xi| \geq 2. \end{cases} \tag{6.35}$$

Then, there exists a positive constant C^* such that $|\Theta'(\xi)| \leq C^*$ and $|\Theta''(\xi)| \leq C^*$ for all $\xi \in \mathbb{R}$.

Lemma 6.8. *Assume that* (1.4)*, and Hypotheses* 6.1 *and* 6.3 *are satisfied,* $\tau \in \mathbb{R}$ *and* $\hat{K} = \{K(\tau) : \tau \in \mathbb{R}\} \in \mathfrak{D}$*. Then, for each* $\varepsilon > 0$ *and for each* $s \in [\tau - 1, \tau]$*, there exists* $k_0 := k_0(\varepsilon, s) \in \mathbb{N}$ *such that the solution* $\mathfrak{q}(\cdot)$ *to system* (2.1) *satisfies*

$$\|\mathfrak{q}(s; \tau - t, \mathfrak{q}_*)\|^2_{\mathbb{L}^2(\Omega^c_k)} \leq \varepsilon, \tag{6.36}$$

for all $\mathfrak{q}_* \in K(\tau - t)$*, for all* $t \geq \mathcal{T} = \max\{\mathcal{T}_1, \mathcal{T}_3\}$ *and for all* $k \geq k_0$*, where* $\Omega_k = \{x \in \Omega : |x| < k\}$*,* $\Omega^c_k = \Omega\backslash\Omega_k$*, and* $\mathcal{T}_1$ *and* $\mathcal{T}_3$ *are the same time given in Lemmas* 6.5 *and* 6.6*, respectively.*

Proof. Taking the divergence of the first equation of (2.1), we obtain

$$
\begin{aligned}
-\Delta\mathbf{P} &= \nabla\cdot\Big[(\mathfrak{q}\cdot\nabla)\mathfrak{q} + \alpha \mathrm{div}((\mathrm{A}(\mathfrak{q}))^2) + \beta \mathrm{div}(|\mathrm{A}(\mathfrak{q})|^2\mathrm{A}(\mathfrak{q}))\Big] - \nabla\cdot\mathfrak{g} \\
&= \nabla\cdot\Big[\nabla\cdot\Big\{(\mathfrak{q}\otimes\mathfrak{q}) + \alpha(\mathrm{A}(\mathfrak{q}))^2\Big\}\Big] + \beta\nabla\cdot\Big[\nabla\cdot\Big\{|\mathrm{A}(\mathfrak{q})|^2\mathrm{A}(\mathfrak{q})\Big\}\Big] - \nabla\cdot\mathfrak{g} \\
&=: -\Delta\mathbf{P}_1 - \Delta\mathbf{P}_2,
\end{aligned}
$$

in the weak sense, where

$$
\begin{aligned}
-\Delta\mathbf{P}_1 &= \nabla\cdot\Big[\nabla\cdot\Big\{(\mathfrak{q}\otimes\mathfrak{q}) + \alpha(\mathrm{A}(\mathfrak{q}))^2\Big\}\Big] - \nabla\cdot\mathfrak{g} \quad \text{and} \\
-\Delta\mathbf{P}_2 &= \beta\nabla\cdot\Big[\nabla\cdot\Big\{|\mathrm{A}(\mathfrak{q})|^2\mathrm{A}(\mathfrak{q})\Big\}\Big].
\end{aligned}
$$

Note that, by Lemmas 6.5 and 6.6, we have for all $t \geq \mathfrak{T} = \max\{\mathfrak{T}_1, \mathfrak{T}_3\}$

$$
e^{\frac{\nu^*\lambda}{2}(\cdot-\tau)}\Big(\nabla\cdot\Big[\nabla\cdot\Big\{(\mathfrak{q}(\cdot)\otimes\mathfrak{q}(\cdot)) + \alpha(\mathrm{A}(\mathfrak{q}(\cdot)))^2\Big\}\Big] - \nabla\cdot\mathfrak{g}\Big) \in \mathrm{L}^2(\tau-t,\tau;\mathbb{H}^{-2}(\Omega))
$$

and

$$
e^{\frac{3\nu^*\lambda}{4}(\cdot-\tau)}\Big(\beta\nabla\cdot\Big[\nabla\cdot\Big\{|\mathrm{A}(\mathfrak{q}(\cdot))|^2\mathrm{A}(\mathfrak{q}(\cdot))\Big\}\Big]\Big) \in \mathrm{L}^{\frac{4}{3}}(\tau-t,\tau;\mathbb{W}^{-2,\frac{4}{3}}(\Omega)),
$$

which gives

$$
e^{\frac{\nu^*\lambda}{2}(\cdot-\tau)}\mathbf{P}_1(\cdot) \in \mathrm{L}^2(\tau-t,\tau;\mathbb{L}^2(\Omega)) \ \text{ and } \ e^{\frac{3\nu^*\lambda}{4}(\cdot-\tau)}\mathbf{P}_2(\cdot) \in \mathrm{L}^{\frac{4}{3}}(\tau-t,\tau;\mathbb{L}^{\frac{4}{3}}(\Omega)).
$$

Also note that an application of Poincaré inequality (6.1) allow us to write

$$
\begin{aligned}
&\lambda\int_\Omega \Theta^2\left(\frac{|x|^2}{k^2}\right)|\mathfrak{q}|^2\,\mathrm{d}x \\
&\leq \int_\Omega\left|\nabla\left(\Theta\left(\frac{|x|^2}{k^2}\right)\mathfrak{q}\right)\right|^2\mathrm{d}x \leq \frac{1}{2}\int_\Omega\left|\mathrm{A}\left(\Theta\left(\frac{|x|^2}{k^2}\right)\mathfrak{q}\right)\right|^2\mathrm{d}x \\
&= \frac{1}{2}\int_\Omega\left|\nabla\left(\Theta\left(\frac{|x|^2}{k^2}\right)\mathfrak{q}\right) + \left[\nabla\left(\Theta\left(\frac{|x|^2}{k^2}\right)\mathfrak{q}\right)\right]^T\right|^2\mathrm{d}x \\
&= \frac{1}{2}\int_\Omega\left|\frac{2}{k^2}\Theta'\left(\frac{|x|^2}{k^2}\right)x^T\mathfrak{q} + \Theta\left(\frac{|x|^2}{k^2}\right)\nabla\mathfrak{q} + \frac{2}{k^2}\Theta'\left(\frac{|x|^2}{k^2}\right)\mathfrak{q}^T x\right. \\
&\quad \left. + \Theta\left(\frac{|x|^2}{k^2}\right)(\nabla\mathfrak{q})^T\right|^2\mathrm{d}x \\
&= \frac{1}{2}\int_\Omega\left|\Theta\left(\frac{|x|^2}{k^2}\right)\mathrm{A}(\mathfrak{q}) + \frac{2}{k^2}\Theta'\left(\frac{|x|^2}{k^2}\right)x^T\mathfrak{q} + \frac{2}{k^2}\Theta'\left(\frac{|x|^2}{k^2}\right)\mathfrak{q}^T x\right|^2\mathrm{d}x \\
&\leq \frac{1}{2}\int_\Omega\Theta^2\left(\frac{|x|^2}{k^2}\right)|\mathrm{A}(\mathfrak{q})|^2\mathrm{d}x + \frac{C}{k}\int_\Omega\Big[|\mathrm{A}(\mathfrak{q})||\mathfrak{q}| + |\mathfrak{q}|^2\Big]\mathrm{d}x
\end{aligned}
$$

$$\leq \frac{1}{2}\int_{\Omega}\Theta^2\left(\frac{|x|^2}{k^2}\right)|\mathrm{A}(\mathfrak{q})|^2\mathrm{d}x + \frac{C}{k}\|\mathrm{A}(\mathfrak{q})\|_2^2,$$

which implies

$$-\frac{1}{2}\int_{\Omega}\Theta^2\left(\frac{|x|^2}{k^2}\right)|\mathrm{A}(\mathfrak{q})|^2\mathrm{d}x \leq -\lambda\int_{\Omega}\Theta^2\left(\frac{|x|^2}{k^2}\right)|\mathfrak{q}|^2\,\mathrm{d}x + \frac{C}{k}\|\mathrm{A}(\mathfrak{q})\|_2^2. \tag{6.37}$$

Taking the inner product of the first equation of (2.1) with $\Theta^2\left(\frac{|x|^2}{k^2}\right)\mathfrak{q}$ in $\mathbb{L}^2(\Omega)$, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\Theta^2\left(\frac{|x|^2}{k^2}\right)|\mathfrak{q}|^2\mathrm{d}x &= \underbrace{\nu\int_{\Omega}(\Delta\mathfrak{q})\Theta^2\left(\frac{|x|^2}{k^2}\right)\mathfrak{q}\mathrm{d}x}_{I_1(k,t)}\underbrace{-b\left(\mathfrak{q},\mathfrak{q},\Theta^2\left(\frac{|x|^2}{k^2}\right)\mathfrak{q}\right)}_{I_2(k,t)}\\ &\quad+\underbrace{\alpha\int_{\Omega}(\mathrm{div}((\mathrm{A}(\mathfrak{q}))^2))\Theta^2\left(\frac{|x|^2}{k^2}\right)\mathfrak{q}\mathrm{d}x}_{I_3(k,t)}+\underbrace{\beta\int_{\Omega}(\mathrm{div}(|\mathrm{A}(\mathfrak{q})|^2\mathrm{A}(\mathfrak{q})))\Theta^2\left(\frac{|x|^2}{k^2}\right)\mathfrak{q}\mathrm{d}x}_{I_4(k,t)}\\ &\quad\underbrace{-\int_{\Omega}(\nabla\mathbf{P})\Theta^2\left(\frac{|x|^2}{k^2}\right)\mathfrak{q}\mathrm{d}x}_{I_5(k,t)}+\underbrace{\int_{\Omega}\mathfrak{g}\Theta^2\left(\frac{|x|^2}{k^2}\right)\mathfrak{q}\mathrm{d}x}_{I_6(k,t)}.\end{aligned} \tag{6.38}$$

Let us now estimate each term on the right hand side of (6.38). Integration by parts, (1.4), and Hölder's, Gagliardo–Nirenberg's (2.2) and Young's inequalities help us to obtain

$$\begin{aligned}I_1(k,t) &= -\frac{\nu}{2}\int_{\Omega}\Theta^2\left(\frac{|x|^2}{k^2}\right)|\mathrm{A}(\mathfrak{q})|^2\mathrm{d}x - \frac{4\nu}{k^2}\int\limits_{\Omega}\Theta\left(\frac{|x|^2}{k^2}\right)\\ &\qquad\Theta'\left(\frac{|x|^2}{k^2}\right)[(x\cdot\nabla)\mathfrak{q}\cdot\mathfrak{q}+(\mathfrak{q}\cdot\nabla)\mathfrak{q}\cdot x]\mathrm{d}x\\ &\leq -\frac{\nu}{2}\int_{\Omega}\Theta^2\left(\frac{|x|^2}{k^2}\right)|\mathrm{A}(\mathfrak{q})|^2\mathrm{d}x + \frac{4\sqrt{2}\nu}{k}\\ &\qquad\int\limits_{\Omega\cap\{k\leq|x|\leq\sqrt{2}k\}}\left|\Theta'\left(\frac{|x|^2}{k^2}\right)\right||\mathfrak{q}|\,|\nabla\mathfrak{q}|\,\mathrm{d}x\\ &\leq -\frac{\nu}{2}\int_{\Omega}\Theta^2\left(\frac{|x|^2}{k^2}\right)|\mathrm{A}(\mathfrak{q})|^2\mathrm{d}x + \frac{C}{k}\int_{\Omega}|\mathfrak{q}|\,|\nabla\mathfrak{q}|\,\mathrm{d}x\\ &\leq -\frac{\nu}{2}\int_{\Omega}\Theta^2\left(\frac{|x|^2}{k^2}\right)|\mathrm{A}(\mathfrak{q})|^2\mathrm{d}x + \frac{C}{k}\left(\|\mathfrak{q}\|_2^2+\|\nabla\mathfrak{q}\|_2^2\right)\\ &\leq -\frac{\nu}{2}\int_{\Omega}\Theta^2\left(\frac{|x|^2}{k^2}\right)|\mathrm{A}(\mathfrak{q})|^2\mathrm{d}x + \frac{C}{k}\|\mathrm{A}(\mathfrak{q})\|_2^2,\end{aligned} \tag{6.39}$$

$$\begin{aligned}|I_2(k,t)| &= \left|\frac{2}{k^2}\int_{\Omega}\Theta\left(\frac{|x|^2}{k^2}\right)\Theta'\left(\frac{|x|^2}{k^2}\right)x\cdot\mathfrak{q}|\mathfrak{q}|^2\mathrm{d}x\right|\\ &= \left|\frac{2}{k^2}\int\limits_{\Omega\cap\{k\leq|x|\leq\sqrt{2}k\}}\Theta\left(\frac{|x|^2}{k^2}\right)\Theta'\left(\frac{|x|^2}{k^2}\right)x\cdot\mathfrak{q}|\mathfrak{q}|^2\mathrm{d}x\right|\end{aligned}$$

$$\leq \frac{2\sqrt{2}}{k} \int\limits_{\Omega\cap\{k\leq|x|\leq\sqrt{2}k\}} \left|\Theta'\left(\frac{|x|^2}{k^2}\right)\right| |\mathfrak{q}|^3 \mathrm{d}x \leq \frac{C}{k}\|\mathfrak{q}\|_3^3 \leq \frac{C}{k}\left[\|\mathfrak{q}\|_4^4 + \|\mathfrak{q}\|_2^2\right]$$

$$\leq \frac{C}{k}\left[\|\mathfrak{q}\|_4^4 + \|\mathrm{A}(\mathfrak{q})\|_2^2\right] \leq \frac{C}{k}\left[\|\mathrm{A}(\mathfrak{q})\|_4^4 + \|\mathfrak{q}\|_2^4 + \|\mathrm{A}(\mathfrak{q})\|_2^2\right], \tag{6.40}$$

$$I_4(k,t) = -\frac{\beta}{2}\int_\Omega \Theta^2\left(\frac{|x|^2}{k^2}\right)|\mathrm{A}(\mathfrak{q})|^4 \mathrm{d}x - \frac{4\beta}{k^2}\int_\Omega \Theta\left(\frac{|x|^2}{k^2}\right)$$

$$\Theta'\left(\frac{|x|^2}{k^2}\right)[|\mathrm{A}(\mathfrak{q})|^2\mathrm{A}(\mathfrak{q}) : (x^T\mathfrak{q})]\mathrm{d}x$$

$$= -\frac{\beta}{2}\int_\Omega \Theta^2\left(\frac{|x|^2}{k^2}\right)|\mathrm{A}(\mathfrak{q})|^4 \mathrm{d}x - \frac{4\beta}{k^2}$$

$$\int\limits_{\Omega\cap\{k\leq|x|\leq\sqrt{2}k\}} \Theta'\left(\frac{|x|^2}{k^2}\right)[|\mathrm{A}(\mathfrak{q})|^2\mathrm{A}(\mathfrak{q}) : (x^T\mathfrak{q})]\mathrm{d}x$$

$$\leq -\frac{\beta}{2}\int_\Omega \Theta^2\left(\frac{|x|^2}{k^2}\right)|\mathrm{A}(\mathfrak{q})|^4 \mathrm{d}x + \frac{C}{k}\int\limits_{\Omega\cap\{k\leq|x|\leq\sqrt{2}k\}} |\mathrm{A}(\mathfrak{q})|^3|\mathfrak{q}|\mathrm{d}x$$

$$\leq -\frac{\beta}{2}\int_\Omega \Theta^2\left(\frac{|x|^2}{k^2}\right)|\mathrm{A}(\mathfrak{q})|^4 \mathrm{d}x + \frac{C}{k}\|\mathrm{A}(\mathfrak{q})\|_4^3\|\mathfrak{q}\|_4$$

$$\leq -\frac{\beta}{2}\int_\Omega \Theta^2\left(\frac{|x|^2}{k^2}\right)|\mathrm{A}(\mathfrak{q})|^4 \mathrm{d}x + \frac{C}{k}\left[\|\mathrm{A}(\mathfrak{q})\|_4^4 + \|\mathfrak{q}\|_4^4\right]$$

$$\leq -\frac{\beta}{2}\int_\Omega \Theta^2\left(\frac{|x|^2}{k^2}\right)|\mathrm{A}(\mathfrak{q})|^4 \mathrm{d}x + \frac{C}{k}\left[\|\mathrm{A}(\mathfrak{q})\|_4^4 + \|\mathfrak{q}\|_2^4\right], \tag{6.41}$$

$$|I_3(k,t)| = \left|\frac{\alpha}{2}\int_\Omega \Theta^2\left(\frac{|x|^2}{k^2}\right)[(\mathrm{A}(\mathfrak{q}))^2 : \mathrm{A}(\mathfrak{q})]\mathrm{d}x + \frac{4\alpha}{k^2}\int_\Omega \Theta\left(\frac{|x|^2}{k^2}\right)\right.$$

$$\left.\Theta'\left(\frac{|x|^2}{k^2}\right)(\mathrm{A}(\mathfrak{q}))^2 : (x^T\mathfrak{q})\mathrm{d}x\right|$$

$$\leq \frac{\nu(1-\varepsilon_0)}{4}\int_\Omega \Theta^2\left(\frac{|x|^2}{k^2}\right)|\mathrm{A}(\mathfrak{q})|^2\mathrm{d}x + \frac{\beta(1-\varepsilon_0)}{2}\int_\Omega$$

$$\Theta^2\left(\frac{|x|^2}{k^2}\right)|\mathrm{A}(\mathfrak{q})|^4\mathrm{d}x + \frac{C}{k}\left[\|\mathrm{A}(\mathfrak{q})\|_4^4 + \|\mathfrak{q}\|_2^2\right]$$

$$\leq \frac{\nu(1-\varepsilon_0)}{4}\int_\Omega \Theta^2\left(\frac{|x|^2}{k^2}\right)|\mathrm{A}(v)|^2\mathrm{d}x + \frac{\beta(1-\varepsilon_0)}{2}\int_\Omega$$

$$\Theta^2\left(\frac{|x|^2}{k^2}\right)|\mathrm{A}(\mathfrak{q})|^4\mathrm{d}x + \frac{C}{k}\left[\|\mathrm{A}(\mathfrak{q})\|_4^4 + \|\mathrm{A}(\mathfrak{q})\|_2^2\right], \tag{6.42}$$

$$|I_5(k,t)| = \left|\frac{4}{k^2}\int_\Omega \mathbf{P}\,\Theta\left(\frac{|x|^2}{k^2}\right)\Theta'\left(\frac{|x|^2}{k^2}\right)(x\cdot\mathfrak{q})\mathrm{d}x\right|$$

$$
\begin{aligned}
&\leq \frac{C}{k} \int\limits_{\Omega\cap\{k\leq|x|\leq\sqrt{2}k\}} |\mathbf{P}|\,|\mathfrak{q}|\,\mathrm{d}x \\
&\leq \frac{C}{k}\int_\Omega \left|(-\Delta)^{-1}\nabla\cdot[\nabla\cdot(\mathfrak{q}\otimes\mathfrak{q})]\right|\,|\mathfrak{q}|\,\mathrm{d}x \\
&\quad + \frac{C}{k}\int_\Omega \left|(-\Delta)^{-1}\nabla\cdot\left[\nabla\cdot\left\{(\mathrm{A}(\mathfrak{q}))^2\right\}\right]\right|\,|\mathfrak{q}|\,\mathrm{d}x \\
&\quad + \frac{C}{k}\int_\Omega \left|(-\Delta)^{-1}\nabla\cdot\left[\nabla\cdot\left\{|\mathrm{A}(\mathfrak{q})|^2\mathrm{A}(\mathfrak{q})\right\}\right]\right|\,|\mathfrak{q}|\,\mathrm{d}x \\
&\quad + \frac{C}{k}\int_\Omega \left|(-\Delta)^{-1}\left[\nabla\cdot\mathfrak{g}\right]\right|\,|\mathfrak{q}|\,\mathrm{d}x \\
&\leq \frac{C}{k}\left[\|\mathfrak{q}\|_4^2\|\mathfrak{q}\|_2 + \|\mathrm{A}(\mathfrak{q})\|_4^2\|\mathfrak{q}\|_2 + \|\mathrm{A}(\mathfrak{q})\|_4^3\|\mathfrak{q}\|_4 + \|\mathfrak{g}\|_2\|\mathfrak{q}\|_2\right] \\
&\leq \frac{C}{k}\left[\|\mathfrak{q}\|_4^4 + \|\mathfrak{q}\|_2^2 + \|\mathrm{A}(\mathfrak{q})\|_4^4 + \|\mathfrak{g}\|_2^2\right] \\
&\leq \frac{C}{k}\left[\|\mathrm{A}(\mathfrak{q})\|_2^2 + \|\mathfrak{q}\|_2^4 + \|\mathrm{A}(\mathfrak{q})\|_4^4 + \|\mathfrak{g}\|_2^2\right],
\end{aligned} \tag{6.43}
$$

$$
|I_6(k,t)| \leq \frac{\nu\lambda\varepsilon_0}{4}\int_\Omega \Theta^2\left(\frac{|x|^2}{k^2}\right)|\mathfrak{q}|^2\mathrm{d}x + C\int_\Omega \Theta^2\left(\frac{|x|^2}{k^2}\right)|\mathfrak{g}|^2\mathrm{d}x. \tag{6.44}
$$

Combining (6.38)–(6.44) and using (6.37), we reach at

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \Theta^2\left(\frac{|x|^2}{k^2}\right)|\mathfrak{q}|^2\mathrm{d}x + \frac{\beta\varepsilon_0}{2}\int_\Omega \Theta^2\left(\frac{|x|^2}{k^2}\right)|\mathrm{A}(\mathfrak{q})|^4\mathrm{d}x \\
&\leq -\frac{\nu}{4}(1+\varepsilon_0)\int_\Omega \Theta^2\left(\frac{|x|^2}{k^2}\right)|\mathrm{A}(\mathfrak{q})|^2\mathrm{d}x + \frac{\nu\lambda\varepsilon_0}{4}\int_\Omega \Theta^2\left(\frac{|x|^2}{k^2}\right)|\mathfrak{q}|^2\mathrm{d}x \\
&\quad + C\int_\Omega \Theta^2\left(\frac{|x|^2}{k^2}\right)|\mathfrak{g}|^2\mathrm{d}x + \frac{C}{k}\left[\|\mathfrak{q}\|_2^4 + \|\mathrm{A}(\mathfrak{q})\|_2^2 + \|\mathrm{A}(\mathfrak{q})\|_4^4 + \|\mathfrak{g}\|_2^2\right] \\
&\leq -\frac{\nu\lambda}{2}(1+\varepsilon_0)\int_\Omega \Theta^2\left(\frac{|x|^2}{k^2}\right)|\mathfrak{q}|^2\mathrm{d}x + \frac{C}{k}\|\mathrm{A}(\mathfrak{q})\|_2^2 + \frac{\nu\lambda\varepsilon_0}{4}\int_\Omega \Theta^2\left(\frac{|x|^2}{k^2}\right)|\mathfrak{q}|^2\mathrm{d}x \\
&\quad + C\int_\Omega \Theta^2\left(\frac{|x|^2}{k^2}\right)|\mathfrak{g}|^2\mathrm{d}x + \frac{C}{k}\left[\|\mathfrak{q}\|_2^4 + \|\mathrm{A}(\mathfrak{q})\|_2^2 + \|\mathrm{A}(\mathfrak{q})\|_4^4 + \|\mathfrak{g}\|_2^2\right],
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \Theta^2\left(\frac{|x|^2}{k^2}\right)|\mathfrak{q}|^2\mathrm{d}x + \nu^*\lambda\int_\Omega \Theta^2\left(\frac{|x|^2}{k^2}\right)|\mathfrak{q}|^2\mathrm{d}x + \beta\varepsilon_0\int_\Omega \Theta^2\left(\frac{|x|^2}{k^2}\right)|\mathrm{A}(\mathfrak{q})|^4\mathrm{d}x \\
&\leq C\int_\Omega \Theta^2\left(\frac{|x|^2}{k^2}\right)|\mathfrak{g}|^2\mathrm{d}x + \frac{C}{k}\left[\|\mathfrak{q}\|_2^4 + \|\mathrm{A}(\mathfrak{q})\|_2^2 + \|\mathrm{A}(\mathfrak{q})\|_4^4 + \|\mathfrak{g}\|_2^2\right].
\end{aligned} \tag{6.45}
$$

Applying variation of constants formula to (6.45), we obtain for all $s \in [\tau-1,\tau]$ for $t \geq \mathfrak{T} = \max\{\mathfrak{T}_1, \mathfrak{T}_3\}$

$$
\int_\Omega \Theta^2\left(\frac{|x|^2}{k^2}\right)|\mathfrak{q}(s;\tau-t,\mathfrak{q}_*)|^2\mathrm{d}x + \beta\varepsilon_0
$$

$$\int_{\tau-t}^{s} e^{\nu^*\lambda(\xi-\tau)} \int_{\Omega} \Theta^2\left(\frac{|x|^2}{k^2}\right) |\mathrm{A}(\mathfrak{q}(\xi;\tau-t,\mathfrak{q}_*))|^4 \mathrm{d}x\mathrm{d}\xi$$
$$\leq e^{-\nu^*\lambda t} \int_{\Omega} \Theta^2\left(\frac{|x|^2}{k^2}\right) |\mathfrak{q}_*(x)|^2 \mathrm{d}x + C \int_{\tau-t}^{s} e^{\nu^*\lambda(\xi-\tau)} \int_{\Omega} \Theta^2\left(\frac{|x|^2}{k^2}\right) |\mathfrak{g}(x,\xi)|^2 \mathrm{d}x\mathrm{d}\xi$$
$$+ \frac{C}{k} \int_{\tau-t}^{s} e^{\nu^*\lambda(\xi-\tau)} \Big[\|\mathfrak{q}(\xi;\tau-t,\mathfrak{q}_*)\|_2^4 + \|\mathrm{A}(\mathfrak{q}(\xi;\tau-t,\mathfrak{q}_*))\|_2^2$$
$$+ \|\mathrm{A}(\mathfrak{q}(\xi;\tau-t,\mathfrak{q}_*))\|_4^4 + \|\mathfrak{g}(\xi)\|_2^2 \Big] \mathrm{d}\xi$$
$$\leq e^{-\nu^*\lambda t} \int_{\Omega} \Theta^2\left(\frac{|x|^2}{k^2}\right) |\mathfrak{q}_*(x)|^2 \mathrm{d}x + C \int_{\tau-t}^{\tau} e^{\nu^*\lambda(\xi-\tau)} \int_{\Omega} \Theta^2\left(\frac{|x|^2}{k^2}\right) |\mathfrak{g}(x,\xi)|^2 \mathrm{d}x\mathrm{d}\xi$$
$$+ \frac{C}{k} \int_{\tau-t}^{\tau} e^{\nu^*\lambda(\xi-\tau)} \Big[\|\mathfrak{q}(\xi;\tau-t,\mathfrak{q}_*)\|_2^4 + \|\mathrm{A}(\mathfrak{q}(\xi;\tau-t,\mathfrak{q}_*))\|_2^2$$
$$+ \|\mathrm{A}(\mathfrak{q}(\xi;\tau-t,\mathfrak{q}_*))\|_4^4 + \|\mathfrak{g}(\xi)\|_2^2 \Big] \mathrm{d}\xi \leq \int_{\Omega\cap\{|x|\geq k\}} |\mathfrak{q}_*(x)|^2 \mathrm{d}x$$
$$+ C \int_{-\infty}^{0} e^{\nu^*\lambda s} \int_{\Omega\cap\{|x|\geq k\}} |\mathfrak{g}(x,\tau+s)|^2 \mathrm{d}x\mathrm{d}s + \frac{C}{k} \to 0 \text{ as } k \to \infty, \tag{6.46}$$

where we have used the bounds obtained in (6.6) and (6.15)–(6.16), and Hypothesis 6.2. Hence, from (6.46), we conclude that for any $\varepsilon > 0$ and for any $s \in [\tau-1,\tau]$, there exists a $k_0 \in \mathbb{N}$ such that

$$\int_{\Omega\cap\{|x|\geq k\}} |\mathfrak{q}(s;\tau-t,\mathfrak{q}_*)|^2 \mathrm{d}x + \int_{\tau-t}^{s} e^{\nu^*\lambda(\xi-\tau)} \int_{\Omega\cap\{|x|\geq k\}} |\mathrm{A}(\mathfrak{q}(\xi;\tau-t,\mathfrak{q}_*))|^4 \mathrm{d}x\mathrm{d}\xi \leq \varepsilon,$$

for all $k \geq k_0$ and $t \geq \mathfrak{T}$. This completes the proof. □

The following lemma provides the $\mathfrak{D}$-pullback asymptotic compactness of Ψ.

Lemma 6.9. *Assume that* (1.4)*, and Hypotheses 6.1 and 6.3 are satisfied. Then, for every* $\tau \in \mathbb{R}$*,* $\hat{K} = \{K(\tau) : \tau \in \mathbb{R}\} \in \mathfrak{D}$*,* $t_m \to \infty$ *and* $\mathfrak{q}_{0m} \in K(\tau - t_m)$*, the sequence* $\Psi(t_m, \tau - t_m, \mathfrak{q}_{0m})$ *or* $\mathfrak{q}(\tau; \tau - t_m, \mathfrak{q}_{0m})$ *of solutions to system* (2.18) *has a convergent subsequence in* $\mathbb{H}$.

Proof. Lemma 6.5 implies that there exists $\mathfrak{T}_1 = \mathfrak{T}_1(\tau, \hat{K}) > 0$ such that for all $t \geq \mathscr{T}$ and $\mathfrak{q}_* \in K(\tau - t)$,

$$\|\mathfrak{q}(\tau-1;\tau-t,\mathfrak{q}_*)\|_2 \leq C, \tag{6.47}$$

where C is a positive constant independent of t and $\mathfrak{q}_*$. Since $t_m \to \infty$, there exists $M_2 = M_2(\tau, \hat{K}) \in \mathbb{N}$ such that $t_m \geq \mathfrak{T}_1$, for all $m \geq M_2$. Since $\mathfrak{q}_{0m} \in K(\tau - t_m)$, (6.47) implies that for all $m \geq M_2$, the sequence

$$\{\mathfrak{q}(\tau-1;\tau-t_m,\mathfrak{q}_{0m})\}_{m\geq M_2} \subset \mathbb{H} \tag{6.48}$$

is bounded in $\mathbb{H}$. We infer from (6.48) and Lemma 6.7 that there exists $s \in (\tau - 1, \tau)$, $\tilde{\mathfrak{p}} \in \mathbb{H}$ and a subsequence (not relabeling) such that for every $k \in \mathbb{N}$

$$\mathfrak{q}(s; \tau - t_m, \mathfrak{q}_{0m}) = \mathfrak{q}(s; \tau - 1, \mathfrak{q}(\tau - 1; \tau - t_m, \mathfrak{q}_{0m})) \to \tilde{\mathfrak{p}} \text{ in } \mathbb{L}^2(\Omega_k). \tag{6.49}$$

as $m \to \infty$. Therefore, we infer from the proof of Lemma 4.1 that there exists a positive constant C_{Lip} such that

$$\|\mathfrak{q}(\tau; s, \mathfrak{q}(s; \tau - t_m, \mathfrak{q}_{0m})) - \mathfrak{q}(\tau; s, \tilde{\mathfrak{p}})\|_2^2 \leq C_{Lip}\|\mathfrak{q}(s; \tau - t_m, \mathfrak{q}_{0m}) - \tilde{\mathfrak{p}}\|_2^2. \tag{6.50}$$

Let us now choose $\eta > 0$ and fix it. Since $\tilde{\mathfrak{p}} \in \mathbb{H}$, there exists $K_1 = K_1(\tau, \eta) > 0$ such that for all $k \geq K_1$,

$$\int_{\Omega \cap \{|x| \geq k\}} |\tilde{\mathfrak{p}}|^2 \mathrm{d}x < \frac{\eta^2}{6C_{Lip}}, \tag{6.51}$$

where $C_{Lip} > 0$ is a constant defined in (6.50). Also, we know from Lemma 6.8 that there exists $M_3 = M_3(s, \hat{K}, \eta) \in \mathbb{N}$ and $K_2 = K_2(s, \eta) \geq K_1$ such that for all $m \geq M_3$ and $k \geq K_2$,

$$\int_{\Omega \cap \{|x| \geq k\}} |\mathfrak{q}(s; \tau - t_m, \mathfrak{q}_{0m})|^2 \mathrm{d}x < \frac{\eta^2}{6C_{Lip}}. \tag{6.52}$$

From (6.49), we have that there exists $M_4 = M_4(\tau, \hat{K}, \eta) > M_3$ such that for all $m \geq M_4$,

$$\int_{\Omega \cap \{|x| < K_2\}} |\mathfrak{q}(s; \tau - t_m, \mathfrak{q}_{0m}) - \tilde{\mathfrak{p}}|^2 \mathrm{d}x < \frac{\eta^2}{3C_{Lip}}. \tag{6.53}$$

Finally, we infer from (6.50) that

$$\begin{aligned}
&\|\mathfrak{q}(\tau; s, \mathfrak{q}(s; \tau - t_m, \mathfrak{q}_{0m})) - \mathfrak{q}(\tau; s, \tilde{\mathfrak{p}})\|_2^2 \\
&\leq C_{Lip}\left[\int\limits_{\Omega \cap \{|x| < K_2\}} |\mathfrak{q}(s; \tau - t_m, \mathfrak{q}_{0m}) - \tilde{\mathfrak{p}}|^2 \mathrm{d}x \right. \\
&\quad \left. + \int\limits_{\Omega \cap \{|x| \geq K_2\}} |\mathfrak{q}(s; \tau - t_m, \mathfrak{q}_{0m}) - \tilde{\mathfrak{p}}|^2 \mathrm{d}x \right] \\
&\leq C_{Lip}\left[\int\limits_{\Omega \cap \{|x| < K_2\}} |\mathfrak{q}(s; \tau - t_m, \mathfrak{q}_{0m}) - \tilde{\mathfrak{p}}|^2 \mathrm{d}x \right. \\
&\quad \left. + 2\int\limits_{\Omega \cap \{|x| \geq K_2\}} (|\mathfrak{q}(s; \tau - t_m, \mathfrak{q}_{0m})|^2 + |\tilde{\mathfrak{p}}|^2) \mathrm{d}x \right] \\
&< \eta^2,
\end{aligned} \tag{6.54}$$

for every $m \geq M_3$, where we have used (6.51)–(6.53). Since $\eta > 0$ is arbitrary, we conclude the proof.

6.2. Existence of a unique global $\mathcal{D}$-pullback attractor

In this subsection, we first give the result on the existence of a $\mathcal{D}$-pullback absorbing set in $\mathbb{H}$ for Ψ. Then, we prove the existence of a unique global $\mathcal{D}$-pullback attractor for Ψ.

Lemma 6.10. *Assume that* (1.4)*, and Hypotheses* 6.1 *and* 6.2 *are satisfied. Then, there exists a closed $\mathcal{D}$-pullback absorbing set $\mathcal{K}_{ubd} = \{\mathcal{K}_{ubd}(\tau) : \tau \in \mathbb{R}\} \in \mathcal{D}$ for Ψ.*

Proof. Let us denote, for given $\tau \in \mathbb{R}$,

$$\mathcal{K}_{ubd}(\tau) = \{\mathfrak{q} \in \mathbb{H} : \|\mathfrak{q}\|_2^2 \leq \mathcal{M}_{ubd}(\tau)\},$$

where

$$\mathcal{M}_{ubd}(\tau) = C_{ubd} \int_{-\infty}^{\tau} e^{-\nu^* \lambda(\tau-\xi)} \|\mathfrak{g}(\xi)\|_2^2 \mathrm{d}\xi.$$

Using a similar argument as in the proof of Lemma 5.3, one can complete the proof (we are not repeating it here). □

Finally, in view of the abstract result introduced in [11], we obtain the main result of this section.

Theorem 6.11. *Assume that* (1.4)*, and Hypotheses* 6.1*,* 6.2 *and* 6.3 *are satisfied. Then, there exists a unique global $\mathcal{D}$-pullback attractor for the continuous ϑ-cocycle Ψ associated with system* (2.18).

Proof. In view of Lemmas 6.9 and 6.10 and Remark 5.2, one can complete the proof (see the proof of Theorem 5.7). □

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Declarations

Conflict of interest The authors declare no conflict of interest.

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Kush Kinra
Center for Mathematics and Applications (NOVA Math)
NOVA School of Science and Technology (NOVA FCT)
Caparica
Portugal
E-mail: kushkinra@gmail.com

Fernanda Cipriano
Center for Mathematics and Applications (NOVA Math) and Department of Mathematics
NOVA School of Science and Technology (NOVA FCT)
Caparica
Portugal
E-mail: cipriano@fct.unl.pt

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