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RESEARCH ARTICLE

Marginal subsemigroups and commutators in inverse semigroups

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Abstract

Marginal subgroups, introduced by P. Hall, are characteristic subgroups induced by group words. The goal of this paper is to extend the notion to inverse semigroups. Our first main result establishes that these marginal subsemigroups are full inverse subsemigroups. We then examine the special case in which the word is the commutator, showing that the induced marginal inverse subsemigroup coincides with the metacenter, which is a normal inverse subsemigroup. In the process we prove some results about commutators in inverse semigroups and in Clifford semigroups. The paper concludes with several open problems.

Keywords Marginal subgroups · Inverse semigroups · Clifford semigroups · Commutators

1 Introduction

Let $\omega(x_1, \ldots, x_n)$ be a group word. An element a of a group G is said to be *marginal for* ω if, for all $i \in \{1, \ldots, n\}$ and for all $x_i \in G$,

$$\omega(x_1, \ldots, ax_i, \ldots, x_n) = \omega(x_1, \ldots, x_i, \ldots, x_n) = \omega(x_1, \ldots, x_i a, \ldots, x_n) \ . \quad (1)$$

This paper is dedicated to Professor John Meakin

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The set $M_\omega(G)$ of all marginal elements of G is a characteristic, hence normal, subgroup of G, called the *marginal subgroup* for ω. Marginal subgroups were introduced by P. Hall in [3]. Following Hall (see, e.g., [13, p. 57]), marginal subgroups are sometimes defined asymmetrically by, say, by the set of the second equations in (1), however these definitions turn out to be equivalent.

An important example of a marginal subgroup of a group G is its *center* $Z(G) = \{a \in G : ax = xa, \forall x \in G\}$, which turns out to be the marginal subgroup for the *commutator* $[x, y] = x^{-1}y^{-1}xy$. One could, in fact, argue that the whole general theory of marginal subgroups is modeled on commutators and centers.

The only use of the notion of marginality in semigroup theory of which we are aware is the work of Moravec on what he called marginal completely regular semigroups [11]. What we do in this paper is in a different direction from Moravec's very interesting work.

In this paper we focus on inverse semigroups. Recall that a semigroup S is said to be an *inverse semigroup* if, for each $x \in S$, there exists a unique inverse $x^{-1} \in S$ satisfying $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. Standard references for inverse semigroup theory are [4, Chap. 5], [9] and [12]. Throughout this paper, we will use well known facts about inverse semigroups without explicit citation. Such facts include: the set $E(S)$ of all idempotents of S forms a commutative subsemigroup, and for all $x, y \in S$, $(x^{-1})^{-1} = x$ and $(xy)^{-1} = y^{-1}x^{-1}$.

Let S be an inverse semigroup and let $\omega(x_1, \ldots, x_n)$ be an inverse semigroup word. For $i \in \{1, \ldots, n\}$, we define an element $a \in S$ to be *i-th partial marginal for* ω if there exist idempotents $e, f \in E(S)$ such that, for all $x_i \in S$,

$$\omega(x_1, \ldots, ax_i, \ldots, x_n) = \omega(x_1, \ldots, ex_i, \ldots, x_n) \tag{2}$$

$$\omega(x_1, \ldots, x_ia, \ldots, x_n) = \omega(x_1, \ldots, x_if, \ldots, x_n)\,. \tag{3}$$

Strictly speaking, we should indicate the dependence of e and f on the index i; however, we will show shortly (Lemma 4) that e and f depend only on a and they are equal to $aa^{-1} = a^{-1}a$. The set $M^i_\omega(S)$ of all such elements a is called *i-th partial margin* for ω. The set $M_\omega(S) = \bigcap_{i=1}^n M^i_\omega(S)$ is called the *margin* for ω and its elements are said to be marginal for ω. (In the group case, the refinement of the notion of marginal element into i-th partial marginal element is due to L.-C. Kappe [5].)

Recall that a subsemigroup of a semigroup is said to be *full* if it contains all idempotents. We can now state our first main result.

Theorem 1 *Let S be an inverse semigroup and let $\omega(x_1, \ldots, x_n)$ be an inverse semigroup word. Then:*

1. *For each $i \in \{1, \ldots, n\}$, $M^i_\omega(S)$ is a full inverse subsemigroup of S;*
2. *$M_\omega(S)$ is a full inverse subsemigroup of S.*

Note that we do not claim that the marginal inverse subsemigroup $M_\omega(S)$ is normal for arbitrary inverse semigroup words; see Problem 11 in §5.

In the special case of the commutator, we are able to characterize the marginal inverse subsemigroup. Recall that the *metacenter* of an inverse semigroup S is

$$Z(S) = \{a \in S : aa^{-1}xa = axa^{-1}a, \ \forall x \in S\}\,.$$

The metacenter is a normal inverse subsemigroup of S [6, 12].

Theorem 2 *Let S be an inverse semigroup and let $\omega(x, y) = [x, y]$ be the commutator. Then*

$$M_\omega(S) = Z(S)\,.$$

Consequently, $M_\omega(S)$ is a normal inverse subsemigroup of S.

In §2, we prove Theorem 1. In §3, we examine commutator identities in inverse semigroups, especially Clifford semigroups. In §4, we prove Theorem 2. Finally in §5, we pose a few problems.

2 Marginal Inverse Subsemigroups

Throughout this section, let S be an inverse semigroup and let $\omega(x_1, \ldots, x_n)$ be an inverse semigroup word.

We start with an easy observation which follows from $E(S)$ being a commutative subsemigroup of S.

Lemma 3 *For all $e_1, \ldots, e_n \in E(S)$,*

$$\omega(e_1, \ldots, e_n) = e_1 \cdots e_n\,.$$

An element a of an inverse semigroup S is *completely regular* if $aa^{-1} = a^{-1}a$, in which case we denote the idempotent $aa^{-1} = a^{-1}a$ by a^0. We now show that i-th partial marginal elements are completely regular.

Lemma 4 *Let S be an inverse semigroup, let $\omega(x_1, \ldots, x_n)$ be an inverse semigroup word, and let $a \in M^i_\omega(S)$. Then a is completely regular and for all $x_i \in S$,*

$$\omega(x_1, \ldots, ax_i, \ldots, x_n) = \omega(x_1, \ldots, a^0x_i, \ldots, x_n) \quad \textit{and} \tag{4}$$

$$\omega(x_1, \ldots, x_ia, \ldots, x_n) = \omega(x_1, \ldots, x_ia^0, \ldots, x_n) \tag{5}$$

Proof Let $e, f \in E(S)$ be as in (2), (3). We will first prove $e = f$, then $aa^{-1} = e$ and $a^{-1}a = e$. These together imply (4), (5).

First, we show

$$\omega(x_1, \ldots, e, \ldots, x_n) = \omega(x_1, \ldots, a, \ldots, x_n) = \omega(x_1, \ldots, f, \ldots, x_n)\,. \tag{6}$$

We compute

$$\begin{aligned}
\omega(x_1, \ldots, a, \ldots, x_n) &= \omega(x_1, \ldots, aa^{-1}a, \ldots, x_n) \\
&= \omega(x_1, \ldots, ea^{-1}a, \ldots, x_n) && \text{by (2)} \\
&= \omega(x_1, \ldots, ea^{-1}ae, \ldots, x_n) \\
&= \omega(x_1, \ldots, aa^{-1}ae, \ldots, x_n) && \text{by (2)}
\end{aligned}$$

$$\begin{aligned} &= \omega(x_1, \ldots, ae, \ldots, x_n) & \\ &= \omega(x_1, \ldots, e^2, \ldots, x_n) & \text{by (2)} \\ &= \omega(x_1, \ldots, e, \ldots, x_n)\,. & \end{aligned}$$

The second equality of (6) follows by a dual calculation.

Now in (6), set each $x_i = e$ and use Lemma 3 to get $e = ef$. Dually, we also have $f = ef$ and thus $e = f$. We will now use this freely when we invoke (3).

Next, starting with Lemma 3, we compute

$$\begin{aligned} aa^{-1}e &= \omega(e, \ldots, aa^{-1}e, \ldots, e) & \\ &= \omega(e, \ldots, aa^{-1}a, \ldots, e) & \text{by (3)} \\ &= \omega(e, \ldots, a, \ldots, e) & \\ &= \omega(e, \ldots, e, \ldots, e) & \text{by (6)} \\ &= e & \text{by Lemma 3} \end{aligned}$$

Together with a dual calculation, we have shown

$$ea^{-1}a = e = aa^{-1}e\,. \tag{7}$$

Now since $a^{-1}ea \in E(S)$,

$$\begin{aligned} ea^{-1}ea &= \omega(e, \ldots, a^{-1}ea, \ldots, e) & \text{by Lemma 3} \\ &= \omega(e, \ldots, a^{-1}e^2, \ldots, e) & \text{by (3)} \\ &= \omega(e, \ldots, a^{-1}e, \ldots, e) & \\ &= \omega(e, \ldots, a^{-1}a, \ldots, e) & \text{by (3)} \\ &= ea^{-1}a & \text{by Lemma 3} \\ &= e & \text{by (7)} \end{aligned}$$

It follows that

$$ea^{-1} = ea^{-1}eaa^{-1} = ea^{-1}aa^{-1}e = ea^{-1}e\,. \tag{8}$$

Finally, we start again with Lemma 3 to get

$$\begin{aligned} aa^{-1} &= \omega(aa^{-1}, \ldots, aa^{-1}, \ldots, aa^{-1}) & \text{by Lemma 3} \\ &= \omega(aa^{-1}, \ldots, ea^{-1}, \ldots, aa^{-1}) & \text{by (2)} \\ &= \omega(aa^{-1}, \ldots, ea^{-1}e, \ldots, aa^{-1}) & \text{by (8)} \\ &= \omega(aa^{-1}, \ldots, aa^{-1}a, \ldots, aa^{-1}) & \text{by (2) and (3)} \\ &= \omega(aa^{-1}, \ldots, a, \ldots, aa^{-1}) & \\ &= \omega(aa^{-1}, \ldots, e, \ldots, aa^{-1}) & \text{by (6)} \end{aligned}$$

$$
\begin{aligned}
&= aa^{-1}e && \text{by Lemma 3}\\
&= e\,, && \text{by (7)}
\end{aligned}
$$

A dual calculation shows $a^{-1}a = e$, and this completes the proof. □

Lemma 5 *If $a \in M^i_\omega(S)$ then $a^{-1} \in M^i_\omega(S)$. In particular, for all $x_i \in S$, $i \in \{1,\dots,n\}$,*

$$
\omega(x_1,\dots,a^{-1}x_i,\dots,x_n) = \omega(x_1,\dots,a^0x_i,\dots,x_n) \quad \textit{and} \tag{9}
$$

$$
\omega(x_1,\dots,x_ia^{-1},\dots,x_n) = \omega(x_1,\dots,x_ia^0,\dots,x_n) \tag{10}
$$

Proof This follows from using (4) twice in succession along with $aa^{-1} = a^{-1}a$:

$$
\begin{aligned}
\omega(x_1,\dots,ax_i,\dots,x_n) &= \omega(x_1,\dots,aa^{-1}x_i,\dots,x_n)\\
&= \omega(x_1,\dots,a^{-1}aa^{-1}x_i,\dots,x_n)\\
&= \omega(x_1,\dots,a^{-1}x_i,\dots,x_n)\,.
\end{aligned}
$$

□

Lemma 6 *If $a \in M^i_\omega(S)$, then a centralizes $E(S)$, that is, $ae = ea$ for all $e \in E(S)$.*

Proof For all $e \in E(S)$,

$$
\begin{aligned}
ea = ea^0a = a^0ea = aa^{-1}eea &= a(ea)^{-1}ea\\
&= a\,\omega(e,\dots,(ea)^{-1}(ea),\dots,e) && \text{by Lemma 3}\\
&= a\,\omega(e,\dots,a^{-1}ea,\dots,e)\\
&= a\,\omega(e,\dots,a^{-1}ea^0,\dots,e) && \text{by (5)}\\
&= a\,\omega(e,\dots,a^{-1}e,\dots,e)\\
&= a\,\omega(e,\dots,a^0e,\dots,e) && \text{by (9)}\\
&= aa^0e && \text{by Lemma 3}\\
&= ae\,,
\end{aligned}
$$

□

We can now prove our first main theorem.

Proof of Theorem 1 Let $a, b \in M^i_\omega(S)$. By Lemma 6, $a^0b = ba^0$. We use this as follows:

$$
(ab)^0 = (ab)^{-1}(ab) = b^{-1}a^{-1}ab = b^{-1}ba^{-1}a = b^0a^0 = a^0b^0\,. \tag{11}
$$

Now for all $x_i \in S$,

$$
\omega(x_1,\dots,abx_i,\dots,x_n) = \omega(x_1,\dots,a^0bx_i,\dots,x_n) \qquad \text{by (4)}
$$

$$\begin{aligned}
&= \omega(x_1, \ldots, ba^0 x_i, \ldots, x_n) \\
&= \omega(x_1, \ldots, b^0 a^0 x_i, \ldots, x_n) && \text{by (4)} \\
&= \omega(x_1, \ldots, (ab)^0 x_i, \ldots, x_n)\,,
\end{aligned}$$

Thus (4) holds for ab, and the proof that (5) holds for ab is similar. Hence $ab \in M^i_\omega(S)$ and therefore $M^i_\omega(S)$ is a subsemigroup of S.

That $M^i_\omega(S)$ is closed under taking inverses is Lemma 5. Finally it is clear that every idempotent is i-th partial marginal, that is, $E(S) \subseteq M^i_\omega(S)$, and thus $M^i_\omega(S)$ is full. This proves part (1) of the theorem, and part (2) immediately follows. □

3 Commutator properties and identities

For the rest of this paper, we focus specifically on commutators $[x, y] = x^{-1}y^{-1}xy$ in inverse semigroups. In groups, commutators satisfy some well-known identities.

1. $[x, 1] = 1$;
2. $[x, y]^{-1} = [y, x]$;
3. $[z^{-1}xz, z^{-1}yz] = z^{-1}[x, y]z$;
4. $[x, yz] = [x, z][x, y]^z$;
5. $y^{-1}xy = x[x, y]$;
6. $[x, y^{-1}] = y[y, x]y^{-1}$;
7. $[[x, y^{-1}], z]^y[[y, z^{-1}], x]^z[[z, x^{-1}], y]^x = 1$;
8. $(xy)^2 = x^2y^2[y, x][[y, x], y]$.

Of course, we cannot expect many of these to hold in arbitrary inverse semigroups without suitable modifications. A couple of them are quite easy.

Lemma 7 *Let S be an inverse semigroup.*

1. *If $e \in E(S)$, then $[x, e] \in E(S)$ for all $x \in S$;*
2. *$[x, y]^{-1} = [y, x]$ for all $x, y \in S$;*

Proof 1. Since $x^{-1}ex \in E(S)$ for all $x \in S$, we have

$$[x, e]^2 = (x^{-1}ex)e(x^{-1}ex)e = (x^{-1}ex)^2e^2 = x^{-1}exe = [x, e]\,.$$

2. We have $[x, y]^{-1} = (x^{-1}y^{-1}xy)^{-1} = y^{-1}x^{-1}yx = [y, x]$. □

For the rest, we do not know if there are any suitable generalizations to arbitrary inverse semigroups; see Problem 14 in §5.

Recall that a *Clifford semigroup* is an inverse semigroup in which every element is completely regular, or equivalently, an inverse semigroup in which every idempotent is central. Clifford semigroups are characterized as semilattices of groups. More precisely, if S is a Clifford semigroup, then there exists a semilattice $(Y, \sqcap)$ and a family of disjoint subgroups S_α ($\alpha \in Y$) such that $S = \bigcup_{\alpha \in Y} S_\alpha$ and $S_\alpha S_\beta \subseteq S_{\alpha \sqcap \beta}$ for all $\alpha, \beta \in Y$.

Not surprisingly, commutators in Clifford semigroups are much better behaved than in general inverse semigroups. They were first studied in detail by Kowol and Mitsch [7]. Among many other results, they showed that Clifford semigroups can be characterized using commutators.

Proposition 8 *(*[7], Cor. 3.2*) Let S be an inverse semigroup. The following are equivalent: (1) For all $x, y \in S$, $[x, y] \in E(S)$ implies $xy = yx$; (2) S is a Clifford semigroup.*

Our goal in this section is to show that group commutator identities hold in Clifford semigroups without change. We start with the following.

Lemma 9 *Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a Clifford semigroup. For $a, b \in S$, assume that for some $\alpha, \beta \in Y$, $a \in S_\alpha$, $b \in S_\beta$ and let $e = e_{\alpha \sqcap b}$ denote the identity element of $S_{\alpha \sqcap \beta}$. Then*

$$[ea, eb] = [a, b]\,.$$

Proof We have that $ea \in S_{(\alpha \sqcap \beta) \sqcap \alpha} = S_{\alpha \sqcap \beta}$ and $eb \in S_{(\alpha \sqcap \beta) \sqcap \beta} = S_{\alpha \sqcap \beta}$. Thus

$$\begin{aligned}
[ea, eb] &= (ea)^{-1_{S_{\alpha \sqcap \beta}}} (eb)^{-1_{S_{\alpha \sqcap \beta}}} (ea)(eb) \\
&= a^{-1_{S_\alpha}} e^{-1_{S_{\alpha \sqcap \beta}}} b^{-1_{S_\beta}} e^{-1_{S_{\alpha \sqcap \beta}}} eaeb \\
&= a^{-1_{S_\alpha}} eb^{-1_{S_\beta}} eeaeb \\
&= a^{-1_{S_\alpha}} b^{-1_{S_\beta}} ab \\
&= [a, b],
\end{aligned}$$

where the last but one equality holds because the element e is the identity in the group $S_{\alpha \sqcap \beta}$. □

Theorem 10 *The following hold in Clifford semigroups.*

1. $[z^{-1}xz, z^{-1}yz] = z^{-1}[x, y]z$;
2. $[xu, xvxw] = [xu, xw](xw)^{-1}[xu, xv]xw$;
3. $y^{-1}xy = x[x, y]$;
4. $[x, zy] = [x, y]y^{-1}[x, z]y$;
5. $[x, y^{-1}] = y[y, x]y^{-1}$;
6. $y^{-1}[[x, y^{-1}], z]yz^{-1}[[y, z^{-1}], x]zx^{-1}[[z, x^{-1}], y]x \in E(S)$;
7. $(xy)^2 = x^2y^2[y, x][[y, x], y]$.

Proof Taking into account Lemma 9 and the properties of commutators in groups, it follows that

1. $[z^{-1}xz, z^{-1}yz] = [ez^{-1}exze, ez^{-1}eyze] = ez^{-1}[ex, ey]ze = z^{-1}[x, y]z$;
2. We have

$$\begin{aligned}
[xu, xvxw] = [exu, exvexw] &= [exu, exw](exw)^{-1}[exu, exv]exw \\
&= [xu, xw](xw)^{-1}[xu, xv]xw;
\end{aligned}$$

3. $y^{-1}xy = (ey)^{-1}exey = ex[ex, ey] = x[x, y]$.
4. $[x, zy] = [ex, ezey] = [ex, ey](ey)^{-1}[ex, ez]ey = [x, y]y^{-1}[x, z]y$.
5. $[x, y^{-1}] = [ex, (ey)^{-1}] = ey[ey, ex](ey)^{-1} = y[y, x]y^{-1}$.
6. We compute

$$\begin{aligned} &y^{-1}[[x, y^{-1}], z]yz^{-1}[[y, z^{-1}], x]zx^{-1}[[z, x], y]x \\ &= (ey)^{-1}[[ex, (ey)^{-1}], ez]ey(ez)^{-1}[[ey, (ez)^{-1}], ex]ez(ex)^{-1}[[ez, (ex)^{-1}], ey]ex \\ &= e. \end{aligned}$$

   It follows that $y^{-1}[[x, y^{-1}], z]yz^{-1}[[y, z^{-1}], x]zx^{-1}[[z, x^{-1}], y]x$ is an idempotent.
7. We compute

$$\begin{aligned} (xy)^2 = (exey)^2 &= (ex)^2(ey)^2[ey, ex][[ey, ex], ey] \\ &= x^2y^2[y, x][[y, x], y]. \end{aligned}$$

□

4 Marginal inverse subsemigroups for commutators

In this section, we prove Theorem 2. Let S be an inverse semigroup and let $\omega(x, y) = [x, y]$ be the commutator. We will prove that $M_\omega(S) = Z(S)$, the metacenter.

Assume first that $a \in Z(S)$ so that

$$axa^{-1}a = aa^{-1}xa \tag{12}$$

for all $x \in S$. Taking $x = a^{-1}a$, we obtain

$$a = a \cdot a^{-1}a \cdot a^{-1}a = aa^{-1} \cdot a^{-1}a \cdot a = a^{-1}a \cdot aa^{-1} \cdot a = a^{-1}aa,$$

using (12) in the second equality. Thus $aa^{-1} = a^{-1}a \cdot aa^{-1}$, that is, $aa^{-1} \leq a^{-1}a$ in the natural order on $E(S)$. On the other hand, $a^{-1} \in Z(S)$ because the metacenter is an inverse subsemigroup of S [6, 12], so we also have $a^{-1}a \leq aa^{-1}$. Thus $aa^{-1} = a^{-1}a$, that is, a is completely regular.

From (12), we obtain

$$a^0xa^0 = a^{-1}xa \tag{13}$$

for all $x \in S$. Thus

$$[ax, y] = (ax)^{-1}y^{-1}axy = x^{-1} \cdot a^{-1}y^{-1}a \cdot xy = x^{-1}a^0y^{-1}a^0xy = [a^0x, y]$$

and

$$[xa, y] = (xa)^{-1}y^{-1}xay = a^{-1}x^{-1}y^{-1}xa \cdot y = a^0x^{-1}y^{-1}xa^0y = [xa^0, y],$$

in both cases using (13) in the third equality. The other two cases, $[x, ay] = [x, a^0 y]$ and $[x, ya] = [x, ya^0]$, are immediately implied by Lemma 7(2). Therefore $a \in M_\omega(S)$.

Now assume $a \in M_\omega(S)$ so that $[ax, y] = [a^0 x, y]$ for all $x, y \in S$. Recall that a is completely regular (Lemma 4) and commutes with every idempotent (Lemma 6). We thus compute

$$\begin{aligned} a^0 xa &= a^0 xx^{-1}xa = a^0 xax^{-1}x = a(a^{-1}xax^{-1})x \\ &= a[a, x^{-1}]x = a[aa^0, x^{-1}]x = a[a^0, x^{-1}]xa \\ &= aa^0 xa^0 x^{-1}x = axx^{-1}xa^0 = axa^0 \,. \end{aligned}$$

Hence $a \in Z(S)$.

Therefore $M_\omega(S) = Z(S)$ as claimed. Finally, as already mentioned in §1, it is known that $Z(S)$ is normal [6, 12]. This completes the proof of Theorem 2.

5 Open problems

We have focused on marginal subsemigroups in this paper, but one could also approach the subject from a different direction. In the group case, a marginal subgroup is normal, hence defines a congruence. If S is a semigroup and ω is a semigroup word, define the *marginal equivalence relation* $\sim_\omega$ on S by $a \sim_\omega b$ if and only if

$$\omega(x_1, \ldots, ax_i, \ldots, x_n) = \omega(x_1, \ldots, bx_i, \ldots, x_n) \tag{14}$$

$$\omega(x_1, \ldots, x_i a, \ldots, x_n) = \omega(x_1, \ldots, x_i b, \ldots, x_n) \tag{15}$$

for all $x_i \in S$. Then define the *marginal congruence* $\equiv_\omega$ to be the smallest congruence on S containing $\sim_\omega$. In special classes of semigroups with additional operations, one would want to use appropriate notions of word and congruence.

In the case of inverse semigroups where ω is the commutator, it turns out the marginal equivalence relation and marginal congruence coincide, that is, $\sim_\omega$ is already a congruence. This can be shown directly, and in this case, the marginal congruence coincides with the central congruence of S [2, 6]. One can also reach the same conclusion starting with our result that the marginal inverse subsemigroup of the commutator coincides with the metacenter, hence is normal.

Problem 11 *Let S be an inverse semigroup and let $\omega(x_1, \ldots, x_n)$ be an inverse semigroup word. Does the marginal congruence $\equiv_\omega$ coincide with the marginal equivalence relation $\sim_\omega$? Equivalently, is the marginal inverse subsemigroup $M_\omega(S)$ normal in S?*

Good candidates for investigation here are the power words $\omega(x) = x^n$.

In a more general direction, we suggest the following.

Problem 12 *Study marginal elements, subsemigroups, congruences, etc. in various classes of semigroups with the appropriate notion of what constitutes a word in that class.*

This might turn out to be particularly interesting in completely regular semigroups, even for commutators. In that case, there is more than one natural choice of commutator word, for example, $x^{-1}y^{-1}xy$ or $(yx)^{-1}xy$. Using Prover9, we have been able to show that in cryptogroups (completely regular semigroups in which Green's $\mathcal{H}$-relation is a congruence), the marginal equivalence relation and marginal congruence for either choice of commutator coincide. However, we do not know what happens for general completely regular semigroups.

Problem 13 *Study marginal elements, subsemigroups, congruences, etc. for commutator words in completely regular semigroups.*

Finally, with reference to §3, we conclude with the following.

Problem 14 *Find appropriate generalizations of well-known group commutator identities to arbitrary inverse semigroups.*

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