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Multiplicity lists by diameter: All trees of diameter < 7 [☆]

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ABSTRACT

This paper considers the problem of determining all multiplicity lists occurring among Hermitian matrices whose graph is a given tree from a new perspective. For all trees of diameter < 7, it is shown how to generate all possible lists. For diameter 5 and 6, this includes many nonlinear trees. In the process, for diameter 4 (double stars), the first succinct and direct description of all ordered lists, for each instance, is given. Observations of topological relationships are helpful.

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1. Introduction

For a simple undirected graph G on n vertices, let $\mathcal{S}(G)$ denote the set of all real symmetric n-by-n matrices, the graph of whose off-diagonal entries is G. When the graph is a tree T, we are interested in the possible multiplicity lists for the eigenvalues occurring among matrices in $\mathcal{S}(T)$. In this case, there is no difference between "real symmetric" and "complex Hermitian" [1]. We are interested both in the multiplicity lists ordered by the numerical values of the underlying eigenvalues (*ordered multiplicity lists*) and the "unordered" multiplicity lists that follow from them. See [1] for background. We measure the *diameter* of a tree T, $\mathrm{d}(T)$, as the number of vertices in a longest induced path of T. Here we categorize trees by diameter and consider trees of diameter < 7. A vertex of a tree is called a *high degree vertex* (HDV) if it has at least 3 neighbors. And, a tree is *linear* if all its HDV's lie on a single induced path. We know how to generate all the multiplicity lists for linear trees [2,3], but generally little is known about direct descriptions of all ordered (or unordered) lists in terms of the tree. As stars are well understood, the first interesting case is *double stars*, the diameter 4 trees with two (adjacent) HDV's. These trees are linear, so that we do know how to generate all the multiplicity lists (see also [4]), but, heretofore, an explicit description has been lacking. In Section 3, we give such a description of all ordered lists.

By diameter 5, nonlinear trees appear and are, in fact, common. Heretofore, no way has been known how to generate ordered or unordered lists for nonlinear trees. (Some individual cases are included in the database, the Appendix B of [5].) In Section 4, we show how to generate all unordered lists for diameter 5 trees. Because of a nice topological relationship with diameter 5, we are able to do the same for diameter 6 in Section 5. Examples are given. The idea here is to use "assignments"; except for multiplicity 1 eigenvalues of edges and stars, this is done by assigning values to vertices (diagonal entries of the corresponding matrix). At diameter 7, there is a known [6] limitation of assignments that prevents the same strategy from working more generally.

In the next section, we give some technical background necessary for this work.

2. Background

Given a graph G on n vertices $1, \ldots, n$, and $A \in \mathcal{S}(G)$ and $\alpha \subseteq \{1, \ldots, n\}$, recall that $A[\alpha]$ (resp. $G[\alpha]$) is the principal submatrix of A (resp. induced subgraph of G) determined by the index subset α and $A(i)$ (resp. $G \setminus i$) denotes the principal submatrix (resp. induced subgraph of G) resulting from deletion of row and column (resp. vertex) i. For $G' = G[\alpha]$, we often write $A[G']$ meaning the principal submatrix $A[\alpha]$. When G is a tree, $A(i)$ is a direct sum, whose summands correspond to components of $G \setminus i$, which we call *branches* of G at v. As usual, we often speak interchangeably about the graph and the matrix, for convenience. We denote by $\mathrm{m}_A(\lambda)$ the (geometric) multiplicity of λ as an eigenvalue of matrix A and $\sigma(A)$ denotes the spectrum of A.

Also recall that, when T is a tree, the maximum possible multiplicity of an eigenvalue among matrices in $\mathcal{S}(T)$, denoted by $\mathrm{M}(T)$, is the *path cover number* of T [7]. Also, the smallest and largest eigenvalue of any matrix in $\mathcal{S}(T)$ both have multiplicity 1 [8].

For a tree T two remarkable facts govern the possible multiplicities of the eigenvalues of matrices in $\mathcal{S}(T)$. The first is the Parter-Wiener, etc. theorem [8] that governs multiple eigenvalues, and even eigenvalues that occur in both A and $A(i)$, for $A \in \mathcal{S}(T)$. Of course the former implies the latter for each of the principal submatrices, by interlacing.

Theorem 1 *(Parter-Wiener, etc. theorem). Let T be a tree and A a matrix in $\mathcal{S}(T)$ and suppose that there is a vertex v of T and a real number λ such that $\lambda \in \sigma(A) \cap \sigma(A(v))$. Then*

1. *there is a vertex u of T such that $\mathrm{m}_{A(u)}(\lambda) = \mathrm{m}_A(u) + 1$;*
2. *if $\mathrm{m}_A(\lambda) \geq 2$, then the prevailing hypothesis is automatically satisfied and u may be chosen so that $\deg_T(u) \geq 3$ and so that there are at least three components T_1, T_2 and T_3 of $T \setminus u$ such that $\mathrm{m}_{A[T_i]}(\lambda) \geq 1$, $i = 1, 2, 3$; and*
3. *if $\mathrm{m}_A(\lambda) = 1$, then u may be chosen so that $\deg_T(u) \geq 2$ and so that there are two components T_1 and T_2 of $T \setminus u$ such that $\mathrm{m}_{A[T_i]}(\lambda) = 1$, $i = 1, 2$.*

A vertex u guaranteed by this theorem (there may be more than one such vertex) is called *Parter* (with respect to λ and A), and such a multiplicity (which could be 1) is called *upward* at u. Similarly, when $\mathrm{m}_{A(u)}(\lambda) = \mathrm{m}_A(u) - 1$, then such vertex is called *downer* and such a multiplicity is called *downer* at u. All multiplicities greater than 1 are upward at some respective Parter vertex, and multiplicities equal to 1 may or may not be upward. There is a simple indicator of whether a vertex is Parter.

Theorem 2. *For a tree T and a matrix $A \in \mathcal{S}(T)$, a vertex v of T is Parter for an eigenvalue $\lambda \in \sigma(A)$ if and only if there is a branch T' of T at v, such that the neighbor u of v in T' is a downer for λ in $A[T']$ (i.e., $\mathrm{m}_{A[T' \setminus u]}(\lambda) = \mathrm{m}_{A[T']}(\lambda) - 1$).)*

Together, Theorems 1 and 2 support the method of assignments to construct matrices with given multiplicities for a given tree [1,6]. For example, if λ appears as a diagonal entry of A, corresponding to a pendent vertex of T, then that vertex is a downer, and its neighbor is Parter for λ in T.

A key fact about stars (trees of diameter 3) is included in [4,9,10]. This will be useful for the analysis of double stars (trees of diameter 4) in Section 3. In a star the (unique) center vertex must be Parter for any eigenvalue that appears on a pendent.

Let $\mathcal{L}(T)$ denote the set of ordered multiplicity lists among matrices in $\mathcal{S}(T)$. This set is also called the *ordered catalog* of T. For stars, from [4,9], we have

Theorem 3. *If T is a star on n vertices then $\mathcal{L}(T)$ consists of all lists $(q_1, \ldots, q_r)$ in which*

1. *$q_1, \ldots, q_r$ are positive integers;*
2. *$q_1 + \cdots + q_r = n$; and*
3. *$q_i > 1$ implies $1 < i < n$ and $q_{i-1} = 1 = q_{i+1}$.*

Let T be a star on n vertices, v be the center vertex of T and $A \in \mathcal{S}(T)$. Since T is a star and v is the center vertex of T, if λ is an eigenvalue of A and $A(v)$ then $\mathrm{m}_{A(v)}(\lambda) = \mathrm{m}_A(\lambda)+1$ [4]. We call each eigenvalue λ of A satisfying $\mathrm{m}_{A(v)}(\lambda) = \mathrm{m}_A(\lambda)+1$, an *upward eigenvalue* of A. The corresponding multiplicity of λ in A is called an *upward multiplicity* of A. If $q = (q_1, \ldots, q_r)$ is the ordered multiplicity list of A, then the *upward multiplicity list* of A, denoted by $\widehat{q}$, is the list with the same entries as q but in which any upward multiplicity q_i is marked as $\widehat{q}_i$ in $\widehat{q}$. Note that we may have upward multiplicities equal 1 and that each multiplicity greater than 1 is, necessarily, an upward multiplicity (at v). Thus, if q_i is marked in $\widehat{q}$, then $1 < i < n$ and neither q_{i-1} nor q_{i+1} can be marked in $\widehat{q}$ (and, therefore, $q_{i-1} = 1 = q_{i+1}$). Let $\widehat{\mathcal{L}}(T)$ denote the set of upward multiplicity lists of T, which we call the *upward catalog* of T. From [4] we have

Theorem 4. *If T is a star on n vertices then $\widehat{\mathcal{L}}(T)$ consists of all lists $\widehat{q} = \widehat{(q_1, \ldots, q_r)}$ in which*

1. *$q_1, \ldots, q_r$ are positive integers;*
2. *$q_1 + \cdots + q_r = n$; and*
3. *if q_i is an upward multiplicity in $\widehat{q}$, then $1 < i < n$ and neither q_{i-1} nor q_{i+1} is an upward multiplicity in $\widehat{q}$.*

A double star consists of two stars, whose center vertices are connected by an edge. We assume one star has k pendent vertices (the star on $k+1$ vertices, S_{k+1}) and the other l (the star on $l+1$ vertices, S_{l+1}), and denote this double star as $D(k,l)$. We also assume wlog $k \geq l \geq 2$.

Since [4], it has been known how to (indirectly) generate all multiplicity lists (the ordered catalog) for a double star, from the upward catalogs of the relevant stars, via a *Superposition Principle* [4] (which is now a special case of the Linear Superposition Principle, LSP [2,3] for linear trees) stated in the next result.

Theorem 5 *(Superposition Principle). Let T be a double star $D(k,l)$ and let $\widehat{b} = \widehat{(b_1, \ldots, b_{s_1})} \in \widehat{\mathcal{L}}(S_{k+1})$ and $\widehat{c} = \widehat{(c_1, \ldots, c_{s_2})} \in \widehat{\mathcal{L}}(S_{l+1})$. Construct any lists $b^+ = (b_1^+, \ldots, b_{s_1+t_1}^+)$ and $c^+ = (c_1^+, \ldots, c_{s_2+t_2}^+)$ subject to the following conditions:*

1. *$t_1, t_2 \in \mathbb{N}_0$ and $s_1 + t_1 = s_2 + t_2$;*
2. *b^+ (resp. c^+) is obtained from $\widehat{b}$ (resp. $\widehat{c}$) by inserting t_1 (resp. t_2) 0's;*
3. *b_i^+ and c_i^+ cannot both be 0; and*
4. *if $b_i^+ > 0$ and $c_i^+ > 0$, at least b_i^+ or c_i^+ must be an upward multiplicity of $\widehat{b}$ or $\widehat{c}$.*

Then $b^+ + c^+ \in \mathcal{L}(T)$*. Moreover,* $a \in \mathcal{L}(T)$ *if and only if there are* $\widehat{b} \in \widehat{\mathcal{L}}(S_{k+1})$ *and* $\widehat{c} \in \widehat{\mathcal{L}}(S_{l+1})$ *such that* $a = b^+ + c^+$*.*

Under the conditions and in the notation of Theorem 5 we say that the pair b^+ and c^+, obtained from the upward multiplicity lists $\widehat{b}$ and $\widehat{c}$, is a *valid pair*. We call the list b^+ (resp. c^+) an *augmented list* for the star S_{k+1} (resp. S_{l+1}) obtained from $\widehat{b}$ (resp. $\widehat{c}$). If a given multiplicity is an upward multiplicity in $\widehat{b}$ or $\widehat{c}$ we still say that the corresponding multiplicity in the augmented list is an upward multiplicity.

3. Multiplicity lists for trees of diameter 4

Our purpose here is to give a more explicit description of the ordered multiplicity lists for double stars. In particular, we make new observations about the structure of such lists and then use these to give, for a proposed list and a given k and l, a yes/no answer to the question of whether the list occurs (as an ordered multiplicity list) for the double star $D(k, l)$.

For this end, we use some special terminology for the assignment structure of double stars. We say that λ is a *split* eigenvalue (for a multiplicity list) of $D(k, l)$ if λ appears on pendent vertices, at least twice, among both k and l. On the other hand λ is a *one-sided* eigenvalue if λ appears on pendents, at least twice, among k or l, but not both. It may appear at most once on the "other side". *One sided* k (resp. l) means that it appears multiply among the k (resp. l) pendents. Note that in either case λ must be an eigenvalue of a corresponding matrix for $D(k, l)$ (and may be multiple).

If λ is a split eigenvalue then the corresponding multiplicity in the ordered multiplicity list is called a *split* multiplicity.

As any ordered multiplicity list for a given tree, an ordered multiplicity list for $D(k, l)$ must begin and end with a 1 [8,11]. Moreover, these two multiplicities 1 are not upward at any vertex of the tree [8]. A contiguous sublist of an ordered multiplicity list is called a *string* if it begins after, and ends before, a 1 and each multiplicity of that sublist is greater than 1. These two 1's are allowed to be the initial and terminal 1's of the entire list. More generally, a sequence L of positive integers, that begins and ends with a 1, with $t \geq 0$ strings, may be represented as

$$(1, \dots, 1, p_{11}, \dots, p_{1s_1}, 1, \dots, 1, p_{21}, \dots, p_{1s_2}, 1, \dots, 1, p_{t1}, \dots, p_{ts_t}, 1, \dots, 1)$$

in which $p_{ij} > 1$, $i = 1, \dots, t$ and $j = 1 \dots, s_i$. If $t > 0$ then, for each $i \in \{1, \dots, t\}$, we call s_i the *length* of string $p_{i1}, \dots, p_{is_i}$. The *size* of L is just the sum of all entries of L.

Often, we focus upon a string of length 1, a *singleton* string. A *multiple* string has length at least 2. As we will see, "alternation" in a multiple string is a necessary condition in an ordered multiplicity list for a double star (Theorem 14). This alternation refers to the situation in which each multiplicity must be one-sided (k or l), and those multiplicities must alternate between the two sides. On the other hand, certain ordered multiplicity lists require one split eigenvalue which implies the existence of a singleton string.

In order to give a yes/no answer to the question of whether a sequence L occurs as an ordered multiplicity list for a given double star (Theorem 14), we now present useful auxiliary results.

From Theorem 4 and the Superposition Principle (Theorem 5) it is simple to conclude

Lemma 6. *Let $L = (m_1, \ldots, m_t)$ be an ordered multiplicity list for a double star $D(k, l)$ and let b^+ and c^+ be a valid pair such that $L = b^+ + c^+$. For each multiplicity $m_i \geq 2$ such that, both multiplicities b_i^+ and c_i^+ are upward and one of them is an upward $\widehat{1}$, then, by replacing that upward $\widehat{1}$ by the non-upward 1, on the relevant augmented list and upward multiplicity list, we obtain a valid pair b'^+ and c'^+ such that $L = b'^+ + c'^+$.*

The next result states that each multiplicity m_i, in an ordered multiplicity list for a double star $T = D(k, l)$, such that $m_i \leq 3$, can always be obtained by an eigenvalue that is upward in, at most, one of the stars S_{k+1} or S_{l+1}. In other words, given an ordered multiplicity list for T, there is a matrix in $\mathcal{S}(T)$ such that, for each multiplicity $m_i \leq 3$, the corresponding eigenvalue is not a split eigenvalue.

Lemma 7. *Let $L = (m_1, \ldots, m_t)$ be an ordered multiplicity list for a double star $D(k, l)$. Then there is a valid pair b^+ and c^+ such that $L = b^+ + c^+$, and this valid pair can be chosen such that, for each $m_i \leq 3$, at most one of the multiplicities b_i^+ and c_i^+ is an upward multiplicity.*

Proof. Let $b^+ = (b_1^+, \ldots, b_t^+)$ and $c^+ = (c_1^+, \ldots, c_t^+)$ be a valid pair such that $L = b^+ + c^+$, guaranteed by the Superposition Principle. Let m_i be a multiplicity in L such that $m_i \leq 3$. If m_i is the sum of two upward multiplicities (b_i^+ and c_i^+) then $m_i = 2$ or $m_i = 3$ and, therefore, at least one of them (b_i^+ or c_i^+) must be an upward $\widehat{1}$. By Lemma 6, replacing that upward $\widehat{1}$ by a non-upward 1 (in the relevant list b^+ or c^+) we still obtain a valid pair whose sum gives L. □

A given multiplicity, in an ordered multiplicity list for a double star $D(k, l)$, is called a *mandatory split* multiplicity if the corresponding eigenvalue must be an eigenvalue of, at least multiplicity 2, of each star S_{k+1} and S_{l+1}. In such case, the corresponding eigenvalue is called a *mandatory split* eigenvalue.

Example 8. Given a double star $T = D(k, l)$, with $k \geq l \geq 3$, there are always multiplicity lists having a mandatory split multiplicity.

Note that $L = (1, k, l, 1)$ is always an ordered multiplicity list for T which is obtained, by the Superposition Principle, from the upward multiplicity lists $\widehat{b} = (1, \widehat{k-1}, 1)$ and $\widehat{c} = (1, \widehat{l-1}, 1)$ for the stars S_{k+1} and S_{l+1}, respectively. For example, $b^+ = (1, \widehat{k-1}, 1, 0)$ and $c^+ = (0, 1, \widehat{l-1}, 1)$ is a valid pair such that $L = b^+ + c^+$. We represent, informally, this process in a diagram as

$$\begin{array}{lcccc} S_{k+1}: & 1 & \widehat{k-1} & 1 & 0 \\ S_{l+1}: & 0 & 1 & \widehat{l-1} & 1 \\ \hline & 1 & k & l & 1 \end{array}$$

in which, we may replace "S_{k+1}" and "S_{l+1}" by "b^+" and "c^+", respectively, or, for simplicity, we may omit this information.

If $k \geq l \geq 3$, then the multiplicity $\mathrm{M}(T) = k + l - 2$ is <u>always</u> a mandatory split multiplicity (because $\mathrm{M}(T) = \mathrm{M}(S_{k+1}) + \mathrm{M}(S_{l+1}) = (k-1) + (l-1)$ and $\mathrm{M}(T) > k$ and $\mathrm{M}(T) > l$) and the corresponding ordered multiplicity list must be $L = (1, 1, \mathrm{M}(T), 1, 1)$. Such list is, again, obtained from the upward multiplicity lists $\widehat{b} = (1, \widehat{k-1}, 1)$ and $\widehat{c} = (1, \widehat{l-1}, 1)$ of the stars S_{k+1} and S_{l+1}, respectively. For example, $b^+ = (0, 1, \widehat{k-1}, 1, 0)$ and $c^+ = (1, 0, \widehat{l-1}, 0, 1)$ is a valid pair such that $L = b^+ + c^+$.

$$\begin{array}{lccccc} S_{k+1}: & 0 & 1 & \widehat{k-1} & 1 & 0 \\ S_{l+1}: & 1 & 0 & \widehat{l-1} & 0 & 1 \\ \hline & 1 & 1 & k+l-2 & 1 & 1 \end{array}$$

As we will see, the position of a multiplicity greater than 1 in a multiplicity list may require, depending on the particular double star, that such multiplicity be a mandatory split multiplicity. Given the importance of this phenomenon in this work, we present some illustrative examples.

Example 9. Let T be the double star $D(7, 7)$.

1. The list $L = (1, 2, 2, 2, 2, 1, 4, 1, 1)$ is an ordered multiplicity list for T in which 4 is a mandatory split multiplicity. There is a valid pair b^+ and c^+ such that $L = b^+ + c^+$, however, by inspection, the multiplicity 4 must be a sum of two upward multiplicities $\widehat{2}$. We must have something like this:

$$\begin{array}{lccccccccc} b^+: & 0 & 1 & \widehat{1} & 1 & \widehat{1} & 1 & \widehat{2} & 0 & 1 \\ c^+: & 1 & \widehat{1} & 1 & \widehat{1} & 1 & 0 & \widehat{2} & 1 & 0 \\ \hline & 1 & 2 & 2 & 2 & 2 & 1 & 4 & 1 & 1 \end{array}$$

   Note that, by the nature of this multiplicity 4, this multiplicity must be located, in L, between two 1's, $\boxed{1, 4, 1}$. Moreover, each of these two 1's must be an "interior" 1 of L, i.e., cannot be the left end 1 or the right end 1 of L.
2. The list $L = (1, 2, 2, 1, 4, 1, 2, 2, 1)$ is also an ordered multiplicity list for T in which 4 is a mandatory split multiplicity. Again, by inspection, the multiplicity 4 must be a sum of two upward multiplicities $\widehat{2}$. We must have something like this:

$$\begin{array}{lccccccccc} b^+: & 0 & 1 & \widehat{1} & 1 & \widehat{2} & 1 & \widehat{1} & 1 & 0 \\ c^+: & 1 & \widehat{1} & 1 & 0 & \widehat{2} & 0 & 1 & \widehat{1} & 1 \\ \hline & 1 & 2 & 2 & 1 & 4 & 1 & 2 & 2 & 1 \end{array}$$

Note again that, the mandatory split multiplicity 4, is located, in L, between two interior 1's, $\boxed{1, 4, 1}$.

3. The list $L = (1, 2, 2, 2, 1, 4, 1, 2, 1)$, with the same multiplicities as in the previous two examples, is now an ordered multiplicity list for T in which 4 is not a mandatory split multiplicity. Note the new position of the sequence $\boxed{1, 4, 1}$ in L. We have, for example,

$$\begin{array}{rccccccccc} b^+: & 0 & 1 & \widehat{1} & 1 & 1 & \widehat{3} & 0 & 1 & 0 \\ c^+: & 1 & \widehat{1} & 1 & \widehat{1} & 0 & 1 & 1 & \widehat{1} & 1 \\ \hline & 1 & 2 & 2 & 2 & 1 & 4 & 1 & 2 & 1 \end{array}$$

or

$$\begin{array}{rccccccccc} b^+: & 0 & 1 & \widehat{1} & 0 & 1 & \widehat{3} & 1 & 1 & 0 \\ c^+: & 1 & \widehat{1} & 1 & \widehat{2} & 0 & 1 & 0 & \widehat{1} & 1 \\ \hline & 1 & 2 & 2 & 2 & 1 & 4 & 1 & 2 & 1 \end{array}$$

or, another possibility,

$$\begin{array}{rccccccccc} b^+: & 0 & 1 & \widehat{1} & 1 & 0 & \widehat{4} & 0 & 1 & 0 \\ c^+: & 1 & \widehat{1} & 1 & \widehat{1} & 1 & 0 & 1 & \widehat{1} & 1 \\ \hline & 1 & 2 & 2 & 2 & 1 & 4 & 1 & 2 & 1 \end{array}$$

4. The list $L = (1, 4, 4, 1, 4, 1, 1)$ is also an ordered multiplicity list for T in which the multiplicity 4, located between two interior 1's, is a mandatory split multiplicity. We must have something like this:

$$\begin{array}{rccccccc} b^+: & 0 & 1 & \widehat{3} & 1 & \widehat{2} & 1 & 0 \\ c^+: & 1 & \widehat{3} & 1 & 0 & \widehat{2} & 0 & 1 \\ \hline & 1 & 4 & 4 & 1 & 4 & 1 & 1 \end{array}$$

As we will see, a mandatory split multiplicity plays an important role on the proposed work.

Lemma 10. *Let $L = (m_1, \ldots, m_t)$ be an ordered multiplicity list for a double star. We have:*

1. *If m_i is a mandatory split multiplicity of L then*
   (a) *$m_i \geq 4$; and*
   (b) *$2 < i < t - 1$ and $m_{i-1} = 1 = m_{i+1}$.*
2. *L has, at most, one mandatory split multiplicity. Moreover, if L has a mandatory split multiplicity, such multiplicity may be chosen among the multiplicities of L, with the highest value, satisfying (1a) and (1b).*

Proof. 1. Lemma 7 justifies (1a). For (1b), let b^+ and c^+ be a valid pair such that $L = b^+ + c^+$. Since, by the Superposition Principle, m_i results from the sum of two upward multiplicities, and an upward multiplicity of a star is located between two non-upward 1's in the corresponding upward multiplicity list for that star, it follows, again by the Superposition Principle, that there are at least two multiplicities on the right (resp. left) of m_i, so that $2 < i < t-1$. Again, by the Superposition Principle, we have either $(b^+_{i+1}, c^+_{i+1}) = (1,0)$ or $(b^+_{i+1}, c^+_{i+1}) = (0,1)$ which implies that $m_{i+1} = 1$ (and $m_{i-1} = 1$, with a similar argument).

2. Let $L = (1,\ldots,1,m_i,1,\ldots,1,m_j,1,\ldots,1)$, in which $m_i \geq 2$ and $m_j \geq 2$, and let b^+ and c^+ be a valid pair such that $L = b^+ + c^+$. Suppose that each multiplicity b^+_i, c^+_i, b^+_j and c^+_j is an upward multiplicity. In a diagram, we have

$$\begin{array}{r|ccccccccc} S_{k+1}: & \cdots & \cdots & b^+_i & \cdots & \cdots & \cdots & b^+_j & \cdots & \cdots \\ S_{l+1}: & \cdots & \cdots & c^+_i & \cdots & \cdots & \cdots & c^+_j & \cdots & \cdots \\ \hline & \cdots & 1 & m_i & 1 & \cdots & 1 & m_j & 1 & \cdots \end{array}$$

Suppose, wlog, that $m_i \geq m_j$. Then we have, necessarily, $b^+_i \geq c^+_j$ or $c^+_i \geq b^+_j$. Suppose, wlog, that $b^+_i \geq c^+_j$. In the augmented list b^+ replace the multiplicity b^+_i by $b^+_i - (c^+_j - 1)$ and replace the multiplicity b^+_j by $b^+_j + c^+_j - 1$, in order to obtain a new list b'^+. Note that

$$[b^+_i - (c^+_j - 1)] + [b^+_j + c^+_j - 1] = b^+_i + b^+_j$$

and, because b^+ is an augmented multiplicity list for S_{k+1}, the same happens with the new list b'^+.

Now, in the augmented list c^+ replace the multiplicity c^+_i by $c^+_i + c^+_j - 1$ and replace the multiplicity c^+_j by a non-upward 1, in order to obtain a new list c'^+. Note that

$$[c^+_i + c^+_j - 1] + 1 = c^+_i + c^+_j$$

and c'^+ is an augmented multiplicity list for S_{l+1}. Moreover, by construction of b'^+ and c'^+ form a valid pair such that $L = b'^+ + c'^+$, in which m_j is <u>not</u> a mandatory split multiplicity. In a diagram we have

$$\begin{array}{ccccccccc} \cdots & \cdots & b^+_i - (c^+_j - 1) & \cdots & \cdots & \cdots & b^+_j + c^+_j - 1 & \cdots & \cdots \\ \cdots & \cdots & c^+_i + c^+_j - 1 & \cdots & \cdots & \cdots & 1 & \cdots & \cdots \\ \hline \cdots & 1 & m_i & 1 & \cdots & 1 & m_j & 1 & \cdots \end{array}$$

Therefore, we may conclude that L may have, at most, one mandatory split multiplicity. □

We say that a mandatory split multiplicity, in an ordered multiplicity list, is located between two *interior* 1's (distinct from the non-upward multiplicities 1 corresponding to the smallest and largest eigenvalues). See Examples 8 and 9. (Note that, a split multiplicity m, in a ordered multiplicity list, must be $m \geq 2$ and, with the same justification for a mandatory split multiplicity, m is also located between two interior 1's.)

In the following two results, given a multiplicity list L of a double star $D(k, l)$, we obtain multiplicity lists of each double star $D(k', l')$, in which $k' \geq k$ and $l' \geq l$, by inserting some multiplicities in L.

Lemma 11. *Let L be an ordered multiplicity list for a double star $D(k, l)$. If L' is a list obtained from L by inserting s ($s \geq 1$) 1's, then L' is an ordered multiplicity list for each double star $D(k', l')$, on $k + l + s + 2$ vertices, such that $k' \geq k$ and $l' \geq l$.*

Proof. Since L is an ordered multiplicity list for $D(k, l)$, there is a valid pair b^+ and c^+ (in which b^+ and c^+ are, respectively, augmented lists for the stars S_{k+1} and S_{l+1}) such that $L = b^+ + c^+$.

Let $s \geq 1$ and let L' be a list obtained from L by inserting s new 1's. Choose any integers $s_1, s_2 \geq 0$ such that $s_1 + s_2 = s$. Among the s new inserted 1's, in L', choose any set of s_1 new 1's and identify the remaining s_2 new 1's in L'.

For each "1" of the chosen s_1 new 1's of L' insert, in the corresponding position, a "1" in b^+ and a "0" in c^+. For each "1" of the chosen s_2 new 1's of L' insert, in the corresponding position, a "0" in b^+ and a "1" in c^+. Note that, from this insertion in b^+ and in c^+ we obtain, respectively, an augmented list ${b'}^+$ for the star S_{k+s_1+1} and an augmented list ${c'}^+$ for the star S_{l+s_2+1}, such that, by the Superposition Principle, $L' = {b'}^+ + {c'}^+$. Therefore, L' is an ordered multiplicity list for $D(k + s_1, l + s_2)$. □

Lemma 12. *Let L be an ordered multiplicity list for a double star $D(k, l)$ and consider a string $p_{11}, \ldots, p_{1s}$ of length $s \geq 1$. Let*

$$(k_1 = \sum_{j \text{ odd}} p_{1j} \text{ and } l_1 = \sum_{j \text{ even}} p_{1j}) \quad \text{or} \quad (k_1 = \sum_{j \text{ even}} p_{1j} \text{ and } l_1 = \sum_{j \text{ odd}} p_{1j}).$$

If L' is the sequence obtained from L by appending the sequence $p_{11}, \ldots, p_{1s}, 1$ at the right end of L, then L' is an ordered multiplicity list for each of the double stars $D(k + k_1, l + l_1 + 1)$ and $D(k + k_1 + 1, l + l_1)$. Moreover, none of the multiplicities $p_{11}, \ldots, p_{1s}$ is a split multiplicity.

Proof. Since L is an ordered multiplicity list for $D(k, l)$, there is a valid pair b^+ and c^+ (in which b^+ and c^+ are, respectively, augmented lists for the stars S_{k+1} and S_{l+1}) such that $L = b^+ + c^+$. Note that the right end entry of L is 1 so that we suppose, wlog, that the right end entry of b^+ (resp. c^+) is a 1 (resp. 0). Now we will extend each list b^+ and c^+ in order to obtain augmented lists ${b'}^+$ and ${c'}^+$, respectively, such that $L' = {b'}^+ + {c'}^+$.

Suppose that $k_1 = \sum_{j \text{ odd}} p_{1j}$ and $l_1 = \sum_{j \text{ even}} p_{1j}$. (For the other possibility the discussion is similar.) We have to consider two cases.

Case 1. We set

$$\begin{array}{lcllllll} b'^{+} & = & b^{+} & \widehat{p_{11}-1} & 1 & \widehat{p_{13}-1} & \cdots \\ c'^{+} & = & c^{+} & 1 & \widehat{p_{12}-1} & 1 & \cdots \\ \hline L' & & L & p_{11} & p_{12} & p_{13} & \cdots \end{array}$$

in which a 0 is appended at the right end of b'^{+} or c'^{+}. By construction, b'^{+} (resp. c'^{+}) is an augmented list for S_{k+1+k_1} (resp. S_{l+1+l_1+1}) and, by the Superposition Principle, we have $L' = b'^{+} + c'^{+}$. Thus, L' is an ordered multiplicity list for $D(k+k_1, l+l_1+1)$.

Case 2. We set

$$\begin{array}{lcllllll} b'^{+} & = & b^{+} & \widehat{p_{11}} & 1 & \widehat{p_{13}-1} & \cdots \\ c'^{+} & = & c^{+} & 0 & \widehat{p_{12}-1} & 1 & \cdots \\ \hline L' & & L & p_{11} & p_{12} & p_{13} & \cdots \end{array}$$

in which a 0 is appended at the right end of b'^{+} or c'^{+}. By construction, b'^{+} (resp. c'^{+}) is an augmented list for S_{k+1+k_1+1} (resp. S_{l+1+l_1}) and, by the Superposition Principle, we have $L' = b'^{+} + c'^{+}$ in which, by construction, none of the multiplicities $p_{11}, \ldots, p_{1s}$ is a split multiplicity. Thus, L' is an ordered multiplicity list for $D(k+k_1+1, l+l_1)$. □

Of course, in Lemma 12, we may replace "appending the sequence $p_{11}, \ldots, p_{1s}, 1$ at the right end of L" by "appending the sequence $1, p_{11}, \ldots, p_{1s}$ at the left end of L".

We now consider a list with t strings and exactly $t+1$ 1's.

Lemma 13. *Let $L = (1, p_{11}, \ldots, p_{1s_1}, 1, p_{21}, \ldots, p_{2s_2} 1, \ldots, p_{t1}, \ldots, p_{ts_t}, 1)$ be a list with $t \geq 1$ strings, in which the string i, $i = 1, \ldots, t$, with length s_i, is $p_{i1}, \ldots, p_{is_i}$. For each string $i \in \{1, \ldots, t\}$ let*

$$(k_i = \sum_{j \text{ odd}} p_{ij} \text{ and } l_i = \sum_{j \text{ even}} p_{ij}) \quad or \quad (k_i = \sum_{j \text{ even}} p_{ij} \text{ and } l_i = \sum_{j \text{ odd}} p_{ij}).$$

Let $k' = k_1 + \cdots + k_t$ and $l' = l_1 + \cdots + l_t$.

Then L is an ordered multiplicity list for each of the double stars $D(k,l)$, on $k'+l'+t+1$ vertices, such that $k \geq k'$ and $l \geq l'$. Moreover, none of the multiplicities is a split multiplicity.

Proof. Note that L has t strings and, if $t > 1$, there is exactly one 1 between two consecutive strings. We prove the result by induction on the number t of strings in L.

If $t = 1$ then $L = (1, p_{11}, \ldots, p_{1s_1}, 1)$ has exactly one string. Suppose, wlog, that

$$k_1 = \sum_{j \text{ odd}} p_{1j} \quad \text{and} \quad l_1 = \sum_{j \text{ even}} p_{1j}$$

and consider the upward multiplicity lists for stars S_{k_1+1} and S_{l_1+1}, respectively,

$$\widehat{b} = (1, \widehat{p_{11} - 1}, 1, \widehat{p_{13} - 1}, \dots, 1) \quad \text{and} \quad \widehat{c} = (1, \widehat{p_{12} - 1}, 1, \widehat{p_{14} - 1}, \dots, 1).$$

Note that the number of terms in the sum of k_1 is, at least, the number of terms of l_1. By inserting a 0 in the left end of $\widehat{c}$ (the upward multiplicity for S_{l_1+1}) and, either inserting a 0 in the right end of $\widehat{c}$ (if the number of parts of l_1 is (one) less than the number of parts of k_1) or inserting a 0 in the right end of $\widehat{b}$ (if the number of parts of l_1 equals the number of parts of k_1), as a result of this insertion we obtain, from $\widehat{b}$ and $\widehat{c}$, respectively, augmented lists b^+ and c^+ for S_{k_1+1} and S_{l_1+1} such that, by the Superposition Principle, $L = b^+ + c^+$. Thus L is an ordered multiplicity list for the double star $D(k_1, l_1)$ and, therefore, we are done with the case $t = 1$. By construction, none of the multiplicities $p_{11}, \dots, p_{1s_1}$ is a split multiplicity.

Let $t > 1$ and suppose the result valid for $t - 1$ strings. Let L' be the list L without the sequence $\boxed{p_{t1}, \dots, p_{ts_t}, 1}$. By the induction hypothesis, for any choice of $k_1, \dots, k_{t-1}$ and $l_1, \dots, l_{t-1}$, setting $k' = k_1 + \cdots + k_{t-1}$ and $l' = l_1 + \cdots + l_{t-1}$, L' is an ordered multiplicity list (with no split multiplicities) for each double star $D(k'', l'')$, on $k' + l' + t$ vertices, and such that $k'' \geq k'$ and $l'' \geq l'$. Choosing

$$(k_t = \sum_{j \text{ odd}} p_{tj} \text{ and } l_t = \sum_{j \text{ even}} p_{tj}) \quad \text{or} \quad (k_t = \sum_{j \text{ even}} p_{tj} \text{ and } l_t = \sum_{j \text{ odd}} p_{tj}),$$

by Lemma 12 we conclude that L is an ordered multiplicity list (with no split multiplicities) for each double star, on $k' + k_t + l' + l_t + t + 1$ vertices,

$$D(k'' + k_t, l'' + l_t + 1) \quad \text{and} \quad D(k'' + k_t + 1, l'' + l_t),$$

such that $k'' \geq k' = k_1 + \cdots + k_{t-1}$ and $l'' \geq l' = l_1 + \cdots + l_{t-1}$. Therefore, L is an ordered multiplicity list for each double star $D(k, l)$, on $k_1 + \cdots + k_t + l_1 + \cdots + l_t + t + 1$ vertices, such that $k \geq k_1 + \cdots + k_t$ and $l \geq l_1 + \cdots + l_t$. □

The main theorem is now

Theorem 14. *Let T be a double star $D(k, l)$ and let L be the list*

$$(1, \dots, 1, p_{11}, \dots, p_{1s_1}, 1, \dots, 1, p_{21}, \dots, p_{1s_2}, 1, \dots, 1, p_{t1}, \dots, p_{ts_t}, 1, \dots, 1)$$

with $t \geq 1$ strings, in which the string i, $i = 1, \dots, t$, with length s_i, is $p_{i1}, \dots, p_{is_i}$. In case there are strings of length 1 in L, whose entry is greater than or equal to 4, located between two interior 1's, let p_{h1} be a string with the highest value among such strings.

For each string $i \in \{1, \dots, t\} \setminus \{h\}$ choose

$$(k_i = \sum_{j \text{ odd}} p_{ij} \text{ and } l_i = \sum_{j \text{ even}} p_{ij}) \quad \text{or} \quad (k_i = \sum_{j \text{ even}} p_{ij} \text{ and } l_i = \sum_{j \text{ odd}} p_{ij}).$$

Then L is an ordered multiplicity list for T if and only if

1. *the size of L is $k+l+2$; and*
2. *there is a choice of k_i's and l_i's such that $\sum_{\substack{i=1\\i\neq h}}^{t} k_i \leq k$ and $\sum_{\substack{i=1\\i\neq h}}^{t} l_i \leq l$.*

Proof. First, the stated conditions are necessary. Condition 1 because T has $k+l+2$ vertices. For condition 2 it suffices to note that: (i) each multiplicity in a string of length 1 in L (except for some multiplicity p_{h1}, if it exists) can be chosen as a non-split multiplicity (Lemma 10), and (ii) a non-split multiplicity m_i uses, at least, m_i pendent vertices of one of the stars of T; and (iii) in a string of length ≥ 2 in L, there are no split multiplicities (by Lemma 10) and adjacent multiplicities, m_i and m_{i+1}, in such string, the multiplicity m_i uses, at least, m_i pendent vertices of one of the stars and the multiplicity m_{i+1} uses, at least, m_{i+1} pendent vertices of the other star of T (by Lemma 10 and Superposition Principle).

We turn now to sufficiency of the stated conditions. If there are strings of length 1 in L, whose entry is greater than or equal to 4, located between two interior 1's, let $p = p_{h1}$ (for some $h \in \{1, \ldots, t\}$) be a string with the highest value among such strings. If such a string does not exist in L we set $h = 0$. We consider two cases.

<u>Case 1</u>: For each string $i \in \{1, \ldots, t\}$ there is a choice of k_i and l_i such that $k' = \sum_{i=1}^{t} k_i \leq k$ and $l' = \sum_{i=1}^{t} l_i \leq l$. By Lemma 13 the list

$$(1, p_{11}, \ldots, p_{1s_1}, 1, p_{21}, \ldots, p_{2s_2} 1, \ldots, p_{t1}, \ldots, p_{ts_t}, 1)$$

is an ordered multiplicity list for each double star $D(\overline{k}, \overline{l})$, on $k' + l' + t + 1$ vertices, such that $\overline{k} \geq k'$ and $\overline{l} \geq l'$. By Lemma 11 we conclude that L is an ordered multiplicity list for each double star $D(\overline{k}, \overline{l})$, on $k+l+2$, such that $\overline{k} \geq k'$ and $\overline{l} \geq l'$. Therefore, because $k \geq k'$ and $l \geq l'$, we conclude that L is an ordered multiplicity of T. (Note that this case includes the case in which $h = 0$.)

<u>Case 2</u>: $h \neq 0$. By hypothesis, there is a choice of k_i's and l_i's such that $k' = \sum_{\substack{i=1\\i\neq h}}^{t} k_i \leq k$ and $l' = \sum_{\substack{i=1\\i\neq h}}^{t} l_i \leq l$. Let $k_h = k - k'$ and $l_h = l - l'$. Since L has size $n = k+l+2$ and $p = p_{h1}$ is a string of length 1, located in L between two interior 1's, we must have $k + l \geq k' + l' + 2 + p$ which implies that $k_h + l_h = (k - k') + (l - l') \geq 2 + p$.

<u>Case 2.1</u>: If $p \leq k_h$ or $p \leq l_h$ then, by Case 1, L is an ordered multiplicity list for T. It suffices to note that, if $p \leq k_h$ (if $\leq l_h$, the argument is similar) then setting $k_h = p$ and

$l_h = 0$, we have $\sum_{i=1}^{t} k_i \leq k$ and $\sum_{i=1}^{t} l_i \leq l$. (Note that this case includes the case $k_h = 0$ or $l_h = 0$.)

Case 2.2: If $p > k_h$ and $p > l_h$. From $p > k_h$ we get $p + l_h > k_h + l_h$ and, because $k_h + l_h \geq 2 + p$, we conclude that $p + l_h \geq 3 + p$, i.e., $l_h \geq 3$. Similarly we conclude that $k_h \geq 3$. Thus, in this case, we have $p > k_h \geq 3$ and $p > l_h \geq 3$ and $p \geq 4$.

Let p_1, p_2 be integers such that $p_1 + p_2 = p$ and $k_h \geq p_1$ and $l_h \geq p_2$. Since $p = p_1 + p_2 > k_h \geq p_1$ and $p = p_1 + p_2 > l_h \geq p_2$ we have $p_1 > 0$ and $p_2 > 0$. (p_1 and p_2 are, respectively, the parts of p that will use "free" vertices among the $k - k'$ "free" vertices of S_{k+1} and the $l - l'$ "free" vertices of S_{l+1}.)

We start this case by considering the list L', obtained from L, by replacing the sequence $\boxed{1, p, 1}$ by the sequence $\boxed{1, 1}$. Let r be the position of the left 1 of the sequence $\boxed{1, 1}$ in L'. The size of L is $n = k + l + 2$ and L' has size $n - p$. By Lemmas 13 and 11, L' is an ordered multiplicity list for each double star $D(\overline{k}, \overline{l})$, on $n - p$ vertices, such that $\overline{k} \geq k'$ and $\overline{l} \geq l'$. Among these double stars we consider a particular double star $D(\overline{k}, \overline{l})$, in which $\overline{k} = k - p_1$ and $\overline{l} = l - p_2$. Note that $\overline{k} \geq k'$ (because $\overline{k} = k - p_1$ and $k_h \geq p_1$) and $\overline{l} \geq l'$ (because $\overline{l} = l - p_2$ and $l_h \geq p_2$).

Since L' is an ordered multiplicity list for $D(\overline{k}, \overline{l})$, there is a valid pair $\overline{b}^+$ and $\overline{c}^+$ (in which $\overline{b}^+$ and $\overline{c}^+$ are, respectively, augmented lists for the stars $S_{\overline{k}+1}$ and $S_{\overline{l}+1}$) such that $L' = \overline{b}^+ + \overline{c}^+$. By Lemmas 13 and 11, we may assume that L' has no split multiplicities.

Now we will insert exactly one entry, in each list $\overline{b}^+$ and $\overline{c}^+$, between the positions r and $r+1$, and make the necessary changes, in order to obtain augmented lists b^+ and c^+, for S_{k+1} and S_{l+1}, respectively, such that b^+ and c^+ form a valid pair and $L = b^+ + c^+$. Focussing the attention on the r and $r + 1$ entry of $\overline{b}^+$, $\overline{c}^+$ and L', in a diagram we have

$$\begin{array}{ccccccc} \overline{b}^+ & = & \cdots & \overline{b}^+_r & \overline{b}^+_{r+1} & \cdots \\ \overline{c}^+ & = & \cdots & \overline{c}^+_r & \overline{c}^+_{r+1} & \cdots \\ \hline L' & & \cdots & \boxed{1} & \boxed{1} & \cdots \end{array}$$

By the Superposition Principle, and avoiding cases of similar discussion, we must have either $\overline{b}^+_r = 1 = \overline{b}^+_{r+1}$ and $\overline{c}^+_r = 0 = \overline{c}^+_{r+1}$ (Case A), or $\overline{b}^+_r = 1 = \overline{c}^+_{r+1}$ and $\overline{c}^+_r = 0 = \overline{b}^+_{r+1}$ (Case B), i.e.,

$$\text{(Case A)} \quad \begin{array}{cccc} \cdots & 1 & 1 & \cdots \\ \cdots & 0 & 0 & \cdots \\ \hline \cdots & 1 & 1 & \cdots \end{array} \quad \text{and} \quad \text{(Case B)} \quad \begin{array}{cccc} \cdots & 1 & 0 & \cdots \\ \cdots & 0 & 1 & \cdots \\ \hline \cdots & 1 & 1 & \cdots \end{array}$$

In the process of constructing the desired valid pair b^+ and c^+, from the valid pair $\overline{b}^+$ and $\overline{c}^+$, given an identified entry of $\overline{b}^+$ (resp. $\overline{c}^+$) it will be important to know the status, as upward or non-upward, of the closest non-zero multiplicity, either on the left or on the right, of that entry. Thus, again avoiding similar cases, in Case A we need to consider two cases, (A.1) and (A.2),

$$(\text{A.1})\ \begin{array}{cccc} \cdots & 1 & 1 & \cdots \\ - & 0 & 0 & - \\ \hline \cdots & 1 & 1 & \cdots \end{array},\quad (\text{A.2})\ \begin{array}{cccc} \cdots & 1 & 1 & \cdots \\ - & 0 & 0 & \frown \\ \hline \cdots & 1 & 1 & \cdots \end{array}$$

(note that, in an augmented list, the non-zero entries closest to a 0, on the right and on the left of that 0, cannot be both upward) and, in Case B, we have three more cases, (B.1), (B.2) and (B.3),

$$(\text{B.1})\ \begin{array}{cccc} \cdots & 1 & 0 & - \\ - & 0 & 1 & \cdots \\ \hline \cdots & 1 & 1 & \cdots \end{array},\quad (\text{B.2})\ \begin{array}{cccc} \cdots & 1 & 0 & \frown \\ - & 0 & 1 & \cdots \\ \hline \cdots & 1 & 1 & \cdots \end{array},\quad (\text{B.3})\ \begin{array}{cccc} \cdots & 1 & 0 & \frown \\ \frown & 0 & 1 & \cdots \\ \hline \cdots & 1 & 1 & \cdots \end{array}$$

in which the symbol $\frown$ (resp. $-$) close to an entry a_i means that the closest non-zero entry, in the direction from a_i to that symbol, is (resp. is not) an upward multiplicity.

In Cases A.1 and B.1 it is enough to insert the upward multiplicities $\widehat{p_1}$ and $\widehat{p_2}$, between positions r and $r+1$, in $\overline{b}^+$ and $\overline{c}^+$, respectively. Note that each inserted upward multiplicities are located between two non-upward multiplicities, producing a valid pair b^+ and c^+ such that $L = b^+ + c^+$. In a diagram we have

$$(\text{A.1})\ \begin{array}{cccc} \cdots & 1 & 1 & \cdots \\ - & 0 & 0 & - \\ \hline \cdots & \boxed{1} & \boxed{1} & \cdots \end{array} \xrightarrow{\text{insertion}} \begin{array}{ccccc} \cdots & 1 & \widehat{p_1} & 1 & \cdots \\ - & 0 & \widehat{p_2} & 0 & - \\ \hline \cdots & \boxed{1} & p & \boxed{1} & \cdots \end{array}$$

and

$$(\text{B.1})\ \begin{array}{cccc} \cdots & 1 & 0 & - \\ - & 0 & 1 & \cdots \\ \hline \cdots & \boxed{1} & \boxed{1} & \cdots \end{array} \xrightarrow{\text{insertion}} \begin{array}{ccccc} \cdots & 1 & \widehat{p_1} & 0 & - \\ - & 0 & \widehat{p_2} & 1 & \cdots \\ \hline \cdots & \boxed{1} & p & \boxed{1} & \cdots \end{array}$$

In Case A.2, we need to consider the cases

$$(\text{A.2.1})\ \begin{array}{cccc} \cdots & 1 & 1 & \frown \\ - & 0 & 0 & \frown \\ \hline \cdots & 1 & 1 & \cdots \end{array},\quad (\text{A.2.2})\ \begin{array}{cccc} \cdots & 1 & 1 & - \\ - & 0 & 0 & \frown \\ \hline \cdots & 1 & 1 & \cdots \end{array}$$

and, each one, have subcases listed below, in which $\widehat{f}$ denotes the relevant upward multiplicity. Recall that L' has no split multiplicities so that, for example, in case $\overline{c}^+_{r+2} = \widehat{f}$, necessarily we must have either $\overline{b}^+_{r+2} = 0$ or $\overline{b}^+_{r+2} = 1$. The subcases of (A.2.1) are

$$(\text{Case A.2.1a})\ \begin{array}{ccccc} \cdots & 1 & 1 & \widehat{f} & \cdots \\ - & 0 & 0 & 0 & \frown \\ \hline \cdots & \boxed{1} & \boxed{1} & f & \cdots \end{array} \xrightarrow[\text{2 changes}]{\substack{\text{insertion} \\ \text{and}}} \begin{array}{cccccc} \cdots & 1 & \widehat{p_1+1} & 1 & \widehat{f-1} & \cdots \\ - & 0 & \widehat{p_2-1} & 0 & 1 & \frown \\ \hline \cdots & \boxed{1} & p & \boxed{1} & f & \cdots \end{array}$$

$$\text{(Case A.2.1b)}\quad \begin{array}{cccccc} \cdots & 1 & 1 & 0 & \widehat{\ } & \\ - & 0 & 0 & \widehat{f} & \cdots & \\ \hline \cdots & \boxed{1} & \boxed{1} & f & \cdots & \end{array} \xrightarrow[\substack{\text{insertion}\\ \text{and}\\ \text{4 changes}}]{} \begin{array}{cccccc} \cdots & 1 & \widehat{p_1} & 0 & 1 & \widehat{\ } \\ - & 0 & \widehat{p_2} & 1 & \widehat{f-1} & \cdots \\ \hline \cdots & \boxed{1} & p & \boxed{1} & f & \cdots \end{array}$$

and the subcases of (A.2.2) are

$$\text{(Case A.2.2a)}\quad \begin{array}{cccccc} \cdots & 1 & 1 & 1 & \cdots \\ - & 0 & 0 & \widehat{f} & \cdots \\ \hline \cdots & \boxed{1} & \boxed{1} & f+1 & \cdots \end{array} \xrightarrow[\substack{\text{insertion}\\ \text{and}\\ \text{2 changes}}]{} \begin{array}{cccccc} \cdots & 1 & \widehat{p_1+1} & 0 & 1 & \cdots \\ - & 0 & \widehat{p_2-1} & 1 & \widehat{f} & \cdots \\ \hline \cdots & \boxed{1} & p & \boxed{1} & f+1 & \cdots \end{array}$$

$$\text{(Case A.2.2b)}\quad \begin{array}{cccccc} \cdots & 1 & 1 & 0 & - \\ - & 0 & 0 & \widehat{f} & \cdots \\ \hline \cdots & \boxed{1} & \boxed{1} & f & \cdots \end{array} \xrightarrow[\substack{\text{insertion}\\ \text{and}\\ \text{2 changes}}]{} \begin{array}{cccccc} \cdots & 1 & \widehat{p_1+1} & 0 & 0 & - \\ - & 0 & \widehat{p_2-1} & 1 & \widehat{f} & \cdots \\ \hline \cdots & \boxed{1} & p & \boxed{1} & f & \cdots \end{array}$$

$$\text{(Case A.2.2c)}\quad \begin{array}{cccccc} \cdots & 1 & 1 & 1 & \cdots \\ - & 0 & 0 & 0 & \widehat{\ } \\ \hline \cdots & \boxed{1} & \boxed{1} & 1 & \cdots \end{array} \xrightarrow[\substack{\text{insertion}\\ \text{and}\\ \text{2 changes}}]{} \begin{array}{cccccc} \cdots & 1 & \widehat{p_1+1} & 0 & 1 & \cdots \\ - & 0 & \widehat{p_2-1} & 1 & 0 & \widehat{\ } \\ \hline \cdots & \boxed{1} & p & \boxed{1} & 1 & \cdots \end{array}$$

in which $\widehat{0}$ means a 0.

We turn now to Case B.2. We need to consider the cases

$$\text{(B.2.1)}\quad \begin{array}{cccc} \cdots & 1 & 0 & \widehat{\ } \\ - & 0 & 1 & - \\ \hline \cdots & 1 & 1 & \cdots \end{array} \quad \text{and} \quad \text{(B.2.2)}\quad \begin{array}{cccc} \cdots & 1 & 0 & \widehat{\ } \\ - & 0 & 1 & \widehat{\ } \\ \hline \cdots & 1 & 1 & \cdots \end{array}.$$

For (Case B.2.1) we have

$$\text{(Case B.2.1)}\quad \begin{array}{cccc} \cdots & 1 & 0 & \widehat{\ } \\ - & 0 & 1 & - \\ \hline \cdots & \boxed{1} & \boxed{1} & \cdots \end{array} \xrightarrow[\substack{\text{insertion}\\ \text{and}\\ \text{2 changes}}]{} \begin{array}{ccccc} - & 1 & \widehat{p_1-1} & 1 & \widehat{\ } \\ - & 0 & \widehat{p_2+1} & 0 & - \\ \hline \cdots & \boxed{1} & p & \boxed{1} & \cdots \end{array}$$

and for (Case B.2.2) we have the subcases (B.2.2a) and (B.2.2b) listed below, depending on the location of the relevant upward multiplicities

$$\text{(Case B.2.2a)}\quad \begin{array}{ccccc} \cdots & 1 & 0 & \widehat{f} & \cdots \\ - & 0 & 1 & 0 & \widehat{\ } \\ \hline \cdots & \boxed{1} & \boxed{1} & f & \cdots \end{array} \xrightarrow[\substack{\text{insertion}\\ \text{and}\\ \text{4 changes}}]{} \begin{array}{cccccc} \cdots & 1 & \widehat{p_1} & 1 & \widehat{f-1} & \cdots \\ - & 0 & \widehat{p_2} & 0 & 1 & \widehat{\ } \\ \hline \cdots & \boxed{1} & p & \boxed{1} & f & \cdots \end{array}$$

$$\text{(Case B.2.2b)}\quad \begin{array}{ccccc} \cdots & 1 & 0 & 0 & \widehat{\ } \\ - & 0 & 1 & \widehat{f} & \cdots \\ \hline \cdots & \boxed{1} & \boxed{1} & f & \cdots \end{array} \xrightarrow[\substack{\text{insertion}\\ \text{and}\\ \text{2 changes}}]{} \begin{array}{cccccc} \cdots & 1 & \widehat{p_1-1} & 0 & 1 & \widehat{\ } \\ - & 0 & \widehat{p_2+1} & 1 & \widehat{f-1} & \cdots \\ \hline \cdots & \boxed{1} & p & \boxed{1} & f & \cdots \end{array}$$

We turn now to Case B.3. We need to consider the cases

$$\text{(B.3.1)}\quad \begin{array}{cccc} \cdots & 1 & 0 & \frown \\ \frown & 0 & 1 & \frown \\ \hline \cdots & 1 & 1 & \cdots \end{array} \quad \text{and} \quad \text{(B.3.2)}\quad \begin{array}{cccc} \cdots & 1 & 0 & \frown \\ \frown & 0 & 1 & - \\ \hline \cdots & 1 & 1 & \cdots \end{array}\ .$$

For the Case (B.3.1) we consider two subcases

$$\text{(B.3.1.1)}\quad \begin{array}{cccc} \frown & 1 & 0 & \frown \\ \frown & 0 & 1 & \frown \\ \hline \cdots & 1 & 1 & \cdots \end{array} \quad \text{and} \quad \text{(B.3.1.2)}\quad \begin{array}{cccc} - & 1 & 0 & \frown \\ \frown & 0 & 1 & \frown \\ \hline \cdots & 1 & 1 & \cdots \end{array}\ .$$

For (B.3.1.1) we have the subcases (B.3.1.1a), (B.3.1.1b) and (B.3.1.1c) listed below, depending on the location of the relevant upward multiplicities $\widehat{f_1}$ and $\widehat{f_2}$

$$\text{(Case B.3.1.1a)}\quad \begin{array}{cccccc} \cdots & \widehat{f_1} & 1 & 0 & \widehat{f_2} & \cdots \\ \frown & 0 & 0 & 1 & 0 & \frown \\ \hline \cdots & f_1 & \boxed{1} & \boxed{1} & f_2 & \cdots \end{array} \quad \underset{\substack{\text{insertion}\\ \text{and}\\ \text{6 changes}}}{\longrightarrow} \quad \begin{array}{ccccccc} \cdots & \widehat{f_1-1} & 1 & \widehat{p_1+1} & 1 & \widehat{f_2-1} & \cdots \\ \frown & 1 & 0 & \widehat{p_2-1} & 0 & 1 & \frown \\ \hline \cdots & f_1 & \boxed{1} & p & \boxed{1} & f_2 & \cdots \end{array}$$

$$\text{(Case B.3.1.1b)}\quad \begin{array}{cccccc} \cdots & \widehat{f_1} & 1 & 0 & 0 & \frown \\ \frown & 0 & 0 & 1 & \widehat{f_2} & \cdots \\ \hline \cdots & f_1 & \boxed{1} & \boxed{1} & f_2 & \cdots \end{array} \quad \underset{\substack{\text{insertion}\\ \text{and}\\ \text{4 changes}}}{\longrightarrow} \quad \begin{array}{ccccccc} \cdots & \widehat{f_1-1} & 1 & \widehat{p_1} & 0 & 1 & \frown \\ \frown & 1 & 0 & \widehat{p_2} & 1 & \widehat{f_2-1} & \cdots \\ \hline \cdots & f_1 & \boxed{1} & p & \boxed{1} & f_2 & \cdots \end{array}$$

$$\text{(Case B.3.1.1c)}\quad \begin{array}{cccccc} \frown & 0 & 1 & 0 & \widehat{f_2} & \cdots \\ \cdots & \widehat{f_1} & 0 & 1 & 0 & \frown \\ \hline \cdots & f_1 & \boxed{1} & \boxed{1} & f_2 & \cdots \end{array} \quad \underset{\substack{\text{insertion}\\ \text{and}\\ \text{8 changes}}}{\longrightarrow} \quad \begin{array}{ccccccc} \frown & 1 & 0 & \widehat{p_1} & 1 & \widehat{f_2-1} & \cdots \\ \cdots & \widehat{f_1-1} & 1 & \widehat{p_2} & 0 & 1 & \frown \\ \hline \cdots & f_1 & \boxed{1} & p & \boxed{1} & f_2 & \cdots \end{array}$$

For (B.3.1.2) we have the subcases (B.3.1.2a) and (B.3.1.2b) listed below, depending on where are the relevant upward multiplicity f

$$\text{(Case B.3.1.2a)}\quad \begin{array}{ccccc} - & 1 & 0 & \widehat{f} & \cdots \\ \frown & 0 & 1 & 0 & \frown \\ \hline \cdots & \boxed{1} & \boxed{1} & f & \cdots \end{array} \quad \underset{\substack{\text{insertion}\\ \text{and}\\ \text{6 changes}}}{\longrightarrow} \quad \begin{array}{cccccc} - & 0 & \widehat{p_1+1} & 1 & \widehat{f-1} & \cdots \\ \frown & 1 & \widehat{p_2-1} & 0 & 1 & \frown \\ \hline \cdots & \boxed{1} & p & \boxed{1} & f & \cdots \end{array}$$

$$\text{(Case B.3.1.2b)}\quad \begin{array}{ccccc} - & 1 & 0 & 0 & \frown \\ \frown & 0 & 1 & \widehat{f} & \cdots \\ \hline \cdots & \boxed{1} & \boxed{1} & f & \cdots \end{array} \quad \underset{\substack{\text{insertion}\\ \text{and}\\ \text{4 changes}}}{\longrightarrow} \quad \begin{array}{cccccc} - & 0 & \widehat{p_1} & 0 & 1 & \frown \\ \frown & 1 & \widehat{p_2} & 1 & \widehat{f-1} & \cdots \\ \hline \cdots & \boxed{1} & p & \boxed{1} & f & \cdots \end{array}$$

We finally turn to Case B.3.2. We need only to consider

$$\text{(Case B.3.2)}\quad \begin{array}{cccc} - & 1 & 0 & \frown \\ \frown & 0 & 1 & - \\ \hline \cdots & \boxed{1} & \boxed{1} & \cdots \end{array} \quad \underset{\substack{\text{insertion}\\ \text{and}\\ \text{4 changes}}}{\longrightarrow} \quad \begin{array}{ccccc} - & 0 & \widehat{p_1} & 1 & \frown \\ \frown & 1 & \widehat{p_2} & 0 & - \\ \hline \cdots & \boxed{1} & p & \boxed{1} & \cdots \end{array}\ .$$

Each displayed small change in $\overline{b}^+$ and $\overline{c}^+$ (and no other changes are performed) guarantees that: (1) an upward multiplicity is inserted between two non-upward multiplicities, and (2) the obtained lists are augmented lists b^+ and c^+ for S_{k+1} and S_{l+1}, respectively, and (3) b^+ and c^+ form a valid pair such that $b^+ + c^+ = L$. Therefore, L is an ordered multiplicity list for the double star $D(k,l)$. □

We now present some illustrative examples of application of Theorem 14.

Example 15. Let T be the double star $D(9,4)$. From Theorem 14 we conclude:

1. $L = (1,3,2,3,2,3,1)$ is an ordered multiplicity list for T and, any other sequence with these multiplicities, distinct of L, is not a an ordered multiplicity of T,
2. (a) $L_1 = (1,2,2,2,2,2,2,1,1)$ is not an ordered multiplicity list for T.
   (b) $L_2 = (1,2,2,2,2,2,1,2,1)$ is an ordered multiplicity list for T.
   (c) $L_3 = (1,2,2,2,2,1,2,2,1)$ is not an ordered multiplicity list for T.
   (d) $L_4 = (1,2,2,2,1,2,2,2,1)$ is an ordered multiplicity list for T.
3. $L = (1,5,3,3,2,1)$ is not an ordered multiplicity list for T and, any other sequence with these multiplicities, is not an ordered multiplicity list for T.
4. $L = (1,2,3,1,2,2,3,1)$ is an ordered multiplicity list for T which can be obtained, by the Superposition Principle, as

$$\begin{array}{rcccccccc} S_{10}: & 0 & 1 & \widehat{3} & 1 & \widehat{1} & 1 & \widehat{2} & 1 \\ S_5: & 1 & \widehat{1} & 0 & 0 & 1 & \widehat{1} & 1 & 0 \\ \hline & 1 & 2 & 3 & 1 & 2 & 2 & 3 & 1 \end{array}$$

or

$$\begin{array}{rcccccccc} S_{10}: & 0 & 1 & \widehat{2} & 1 & \widehat{2} & 1 & \widehat{2} & 1 \\ S_5: & 1 & \widehat{1} & 1 & 0 & 0 & \widehat{1} & 1 & 0 \\ \hline & 1 & 2 & 3 & 1 & 2 & 2 & 3 & 1 \end{array}$$

In the following example we focus upon proposed lists with a singleton string.

Example 16. Let T be the double star $D(7,7)$.

1. $L = (1,2,2,2,2,1,5,1)$ is not an ordered multiplicity list for T.
2. $L = (1,2,2,2,2,1,1,4,1)$ is not an ordered multiplicity list for T.
3. $L = (1,2,2,2,2,1,4,1,1)$ is an ordered multiplicity list for T in which 4 is a mandatory split multiplicity (see Example 9).
4. $L = (1,2,2,2,1,4,1,2,1)$ is an ordered multiplicity list for T but the multiplicity 4 is not a mandatory split multiplicity (see Example 9).

5. $L = (1, 2, 2, 1, 4, 1, 2, 2, 1)$ is also an ordered multiplicity list for T in which 4 is a mandatory split multiplicity (see Example 9). Note the new position of the sequence $\boxed{1, 4, 1}$ in L.
6. $L = (1, 4, 4, 1, 4, 1, 1)$ is an ordered multiplicity list for T in which the multiplicity 4 is a mandatory split multiplicity (see Example 9).

The following example is about ordered multiplicity lists with several strings and with a singleton string corresponding to a mandatory split multiplicity.

Example 17. We may have double stars with an ordered multiplicity list, with several strings, and a mandatory split multiplicity.

1. $(1, 6, 2, 1, 4, 1, 1)$ is an ordered multiplicity list for the double star $D(9, 5)$ in which 4 is a mandatory split multiplicity.
   Since we have the string $\boxed{6, 2}$ and $k = 9$ and $l = 5$, the multiplicity 6 must be one sided k (because $6 > l$) and the multiplicity 2 must be one sided l. Thus, the multiplicity 4 of the singleton string must be a mandatory split multiplicity and the corresponding eigenvalue must appears on 3 pendent vertices among k and l.
2. With a similar analysis to the previous example, we conclude that $(1, 12, 2, 1, 12, 2, 1, 12, 2, 1, 6, 1, 1)$ is an ordered multiplicity list for the double star $D(41, 11)$ in which 6 is a mandatory split multiplicity.
3. More generally, for any positive integer j, setting $l = 2 + 3j$ and $k = 2 + 4j + 3j^2$ and $p = 3 + j$, the list

$$L = (1, l + 1, 2, 1, l + 1, 2, 1, \dots, l + 1, 2, 1, p, 1, 1),$$

   with size $k + l + 2$, in which the string $\boxed{l + 1, 2}$ appears j times, is an ordered multiplicity list for the double star $D(k, l)$, in which the multiplicity p is a mandatory split multiplicity.
   Since multiplicity $l + 1$ in each string $\boxed{l + 1, 2}$ of L is greater than the number l of pendent vertices of the star S_{l+1}, and that string occurs j times in L, we must have j eigenvalues one sided k (each eigenvalue with multiplicity $l + 1$) and j eigenvalues one sided l (each eigenvalue with multiplicity 2). Setting $k' = j(l + 1) = j(3 + 3j) = 3j + 3j^2$ and $l' = 2j$, we have $\overline{k} = k - k' = (2 + 4j + 3j^2) - (3j + 3j^2) = 2 + j$ and $\overline{l} = l - l' = (2 + 3j) - 2j = 2 + j$. Because $p = 3 + j > \overline{k} = 2 + j$ and $p = 3 + j > \overline{l} = 2 + j$, we conclude that the multiplicity p can occur because $p + 2 = (3 + j) + 2 = 5 + j \leq \overline{k} + \overline{l} = 4 + 2j$ and, therefore, this multiplicity must occur in L, as claimed, as a mandatory split multiplicity.
   Finally, we show that the described list L has size $k + l + 2 = (2 + 4j + 3j^2) + (2 + 3j) + 2 = 3j^2 + 7j + 6$. The size of L is the sum of the size of j strings $\boxed{l + 1, 2}$, plus $j + 3$ 1's, and the multiplicity $p = 3 + j$, which gives $j(l + 1 + 2) + (j + 3) + p = j(2 + 3j + 3) + 2(j + 3) = 3j^2 + 7j + 6$.

4. Diameter 5 trees

We now turn to trees of diameter 5 and ask a somewhat different question. Since diameter 5 permits nonlinear trees and very little is known about even unordered multiplicity lists for nonlinear trees (save the specific cases contained in the database [5, Appendix B]), we show how to generate all unordered multiplicity lists for diameter 5. As it turns out, there is a relationship between diameter 5 and diameter 6 trees that permits a similar analysis for diameter 6. And so diameter 6 is presented in the next section. Our method is to use assignments, which is almost the only method generally available. Unfortunately, because an example for diameter 7 that shows a limitation of assignments [6, example 2.3], our methods do not generally work past diameter 6. Of course, the linear trees of diameter 5 (and 6) could be analyzed via superposition [4,2,3], but these are a modest portion of diameter 5 and 6 trees [12], so we simply consider all diameter 5 (and then 6) trees at once. Previously, there was no way to generate all lists for all such trees.

An important observation is that any tree T of diameter 5 must have a (unique) center vertex c that is the middle vertex of every diameter. Adjacent to c are vertices of only 3 types: (1) centers of nontrivial stars; (2) vertices of pendent edges; and (3) pendent vertices. Call the stars $S_1, \dots, S_{k_1}$ with centers, respectively, $b_1, \dots, b_{k_1}$; call the edges $E_{k_1+1}, \dots, E_{k_1+k_2}$ with vertices adjacent to c, respectively, $b_{k_1+1}, \dots, b_{k_1+k_2}$, and call the pendent vertices at c by $b_{k_1+k_2+1}, \dots, b_{k_1+k_2+k_3}$. If star S_i, $i = 1, \dots, k_1$, has $p_i \geq 2$ pendents, we have a total of $\sum_{i=1}^{k_1} p_i + k_2 + k_3$ pendents in all and a total of $\sum_{i=1}^{k_1} p_i + k_1 + 2k_2 + k_3 + 1$ vertices in all. Star S_i, $i = 1, \dots, k_1$, has a total of $p_i + 1$ eigenvalues; edge E_i, $i = k_1 + 1, \dots, k_1 + k_2$, has 2 eigenvalues; and pendent b_i, $i = k_1 + k_2 + 1, \dots, k_1 + k_2 + k_3$, has 1 eigenvalue, the number that appears on the vertex. Any multiple eigenvalue λ of $A \in \mathcal{S}(T)$ must have at least one of $c, b_1, \dots, b_{k_1}$, as a Parter vertex.

Each pendent vertex of star S_i may be assigned by an eigenvalue and, additionally, each star may be assigned by two multiplicity 1 eigenvalues. Each edge may be assigned by two multiplicity 1 eigenvalues, and each pendent from c, by one eigenvalue of multiplicity 1. These assignments are essentially independent, except, of course, that the two additional multiplicity 1 eigenvalues of star S_i must be one above and the other below the assignments to its pendents. There are no other constraints on the numerical order of the eigenvalues, and each eigenvalue may be assigned independently of the others. Of course, there are only so many "slots" for eigenvalues, and when they are exhausted, no more assignments may be made.

For an eigenvalue λ to be multiple, it must have a Parter vertex (possible several) from among $c, b_1, \dots, b_{k_1}$. There are three possibilities to consider: (i) The Parter vertices for λ are a nonempty subset of $\{b_1, \dots, b_{k_1}\}$ and not c; (ii) c alone; or (iii) a nonempty subset of $\{b_1, \dots, b_{k_1}\}$, together with c.

We may now determine the multiplicity of each eigenvalue λ, one at a time, i.e., $\mathrm{m}_A(\lambda)$ for $A \in \mathcal{S}(T)$.

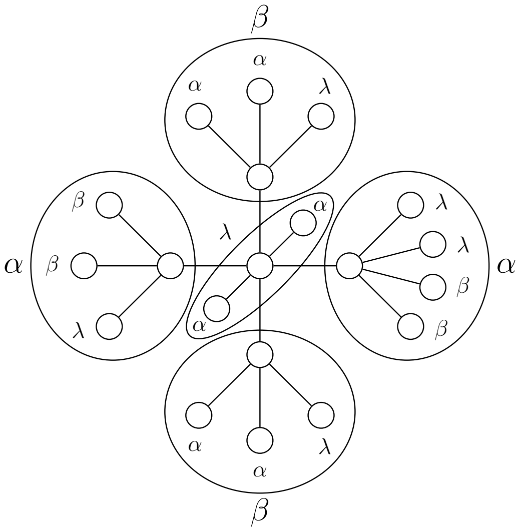

Fig. 1. First illustration of Example 18.

In Case (i), we count the number of pendent vertices on which λ appears among the stars $S_1, \ldots, S_{k_1}$ and subtract the number of such stars. If λ is an eigenvalue (once) of the residual tree left after removal of the indicated stars, we add 1 to the multiplicity of λ. This is the total multiplicity of λ in T. In the residual tree λ cannot occur more than once and would be non-upward, else there would be another Parter vertex among HDV's in the residual tree.

In Case (ii), we count the number of stars among $S_1, \ldots, S_{k_1}$, add the number of edges among $E_{k_1+1}, \ldots, E_{k_1+k_2}$ on which λ appears, add the number of pendents among $b_{k_1+k_2+1}, \ldots, b_{k_1+k_2+k_3}$ on which λ appears, and subtract 1 (for c) and this is the total multiplicity for λ.

In Case (iii), we first count the number of pendent vertices of stars on which λ appears and subtract the number of such stars, as in Case (i). Then, we add the number of stars on which λ appears non-upwardly, plus the number of vertices and edges pendent from c on which λ appears and subtract 1 (for c). This is the total multiplicity of λ in Case (iii).

Finally, if we continue the same process for each of the other eigenvalues, add the total of all multiplicities found, and subtract this quantity from the total number of vertices in T, we have the number of 1's that occur in our list. We can then form an unordered multiplicity list for T by combining the multiplicities found for each eigenvalue.

Example 18. Fig. 1 gives an example of how our process can be used to find a multiplicity list for a diameter 5 nonlinear tree. Using our process, we note the following multiplicities: $\mathrm{m}(\lambda) = 6 - 4 = 2$, $\mathrm{m}(\alpha) = 8 - 3 = 5$, $\mathrm{m}(\beta) = 6 - 3 = 3$. Also, the number of "1"s in our list will be $20 - 10 = 10$. Thus $(5, 3, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$ will be an unordered multiplicity list for this tree.

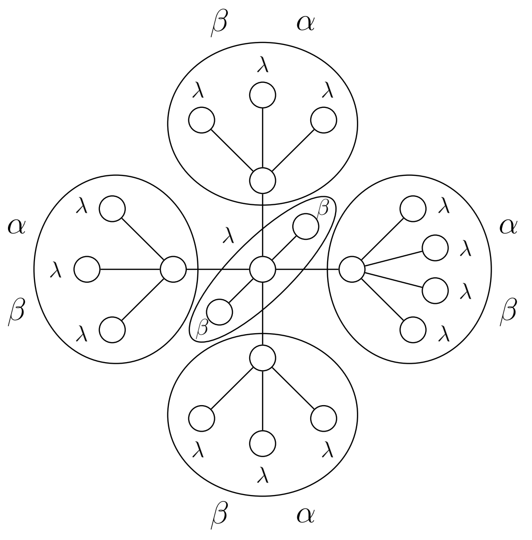

Fig. 2. Second illustration of Example 18.

Fig. 2 shows the occurrence of a different multiplicity list on the same diameter 5 nonlinear tree. Using our process, we note that $\mathrm{m}(\lambda) = 14 - 4 = 10$, $\mathrm{m}(\alpha) = 4 - 1 = 3$, and $\mathrm{m}(\beta) = 6 - 1 = 5$. Also, there will be $20 - 18 = 2$ "1"s in our list. Thus $(10, 5, 3, 1, 1)$ will be another unordered multiplicity list for this tree.

5. Diameter 6 trees

Again, for diameter 6, a tree may be linear or nonlinear and the linear case is again covered by the LSP. Nonetheless, it is simplest to consider all diameter 6 trees in the same way. There is little known about the nonlinear case (which is the majority [12]) and we focus upon unordered lists. Fortunately, the form of diameter 6 trees is only slightly more complicated than the diameter 5 case. In place of the center vertex c is a center edge with endpoints $\{c_1, c_2\}$, i.e., a diameter 6 tree is a vertex partition of a diameter 5 one. Our procedure is similar to the diameter 5 case. The center vertex c is partitioned into 2 adjacent vertices c_1 and c_2 and the neighbors of c are partitioned between c_1 and c_2. So, we now label these analogously. The stars $S_{i1}, \ldots, S_{ik_{i1}}$ pendent at c_i, $i = 1, 2$, have centers $b_{i1}, \ldots, b_{ik_{i1}}$, etc.

As in diameter 5 case, we wish to find multiplicity of each λ, one at a time. Now, there are a few more combinations of the possible Parter vertices: $c_1, c_2, b_{11}, \ldots, b_{1k_{11}}, b_{21}, \ldots, b_{2k_{21}}$ to consider. The combinations are a subset of $C = \{c_1, c_2\}$ with a subset of $B_1 = \{b_{11}, \ldots, b_{1k_{11}}\}$ with a subset of $B_2 = \{b_{21}, \ldots, b_{2k_{21}}\}$, at least one of which is nonempty. But there are obvious symmetries in the cases, and most are analogous to diameter 5 cases.

A subset of B_1 or B_2 or of both is exactly analogous. The set C is analogous, and the sets $\{c_1\}$ or $\{c_2\}$ are similar, except that, in each case, the residual tree (diameter 5 with

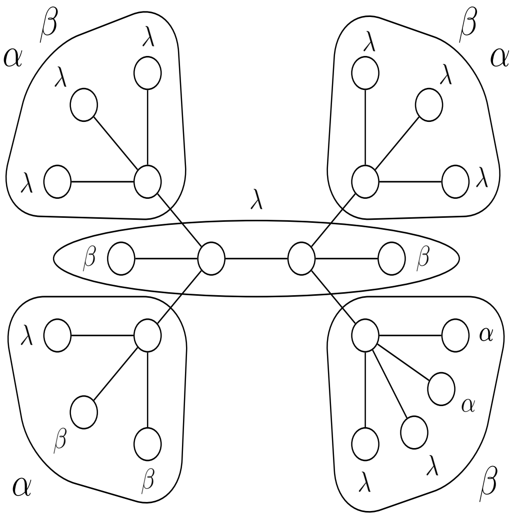

Fig. 3. First illustration of Example 19.

the other of c_1 or c_2 as centers) may add 1 to the multiplicity of λ if λ is a multiplicity 1 (non-upward) eigenvalue of the residual tree.

This leaves subsets of B_1 and/or B_2 combined with a subset of C. Up to obvious symmetries, the possible combinations are

(i) $B_1, \{c_1\}$,
(ii) $B_1, \{c_2\}$,
(iii) $B_1, B_2, \{c_1\}$, and
(iv) B_1, B_2, C.

Case (i) is analogous to Case (iii) in the diameter 5 case, except that the diameter 5 subtree with center c_2 may contribute an additional 1 to the multiplicity of λ. Case (ii), if B_1 includes all the star centers pendent at c_1, is simply the union of Cases (i) and (ii) from the diameter 5 case. In Case (ii) if some star centers are absent from B_1, the residual diameter 5 tree with center c_2 may contribute an additional 1 to the multiplicity of λ. Case (iii) above is analogous to the diameter 5 case (iii), except that, again, the residual tree (as well as each of the vertices, edges and other stars pendent at c_1) may add 1 to the multiplicity. Case (iv) is exactly analogous to the diameter 5 case (iii). This completes the analysis necessary for diameter 6.

Finally, as in the diameter 5 case, we continue the same process for each of the other eigenvalues, add the total number of all multiplicities found, and subtract this quantity from the total number of vertices in T. We then have the number of non-upward 1's that occur in our list. We can then form a multiplicity list for T by combining the multiplicities found for each eigenvalue.

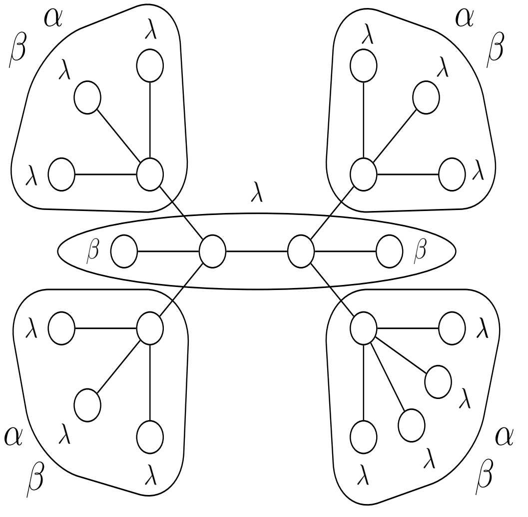

Fig. 4. Second illustration of Example 19.

Example 19. Fig. 3 gives an example of how our process can be used to find a multiplicity list for a diameter 6 nonlinear tree. Using our process, we note the following multiplicities: $\mathrm{m}(\lambda) = 10-4 = 6$, $\mathrm{m}(\alpha) = 5-3 = 2$, $\mathrm{m}(\beta) = 7-3 = 4$. Also, the number of 1's in our list will be $21 - 12 = 9$. Thus $(6, 4, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1)$ will be an unordered multiplicity list for this tree.

Finally, Fig. 4 shows the occurrence of a different multiplicity list on the same diameter 6 nonlinear tree. Using our process, we note that $\mathrm{m}(\lambda) = 14 - 4 = 10$, $\mathrm{m}(\alpha) = 4 - 2 = 2$, and $\mathrm{m}(\beta) = 6 - 2 = 4$. There will be $21 - 16 = 5$ "1"s in our list and so $(10, 4, 2, 1, 1, 1, 1, 1)$ will be another unordered multiplicity list for this tree.

Declaration of competing interest

The authors declare that they have no competing interests.

Data availability

No data was used for the research described in the article.

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