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Fundamental graphs for the maximum multiplicity of an eigenvalue among Hermitian matrices with a given graph

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Abstract

Our purpose is to identify the graphs that are "fundamental" for the maximum multiplicity problem for Hermitian matrices with a given undirected simple graph. Like paths for trees, these are the special graphs to which the maximum multiplicity problem may be reduced. These are the graphs for which maximum multiplicity implies that all vertices are downers. Examples include cycles and complete graphs, and several more are identified, using the theory developed herein. All the unicyclic graphs that are fundamental, are explicitly identified. We also list those graphs with two edges added to a tree, and their maximum multiplicities, which we have found so far to be fundamental. A formula for maximum multiplicity is given based on fundamental graphs.

Keywords Eigenvalue multiplicity · Fundamental graph · Graph of an Hermitian matrix · Maximum multiplicity · Trees

Mathematics Subject Classification 15A18 · 15B57 · 05C50

Dedicated to my long time friend, collaborator and co-mentor, Ilya Spitkovsky, whom I value greatly. (C.J.)

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1 Introduction

Throughout G is a simple undirected graph on n vertices, and $\mathcal{H}(G)$ denotes the set of all Hermitian matrices whose graph is G. No restriction is placed by G on the diagonal entries of $A \in \mathcal{H}(G)$, besides reality. We denote by $\mathrm{m}_A(\lambda)$ the multiplicity of λ as an eigenvalue of A and by $\sigma(A)$ the spectrum of A.

Let $\mathrm{M}(G)$ denote the maximum multiplicity, $\max \mathrm{m}_A(\lambda)$, over $\lambda \in \sigma(A)$ and $A \in \mathcal{H}(G)$ [6]. Of course, $\mathrm{M}(G) = n - \min \operatorname{rank}(A)$, over $A \in \mathcal{H}(G)$. $\mathrm{M}(G)$, and how to determine it, has long been of interest from a variety of perspectives. Ours is the general multiplicity problem for Hermitian matrices with a given graph [1, 2, 6–9, 11, 13]. When G is a tree T, there is an elegant solution to this problem [6], $\mathrm{M}(T) = \mathrm{P}(T) = \max(p - q)$, in which $\mathrm{P}(T)$ is the *path cover number* of T, and p is the number of paths remaining when sufficiently many vertices q have been removed from T to leave only paths. For general graphs there is sometimes, but seldom, a similar solution. For trees, then, paths are the fundamental building blocks for $\mathrm{M}(G)$. Is there anything similar for general graphs?

A vertex v of G, relative to an eigenvalue λ of $A \in \mathcal{H}(G)$, is called *Parter* [15] (resp. *neutral*, *downer*) if $\mathrm{m}_{A(v)}(\lambda) = \mathrm{m}_A(\lambda) + 1$ (resp. $\mathrm{m}_{A(v)}(\lambda) = \mathrm{m}_A(\lambda)$, $\mathrm{m}_{A(v)}(\lambda) = \mathrm{m}_A(\lambda) - 1$). Here $A(v)$ means, as usual, the principal submatrix of A resulting from deletion of row and column v. (Of course, deletion of v from $G = G(A)$ corresponds to $A(v)$, and we often view these interchangeably.) Because of the interlacing inequalities [5], these are the only possibilities, and all three may occur. Our purpose here is to introduce an additional kind of building block useful for general graphs in a way similar to paths for trees. In case G is a tree T (that is not a path), whenever $\mathrm{M}(T)$ is attained there will be Parter vertices (of degree at least 3) [10], and there will be a sequential removal of these, leaving only paths, that will give $\max(p - q)$. A path on t vertices, P_t, has $\mathrm{M}(P_t) = 1$. If $t \geq 3$, the path P_t could have Parter vertices, but generically will not.

We call a graph G, that is not a tree, a *fundamental graph* (FG) if whenever $\mathrm{M}(G)$ is attained by $A \in \mathcal{H}(G)$, the corresponding eigenvalue has no Parter vertices. Except for paths, which we view separately, no tree is fundamental by this definition. Some simple examples are (1) the complete graph K_n ($\mathrm{M}(K_n) = n - 1$, and each vertex is a downer for any maximum multiplicity eigenvalue) and (2) a cycle C_n ($\mathrm{M}(C_n) = 2$ [3, 4], and removal of any vertex leaves a path P_{n-1} with $\mathrm{M}(P_{n-1}) = 1$ [4]). We will discover many other examples.

It is no accident that in the examples above all vertices had to be downers. From [13] we have the following lemma.

Lemma 1.1 *Let G be any graph, $A \in \mathcal{H}(G)$ and $\lambda \in \sigma(A)$ be such that* $\mathrm{m}_A(\lambda) = \mathrm{M}(G)$. *Then no vertex of G is neutral in G for the eigenvalue λ relative to A.*

Thus, we have the following theorem.

Theorem 1.2 *If G is a graph, then G is an* FG *if and only if every vertex of G is a downer for any eigenvalue that attains* $\mathrm{M}(G)$.

Proof It suffices to note that, if G is an FG then, by definition, there can be no Parter vertices and, by Lemma 1.1, there are no neutral vertices in G for any eigenvalue attaining $\mathrm{M}(G)$. Thus, every vertex of an FG is a downer for any maximum multiplicity eigenvalue. On the other hand, if every vertex is a downer for any eigenvalue attaining $\mathrm{M}(G)$, there are no Parter vertices for such eigenvalues, and, by definition, G is an FG. □

2 Characterization of fundamental graphs

If G is a graph on n vertices, there are n, single vertex deleted subgraphs $G_1, \ldots, G_n$ and each of these has a maximum multiplicity

$$M_1 = \mathrm{M}(G_1), \ldots, M_n = \mathrm{M}(G_n).$$

Again, because of interlacing, these positive integers may not be too far apart. The values for $\max_{1 \leq i,j \leq n}(M_i - M_j)$ can be **0**, **1**, or **2**. We index these cases by the maximum difference. In **Case 0**, all M_i's are the same; in **Case 1**, there are two different, but consecutive, values; in **Case 2**, there are two or three different values. We begin with a lemma we shall use repeatedly. (See [14] for a proof of the inequalities for rank.)

Lemma 2.1 *If G is a simple graph and $G - v$ is the subgraph of G resulting from the removal of vertex v of G, then*

$$\mathrm{M}(G) - 1 \leq \mathrm{M}(G - v) \leq \mathrm{M}(G) + 1,$$

or equivalently,

$$\mathrm{M}(G - v) - 1 \leq \mathrm{M}(G) \leq \mathrm{M}(G - v) + 1.$$

In particular, if v is Parter for an eigenvalue of an $A \in \mathcal{H}(G)$ that attains $\mathrm{M}(G)$ then $\mathrm{M}(G - v) = \mathrm{M}(G) + 1$.

Proof Let v be a vertex of G and $A \in \mathcal{H}(G)$ be a matrix that has λ as an eigenvalue. For the inequalities, it suffices to note that if $\mathrm{m}_{A(v)}(\lambda) = \mathrm{M}(G - v)$, then, by interlacing, we have $\mathrm{m}_A(\lambda) \geq \mathrm{M}(G - v) - 1$ and, therefore, $\mathrm{M}(G) \geq \mathrm{M}(G - v) - 1$. Similarly, if $\mathrm{m}_A(\lambda) = \mathrm{M}(G)$ then, again by interlacing, we have $\mathrm{m}_{A(v)}(\lambda) \geq \mathrm{M}(G) - 1$ and, therefore, $\mathrm{M}(G - v) \geq \mathrm{M}(G) - 1$. Thus, we have the claimed inequalities.

Finally, if $\mathrm{m}_A(\lambda) = \mathrm{M}(G)$ and $\mathrm{m}_{A(v)}(\lambda) = \mathrm{M}(G) + 1$, then $\mathrm{M}(G - v) \geq \mathrm{M}(G) + 1$ and, since $\mathrm{M}(G - v) \leq \mathrm{M}(G) + 1$, it follows that $\mathrm{M}(G - v) = \mathrm{M}(G) + 1$. □

We say that a matrix A is a *maximizing* matrix for $\mathrm{M}(G)$ when $A \in \mathcal{H}(G)$ and A has an eigenvalue of multiplicity $\mathrm{M}(G)$. If v is a vertex of G and A is a maximizing matrix for $\mathrm{M}(G)$, we further say that v is Parter (resp. neutral, downer) for a maximizing matrix A, meaning that v is Parter (resp. neutral, downer) for the relevant eigenvalue of A that achieves the multiplicity $\mathrm{M}(G)$. (Note that we may have a graph G and a maximizing matrix A for $\mathrm{M}(G)$ which has, for example, two eigenvalues of multiplicity $\mathrm{M}(G)$. Each of these eigenvalues may have a different set of Parter vertices. A simple example is the tree obtained by connecting the interior vertices of two paths on three vertices with an edge. This tree is a double star, has maximum multiplicity 2 and admits two eigenvalues of multiplicity 2. Each vertex of degree 3 is Parter for only one of these eigenvalues.)

Returning to the discussion prior to Lemma 2.1, we study now the **Cases 0**, **1** and **2**. Note that, by Lemma 2.1, we have, for each G_i,

$$\mathrm{M}(G_i) - 1 \le \mathrm{M}(G) \le \mathrm{M}(G_i) + 1.$$

Theorem 2.2 0. *In* **Case 0***, G is an* FG *if and only if either* $\mathrm{M}(G) = M_i + 1$ *or* $\mathrm{M}(G) = M_i$*, and either may occur.*

1. *In* **Case 1***, we have either* $\mathrm{M}(G) = \min_{1\le i\le n} M_i$ *or* $\mathrm{M}(G) = \max_{1\le i\le n} M_i$*. The former case occurs if and only if G is not an* FG*; and the latter case occurs if and only if G is an* FG.
2. *In* **Case 2***, we have* $\mathrm{M}(G) = \min_{1\le i\le n} M_i + 1$ *and G is not be an* FG.

Proof **Case 0**: By Lemma 2.1 we may have $\mathrm{M}(G) = M_i + 1$ or $\mathrm{M}(G) = M_i$ or $\mathrm{M}(G) = M_i - 1$.

In the event that $\mathrm{M}(G) \ge M_i$, since there are no neutral vertices in G for the eigenvalue attaining maximum multiplicity $\mathrm{M}(G)$ (Lemma 1.1), it implies that any index of G will be a downer for any maximizing matrix for $\mathrm{M}(G)$, which means that G is an FG. (The statement "either may occur" will be shown in Examples 2.3.)

In the event that $\mathrm{M}(G) = M_i - 1$ choose any subgraph G_i and consider a matrix $A \in \mathcal{H}(G)$ such that $A(i)$ is a maximizing matrix for M_i. Since $\mathrm{M}(G) = M_i - 1$ we conclude that A is a maximizing matrix for $\mathrm{M}(G)$. This means that i is Parter for the maximizing matrix A and, therefore, G is not an FG.

Case 1: In this case we have $\min_{1\le i\le n} M_i = \max_{1\le i\le n} M_i - 1$. Since, by Lemma 2.1, we have $\max_{1\le i\le n} M_i - 1 \le \mathrm{M}(G) \le \min_{1\le i\le n} M_i + 1$, it implies that $\mathrm{M}(G) = \min_{1\le i\le n} M_i$ or $\mathrm{M}(G) = \min_{1\le i\le n} M_i + 1 = \max_{1\le i\le n} M_i$.

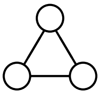
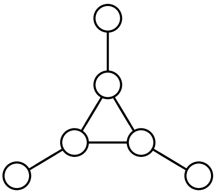
In the event that $\mathrm{M}(G) = \min_{1\le i\le n} M_i$, consider a subgraph G_j such that $\mathrm{M}(G_j) = \max_{1\le i\le n} M_i$ and let $A \in \mathcal{H}(G)$ be a matrix such that $A(j)$ is a maximizing matrix for $\mathrm{M}(G_j)$. Since in this case $\mathrm{M}(G_j) = \mathrm{M}(G)+1$, by interlacing A is a maximizing matrix for $\mathrm{M}(G)$. This means that j is Parter for the maximizing matrix A and, therefore, G is not an FG. Conversely, if G is not an FG then, by Lemma 2.1, there is an index j such that $M_j = \mathrm{M}(G)+1$. Since we are in **Case 1** we conclude that $\mathrm{M}(G) = \min_{1\le i\le n} M_i$.

Similarly, in the event that $\mathrm{M}(G) = \max_{1\le i\le n} M_i$, we may conclude that every index of G is a downer for any maximizing matrix for $\mathrm{M}(G)$, which means that G is an FG.

Case 2: In this case we have $\min_{1\le i\le n} M_i = \max_{1\le i\le n} M_i - 2$. Since, by Lemma 2.1, we have $\max_{1\le i\le n} M_i - 1 \le \mathrm{M}(G) \le \min_{1\le i\le n} M_i + 1$, we conclude that $\mathrm{M}(G) = \max_{1\le i\le n} M_i - 1$ (or, equivalently, $\mathrm{M}(G) = \min_{1\le i\le n} M_i + 1$).

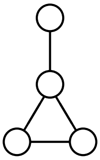
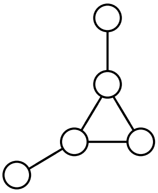
Let G_j be such that $\mathrm{M}(G_j) = \max_{1\le i\le n} M_i$ and let $A \in \mathcal{H}(G)$ be a matrix such that $A(j)$ is a maximizing matrix for $\mathrm{M}(G_j)$. By interlacing A is a maximizing matrix for $\mathrm{M}(G)$. This means that j is Parter for the maximizing matrix A and, therefore, G is not an FG. □

Examples 2.3 We first mention some graphs of **Case 0**.

$G =$  and $G' =$ 

are examples of FG's from **Case 0**. Regarding the graph G, we have $\mathrm{M}(G) = 2$ while, for each subgraph G_i, we have $\mathrm{M}(G_i) = 1$. For the graph G', we have $\mathrm{M}(G') = 2$ and, for each subgraph G'_i, we have $\mathrm{M}(G'_i) = 2 = \mathrm{M}(G')$.

Next, the FG's from **Case 1** include

$G =$  and $G' =$  .

We have $\min_{1\le i\le 4} \mathrm{M}(G_i) = 1$ and $\max_{1\le i\le 4} \mathrm{M}(G_i) = 2 = \mathrm{M}(G)$. Similarly, $\min_{1\le i\le 5} \mathrm{M}(G'_i) = 1$ and $\max_{1\le i\le 5} \mathrm{M}(G'_i) = 2 = \mathrm{M}(G')$.

From Theorem 2.2 we obtain a characterization of FG's.

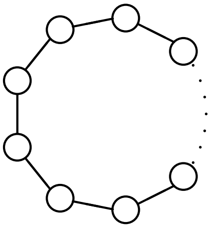
Theorem 2.4 *If G is a graph on n vertices, then G is an* FG *if and only if all* $\mathrm{M}(G_i)$ *are the same and* $\mathrm{M}(G) \ge \mathrm{M}(G_i)$*, or G is in* **Case 1** *with* $\mathrm{M}(G) = \max_{1\le i\le n} \mathrm{M}(G_i)$.

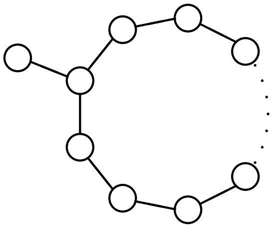
As an alternative, we may state this result as follows.

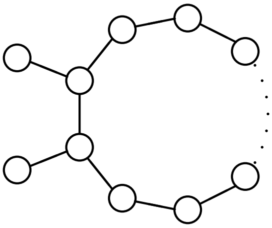
Theorem 2.5 *If G is a graph on n vertices, then G is an* FG *if and only if no M_i exceeds* $\mathrm{M}(G)$.

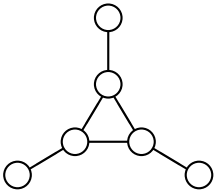
3 Small fundamental graphs

We view paths as "honorary" FG's. Otherwise, no trees are FG's. The graphs G with one more edge than a tree that are FG's are





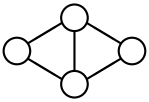




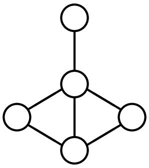
All have $\mathrm{M}(G) = 2$ (see, for example, [12]). They are a 3-cycle with arbitrary edge subdivision allowed; a 3-cycle with a single pendent vertex at one vertex of the cycle (edge subdivision is allowed only on all edges of the cycle); a 3-cycle with pendent vertices at two adjacent vertices of the cycle (edge subdivision is allowed outside the 3-path that includes the two pendents); and a 3-cycle with pendents at each vertex (no edge subdivision allowed). (By Theorem 2.5, we conclude that all these graphs are fundamental. That these are all of the fundamental unicyclic graphs follows from the fact that in all other unicyclic cases, removal of some vertices results in a nontrivial tree that necessarily has Parter vertices. We plan to include a more general result, of which this is a special case, in a planned future work.)

For two edges additions to a tree, again by Theorem 2.5, we may conclude that the following graphs are fundamental

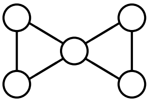
$H_1 =$



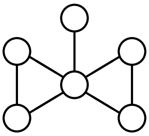
$H_2 =$



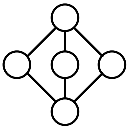
$H_3 =$



$H_4 =$



$H_5 =$



in which $\mathrm{M}(H_1) = \mathrm{M}(H_2) = 2$ and $\mathrm{M}(H_i) = 3$, $i = 3, 4, 5$. All are edge additions to a star or generalized star. (By [12] we conclude $\mathrm{M}(H_1) = \mathrm{M}(H_2) = 2$ and $\mathrm{M}(H_i) \geq 3$, $i = 3, 4, 5$. Since, for each $i \in \{3, 4, 5\}$, there is a vertex v in H_i such that $\mathrm{M}(H_i - v) = 2$ we conclude, by interlacing, that $\mathrm{M}(H_i) \leq 3$.)

4 General maximum multiplicity

Suppose that G is a simple graph on n vertices and we wish to know $\mathrm{M}(G)$. If we remove k vertices in such a way as to leave only r FG's and s paths, say, respectively, $H_1, \ldots, H_r, P_1, \ldots, P_s$, then

$$\mathrm{M}(G) \geq \sum_{i=1}^{r} \mathrm{M}(H_i) + s - k\ . \tag{4.1}$$

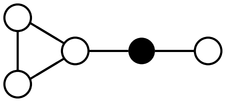
We wish to show that maximization of the right-hand side, over k and the choice of vertices, is $\mathrm{M}(G)$. If G is itself an FG, there is nothing to do. If not, there must be a Parter vertex in G, relative to $\mathrm{M}(G)$. Such a vertex may be removed, leaving one or more components, and the process continues with each component. When a point is reached at which all components are paths or FG's, we are finished, and $\mathrm{M}(G)$ is determined by (4.1) with equality. We can avoid worrying about which vertices may be Parter, by simply removing vertices in all possible ways, until all remaining components are paths or FG's. Then maximization of the right-hand side of (4.1) gives $\mathrm{M}(G)$. In any event, we have, for a general simple graph G,

Theorem 4.1 *For each simple graph G, $\mathrm{M}(G)$ can be determined from the maximum multiplicities of the subgraphs of G that are* FG*'s.*

We have been a bit informal about viewing paths as FG's for trees. Technically, they are not, as a path on more than two vertices may have an (interior) Parter vertex for an eigenvalue with maximum multiplicity. This is a minor issue, but it may be overcome by removing interior vertices from paths (as though they were Parter) to leave only vertices and edges (2-paths, i.e., paths on two vertices), which are, strictly speaking, fundamental. All prior and current results are the same, and Theorem 4.1 is valid for trees, as well.

Remark. Regarding the subtlety of the notion of FG, we also note that there are graphs G for which there are matrices achieving $\mathrm{M}(G)$ that have Parter vertices, while there are also matrices achieving $\mathrm{M}(G)$ without any Parter vertices. We do not classify these graphs as fundamental. The simplest example is the path on three vertices. But there are several other examples, such as

$G =$



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$\mathrm{M}(G) = 2$, which may be achieved with the darkened vertex being Parter for the maximum multiplicity eigenvalue, or with no vertex Parter. For example,

$$A = \begin{bmatrix} -5 & -5 & 10 & 0 & 0 \\ -5 & -5 & 10 & 0 & 0 \\ 10 & 10 & -4 & 8 & 0 \\ 0 & 0 & 8 & 0 & 4 \\ 0 & 0 & 0 & 4 & -4 \end{bmatrix} \in \mathcal{H}(G)$$

has 0 as an eigenvalue of multiplicity 2 (as $\operatorname{rank}(A) = 3$) with no Parter vertices. However,

$$B = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \in \mathcal{H}(G)$$

has also 0 as an eigenvalue of multiplicity 2 with the darkened vertex, and no other, being Parter.

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