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A local characterization of quasi-crystal graphs[☆]

Alan J. Cain [b], António Malheiro [a], Fátima Rodrigues [a], Inês Rodrigues [b]

[a] *Center for Mathematics and Applications (NOVA Math) and Department of Mathematics, NOVA School of Science and Technology (NOVA FCT), 2829–516 Caparica, Portugal*
[b] *Center for Mathematics and Applications (NOVA Math), NOVA School of Science and Technology (NOVA FCT), 2829–516 Caparica, Portugal*

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ABSTRACT

A local characterization of quasi-crystal graphs of type A_{n-1} is provided, by presenting a set of local axioms, similar to the ones introduced by Stembridge for crystal graphs of simply-laced root systems, but restricted to type A_{n-1}. It is also shown that quasi-crystal graphs satisfying these axioms are closed under the tensor product recently introduced by Cain, Guilherme and Malheiro. It is deduced that each connected component of such a graph has a unique highest weight element, whose weight is a composition, and it is isomorphic to a quasi-crystal graph of semistandard quasi-ribbon tableaux.

1. Introduction

In the 1980's, Drinfeld [6] and Jimbo [12] independently introduced quantum groups, which are quantized deformations of universal enveloping algebras of semisimple Lie algebras, and which played a significant role in theoretical physics. Kashiwara, building upon their work, developed the theory of crystal bases and crystal graphs as a framework for studying representations of quantum groups [13–15], at $q = 0$ limit. Crystal bases are (informally) combinatorial objects associated with representations, and crystal graphs are directed graphs that encode the combinatorial data of crystal

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E-mail addresses: a.j.cain@gmail.com (A.J. Cain), ajm@fct.unl.pt (A. Malheiro), mfsr@fct.unl.pt (F. Rodrigues), ima.rodrigues@fct.unl.pt (I. Rodrigues).

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bases. Kashiwara showed that the crystal graph structure has very interesting properties such as being stable under tensor products [13].

The theory of crystal bases has proven to be a valuable tool in representation theory, with applications in various fields such as algebraic geometry [19], mathematical physics [17], and combinatorics [23]. Indeed, crystal bases are related to Young tableaux through a categorification process that involves replacing the crystal operators of a crystal base by certain combinatorial operations, known as the Kashiwara operators. These operators correspond to edges in the crystal graph between different tableaux of the same shape. Through this connection, the combinatorial properties of Young tableaux, such as their shapes, content, and row and column insertion operations, can be related to the combinatorial structures of crystal bases [18].

The monoid whose elements are identified with Young tableaux is called the plactic monoid. It has its origin in the works of Schensted [28] and Knuth [20], and was later studied in depth by Lascoux and Schützenberger [22]. Kashiwara and Nakashima showed that the plactic monoid arises from crystal bases associated with the vector representation of the quantized universal enveloping general linear Lie algebra. It emerges as the quotient of a free monoid on a given alphabet $A_n = \{1, \ldots, n\}$ by a congruence that identifies words that have the same position in isomorphic components of the crystal graph [18].

The plactic monoid also plays a significant role in the theory of symmetric polynomials, particularly in connection with Schur polynomials. These polynomials, which serve as the irreducible polynomial characters of the general linear group $\mathrm{GL}_n(\mathbb{C})$, are indexed by partitions with at most n parts. They form a basis for the ring of symmetric polynomials in n indeterminates. The application of the plactic monoid yielded the first rigorous proof of the Littlewood–Richardson rule, a combinatorial rule that expresses a product of two Schur polynomials as a linear combination of Schur polynomials [24].

In addition to the classical plactic monoid, there exists another monoid known as the hypoplactic monoid that emerges in the realm of quasi-symmetric functions and non-commutative symmetric functions. It was first introduced by Krob and Thibon [21] and studied in depth by Novelli [27]. The hypoplactic monoid is an analogue of the classical plactic monoid, but with quasi-ribbon tableaux as elements. The quasi-ribbon polynomials, also known as fundamental quasi-symmetric polynomials, serve as a basis for the ring of quasi-symmetric polynomials [7], just as Schur polynomials form a basis for the ring of symmetric polynomials.

Cain and Malheiro [5] introduced a purely combinatorial quasi-crystal structure for the hypoplactic monoid, similar to the crystal structure for the plactic monoid. They show that many of the intriguing connections observed between the crystal graph, Kashiwara operators, Young tableaux, and the plactic monoid are mirrored in the interaction of the analogous quasi-crystal graph, quasi-Kashiwara operators, quasi-ribbon tableaux, and the hypoplactic monoid. In particular, the hypoplactic monoid is defined as the quotient of the free monoid on the alphabet A_n by the congruence that identifies words in the same position in isomorphic connected components of the quasi-crystal graph. Recently, Maas-Gariépy [26] independently introduced an equivalent quasi-crystal structure by considering the decomposition of Schur functions into fundamental quasi-symmetric functions.

The first two authors, together with Guilherme [4], introduced the concept of a hypoplactic congruence that can be defined for any seminormal quasi-crystal, leading to a broader notion of a hypoplactic monoid. They demonstrated that the hypoplactic monoid construction proposed by Cain and Malheiro can be viewed within the context of the hypoplactic monoid associated with the general linear Lie algebra. In the same paper, they also defined a quasi-tensor product of quasi-crystals, in a similar way as it was done for crystals.

In this paper, the authors aim to further advance the theory of quasi-crystals in parallel with the existing theory of crystals. This endeavour is motivated by the desire to extend the rich interplay between the crystal graphs, Kashiwara operators, Young tableaux, and the plactic monoid to the realm of quasi-crystals.

In [16], Kashiwara presented an abstract notion of crystal associated to a root system as an edge-coloured graph satisfying certain specific axioms, levering this notion to a more general setting that goes beyond crystals of representations — see [2]. While explicit constructions of crystals exist for certain quantum algebras, such as those described in [18,23], the characterization of those arising from representations was obtained by Stembridge in [29]. More specifically, Stembridge gave a set of local structural properties on crystal graphs of simply-laced root systems that permit to identify which crystal graphs correspond to the crystal of a representation. The simply-laced cases include all quantum Kac–Moody algebras having a Cartan matrix with off-diagonal entries of 0 or -1. It is worth noting that these simply-laced crystals are significant as they include all highest weight crystals of finite or affine type, which are of immense interest in the field.

Moreover, this characterization provides another way to prove the Schur positivity of a symmetric function, as follows. Given the monomial expansion of a symmetric function indexed by a set of combinatorial objects, define a crystal structure on that set, by defining crystal operators that satisfy the Stembridge axioms. Then, since the character of each connected component is a Schur function, it implies that the weight generating function is Schur positive. Additionally, while the Schur decomposition for the plethysm of Schur functions or for the characters of Lie modules (Thrall's problem) remain open problems, their fundamental quasi-symmetric decompositions are known [9,25]. Understanding the crystal decomposition into quasi-crystals may help in understanding these Schur expansions.

Other crystal-like structures also exhibit local characterizations. For instance, Gillespie and Levinson [11] provided local axioms for a crystal of shifted tableaux, Gillespie, Hawkes, Poh and Schilling [10] gave a characterization of crystals for the quantum queer superalgebra, building on conjectural local axioms introduced by Assaf and Oğuz [1], and Tsuchioka [30] introduced a local characterization of B_2 regular crystals.

Parallelling the previous work, the authors present in this paper a set of local axioms, similar to those presented by Stembridge for crystals, that characterize quasi-crystal graphs of simply-laced root systems that arise from the quasi-crystal of type A_n. This characterization answers a question (Question 1) posed by the referee of [5]. Furthermore, since the character of a connected quasi-crystal graph is a fundamental quasi-symmetric function, this characterization may have applications in proving the positivity of a quasi-symmetric function in the basis of fundamental quasi-symmetric functions.

This paper is organized as follows. In Section 2 we recall the notion of crystals, focusing on Stembridge crystals, and quasi-crystals. In Section 3, we introduce local axioms for quasi-crystal graphs (Definition 3.1), and prove that connected quasi-crystals satisfying these axioms are completely characterized by their unique highest weight elements (Theorems 3.9 and 3.10). We also prove that the quasi-crystal graphs satisfying the said axioms are closed under the quasi-tensor product introduced in [4]. In Section 4, we introduce an algorithm to obtain quasi-crystal graphs satisfying the local axioms from a connected Stembridge crystal (Theorem 4.6).

2. Background

We begin by recalling the notion of (abstract) crystals and Stembridge crystals, following mainly [2]. Then, we recall the notion of quasi-crystals, first introduced in [5] and further developed in [4].

2.1. Crystals

Definition 2.1. Let Φ be a root system, with weight lattice Λ, index set I and simple roots α_i, for $i \in I$. A *crystal* of type Φ is a non-empty set $\mathcal{C}$ together with maps $\widetilde{e}_i, \widetilde{f}_i : \mathcal{C} \longrightarrow \mathcal{C} \sqcup \{\perp\}$, $\widetilde{\varepsilon}_i, \widetilde{\varphi}_i : \mathcal{C} \longrightarrow \mathbb{Z} \sqcup \{-\infty\}$, and $\mathrm{wt} : \mathcal{C} \longrightarrow \Lambda$ for $i \in I := \{1, \dots, n-1\}$, satisfying the following:

C1. For any $x, y \in \mathcal{C}$, $\widetilde{e}_i(x) = y$ if and only if $x = \widetilde{f}_i(y)$. Moreover, if $\widetilde{e}_i(x) = y$, we have

$$\begin{aligned} \mathrm{wt}(y) &= \mathrm{wt}(x) + \alpha_i, \\ \widetilde{\varepsilon}_i(y) &= \widetilde{\varepsilon}_i(x) - 1, \\ \widetilde{\varphi}_i(y) &= \widetilde{\varphi}_i(x) + 1, \end{aligned}$$

C2. $\widetilde{\varphi}_i(x) = \widetilde{\varepsilon}_i(x) + \langle \mathrm{wt}(x), \alpha_i^\vee \rangle$.

C3. If $\widetilde{\varepsilon}_i(x) = -\infty$, then $\widetilde{e}_i(x) = \widetilde{f}_i(x) = \perp$.

The maps $\widetilde{e}_i$ and $\widetilde{f}_i$ are called the *Kashiwara operators* (respectively, the *raising* and *lowering* operators), $\widetilde{\varepsilon}_i$ and $\widetilde{\varphi}_i$ are the *length functions* and wt is the *weight function*. Whenever $\widetilde{e}_i(x) = \perp$ (respectively $\widetilde{f}_i(x) = \perp$), for $x \in \mathcal{C}$, we say that $\widetilde{e}_i$ (respectively $\widetilde{f}_i$) is *undefined* on x. Otherwise, the operators are *defined*. We set $\widetilde{e}_i^0$ and $\widetilde{f}_i^0$ to be the identity map, and $\widetilde{e}_i^k$ (respectively $\widetilde{f}_i^k$) is defined recursively as $\widetilde{e}_i\widetilde{e}_i^{k-1}$ (respectively $\widetilde{f}_i\widetilde{f}_i^{k-1}$).

A crystal is said to be *seminormal* if

$$\begin{aligned} \widetilde{\varepsilon}_i(x) &= \max\{k : \widetilde{e}_i^k(x) \neq \perp\}, \\ \widetilde{\varphi}_i(x) &= \max\{k : \widetilde{f}_i^k(x) \neq \perp\}, \end{aligned} \tag{2.1}$$

for all $i \in I$, $x \in \mathcal{C}$. In particular, in a seminormal crystal, we have $\widetilde{\varepsilon}_i(x), \widetilde{\varphi}_i(x) \geq 0$, for all $x \in \mathcal{C}$, $i \in I$. We say that $x \in \mathcal{C}$ is a *highest weight element* if $\widetilde{e}_i(x) = \perp$, for all $i \in I$. If $\mathcal{C}$ is seminormal, this is equivalent to having $\widetilde{\varepsilon}_i(x) = 0$, for all $i \in I$. Whenever there is no risk of ambiguity, we will refer to the crystal by its underlying set $\mathcal{C}$. We associate a crystal to its *crystal graph*, a directed graph, whose edges are labelled in I and whose vertices are weighted in Λ, and such that there exists an i-labelled edge $y \xrightarrow{i} x$ if and only if $\widetilde{f}_i(y) = x$.

We recall the A_{n-1} root system, where the weight lattice is $\Lambda = \mathbb{Z}^n$, the index set $I = \{1 < \cdots < n-1\}$, and the simple roots are $\alpha_i = (0, \ldots, 0, 1, -1, 0, \ldots, 0)$, which satisfy $\alpha_i^\vee = \alpha_i$, for $i \in I$. In particular, for type A_{n-1}, we have $\langle \alpha_i, \alpha_j \rangle = -1$ if $|i-j| = 1$, and 0 otherwise.

Example 2.2. Consider the alphabet $\mathcal{A}_n := \{1 < \cdots < n\}$ and the free monoid $\mathcal{A}_n^*$. Given a word $w \in \mathcal{A}_n^*$, the operators $\widetilde{e}_i$ and $\widetilde{f}_i$ are defined as follows. Replace each letter i with the symbol $+$, each letter $i+1$ with the symbol $-$, while ignoring the letters that are not i or $i+1$. After cancelling all the subwords $-+$, $\widetilde{e}_i(w)$ is obtained by changing the leftmost symbol $-$ to $+$ (or equivalently, changing the leftmost unpaired letter $i+1$ to i), while $\widetilde{f}_i(w)$ is obtained by changing the rightmost symbol $+$ to $-$. For instance,

$$\widetilde{e}_1(2112) = 2111, \qquad \widetilde{f}_1(2112) = 2122.$$

Additionally, define $\widetilde{\varepsilon}_i(w)$ and $\widetilde{\varphi}_i(w)$ as in (2.1) and $\mathrm{wt}(w)$ as the content of the word w. Then, $\mathcal{A}_n^*$, together with the previous maps, is a type A_{n-1} crystal, denoted by plac_n. An example is illustrated on Fig. 1.

Moreover, isomorphic connected components of plac_n are isomorphic, via the classical Robinson–Schensted–Knuth insertion, to a unique connected crystal graph of Young tableaux. We refer to [5, §3], and the references therein, for further details.

Next, we recall the Stembridge axioms, describing a local structure of certain crystal graphs. These structures are depicted in Figs. 2 and 3.

Definition 2.3 (*[2, §4.2]*)**.** *Let Φ be a simply-laced root system. A crystal $\mathcal{C}$ of type Φ is a weak Stembridge crystal if the following axioms are satisfied, for all $i, j \in I$ such that $i \neq j$:*

S1. If $\widetilde{e}_i(x) = y$, then $\widetilde{\varepsilon}_j(y)$ is equal to either $\widetilde{\varepsilon}_j(x)$ or $\widetilde{\varepsilon}_j(x) + 1$, and the latter happens only if α_i and α_j are not orthogonal roots.

S2. If $\widetilde{e}_i(x) = y$ and $\widetilde{\varepsilon}_j(y) = \widetilde{\varepsilon}_j(x) > 0$, then

$$\widetilde{e}_i\widetilde{e}_j(x) = \widetilde{e}_j\widetilde{e}_i(x) \neq \perp,$$

and $\widetilde{\varphi}_i(\widetilde{e}_j(x)) = \widetilde{\varphi}_i(x)$.

S2′. If $\widetilde{f}_i(x) = y$ and $\widetilde{\varphi}_j(y) = \widetilde{\varphi}_j(x) > 0$, then

$$\widetilde{f}_i\widetilde{f}_j(x) = \widetilde{f}_j\widetilde{f}_i(x) \neq \perp,$$

and $\widetilde{\varepsilon}_i(\widetilde{f}_j(x)) = \widetilde{\varepsilon}_i(x)$.

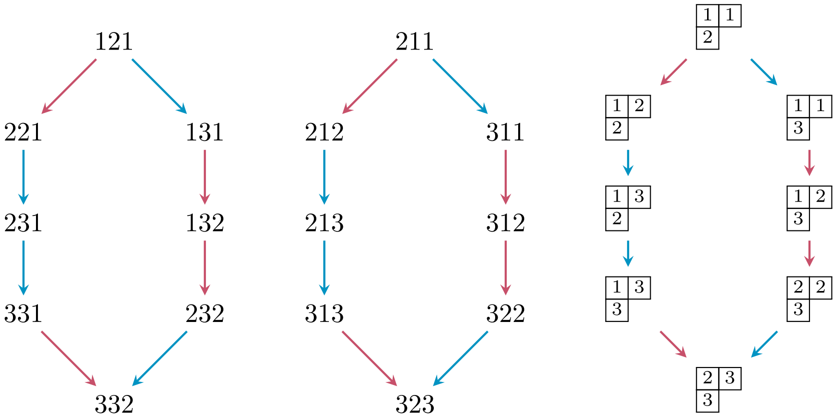


Fig. 1. Two isomorphic components of the crystal graph of plac_3, and the unique crystal graph of Young tableaux isomorphic to them. The red arrows denote $\widetilde{f}_1$ and the blue ones $\widetilde{f}_2$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

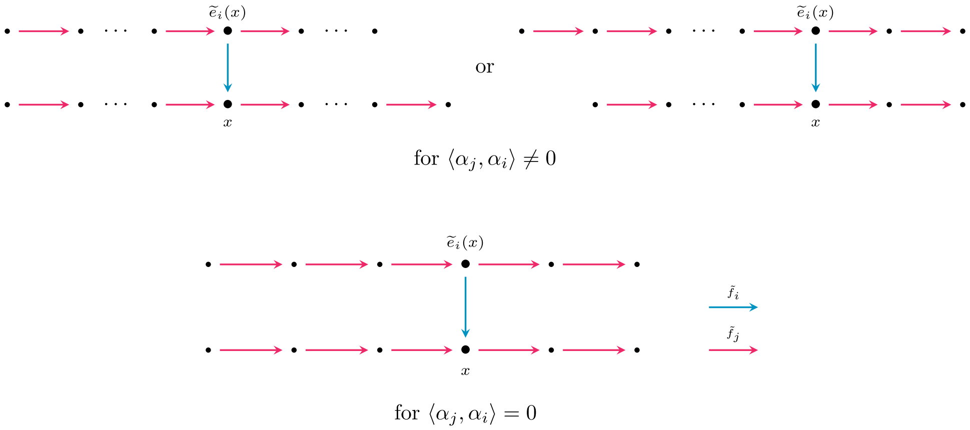


Fig. 2. Illustration of axiom **S1**.

S3. If $\widetilde{e}_i(x)=y$ and $\widetilde{e}_j(x)=z$, and $\widetilde{\varepsilon}_j(y)=\widetilde{\varepsilon}_j(x)+1$ and $\widetilde{\varepsilon}_i(z)=\widetilde{\varepsilon}_i(x)+1$, then

$$\widetilde{e}_i\widetilde{e}_j^{\,2}\widetilde{e}_i(x)=\widetilde{e}_j\widetilde{e}_i^{\,2}\widetilde{e}_j(x)\neq\perp,$$

and $\widetilde{\varphi}_i(\widetilde{e}_j(x))=\widetilde{\varphi}_i(\widetilde{e}_j^{\,2}\widetilde{e}_i(x))$ and $\widetilde{\varphi}_j(\widetilde{e}_i(x))=\widetilde{\varphi}_j(\widetilde{e}_i^{\,2}\widetilde{e}_j(x))$.

S3′. If $\widetilde{f}_i(x)=y$ and $\widetilde{f}_j(x)=z$, and $\widetilde{\varphi}_j(y)=\widetilde{\varphi}_j(x)+1$ and $\widetilde{\varphi}_i(z)=\widetilde{\varphi}_i(x)+1$, then

$$\widetilde{f}_i\widetilde{f}_j^{\,2}\widetilde{f}_i(x)=\widetilde{f}_j\widetilde{f}_i^{\,2}\widetilde{f}_j(x)\neq\perp,$$

and $\widetilde{\varepsilon}_i(\widetilde{f}_j(x))=\widetilde{\varepsilon}_i(\widetilde{f}_j^{\,2}\widetilde{f}_i(x))$ and $\widetilde{\varepsilon}_j(\widetilde{f}_i(x))=\widetilde{\varepsilon}_j(\widetilde{f}_i^{\,2}\widetilde{f}_j(x))$.

A *Stembridge crystal* is a weak Stembridge crystal that is also seminormal.

Remark 2.4. Note that Definition 2.3 was stated in terms of crystals, but we could have an equivalent definition using certain directed graphs. Following the notation of Definition 2.1, let G be a directed graph, with vertices weighted in Λ and edges labelled in I, satisfying the previous axioms and additionally:

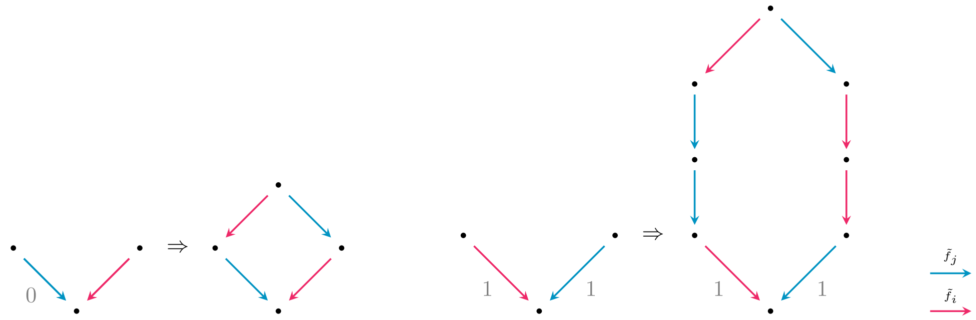


Fig. 3. Illustration of axioms **S2** (left) and **S3** (right). The 0 below the blue edge on the left indicates that $\widetilde{\varepsilon}_j(x) = \widetilde{\varepsilon}_j(y)$ and, likewise, the 1 below the edges on the right indicates that $\widetilde{\varepsilon}_j(y) = \widetilde{\varepsilon}_j(x) + 1$ and $\widetilde{\varepsilon}_i(z) = \widetilde{\varepsilon}_i(x) + 1$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

- For any vertex $x \in G$ and $i \in I$, there is at most one i-labelled edge starting at x and at most one i-labelled edge ending at x.
- For all $i \in I$, all paths made up of i-labelled edges are finite.

Then, we may define a seminormal crystal $\mathcal{C}$ such that G is its crystal graph, as follows. For all vertices $v \in G$ and $i \in I$, define $\widetilde{f}_i(v) = u$, if there exists an i-labelled directed edge from v to u, and otherwise set $\widetilde{f}_i(v) = \perp$. Define $\widetilde{e}_i$ to be the partial inverse of $\widetilde{f}_i$, and $\widetilde{\varepsilon}_i(v)$, $\widetilde{\varphi}_i(v)$ and $\mathrm{wt}(v)$ such that they satisfy axioms **C1** and **C2**.

These previous conditions coincide with (P1) and (P2) in [29], and we remark that, if G is a crystal graph of a seminormal crystal $\mathcal{C}$, then these conditions follow from the axioms of Definition 2.1.

Therefore, a seminormal crystal is completely determined by its crystal graph. We refer to [3, §2.2] for further details.

Example 2.5. The type A_{n-1} crystal plac_n defined in Example 2.2 is a Stembridge crystal.

Proposition 2.6 ([2, Proposition 4.5])**.** *Let $\mathcal{C}$ be a crystal graph satisfying axiom* ***S1*** *and let $\widetilde{e}_i(x) = y$. Then, exactly one of the following possibilities is true:*

1. $\widetilde{\varepsilon}_j(y) = \widetilde{\varepsilon}_j(x)$, $\widetilde{\varphi}_j(y) = \widetilde{\varphi}_j(x) - 1$, *for* $\langle \alpha_i, \alpha_j \rangle = -1$.
2. $\widetilde{\varepsilon}_j(y) = \widetilde{\varepsilon}_j(x) + 1$, $\widetilde{\varphi}_j(y) = \widetilde{\varphi}_j(x)$, *for* $\langle \alpha_i, \alpha_j \rangle = -1$.
3. $\widetilde{\varepsilon}_j(y) = \widetilde{\varepsilon}_j(x)$, $\widetilde{\varphi}_j(y) = \widetilde{\varphi}_j(x)$, *for* $\langle \alpha_i, \alpha_j \rangle = 0$.

Remark 2.7. Given a Stembridge crystal $\mathcal{C}$, if $\widetilde{e}_i(x) = y \in \mathcal{C}$, then it follows from the definition that $\widetilde{\varepsilon}_j(y) \geq \widetilde{\varepsilon}_j(x)$ and $\widetilde{\varphi}_j(y) \leq \widetilde{\varphi}_j(x)$, for $j \neq i$.

2.2. Quasi-crystals

Consider $\mathbb{Z} \sqcup \{-\infty, +\infty\}$ the set of integers with two additional symbols, a minimal element $-\infty$ and a maximal element $+\infty$. In this set consider also the usual addition between integers, and set $m + (-\infty) = (-\infty) + m = -\infty$ and $m + (+\infty) = (+\infty) + m = +\infty$, for all $m \in \mathbb{Z}$. We will now recall the definition of quasi-crystal, whose corresponding graph allows loops. The symbol $+\infty$ will keep track of these loops.

Definition 2.8 ([4, *Definition 3.1*])**.** Let Φ be a root system, with weight lattice Λ, index set I and simple roots α_i, for $i \in I$. A *quasi-crystal* of type Φ is a non-empty set $\mathcal{Q}$ together with maps $\ddot{e}_i, \ddot{f}_i : \mathcal{Q} \longrightarrow \mathcal{Q} \sqcup \{\perp\}$, $\ddot{\varepsilon}_i, \ddot{\varphi}_i : \mathcal{Q} \longrightarrow \mathbb{Z} \sqcup \{-\infty, +\infty\}$, and $\mathrm{wt} : \mathcal{Q} \longrightarrow \Lambda$ for $i \in I := \{1, \ldots, n-1\}$, satisfying the following:

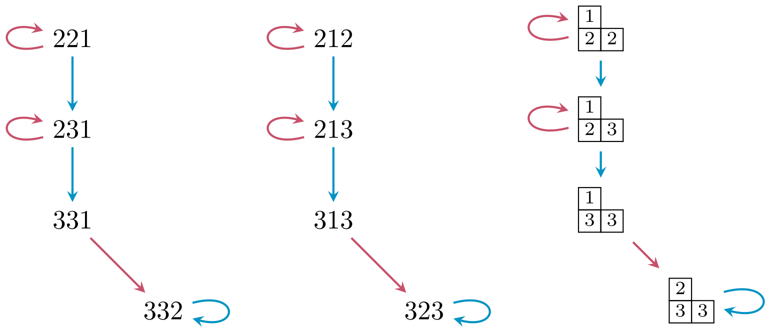


Fig. 4. Two isomorphic components of the quasi-crystal graph of hypo_3, and the unique component of quasi-ribbon tableaux isomorphic to them. The red arrows denote $\ddot{f}_1$ and the blue ones $\ddot{f}_2$. The red and blue loops denote, respectively, the 1- and 2-inversions. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Q1. For any $x, y \in \mathcal{Q}$, $\ddot{e}_i(x) = y$ if and only if $x = \ddot{f}_i(y)$. Moreover, whenever $\ddot{e}_i(x) = y$, we have

$$\begin{aligned} \mathrm{wt}(y) &= \mathrm{wt}(x) + \alpha_i, \\ \ddot{\varepsilon}_i(y) &= \ddot{\varepsilon}_i(x) - 1, \\ \ddot{\varphi}_i(y) &= \ddot{\varphi}_i(x) + 1. \end{aligned}$$

Q2. $\ddot{\varphi}_i(x) = \ddot{\varepsilon}_i(x) + \langle \mathrm{wt}(x), \alpha_i^\vee \rangle$.

Q3. If $\ddot{\varepsilon}_i(x) = -\infty$, then $\ddot{e}_i(x) = \ddot{f}_i(x) = \perp$.

Q4. If $\ddot{\varepsilon}_i(x) = +\infty$, then $\ddot{e}_i(x) = \ddot{f}_i(x) = \perp$.

Example 2.9. Consider the free monoid $\mathcal{A}_n^*$ as in Example 2.2, and let $w \in \mathcal{A}_n^*$. Define the following operators $\ddot{f}_i$ and $\ddot{e}_i$, for $i \in [n-1]$. As before, replace each letter i with the symbol $+$, and each letter $i+1$ with $-$, ignoring the letters that are not i or $i+1$. If the subword $-+$ occurs in w, we say that w has an *i-inversion*. In this case, we set $\ddot{f}_i(w) = \ddot{e}_i(w) = \perp$ and say that these operators are *undefined* on w. Otherwise, $\ddot{f}_i(w)$ and $\ddot{e}_i(w)$ are obtained using the same process of Example 2.2. For example,

$$\ddot{e}_1(2112) = \perp, \quad \ddot{e}_1(1122) = 1112.$$

Additionally, $\ddot{\varepsilon}_i(w)$ and $\ddot{\varphi}_i(w)$ are set to be $+\infty$ whenever w has as i-inversion, and defined as in (2.1) otherwise. For instance,

$$\ddot{\varepsilon}_1(2112) = +\infty, \quad \ddot{\varepsilon}_1(1122) = 2.$$

Then, $\mathcal{A}_n^*$, together with the previous operators and the usual weight wt, is a quasi-crystal of type A_{n-1}, denoted by hypo_n. An example is illustrated on Fig. 4.

Similar to the case of plac_n, isomorphic components of hypo_n are isomorphic, via the Krob–Thibon insertion [21], to a unique connected quasi-crystal graph of quasi-ribbon tableaux. We refer to [5, §5], and the references therein, for further details.

We remark that the previous definition extends the one of quasi-crystal provided in [5], which concerns only type A_{n-1} and omits the symbol $+\infty$. In the more general definition introduced in [4], the new symbol $+\infty$ is used to signal the i-inversions in the original quasi-crystal definition.

We use the same terminology as of crystals, except the maps $\ddot{e}_i$ and $\ddot{f}_i$ are called the *quasi-Kashiwara operators*. Similarly to the Kashiwara operators, we set $\ddot{e}_i^0$ and $\ddot{f}_i^0$ to be the identity map, and $\ddot{e}_i^k$ (respectively $\ddot{f}_i^k$) is defined recursively as $\ddot{e}_i\ddot{e}_i^{k-1}$ (respectively $\ddot{f}_i\ddot{f}_i^{k-1}$).

A quasi-crystal is said to be *seminormal* if

$$\ddot{\varepsilon}_i(x) = \max\{k : \ddot{e}_i^k(x) \neq \perp\},$$
$$\ddot{\varphi}_i(x) = \max\{k : \ddot{f}_i^k(x) \neq \perp\},$$

for all $i \in I$, $x \in \mathcal{C}$, whenever $\ddot{\varepsilon}_i(x) \neq +\infty$. In particular, in a seminormal quasi-crystal $\ddot{\varepsilon}_i(x), \ddot{\varphi}_i(x) \in \mathbb{Z}_{\geq 0} \sqcup \{+\infty\}$. We remark that it is possible for this maximum to exist and be finite, while $\ddot{\varepsilon}_i(x)$ is $+\infty$.

It follows from Definition 2.8 that a crystal is a quasi-crystal such that $\ddot{\varepsilon}_i(x), \ddot{\varphi}_i(x) \neq +\infty$, for all $x \in \mathcal{C}$, $i \in I$ [4, Remark 3.2]. We say that $x \in \mathcal{Q}$ is a *highest weight element* if $\ddot{e}_i(x) = \perp$ for all $i \in I$. Note that, in a seminormal quasi-crystal, this is not equivalent to have $\ddot{\varepsilon}_i(x) = 0$ for all $i \in I$, as one might have $\ddot{e}_j(x) = \perp$ and $\ddot{\varepsilon}_j(x) = +\infty$.

Similarly to crystals, we associate a quasi-crystal with its *quasi-crystal graph*, defined in the same way as the crystal graph, but such that $x \in \mathcal{Q}$ has an i-labelled loop if and only if $\ddot{\varepsilon}_i(x) = \ddot{\varphi}_i(x) = +\infty$.

Remark 2.10. Similarly to seminormal crystals (see Remark 2.4), a seminormal quasi-crystal is completely determined by its quasi-crystal graph. We refer to [4, §4] for further details.

3. Local axioms for quasi-crystals

In what follows, we will focus our study on type A_{n-1} quasi-crystals, because, unlike the crystal case, the other simply-laced types do not exhibit the same nice properties. For instance, connected quasi-crystals of type D_n might have more that one highest weight element [4]. The type A_{n-1} quasi-crystals have a natural interpretation in terms of quasi-ribbon tableaux [5] and this definition coincides with the one obtained using the quasi-tensor product in [4]. For other Cartan types, we do not have an obvious interpretation, and thus, a different tensor product may be required to obtain quasi-crystals of simply-laced types with the desired properties, and consequently, to extend the local characterization.

We remark that, for seminormal crystals (or, more generally, for upper seminormal ones, where it is only required that $\ddot{\varepsilon}_i(x) = \max\{k : \ddot{e}_i^k(x) \neq \perp\}$), the condition $\widetilde{\varepsilon}_i(x) > 0$ is equivalent to $\widetilde{e}_i(x) \neq \perp$. For seminormal quasi-crystals, one may have $\ddot{\varepsilon}_i(x) = +\infty > 0$ and, by **Q4**, $\ddot{e}_i(x) = \perp$. Therefore, in the following statements, differing from [2], we will require specifically that $\ddot{e}_i(x) \neq \perp$, instead of $\ddot{\varepsilon}_i(x) > 0$. For this reason, for certain results we do not require that the quasi-crystal graphs are seminormal, but rather that $\ddot{\varepsilon}_i(x), \ddot{\varphi}_i(x) \in \mathbb{Z}_{\geq 0} \sqcup \{+\infty\}$ (which is satisfied by seminormal quasi-crystals).

Definition 3.1 (*Local Quasi-crystal Axioms*)**.** Let $\mathcal{Q}$ be a quasi-crystal. Let $i, j \in I$ and $x, y \in \mathcal{Q}$. We say that $\mathcal{Q}$ satisfies the *local quasi-crystal axioms* if the following hold.

LQ1. For $i \in \{1, \dots, n-2\}$, $\ddot{\varepsilon}_i(x) = 0$ if and only if $\ddot{\varphi}_{i+1}(x) = 0$.

LQ2. If $\ddot{e}_i(x) = y$, then:

- **LQ2 (1).** For $|i-j| > 1$, $\ddot{\varepsilon}_j(x) = \ddot{\varepsilon}_j(y)$.
- **LQ2 (2).** For $i \in \{1, \dots, n-2\}$, $\ddot{\varepsilon}_{i+1}(x) \neq \ddot{\varepsilon}_{i+1}(y)$ if and only if $\big(\ddot{\varepsilon}_{i+1}(x) = +\infty \wedge \ddot{\varepsilon}_i(y) = 0\big)$. Furthermore, whenever $\ddot{\varepsilon}_{i+1}(x) \neq \ddot{\varepsilon}_{i+1}(y)$, we have $\ddot{\varepsilon}_{i+1}(y) \neq 0$.
- **LQ2 (3).** For $i \in \{2, \dots, n-1\}$, $\ddot{\varphi}_{i-1}(x) \neq \ddot{\varphi}_{i-1}(y)$ if and only if $\big(\ddot{\varphi}_{i-1}(y) = +\infty \wedge \ddot{\varphi}_i(x) = 0\big)$. Furthermore, whenever $\ddot{\varphi}_{i-1}(x) \neq \ddot{\varphi}_{i-1}(y)$, we have $\ddot{\varphi}_{i-1}(x) \neq 0$.

LQ3. For $i \neq j$, if both $\ddot{e}_i(x)$ and $\ddot{e}_j(x)$ are defined, then $\ddot{e}_i\ddot{e}_j(x) = \ddot{e}_j\ddot{e}_i(x) \neq \perp$.

LQ3′. For $i \neq j$, if both $\ddot{f}_i(x)$ and $\ddot{f}_j(x)$ are defined, then $\ddot{f}_i\ddot{f}_j(x) = \ddot{f}_j\ddot{f}_i(x) \neq \perp$.

Remark 3.2. Axiom **LQ1** may be described as follows. If w is the highest weight element of its unique i-labelled component, then w has no $(i+1)$-loops nor it is the initial vertex of an $(i+1)$-labelled edge, and likewise, if w is the lowest weight element of its unique $(i+1)$-labelled component, then w has no i-loops nor it is ending vertex of an i-labelled edge.

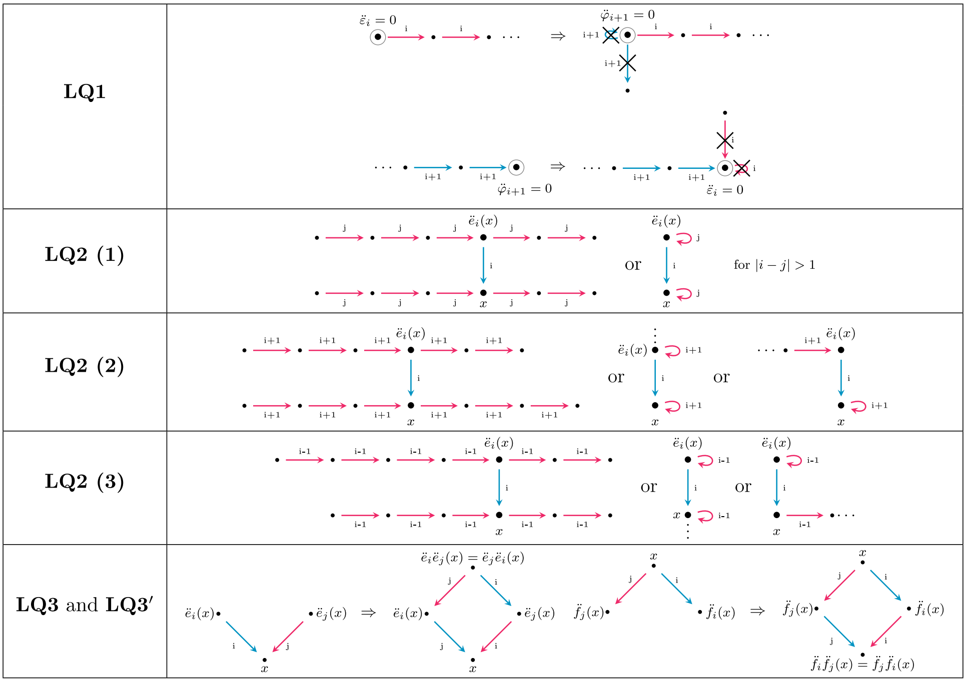


Fig. 5. Illustration of the axioms of Definition 3.1 and Lemma 3.5.

Regarding **LQ2**, we remark that, unlike the case for Stembridge axioms for type A_{n-1}, the two cases corresponding to edges labelled with i and $j = i \pm 1$ (more generally, corresponding to non-orthogonal roots) now depend on whether j is $i+1$ or $i-1$, and their roles are not symmetric.

For axiom **LQ3**, notice that, unlike the case for Stembridge axioms, the existence of two differently labelled edges is sufficient to ensure the closing square.

We remark that in Definition 3.1, the condition in **LQ2 (1)** can be replaced with $\ddot{\varphi}_j$. We prove this equivalence in Lemma 3.5. The previous axioms, and the additional structure described in Lemma 3.5, are depicted in the table of Fig. 5.

Remark 3.3. It was proven in [26, Theorem 2] that type A quasi-crystal graphs are isomorphic, as oriented, unlabelled graphs, to crystal graphs of single row tableaux. Since this isomorphism does not consider the labels on edges, applying the Stembridge axioms in this setting is not sufficient to obtain the local quasi-crystal axioms in Definition 4.3. This result implies the uniqueness of the highest weight element, a fact that was already known for type A [5, Proposition 6], and is here obtained using the quasi-crystal axioms in Theorem 3.9.

Proposition 3.4. *The quasi-crystal* hypo_n *defined in Example 2.9 satisfies the local quasi-crystal axioms.*

Proof. To verify **LQ1**, let $x \in \mathcal{A}_n^*$ be such that $\ddot{\varepsilon}_i(x) = 0$. Then, by the definition of hypo_n, x has no i-inversions and $\ddot{e}_i(x)$ is undefined, meaning that x has no occurrences of $i+1$. Consequently, x must have no $(i+1)$-inversions, and $\ddot{f}_{i+1}(x)$ is undefined, as there are no $i+1$ to be changed to $i+1$. Therefore, $\ddot{\varphi}_{i+1}(x) = 0$. The other implication is proved analogously.

We now verify that hypo_n satisfies **LQ2**. Let $x, y \in \mathcal{A}_n$ be such that $\ddot{e}_i(x) = y$. If $|i-j| > 1$, since $\ddot{e}_i$ only acts on letters $i, i+1$, leaving the others unchanged, we have $\ddot{\varepsilon}_j(x) = \ddot{\varepsilon}_j(y)$. Thus, hypo_n satisfies **LQ2 (1)**. To verify **LQ2 (2)**, we consider two cases. The proof for **LQ2 (3)** is analogous.

- Suppose that $\ddot{\varepsilon}_{i+1}(x) \neq +\infty$. Then, x must have no i-inversions and thus $\ddot{\varepsilon}_{i+1}(x)$ is the number of occurrences of $i+2$ in x. Since $\ddot{e}_i$ only alters the letters $i, i+1$, we get that y has the same occurrences of $i+2$ as x, and thus $\ddot{\varepsilon}_{i+1}(y) = \ddot{\varepsilon}_{i+1}(x)$.
- Now suppose that $\ddot{\varepsilon}_{i+1}(x) = +\infty$. If $\ddot{\varepsilon}_{i+1}(x) \neq \ddot{\varepsilon}_{i+1}(y)$, then we must have $\ddot{\varepsilon}_{i+1}(y) < +\infty$, which means that x has an $(i+1)$-inversion, while y has no $(i+1)$-inversions. For this to happen, considering that x must have at least one occurrence of $i+1$ (that is changed to i in y by $\ddot{e}_i$), that $i+1$ in x must be to the right of an $i+2$ and must be the unique $i+1$ in x, otherwise y would have an $(i+1)$-inversion. Therefore, y has no occurrences of $i+1$, and we get $\ddot{\varepsilon}_i(y) = 0$. If $\ddot{\varepsilon}_i(y) = 0$, then $\ddot{e}_i(y)$ is undefined and y has no i-inversions. Therefore, y has no occurrences of $i+1$, and consequently, it has no $(i+1)$-inversions. Thus, $\ddot{\varepsilon}_{i+1}(y) \neq \ddot{\varepsilon}_{i+1}(x) = +\infty$.

We now verify **LQ3**, the proof for **LQ3**′ is similar. The case where $|i-j| > 1$ is straightforward. If $j = i+1$ (the case where $j = i-1$ is similar), then, as $\ddot{e}_i(x)$ and $\ddot{e}_{i+1}(x)$ are both defined, then all occurrences of i in x are to the left of all occurrences of $i+1$, and these occurrences of $i+1$ are all to the left of all $i+2$. Then, it easy to check that $\ddot{e}_i\ddot{e}_{i+1}(x) = \ddot{e}_{i+1}\ddot{e}_i(x)$. □

Lemma 3.5. *Let $\mathcal{Q}$ be a quasi-crystal graph. Let $i, j \in I$ and $x, y \in \mathcal{Q}$, and suppose that $\ddot{e}_i(x) = y$.*

1. *If $|i-j| > 1$, then $\ddot{\varepsilon}_j(y) = \ddot{\varepsilon}_j(x)$ if and only if $\ddot{\varphi}_j(y) = \ddot{\varphi}_j(x)$.*
2. *If $\ddot{\varepsilon}_{i+1}(x) \neq +\infty$, then $\ddot{\varepsilon}_{i+1}(y) = \ddot{\varepsilon}_{i+1}(x)$ if and only if $\ddot{\varphi}_{i+1}(y) = \ddot{\varphi}_{i+1}(x) - 1$.*
3. *If $\ddot{\varepsilon}_{i-1}(x) \neq +\infty$, then $\ddot{\varepsilon}_{i-1}(y) = \ddot{\varepsilon}_{i-1}(x) + 1$ if and only if $\ddot{\varphi}_{i-1}(y) = \ddot{\varphi}_{i-1}(x)$.*

Proof. Suppose that $|i-j| > 1$. Then, since $\mathcal{Q}$ is of type A_{n-1}, we have $\langle \alpha_i, \alpha_j \rangle = 0$. Suppose that $\ddot{\varepsilon}_j(y) = \ddot{\varepsilon}_j(x)$. Then,

$$\begin{aligned}
\ddot{\varphi}_j(y) &= \ddot{\varepsilon}_j(y) + \langle \mathrm{wt}(y), \alpha_j \rangle && \text{(by } \mathbf{Q2} \text{ and since } \alpha_i^\vee = \alpha_i) \\
&= \ddot{\varepsilon}_j(x) + \langle \mathrm{wt}(x) + \alpha_i, \alpha_j \rangle && \text{(by } \mathbf{Q1}) \\
&= \ddot{\varepsilon}_j(x) + \langle \mathrm{wt}(x), \alpha_j \rangle + \langle \alpha_i, \alpha_j \rangle \\
&= \ddot{\varepsilon}_j(x) + \langle \mathrm{wt}(x), \alpha_j \rangle \\
&= \ddot{\varphi}_j(x) && \text{(by } \mathbf{Q2} \text{ and since } \alpha_i^\vee = \alpha_i).
\end{aligned}$$

Similarly, $\ddot{\varphi}_j(y) = \ddot{\varphi}_j(x)$, implies that $\ddot{\varepsilon}_j(y) = \ddot{\varepsilon}_j(x)$.

Now suppose that $\ddot{\varepsilon}_{i+1}(x) \neq +\infty$, and suppose that $\ddot{\varepsilon}_{i+1}(y) = \ddot{\varepsilon}_{i+1}(x)$. Then,

$$\begin{aligned}
\ddot{\varphi}_{i+1}(y) &= \ddot{\varepsilon}_{i+1}(y) + \langle \mathrm{wt}(y), \alpha_{i+1} \rangle && \text{(by } \mathbf{Q2} \text{ and since } \alpha_i^\vee = \alpha_i) \\
&= \ddot{\varepsilon}_{i+1}(x) + \langle \mathrm{wt}(x) + \alpha_i, \alpha_{i+1} \rangle && \text{(by } \mathbf{Q1}) \\
&= \ddot{\varepsilon}_{i+1}(x) + \langle \mathrm{wt}(x), \alpha_{i+1} \rangle + \langle \alpha_i, \alpha_{i+1} \rangle \\
&= \ddot{\varepsilon}_{i+1}(x) + \langle \mathrm{wt}(x), \alpha_{i+1} \rangle - 1 \\
&= \ddot{\varphi}_{i+1}(x) - 1 && \text{(by } \mathbf{Q2} \text{ and since } \alpha_i^\vee = \alpha_i)
\end{aligned}$$

and similarly, one shows that $\ddot{\varphi}_{i+1}(y) = \ddot{\varphi}_{i+1}(x) - 1$ implies that $\ddot{\varepsilon}_{i+1}(y) = \ddot{\varepsilon}_{i+1}(x)$. The proof for the third case is analogous. □

Proposition 3.6. *Let $\mathcal{Q}$ be a quasi-crystal graph satisfying* **LQ1** *and* **LQ2** *and such that $\ddot{\varepsilon}_i(x), \ddot{\varphi}_i(x) \in \mathbb{Z}_{\geq 0} \sqcup \{+\infty\}$, for all $x \in Q$, $i \in I$. Let $i, j \in I$ and $x, y \in \mathcal{Q}$, and suppose that $\ddot{e}_i(x) = y$. Then:*

1. *If $|i-j| > 1$, $\ddot{\varphi}_j(y) = \ddot{\varphi}_j(x)$.*
2. *If $j = i+1$, then:*
    (1) *If $\ddot{\varepsilon}_{i+1}(x) \neq +\infty$, then $\ddot{\varepsilon}_{i+1}(y) = \ddot{\varepsilon}_{i+1}(x)$ and $\ddot{\varphi}_{i+1}(y) = \ddot{\varphi}_{i+1}(x) - 1$.*
    (2) *If $\ddot{\varepsilon}_{i+1}(x) = +\infty$ and $\ddot{\varepsilon}_i(y) > 0$, then $\ddot{\varepsilon}_{i+1}(y) = \ddot{\varepsilon}_{i+1}(x) = +\infty$ and $\ddot{\varphi}_{i+1}(y) = \ddot{\varphi}_{i+1}(x) = +\infty$.*
    (3) *If $\ddot{\varepsilon}_{i+1}(x) = +\infty$ and $\ddot{\varepsilon}_i(y) = 0$, then $\ddot{\varepsilon}_{i+1}(y) = -\langle \mathrm{wt}(y), \alpha_{i+1} \rangle > 0$ and $\ddot{\varphi}_{i+1}(y) = 0$.*
3. *If $j = i-1$, then:*

1. *If $\ddot{\varphi}_{i-1}(y) \neq +\infty$, then $\ddot{\varepsilon}_{i-1}(x) = \ddot{\varepsilon}_{i-1}(y) - 1$ and $\ddot{\varphi}_{i-1}(x) = \ddot{\varphi}_{i-1}(y)$.*
2. *If $\ddot{\varphi}_{i-1}(y) = +\infty$ and $\ddot{\varphi}_i(x) > 0$, then $\ddot{\varepsilon}_{i-1}(x) = \ddot{\varepsilon}_{i-1}(y) = +\infty$ and $\ddot{\varphi}_{i-1}(x) = \ddot{\varphi}_{i-1}(y) = +\infty$.*
3. *If $\ddot{\varphi}_{i-1}(y) = +\infty$ and $\ddot{\varphi}_i(x) = 0$, then $\ddot{\varepsilon}_{i-1}(x) = 0$ and $\ddot{\varphi}_{i-1}(x) = \langle \mathrm{wt}(x), \alpha_{i-1}\rangle > 0$.*

We remark that in Proposition 3.6 exactly one of the cases holds.

Proof of Proposition 3.6. Since $\ddot{e}_i(x) = y$, we have $\mathrm{wt}(y) = \mathrm{wt}(x) + \alpha_i$, by **Q1**.

The first condition is a direct consequence of Lemma 3.5.

Now suppose that $j = i + 1$ (the proof for $j = i - 1$ is analogous). Hence, $\langle \alpha_i, \alpha_{i+1}\rangle = -1$. If $\ddot{\varepsilon}_{i+1}(x) \neq +\infty$, then, by **LQ2 (2)**, we have $\ddot{\varepsilon}_{i+1}(x) = \ddot{\varepsilon}_{i+1}(y)$. It follows from Lemma 3.5 that $\ddot{\varphi}_{i+1}(y) = \ddot{\varphi}_{i+1}(x) - 1$. If $\ddot{\varepsilon}_{i+1}(x) = +\infty$, and since $\ddot{\varepsilon}_i(y) \geq 0$, we have two cases to consider:

1. If $\ddot{\varepsilon}_i(y) > 0$, axiom **LQ2** implies that $\ddot{\varepsilon}_{i+1}(y) = \ddot{\varepsilon}_{i+1}(x) = +\infty$. Consequently, by **Q2**, we have $\ddot{\varphi}_{i+1}(y) = +\infty$.
2. If $\ddot{\varepsilon}_i(y) = 0$, then **LQ2** implies that $\ddot{\varepsilon}_{i+1}(x) \neq \ddot{\varepsilon}_{i+1}(y)$ and $\ddot{\varepsilon}_{i+1}(y) > 0$. By axiom **LQ1**, $\ddot{\varepsilon}_i(y) = 0$ implies that $\ddot{\varphi}_{i+1}(y) = 0$. In particular, by **Q2**, we have that $\ddot{\varepsilon}_{i+1}(y)$ and $\ddot{\varphi}_{i+1}(y)$ are finite, and thus, since $\alpha_i^\vee = \alpha_i$,
$$\ddot{\varepsilon}_{i+1}(y) = \ddot{\varphi}_{i+1}(y) - \langle \mathrm{wt}(y), \alpha_{i+1}\rangle = -\langle \mathrm{wt}(y), \alpha_{i+1}\rangle. \quad \square$$

The next result is illustrated in Fig. 6.

Corollary 3.7. *Let $\mathcal{Q}$ be a seminormal quasi-crystal graph satisfying* **LQ1** *and* **LQ2**, *and suppose that $\ddot{e}_i(x) = y$, for some $x, y \in \mathcal{Q}$ and $i \in I$.*

1. *Let $i \in \{1, \dots, n-2\}$. If $\ddot{\varepsilon}_{i+1}(y) = +\infty$, then $\ddot{\varepsilon}_{i+1}(x) = +\infty$ and, there exists $k > 0$ such that $\ddot{\varepsilon}_{i+1}(\ddot{e}_i^k(y)) \notin \{0, +\infty\}$, and, for $0 \leq l < k$, $\ddot{e}_i^l(y)$ is defined and $\ddot{\varepsilon}_{i+1}(\ddot{e}_i^l(y)) = +\infty$.*
2. *Let $i \in \{2, \dots, n-1\}$. If $\ddot{\varphi}_{i-1}(x) = +\infty$, then $\ddot{\varphi}_{i-1}(y) = +\infty$ and there exists $k > 0$ such that $\ddot{\varphi}_{i-1}(\ddot{f}_i^k(x)) \notin \{0, +\infty\}$, and, for $0 \leq l < k$, $\ddot{f}_i^l(y)$ is defined and $\ddot{\varphi}_{i-1}(\ddot{f}_i^l(y)) = +\infty$.*

Proof. We prove the first statement; the second one is proved similarly. Suppose that $\ddot{\varepsilon}_{i+1}(y) = +\infty$. If $\ddot{\varepsilon}_{i+1}(x) \neq +\infty$, then axiom **LQ2 (2)** implies that $\ddot{\varepsilon}_{i+1}(y) = \ddot{\varepsilon}_{i+1}(x) \neq +\infty$, which is a contradiction. Therefore, $\ddot{\varepsilon}_{i+1}(x) = +\infty$.

Now suppose that $\ddot{\varepsilon}_i(y) = 0$. By **LQ1**, we have $\ddot{\varphi}_{i+1}(y) = 0$, which implies that $\ddot{\varepsilon}_{i+1}(y)$ is finite, contradicting the hypothesis. Thus, $\ddot{\varepsilon}_i(y) \neq 0$. By **Q1**, $\ddot{e}_i(x) = y$ implies that $\ddot{f}_i(y) = x$, hence $\ddot{f}_i(y) \neq \perp$, and by **Q4**, $\ddot{\varepsilon}_i(y) \neq +\infty$. Therefore, $\ddot{\varepsilon}_i(y) \notin \{0, +\infty\}$, and thus, there exists $k > 0$ such that $\ddot{\varepsilon}_i(y) = k$. Since $\mathcal{Q}$ is seminormal, we have $\ddot{e}_i^l(y) \neq \perp$, for all $l \leq k$ and $\ddot{e}_i^s(y) = \perp$, for $s > k$. Therefore, $\ddot{\varepsilon}_i(\ddot{e}_i^l(y)) \notin \{0, +\infty\}$, for $l < k$. Now, iteratively, applying **LQ2 (2)**, we obtain $\ddot{\varepsilon}_{i+1}(\ddot{e}_i^l(y)) = \ddot{\varepsilon}_{i+1}(y) = +\infty$, for $1 \leq l < k$. Finally, since $\ddot{\varepsilon}_i(\ddot{e}_i^k(y)) = 0$ and $\ddot{\varepsilon}_{i+1}(\ddot{e}_i^{k-1}(y)) = +\infty$ we conclude by **LQ2 (2)**, that $\ddot{\varepsilon}_{i+1}(\ddot{e}_i^k(y)) \notin \{0, +\infty\}$. $\square$

We have recalled that a crystal $\mathcal{C}$ is a quasi-crystal such that $\ddot{\varepsilon}_i(x) \neq +\infty$ for all $x \in \mathcal{C}, i \in I$ [4]. Therefore, we have the following result.

Proposition 3.8. *Let $\mathcal{Q}$ be a crystal graph satisfying the local quasi-crystal axioms of Definition 3.1. Then $\mathcal{Q}$ is a weak Stembridge crystal.*

Proof. Suppose that $\ddot{e}_i(x) = y$, for $x, y \in \mathcal{Q}$, $i \in I$ and let $j \in I$ be such that $j \neq i$. Since $\mathcal{Q}$ is a crystal, then the length functions never take the value $+\infty$. Thus, if $|i - j| > 1$ or $j = i + 1$, by axioms **LQ2 (1)** and **LQ2 (2)** we have $\ddot{\varepsilon}_j(y) = \ddot{\varepsilon}_j(x)$. If $j = i - 1$ (and, in this case, α_j and α_i are non-orthogonal roots), we have $\ddot{\varepsilon}_j(y) = \ddot{\varepsilon}_j(x) + 1$ by axiom **LQ2 (3)** and Lemma 3.5 (3). Therefore, $\mathcal{Q}$ satisfies axiom **S1**. Now suppose that $\ddot{e}_i(x) = y$ and $\ddot{e}_j(x) = z$ are both defined. Then, axiom **LQ3** ensures that $\ddot{e}_i\ddot{e}_j(x) = \ddot{e}_j\ddot{e}_i(x) \neq \perp$. We now consider three cases:

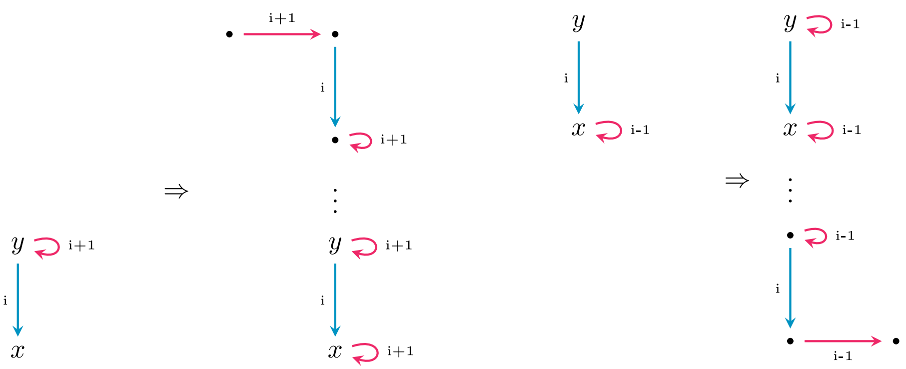


Fig. 6. Illustration of Corollary 3.7.

1. If $|i-j| > 1$, we have $\ddot{\varepsilon}_i(z) = \ddot{\varepsilon}_i(x) > 0$ and $\ddot{\varepsilon}_j(y) = \ddot{\varepsilon}_j(x) > 0$, and thus, by Lemma 3.5, we have $\ddot{\varphi}_i(z) = \ddot{\varphi}_i(x)$ and $\ddot{\varphi}_j(y) = \ddot{\varphi}_j(x)$.
2. If $j = i + 1$, and since $\ddot{e}_i(x) = y$ and $\ddot{\varepsilon}_j(x) \neq +\infty$ (because $\mathcal{Q}$ is a crystal and thus $\ddot{\varepsilon}_i(x) \neq +\infty$, for any $x \in Q$, $i \in I$), it follows from axiom **LQ2 (2)** that $\ddot{\varepsilon}_j(y) = \ddot{\varepsilon}_j(x) > 0$. By the same reasoning, since $\ddot{e}_j(x) = z$ and $\ddot{\varphi}_i(z) \neq +\infty$, axiom **LQ2 (3)** ensures that $\ddot{\varphi}_i(z) = \ddot{\varphi}_i(x)$.
3. If $j = i - 1$, since $\ddot{e}_i(x) = y$ and $\ddot{\varphi}_j(y) \neq +\infty$, axiom **LQ2 (3)** implies that $\ddot{\varphi}_j(y) = \ddot{\varphi}_j(x)$, and thus, by Lemma 3.5, we have $\ddot{\varepsilon}_j(y) = \ddot{\varepsilon}_j(x) + 1$. Similarly, since $\ddot{e}_j(x) = z$ and $\ddot{\varepsilon}_i(x) \neq +\infty$, axiom **LQ2 (2)** implies that $\ddot{\varepsilon}_i(z) = \ddot{\varepsilon}_i(x) > 0$.

In all cases, $\mathcal{Q}$ satisfies axiom **S2**. Moreover, the conditions $\ddot{\varepsilon}_i(z) = \ddot{\varepsilon}_i(x) + 1$ and $\ddot{\varepsilon}_j(y) = \ddot{\varepsilon}_j(x) + 1$ do not occur simultaneously, hence the hypotheses of axiom **S3** never occur. With similar reasoning, we obtain the same conclusions for the dual axioms. □

Recall that a highest weight element in a quasi-crystal is an element x such that $\ddot{e}_i(x) = \perp$, for all $i \in I$. We define a partial order in $\mathcal{Q}$ where $x \prec y$ if there exists a sequence $i_1, \ldots, i_N$ in I such that $\ddot{e}_{i_1} \ldots \ddot{e}_{i_N}(x) = y$. Similarly to the crystal case, we say that a quasi-crystal graph $\mathcal{Q}$ is *bounded above* if, for every $x \in \mathcal{Q}$, there exists a highest weight element $x_h \in \mathcal{Q}$ such that $x \preceq x_h$.

Theorem 3.9. *Let $\mathcal{Q}$ be a non-empty and bounded above connected quasi-crystal graph satisfying the local quasi-crystal axioms. Then, $\mathcal{Q}$ has a unique highest weight element.*

Proof. Let S be the subset of vertices $w \in \mathcal{Q}$ for which there exist distinct highest weight elements x and y, such that $w \prec x$ and $w \prec y$.

Suppose that $S \neq \emptyset$ and let w_0 be a maximal element of S. By the definition of S, there exist distinct x_h and y_h highest weight elements, with $w_0 \prec x_h$ and $w_0 \prec y_h$. Therefore, $x_h = \ddot{e}_{i_k} \ldots \ddot{e}_{i_1}(w_0)$ and $y_h = \ddot{e}_{j_s} \ldots \ddot{e}_{j_1}(w_0)$, for $i_1, \ldots, i_k, j_1, \ldots, j_s \in I$. By the maximality of w_0, we have $x_0 := \ddot{e}_{i_1}(w_0)$ and $y_0 := \ddot{e}_{j_1}(w_0)$ are not in S. By **LQ3**, $z := \ddot{e}_{i_1}\ddot{e}_{j_1}(w_0) = \ddot{e}_{i_1}\ddot{e}_{j_1}(w_0)$. Since $\mathcal{Q}$ is bounded above, there exists a highest weight element z_h such that $z \preceq z_h$. Therefore, $x_0 \prec z \preceq z_h$ and $y_0 \prec z \preceq z_h$. Since $x_0, y_0 \notin S$, and $x_0 \prec x_h, x_0 \prec z_h$ and $y_0 \prec y_h, y_0 \prec z_h$, we have $x_h = z_h$ and $y_h = z_h$, contradicting x_h and y_h being distinct. □

Theorem 3.10. *Let $\mathcal{Q}$ and $\mathcal{Q}'$ be connected components of seminormal quasi-crystal graphs satisfying the local quasi-crystal axioms, with highest weight elements u and u', respectively. Then, if* $\mathrm{wt}(u) = \mathrm{wt}(u')$, *there exists an isomorphism between $\mathcal{Q}$ and $\mathcal{Q}'$.*

To prove Theorem 3.10, we recall the notion of *rank* of $x \in \mathcal{Q}$ [2, §4.4], where $\mathcal{Q}$ is a connected component of a quasi-crystal graph with unique highest weight element u, which is defined as $\mathrm{rank}(x) := \langle \mathrm{wt}(u) - \mathrm{wt}(x), \rho \rangle$, where ρ is any vector such that $\langle \alpha_i, \rho \rangle = 1$, for all $i \in I$. This is well defined, and, in particular, if $\ddot{e}_{i_N}\cdots\ddot{e}_{i_1}(x) = u$, and all $\ddot{e}_{i_1}(x), \ddot{e}_{i_2}\ddot{e}_{i_1}(x), \dots, \ddot{e}_{i_N}\cdots\ddot{e}_{i_1}(x)$ are defined, then $\mathrm{rank}(x) = N$.

Proof of Theorem 3.10. Let Ω be the set of subsets $S \subseteq \mathcal{Q}$ such that:

- $u \in S$.
- If $x \in S$ and $\ddot{e}_i(x) \in Q$, then $\ddot{e}_i(x) \in S$.
- There exists a subset $S' \subseteq \mathcal{Q}'$ and a bijection $\theta : S \longrightarrow S'$ such that $\theta(u) = u'$ and, given $x \in S$, then, for every $i \in I$,

$$\big(\ddot{e}_i(x) \neq \bot \Leftrightarrow \ddot{e}_i(\theta(x)) \neq \bot\big) \wedge \big(\ddot{e}_i(x) \neq \bot \Rightarrow \theta(\ddot{e}_i(x)) = \ddot{e}_i(\theta(x))\big).$$

We have $\Omega \neq \emptyset$, since $\{u\} \in \Omega$. Let S_0 be a maximal element of Ω, with respect to set inclusion. We will show that $S_0 = \mathcal{Q}$. We claim that θ preserves the length and weight functions, that is, for all $x \in S_0$ and $i \in I$, $\ddot{\varepsilon}_i(\theta(x)) = \ddot{\varepsilon}_i(x)$, $\ddot{\varphi}_i(\theta(x)) = \ddot{\varphi}_i(x)$ and $\mathrm{wt}(\theta(x)) = \mathrm{wt}(x)$. Given $x \in S_0$ and $i \in I$, there exist $i_1, \dots, i_N \in I$, for some N, such that

$$\ddot{e}_{i_N}\cdots\ddot{e}_{i_1}(x) = u.$$

By the definition of S_0, we have

$$u' = \theta(u) = \theta(\ddot{e}_{i_N}\cdots\ddot{e}_{i_1}(x)) = \ddot{e}_{i_N}\cdots\ddot{e}_{i_1}(\theta(x))$$

and thus,

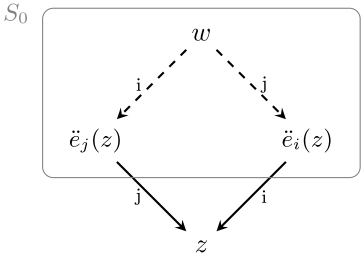
$$\mathrm{wt}(\theta(x)) = \mathrm{wt}(u') - \sum_{j=1}^{N}\alpha_{i_j} = \mathrm{wt}(u) - \sum_{j=1}^{N}\alpha_{i_j} = \mathrm{wt}(x).$$

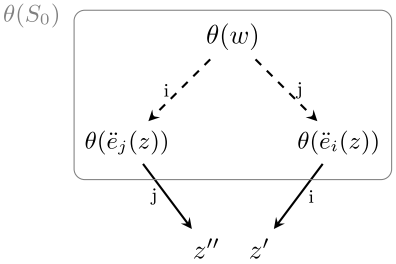
Given $x \in S_0$, since $\mathcal{Q}$ is seminormal, if $\ddot{\varepsilon}_i(x) \neq +\infty$, then $\ddot{\varepsilon}_i(x) = \max\{k : \ddot{e}_i^k(x) \neq \bot\}$. By the definition of S_0, we have $\ddot{e}_i^k(x) \in S_0$, for $k = 1, \dots, \ddot{\varepsilon}_i(x)$. In particular, $\ddot{e}_i^k(\theta(x)) = \theta(\ddot{e}_i^k(x))$. Thus, we have

$$\begin{aligned}\ddot{\varepsilon}_i(\theta(x)) &= \max\{k : \ddot{e}_i^k(\theta(x)) \neq \bot\}\\ &= \max\{k : \theta(\ddot{e}_i^k(x)) \neq \bot\}\\ &= \max\{k : \ddot{e}_i^k(x) \neq \bot\} = \ddot{\varepsilon}_i(x).\end{aligned}$$

Consequently, $\ddot{\varepsilon}_i(x) = +\infty$ if and only if $\ddot{\varepsilon}_i(\theta(x)) = +\infty$, and therefore, $\ddot{\varepsilon}_i(x) = \ddot{\varepsilon}_i(\theta(x))$. From **Q2**, we get $\ddot{\varphi}_i(x) = \ddot{\varphi}_i(\theta(x))$.

Now suppose that $\mathcal{Q} \neq S_0$, and let $z \in \mathcal{Q} \setminus S_0$ be an element of minimal rank. Since $z \notin S_0$, then $z \neq u$, and thus, there exists $i \in I$ such that $\ddot{e}_i(z) \neq \bot$. Moreover, $\mathrm{rank}(\ddot{e}_i(z)) < \mathrm{rank}(z)$, hence $\ddot{e}_i(z) \in S_0$, as z has minimal rank. Therefore, $\ddot{\varphi}_i(\theta(\ddot{e}_i(z))) = \ddot{\varphi}_i(\ddot{e}_i(z)) > 0$, and thus, there exists $z' \in \mathcal{Q}'$ such that $\ddot{e}_i(z') = \theta(\ddot{e}_i(z))$, as depicted in the following diagram:





We will show that z' does not depend on the choice of i, that is, if there exists $z'' \in \mathcal{Q}'$ such that $\ddot{e}_j(z'') = \theta(\ddot{e}_j(z))$, with $j \neq i$, then $z' = z''$. Since $\ddot{e}_i(z)$ and $\ddot{e}_j(z)$ are both defined, axiom **LQ3** implies that there exists $w \in S_0$ such that

$$w = \ddot{e}_i\ddot{e}_j(z) = \ddot{e}_j\ddot{e}_i(z).$$

Therefore, we have $\theta(w) = \theta\ddot{e}_i\ddot{e}_j(z) = \theta\ddot{e}_j\ddot{e}_i(z)$, and by definition of S_0, $\theta(w) = \ddot{e}_i\theta\ddot{e}_j(z) = \ddot{e}_j\theta\ddot{e}_i(z)$. Thus, we have $\ddot{f}_i(\theta(w)) = \theta(\ddot{e}_j(z))$ and $\ddot{f}_j(\theta(w)) = \theta(\ddot{e}_i(z))$. It then follows from axiom **LQ3**′ that

$$z' = \ddot{f}_i\ddot{f}_j(\theta(w)) = \ddot{f}_j\ddot{f}_i(\theta(w)) = z''.$$

Then, if $Q \neq S_0$, the map θ may be extend by defining $\theta(z) := z'$. This contradicts the maximality of S_0. Therefore, $Q = S_0$ and the map θ is a weight-preserving isomorphism. □

Recall that the *outdegree* of a vertex x in a directed graph is the number of edges with initial vertex x.

Proposition 3.11. *Let $\mathcal{Q}$ be a connected component of a seminormal quasi-crystal graph satisfying the local axioms, with highest weight element u. Then, u has outdegree at most one.*

Proof. If $\mathcal{Q}$ has only one vertex, then the result is trivial. Suppose that $\mathcal{Q}$ has at least two vertices. Therefore, there exists $k \in I$ such that $\ddot{f}_k(u) \neq \bot$, and consequently, $\ddot{\varphi}_k(u), \ddot{\varepsilon}_k(u) \neq +\infty$, by **Q4**. Let k_0 be the smallest index such that this happens. Since u is the highest weight element, we have $\ddot{e}_{k_0}(u) = \bot$, and since $\mathcal{Q}$ is seminormal, this implies that $\ddot{\varepsilon}_{k_0}(u) = 0$.

We claim that $\ddot{\varphi}_j(u) = 0$, for $j > k_0$. If $k_0 = n-1$, the result is trivial. Otherwise, axiom **LQ1** implies that $\ddot{\varphi}_{k_0+1}(u) = 0$, and thus, $\ddot{\varphi}_{k_0+1}(u) \neq +\infty$. Thus, by **Q2**, $\ddot{\varepsilon}_{k_0+1}(u) \neq +\infty$, and since u is the highest weight element and $\mathcal{Q}$ is seminormal, we get $\ddot{\varepsilon}_{k_0+1}(u) = 0$. Then, by axiom **LQ1** we obtain $\ddot{\varphi}_{k_0+2}(u) = 0$, and applying the same reasoning, we have $\ddot{\varphi}_j(u) = 0$, for $j > k_0$. Therefore, $\ddot{f}_j(u) = \bot$, for $j > k_0$. □

3.1. Quasi-tensor products

We recall the notion of quasi-tensor product introduced by Cain, Guilherme and Malheiro [4]. Throughout this section, we will only consider seminormal quasi-crystals.

Definition 3.12 (*[4, Theorem 5.1]*)**.** Let $\mathcal{Q}$ and $\mathcal{Q}'$ be seminormal quasi-crystals of type A_{n-1}. The quasi-tensor product $\mathcal{Q} \ddot{\otimes} \mathcal{Q}'$ is the Cartesian product $\mathcal{Q} \times \mathcal{Q}'$ together with the maps defined by:

1. $\operatorname{wt}(x \ddot{\otimes} x') = \operatorname{wt}(x) + \operatorname{wt}(x')$, for $x \in \mathcal{Q}$, $x' \in \mathcal{Q}'$.
2. If $\ddot{\varphi}_i(x) > 0$ and $\ddot{\varepsilon}_i(x') > 0$, for $i \in I$,

$$\ddot{e}_i(x \ddot{\otimes} x') = \ddot{f}_i(x \ddot{\otimes} x') = \bot \qquad \text{and} \qquad \ddot{\varepsilon}_i(x \ddot{\otimes} x') = \ddot{\varphi}_i(x \ddot{\otimes} x') = +\infty.$$

3. Otherwise,

$$\ddot{e}_i(x \ddot{\otimes} x') = \begin{cases} \ddot{e}_i(x) \ddot{\otimes} x' & \text{if } \ddot{\varphi}_i(x) \geq \ddot{\varepsilon}_i(x') \\ x \ddot{\otimes} \ddot{e}_i(x') & \text{if } \ddot{\varphi}_i(x) < \ddot{\varepsilon}_i(x') \end{cases}$$

$$\ddot{f}_i(x \ddot{\otimes} x') = \begin{cases} \ddot{f}_i(x) \ddot{\otimes} x' & \text{if } \ddot{\varphi}_i(x) > \ddot{\varepsilon}_i(x') \\ x \ddot{\otimes} \ddot{f}_i(x') & \text{if } \ddot{\varphi}_i(x) \leq \ddot{\varepsilon}_i(x') \end{cases}$$

and

$$\ddot{\varepsilon}_i(x \ddot{\otimes} x') = \max\{\ddot{\varepsilon}_i(x), \ddot{\varepsilon}_i(x') - \langle \operatorname{wt}(x), \alpha_i \rangle\}$$
$$\ddot{\varphi}_i(x \ddot{\otimes} x') = \max\{\ddot{\varphi}_i(x) + \langle \operatorname{wt}(x'), \alpha_i \rangle, \ddot{\varphi}_i(x')\}$$

where $x \ddot{\otimes} \bot = \bot \ddot{\otimes} x' = \bot$.

The quasi-tensor product $\mathcal{Q} \ddot{\otimes} \mathcal{Q}'$ is a seminormal quasi-crystal of type A_{n-1} [4, Theorem 5.1]. We remark that, with this convention, $x \ddot{\otimes} y$ is identified with the word yx. The following Lemma will be often used and is a direct consequence of [4, Theorem 5.1].

Lemma 3.13. *Let $x \in \mathcal{Q}$ and $x' \in \mathcal{Q}'$. If $\ddot{\varepsilon}_i(x \ddot{\otimes} x') = +\infty$, then $\ddot{\varphi}_i(x) > 0$ or $\ddot{\varepsilon}_i(x') > 0$.*

Proposition 3.14 ([4, Proposition 5.3]). *Let $\mathcal{Q}$ and $\mathcal{Q}'$ be seminormal quasi-crystals of the same type, and let $x \in \mathcal{Q}$ and $x' \in \mathcal{Q}'$ be such that $\ddot{\varphi}_i(x) = 0$ or $\ddot{\varepsilon}_i(x') = 0$. Then:*

$$\ddot{e}_i(x \mathbin{\ddot{\otimes}} x') = \begin{cases} \ddot{e}_i(x) \mathbin{\ddot{\otimes}} x' & \text{if } \ddot{\varepsilon}_i(x') = 0 \\ x \mathbin{\ddot{\otimes}} \ddot{e}_i(x') & \text{if } \ddot{\varepsilon}_i(x') > 0 \end{cases} \qquad \ddot{f}_i(x \mathbin{\ddot{\otimes}} x') = \begin{cases} \ddot{f}_i(x) \mathbin{\ddot{\otimes}} x' & \text{if } \ddot{\varphi}_i(x) > 0 \\ x \mathbin{\ddot{\otimes}} \ddot{f}_i(x') & \text{if } \ddot{\varphi}_i(x) = 0 \end{cases}$$

and $\ddot{\varepsilon}_i(x \mathbin{\ddot{\otimes}} x') = \ddot{\varepsilon}_i(x) + \ddot{\varepsilon}_i(x')$, $\ddot{\varphi}_i(x \mathbin{\ddot{\otimes}} x') = \ddot{\varphi}_i(x) + \ddot{\varphi}_i(x')$.

We may now state the following result.

Theorem 3.15. *Let $\mathcal{Q}$ and $\mathcal{Q}'$ be seminormal quasi-crystal graphs satisfying the local quasi-crystal axioms of Definition 3.1. Then, $\mathcal{Q} \mathbin{\ddot{\otimes}} \mathcal{Q}'$ satisfies the same axioms.*

Proof. **Claim: $\mathcal{Q} \mathbin{\ddot{\otimes}} \mathcal{Q}'$ satisfies axiom LQ1.** Suppose that $\ddot{\varepsilon}_i(x \mathbin{\ddot{\otimes}} x') = 0$. Then, by Definition 3.12, we must have $\ddot{\varphi}_i(x) = 0$ or $\ddot{\varepsilon}_i(x') = 0$. Proposition 3.14 then implies that $\ddot{\varepsilon}_i(x \mathbin{\ddot{\otimes}} x') = \ddot{\varepsilon}_i(x) + \ddot{\varepsilon}_i(x') = 0$. Since $\mathcal{Q}$ and $\mathcal{Q}'$ are seminormal, we have $\ddot{\varepsilon}_i(x), \ddot{\varepsilon}_i(x') \geq 0$. Therefore, we must have $\ddot{\varepsilon}_i(x) = \ddot{\varepsilon}_i(x') = 0$, and then, axiom **LQ1** implies that $\ddot{\varphi}_{i+1}(x) = \ddot{\varphi}_{i+1}(x') = 0$. In particular, since $\ddot{\varphi}_{i+1}(x) = 0$, by Proposition 3.14, we have $\ddot{\varphi}_{i+1}(x \mathbin{\ddot{\otimes}} x') = \ddot{\varphi}_{i+1}(x) + \ddot{\varphi}_{i+1}(x') = 0$. The reverse implication is proved similarly.

Claim: $\mathcal{Q} \mathbin{\ddot{\otimes}} \mathcal{Q}'$ satisfies LQ2. Suppose that $\ddot{e}_i(x \mathbin{\ddot{\otimes}} x') = y \mathbin{\ddot{\otimes}} y'$. Then, by Definition 3.12, we have $\ddot{\varphi}_i(x) = 0$ or $\ddot{\varepsilon}_i(x') = 0$. We first prove that $\mathcal{Q} \mathbin{\ddot{\otimes}} \mathcal{Q}'$ satisfies **LQ2 (1)**. Let $i, j \in I$ be such that $|i - j| > 1$. Then, by Proposition 3.14, we have two cases. We prove the case where $\ddot{\varepsilon}_i(x') = 0$. The case where $\ddot{\varepsilon}_i(x') > 0$ is proved similarly. Suppose that $\ddot{\varepsilon}_i(x') = 0$. Then, $\ddot{e}_i(x \mathbin{\ddot{\otimes}} x') = \ddot{e}_i(x) \mathbin{\ddot{\otimes}} x' = y \mathbin{\ddot{\otimes}} y'$, and we have

$$\ddot{e}_i(x) = y, \quad x' = y'.$$

If $\ddot{\varphi}_j(y), \ddot{\varepsilon}_j(y') > 0$, then, as $\ddot{e}_i(x) = y$, we have $\ddot{\varphi}_j(x) = \ddot{\varphi}_j(y)$ by Proposition 3.6, and since $x' = y'$, we have $\ddot{\varepsilon}_j(x') = \ddot{\varepsilon}_j(y')$. Then, by Definition 3.12, we have $\ddot{\varepsilon}_j(y \mathbin{\ddot{\otimes}} y') = \ddot{\varepsilon}_j(x \mathbin{\ddot{\otimes}} x') = +\infty$. Otherwise, suppose that $\ddot{\varphi}_j(y) = 0$ or $\ddot{\varepsilon}_j(y') = 0$. Then, by axiom **LQ2 (1)**, we have $\ddot{\varepsilon}_j(x) = \ddot{\varepsilon}_j(y)$, and since $x' = y'$, we have $\ddot{\varepsilon}_j(x') = \ddot{\varepsilon}_j(y')$. Thus, by Proposition 3.14, we have

$$\ddot{\varepsilon}_j(y \mathbin{\ddot{\otimes}} y') = \ddot{\varepsilon}_j(y) + \ddot{\varepsilon}_j(y') = \ddot{\varepsilon}_j(x) + \ddot{\varepsilon}_j(x') = \ddot{\varepsilon}_j(x \mathbin{\ddot{\otimes}} x').$$

We now prove that $\mathcal{Q} \mathbin{\ddot{\otimes}} \mathcal{Q}'$ satisfies **LQ2 (2)**. The proof for **LQ2 (3)** is analogous.

1. We first suppose that $\ddot{\varepsilon}_{i+1}(x \mathbin{\ddot{\otimes}} x') \neq +\infty$, and we prove that $\ddot{\varepsilon}_{i+1}(x \mathbin{\ddot{\otimes}} x') = \ddot{\varepsilon}_{i+1}(y \mathbin{\ddot{\otimes}} y')$. By Definition 3.12, we have $\ddot{\varphi}_{i+1}(x) = 0$ or $\ddot{\varepsilon}_{i+1}(x') = 0$, and by Proposition 3.14, we have $\ddot{\varepsilon}_{i+1}(x \mathbin{\ddot{\otimes}} x') = \ddot{\varepsilon}_{i+1}(x) + \ddot{\varepsilon}_{i+1}(x')$. In particular, $\ddot{\varepsilon}_{i+1}(x)$ and $\ddot{\varepsilon}_{i+1}(x')$ must be finite. Moreover, since $\ddot{e}_i(x \mathbin{\ddot{\otimes}} x') = y \mathbin{\ddot{\otimes}} y'$, we have $\ddot{\varphi}_i(x) = 0$ or $\ddot{\varepsilon}_i(x') = 0$. Following Proposition 3.14, we have two cases to consider:

   Case 1.1. Suppose that $\ddot{\varepsilon}_i(x') = 0$. Then,

   $$\ddot{e}_i(x) = y, \quad x' = y'.$$

   Since $\ddot{\varepsilon}_{i+1}(x)$ is finite, Proposition 3.6 implies that

   $$\ddot{\varepsilon}_{i+1}(y) = \ddot{\varepsilon}_{i+1}(x), \quad \ddot{\varphi}_{i+1}(y) = \ddot{\varphi}_{i+1}(x) - 1. \tag{3.1}$$

   Therefore, since $\mathcal{Q}$ is seminormal, we have $\ddot{\varphi}_{i+1}(x) > 0$, and consequently, $\ddot{\varepsilon}_{i+1}(x') = 0$. Since $x' = y'$, we have

   $$\ddot{\varepsilon}_{i+1}(x') = \ddot{\varepsilon}_{i+1}(y') = 0. \tag{3.2}$$

   Then, by Proposition 3.14, and Eqs. (3.1) and (3.2), we have

   $$\begin{aligned} \ddot{\varepsilon}_{i+1}(y \mathbin{\ddot{\otimes}} y') &= \ddot{\varepsilon}_{i+1}(y) + \ddot{\varepsilon}_{i+1}(y') \\ &= \ddot{\varepsilon}_{i+1}(x) + \ddot{\varepsilon}_{i+1}(x') \\ &= \ddot{\varepsilon}_{i+1}(x \mathbin{\ddot{\otimes}} x') \end{aligned}$$

Case 1.2. Now suppose that $\ddot{\varepsilon}_i(x') > 0$. Then, we have $\ddot{\varphi}_i(x) = 0$ and

$$x = y, \quad \ddot{e}_i(x') = y'.$$

Since $\ddot{\varepsilon}_{i+1}(x')$ is finite, by axiom **LQ2 (2)** we have

$$\ddot{\varepsilon}_{i+1}(x') = \ddot{\varepsilon}_{i+1}(y'). \tag{3.3}$$

If $\ddot{\varphi}_{i+1}(x) = 0$, since $x = y$, we get $\ddot{\varphi}_{i+1}(y) = 0$. Otherwise, we must have $\ddot{\varepsilon}_{i+1}(x') = 0$, and by (3.3), we have $\ddot{\varepsilon}_{i+1}(y') = 0$. Thus, we have $\ddot{\varphi}_{i+1}(y) = 0$ or $\ddot{\varepsilon}_{i+1}(y') = 0$, and by Proposition 3.14,

$$\begin{aligned}\ddot{\varepsilon}_{i+1}(y \mathbin{\ddot{\otimes}} y') &= \ddot{\varepsilon}_{i+1}(y) + \ddot{\varepsilon}_{i+1}(y') \\ &= \ddot{\varepsilon}_{i+1}(x) + \ddot{\varepsilon}_{i+1}(x') \\ &= \ddot{\varepsilon}_{i+1}(x \mathbin{\ddot{\otimes}} x')\end{aligned}$$

2. Now suppose that $\ddot{\varepsilon}_{i+1}(x \mathbin{\ddot{\otimes}} x') = +\infty$. By Lemma 3.13, we have $\ddot{\varphi}_{i+1}(x) > 0$ or $\ddot{\varepsilon}_{i+1}(x') > 0$. We will show that $\ddot{\varepsilon}_{i+1}(x \mathbin{\ddot{\otimes}} x') \neq \ddot{\varepsilon}_{i+1}(y \mathbin{\ddot{\otimes}} y')$ if, and only if, $\ddot{\varepsilon}_i(y \mathbin{\ddot{\otimes}} y') = 0$.
Suppose that $\ddot{\varepsilon}_i(y \mathbin{\ddot{\otimes}} y') = 0$. Since $\mathcal{Q} \mathbin{\ddot{\otimes}} \mathcal{Q}'$ satisfies **LQ1**, we have $\ddot{\varphi}_{i+1}(y \mathbin{\ddot{\otimes}} y') = 0 \neq +\infty$. Therefore, by **Q2**, $\ddot{\varepsilon}_{i+1}(y \mathbin{\ddot{\otimes}} y') \neq +\infty$ and hence $\ddot{\varepsilon}_{i+1}(y \mathbin{\ddot{\otimes}} y') \neq \ddot{\varepsilon}_{i+1}(x \mathbin{\ddot{\otimes}} x') = +\infty$.
Now suppose that $\ddot{\varepsilon}_{i+1}(x \mathbin{\ddot{\otimes}} x') \neq \ddot{\varepsilon}_{i+1}(y \mathbin{\ddot{\otimes}} y')$. Then, $\ddot{\varepsilon}_{i+1}(y \mathbin{\ddot{\otimes}} y') \neq +\infty$, which implies, in particular, that $\ddot{\varepsilon}_{i+1}(y)$ and $\ddot{\varepsilon}_{i+1}(y')$ are finite, and thus, $\ddot{\varphi}_{i+1}(y) = 0$ or $\ddot{\varepsilon}_{i+1}(y') = 0$. By Proposition 3.14, we have two cases to consider.

Case 2.1. Suppose that $\ddot{\varepsilon}_i(x') = 0$. Then,

$$\ddot{e}_i(x) = y, \quad x' = y'.$$

This implies that $\ddot{\varepsilon}_i(y') = \ddot{\varepsilon}_i(x') = 0$, and thus, by Proposition 3.14, we have

$$\ddot{\varepsilon}_i(y \mathbin{\ddot{\otimes}} y') = \ddot{\varepsilon}_i(y) + \ddot{\varepsilon}_i(y') = \ddot{\varepsilon}_i(y). \tag{3.4}$$

If $\ddot{\varphi}_{i+1}(y) = 0$, then axiom **LQ1** implies that $\ddot{\varepsilon}_i(y) = 0$, and by (3.4) we get $\ddot{\varepsilon}_i(y \mathbin{\ddot{\otimes}} y') = 0$. Otherwise, if $\ddot{\varphi}_{i+1}(y) > 0$, we must have $\ddot{\varepsilon}_{i+1}(y') = 0$. Since $x' = y'$, we have $\ddot{\varepsilon}_{i+1}(x') = 0$, and by Proposition 3.14,

$$\ddot{\varepsilon}_{i+1}(x \mathbin{\ddot{\otimes}} x') = \ddot{\varepsilon}_{i+1}(x) + \ddot{\varepsilon}_{i+1}(x') = \ddot{\varepsilon}_{i+1}(x) = +\infty.$$

Since $\ddot{\varepsilon}_{i+1}(y) \neq +\infty$, we have $\ddot{\varepsilon}_{i+1}(x) \neq \ddot{\varepsilon}_{i+1}(y)$, and then, axiom **LQ2 (2)** implies that $\ddot{\varepsilon}_i(y) = 0$. Thus, by (3.4), $\ddot{\varepsilon}_i(y \mathbin{\ddot{\otimes}} y') = 0$.

Case 2.2. Now suppose that $\ddot{\varepsilon}_i(x') > 0$. Then, we have $\ddot{\varphi}_i(x) = 0$ and

$$x = y, \quad \ddot{e}_i(x') = y'.$$

Since $\ddot{\varepsilon}_{i+1}(y \mathbin{\ddot{\otimes}} y') \neq +\infty$, we have $\ddot{\varphi}_{i+1}(y) = 0$ or $\ddot{\varepsilon}_{i+1}(y') = 0$. We claim that $\ddot{\varepsilon}_{i+1}(y') > 0$. If $\ddot{\varepsilon}_{i+1}(y') = 0$, then, as $\ddot{e}_i(x') = y'$, axiom **LQ2 (2)** implies that $\ddot{\varepsilon}_{i+1}(x') = \ddot{\varepsilon}_{i+1}(y') = 0$, and by Proposition 3.14,

$$\ddot{\varepsilon}_{i+1}(x) = \ddot{\varepsilon}_{i+1}(x) + \ddot{\varepsilon}_{i+1}(x') = \ddot{\varepsilon}_{i+1}(x \mathbin{\ddot{\otimes}} x') = +\infty.$$

But since $x = y$, we have $\ddot{\varepsilon}_{i+1}(y) = +\infty$, and thus $\ddot{\varepsilon}_{i+1}(y \mathbin{\ddot{\otimes}} y') = +\infty$, which contradicts the hypothesis that $\ddot{\varepsilon}_{i+1}(y \mathbin{\ddot{\otimes}} y') \neq +\infty$. Thus, we have $\ddot{\varepsilon}_{i+1}(y') > 0$ and consequently, $\ddot{\varphi}_{i+1}(y) = 0$. Axiom **LQ1** then implies that $\ddot{\varepsilon}_i(y) = 0$. Since $\ddot{\varphi}_i(x) = 0$ and $x = y$, we get $\ddot{\varphi}_i(y) = 0$. Thus, by Proposition 3.14, we have

$$\ddot{\varepsilon}_i(y \mathbin{\ddot{\otimes}} y') = \ddot{\varepsilon}_i(y) + \ddot{\varepsilon}_i(y') = \ddot{\varepsilon}_i(y'). \tag{3.5}$$

Moreover, since $\ddot{\varphi}_{i+1}(y) = 0$ and $x = y$, we have $\ddot{\varphi}_{i+1}(x) = 0$, and by Proposition 3.14,

$$\ddot{\varepsilon}_{i+1}(x) + \ddot{\varepsilon}_{i+1}(x') = \ddot{\varepsilon}_{i+1}(x \mathbin{\ddot{\otimes}} x') = +\infty, \tag{3.6}$$

which implies that $\ddot{\varepsilon}_{i+1}(x) = +\infty$ or $\ddot{\varepsilon}_{i+1}(x') = +\infty$. Since $\ddot{\varphi}_{i+1}(x) = 0$, we get that $\ddot{\varepsilon}_{i+1}(x)$ is finite. Thus, (3.6) implies that $\ddot{\varepsilon}_{i+1}(x') = +\infty$. Since $\ddot{\varepsilon}_{i+1}(y') \neq +\infty$, we have

$\ddot{\varepsilon}_{i+1}(x') \neq \ddot{\varepsilon}_{i+1}(y')$, and thus, by axiom **LQ2** (2), we have $\ddot{\varepsilon}_i(y') = 0$. Therefore, by (3.5), we have $\ddot{\varepsilon}_i(y \mathbin{\ddot{\otimes}} y') = 0$.

3. Finally, we show that $\ddot{\varepsilon}_{i+1}(y \mathbin{\ddot{\otimes}} y') = 0$ implies that $\ddot{\varepsilon}_{i+1}(x \mathbin{\ddot{\otimes}} x') = \ddot{\varepsilon}_{i+1}(y \mathbin{\ddot{\otimes}} y')$. If $\ddot{\varepsilon}_{i+1}(y \mathbin{\ddot{\otimes}} y') = 0$, then, in particular, $\ddot{\varepsilon}_{i+1}(y \mathbin{\ddot{\otimes}} y')$ is finite, and thus $\ddot{\varphi}_{i+1}(y) = 0$ or $\ddot{\varepsilon}_{i+1}(y') = 0$. Thus, by Proposition 3.14 we have

$$\ddot{\varepsilon}_{i+1}(y) + \ddot{\varepsilon}_{i+1}(y') = \ddot{\varepsilon}_{i+1}(y \mathbin{\ddot{\otimes}} y') = 0.$$

This implies, since $\mathcal{Q}$ and $\mathcal{Q}'$ are seminormal, that $\ddot{\varepsilon}_{i+1}(y) = \ddot{\varepsilon}_{i+1}(y') = 0$. If $\ddot{\varepsilon}_i(x') = 0$, we get $\ddot{e}_i(x) = y$ and $x' = y'$. Since $\ddot{\varepsilon}_{i+1}(y) = 0$, axiom **LQ2** (2) implies that $\ddot{\varepsilon}_{i+1}(x) = \ddot{\varepsilon}_{i+1}(y) = 0$. And since $x' = y'$, we have $\ddot{\varepsilon}_{i+1}(x') = \ddot{\varepsilon}_{i+1}(y') = 0$. Therefore, by Proposition 3.14, we have

$$\ddot{\varepsilon}_{i+1}(x \mathbin{\ddot{\otimes}} x') = \ddot{\varepsilon}_{i+1}(x) + \ddot{\varepsilon}_{i+1}(x') = \ddot{\varepsilon}_{i+1}(y) + \ddot{\varepsilon}_{i+1}(y') = \ddot{\varepsilon}_{i+1}(y \mathbin{\ddot{\otimes}} y').$$

If $\ddot{\varepsilon}_i(x') > 0$, then $x = y$ and $\ddot{e}_i(x') = y'$. Since $\ddot{\varepsilon}_{i+1}(y') = 0$, as before, axiom **LQ2** (2) implies that $\ddot{\varepsilon}_{i+1}(x') = \ddot{\varepsilon}_{i+1}(y') = 0$, and as $x = y$, we get $\ddot{\varepsilon}_{i+1}(x) = \ddot{\varepsilon}_{i+1}(y) = 0$. Reasoning as before, we conclude that $\ddot{\varepsilon}_{i+1}(x \mathbin{\ddot{\otimes}} x') = \ddot{\varepsilon}_{i+1}(y \mathbin{\ddot{\otimes}} y')$.

Claim: $\mathcal{Q} \mathbin{\ddot{\otimes}} \mathcal{Q}'$ satisfies LQ3 and LQ3′. We will show that $\mathcal{Q} \mathbin{\ddot{\otimes}} \mathcal{Q}'$ satisfies **LQ3**, the proof for **LQ3′** is analogous. Let $i, j \in I$, with $i \neq j$, and suppose that $\ddot{e}_i(x \mathbin{\ddot{\otimes}} x')$ and $\ddot{e}_j(x \mathbin{\ddot{\otimes}} x')$ are both defined. By Proposition 3.14, we have four cases to consider.

Case 1. Suppose that $\ddot{\varepsilon}_i(x') = 0$ and $\ddot{\varepsilon}_j(x') = 0$. Then, by Proposition 3.14, we have

$$\ddot{e}_i(x \mathbin{\ddot{\otimes}} x') = \ddot{e}_i(x) \mathbin{\ddot{\otimes}} x', \quad \ddot{e}_j(x \mathbin{\ddot{\otimes}} x') = \ddot{e}_j(x) \mathbin{\ddot{\otimes}} x'. \tag{3.7}$$

Since $\ddot{e}_i(x) = y$ and $\ddot{e}_j(x) = z$ are both defined, by axiom **LQ3** we have

$$\ddot{e}_i\ddot{e}_j(x) = \ddot{e}_j\ddot{e}_i(x) \neq \perp. \tag{3.8}$$

Then, since $\ddot{\varepsilon}_i(x') = \ddot{\varepsilon}_j(x') = 0$, we have

$$\begin{aligned}
\ddot{e}_i\ddot{e}_j(x \mathbin{\ddot{\otimes}} x') &= \ddot{e}_i(\ddot{e}_j(x) \mathbin{\ddot{\otimes}} x') && \text{(by (3.7))}\\
&= \ddot{e}_i\ddot{e}_j(x) \mathbin{\ddot{\otimes}} x' && \text{(by Proposition 3.14)}\\
&= \ddot{e}_j\ddot{e}_i(x) \mathbin{\ddot{\otimes}} x' && \text{(by (3.8))}\\
&= \ddot{e}_j(\ddot{e}_i(x) \mathbin{\ddot{\otimes}} x') && \text{(by Proposition 3.14)}\\
&= \ddot{e}_j\ddot{e}_i(x \mathbin{\ddot{\otimes}} x'). && \text{(by (3.7))}
\end{aligned}$$

Case 2. Suppose that $\ddot{\varepsilon}_i(x') = 0$ and $\ddot{\varepsilon}_j(x') > 0$. Then, we have $\ddot{\varphi}_j(x) = 0$ and

$$\ddot{e}_i(x \mathbin{\ddot{\otimes}} x') = \ddot{e}_i(x) \mathbin{\ddot{\otimes}} x', \quad \ddot{e}_j(x \mathbin{\ddot{\otimes}} x') = x \mathbin{\ddot{\otimes}} \ddot{e}_j(x'). \tag{3.9}$$

We claim that $j \neq i+1$. If $j = i+1$, then, since $\ddot{e}_i(x)$ is defined, we have $\ddot{\varepsilon}_i(x) > 0$, and axiom **LQ1** implies that $\ddot{\varphi}_{i+1}(x) > 0$. And since $\ddot{e}_j(x') = \ddot{e}_{i+1}(x')$ is defined, we have $\ddot{\varepsilon}_{i+1}(x') > 0$. Thus, by Definition 3.12, we have $\ddot{e}_j(x \mathbin{\ddot{\otimes}} x') = \ddot{e}_{i+1}(x \mathbin{\ddot{\otimes}} x') = \perp$, which contradicts the hypothesis. Thus, $j \neq i+1$, and we have two cases to consider.

Case 2.1 Suppose that $|i-j| > 1$. By axiom **LQ2 (1)**, we have $\ddot{\varepsilon}_i(\ddot{e}_j(x')) = \ddot{\varepsilon}_i(x') = 0$. Thus, by Proposition 3.14,

$$\ddot{e}_i\ddot{e}_j(x \mathbin{\ddot{\otimes}} x') = \ddot{e}_i(x \mathbin{\ddot{\otimes}} \ddot{e}_j(x')) = \ddot{e}_i(x) \mathbin{\ddot{\otimes}} \ddot{e}_j(x'). \tag{3.10}$$

Axiom **LQ2 (1)** also implies that $\ddot{\varepsilon}_j(\ddot{e}_i(x)) = \ddot{\varepsilon}_j(x) > 0$, and therefore, by Proposition 3.14,

$$\ddot{e}_j\ddot{e}_i(x \mathbin{\ddot{\otimes}} x') = \ddot{e}_j(\ddot{e}_i(x) \mathbin{\ddot{\otimes}} x') = \ddot{e}_i(x) \mathbin{\ddot{\otimes}} \ddot{e}_j(x'). \tag{3.11}$$

From (3.10) and (3.11), we get $\ddot{e}_i\ddot{e}_j(x \mathbin{\ddot{\otimes}} x') = \ddot{e}_j\ddot{e}_i(x \mathbin{\ddot{\otimes}} x')$.

Case 2.2 Now suppose that $j = i-1$. We claim that $\ddot{\varphi}_{i-1}(\ddot{e}_i(x))$ is finite. If $\ddot{\varphi}_{i-1}(\ddot{e}_i(x)) = +\infty$, then, as $\ddot{\varphi}_{i-1}(x) = \ddot{\varphi}_j(x) = 0$, we would have $\ddot{\varphi}_{i-1}(\ddot{e}_i(x)) \neq \ddot{\varphi}_{i-1}(x)$. Thus, by axiom **LQ2 (3)**, we would have $\ddot{\varphi}_{i-1}(x) > 0$, which is a contradiction. Therefore, we have $\ddot{\varphi}_{i-1}(\ddot{e}_i(x)) \neq +\infty$, and then,

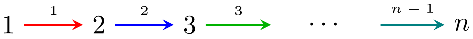


Fig. 7. The standard crystal $\mathbb{B}_n$ of type A_{n-1}.

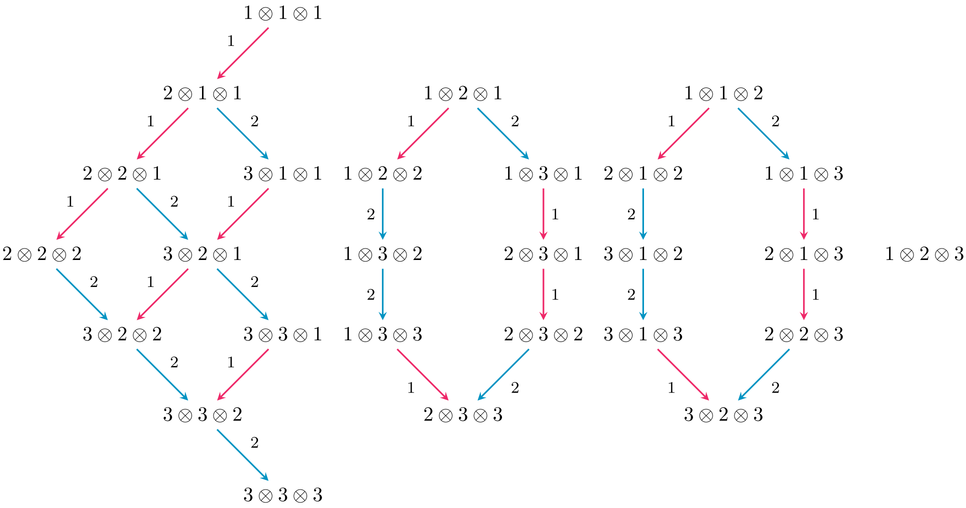


Fig. 8. The connected components of the usual tensor product $\mathbb{B}_3^{\otimes 3}$.

axiom **LQ2 (3)** implies that $\ddot{\varphi}_{i-1}(\ddot{e}_i(x)) = \ddot{\varphi}_{i-1}(x) = 0$. Consequently, $\ddot{e}_{i-1}(\ddot{e}_i(x) \ddot{\otimes} x')$ is defined. Moreover, $\ddot{\varepsilon}_j(x') = \ddot{\varepsilon}_{i-1}(x') > 0$, and thus Proposition 3.14 implies that

$$\ddot{e}_{i-1}\ddot{e}_i(x \ddot{\otimes} x') = \ddot{e}_{i-1}(\ddot{e}_i(x) \ddot{\otimes} x') = \ddot{e}_i(x) \ddot{\otimes} \ddot{e}_{i-1}(x'). \tag{3.12}$$

Since $\ddot{\varepsilon}_i(x') = 0$, in particular, $\ddot{\varepsilon}_i(x') \neq +\infty$. Thus, by axiom **LQ2 (2)**, we have $\ddot{\varepsilon}_i(\ddot{e}_{i-1}(x')) = \ddot{\varepsilon}_i(x') = 0$, and by Proposition 3.14,

$$\ddot{e}_i\ddot{e}_{i-1}(x \ddot{\otimes} x') = \ddot{e}_i(x \ddot{\otimes} \ddot{e}_{i-1}(x')) = \ddot{e}_i(x) \ddot{\otimes} \ddot{e}_{i-1}(x'). \tag{3.13}$$

From (3.12) and (3.13), we get $\ddot{e}_{i-1}\ddot{e}_i(x \ddot{\otimes} x') = \ddot{e}_i\ddot{e}_{i-1}(x \ddot{\otimes} x')$.

Case 3. Suppose that $\ddot{\varepsilon}_i(x') > 0$ and $\ddot{\varepsilon}_j(x') = 0$. This is similar to the previous case, except now we have $j \neq i - 1$.

Case 4. Suppose that $\ddot{\varepsilon}_i(x') > 0$ and $\ddot{\varepsilon}_j(x') > 0$. Then, we have $\ddot{\varphi}_i(x) = \ddot{\varphi}_j(x) = 0$, and

$$\ddot{e}_i(x \ddot{\otimes} x') = x \ddot{\otimes} \ddot{e}_i(x'), \quad \ddot{e}_j(x \ddot{\otimes} x') = x \ddot{\otimes} \ddot{e}_j(x'). \tag{3.14}$$

Since $\ddot{e}_i(x')$ and $\ddot{e}_j(x')$ are both defined, **LQ3** implies that $\ddot{e}_i\ddot{e}_j(x') = \ddot{e}_j\ddot{e}_i(x') \neq \perp$. Therefore, $\ddot{\varepsilon}_i(\ddot{e}_j(x')) > 0$ and $\ddot{\varepsilon}_j(\ddot{e}_i(x')) > 0$. Thus, by Proposition 3.14 we have

$$\begin{aligned}
\ddot{e}_i\ddot{e}_j(x \ddot{\otimes} x') &= \ddot{e}_i(x \ddot{\otimes} \ddot{e}_j(x')) \\
&= x \ddot{\otimes} \ddot{e}_i\ddot{e}_j(x') \\
&= x \ddot{\otimes} \ddot{e}_j\ddot{e}_i(x') \\
&= \ddot{e}_j(x \ddot{\otimes} \ddot{e}_i(x')) \\
&= \ddot{e}_j\ddot{e}_i(x \ddot{\otimes} x'). \quad \square
\end{aligned}$$

Recall that the *standard crystal* $\mathbb{B}_n$ of type A_{n-1}, depicted in Fig. 7, is the crystal structure on $\{1 < \cdots < n\}$ in which $\widetilde{f}_i(i) = i + 1$, $\widetilde{f}_i(j) = \perp$, for $j \neq i$, and $\widetilde{\varphi}_i(j) = \delta_{i,j}$, for $j \in I$.

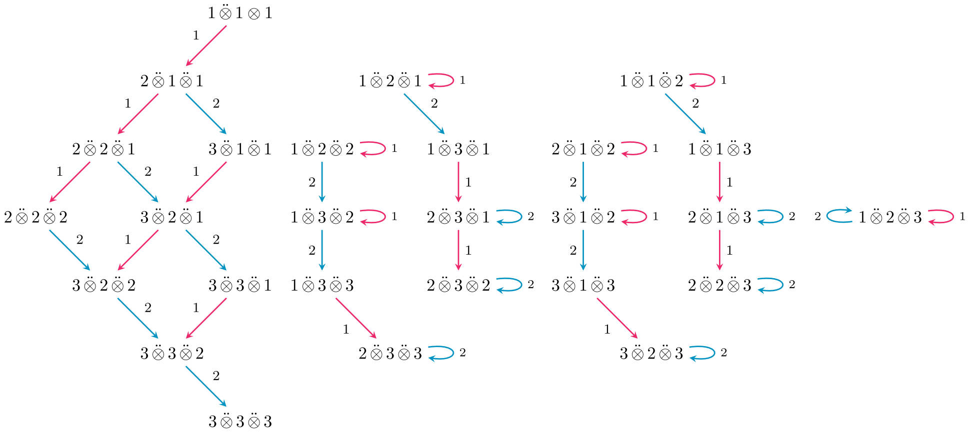


Fig. 9. The connected components of the quasi-tensor product $\mathbb{B}_3^{\ddot{\otimes}3}$. Notice this may be obtained from $\mathbb{B}_3^{\otimes 3}$ be removing certain edges and adding i-labelled loops on vertices with i-inversions.

Figs. 8 and 9 illustrate the differences between the usual tensor product of crystals and the quasi-tensor product, on the standard crystal $\mathbb{B}_3$ of type A_2. We remark that the structures in both figures are quite similar, and that the latter may be obtained from the former by removing certain edges and adding i-labelled loops to vertices with i-inversions.[1] This construction will be detailed on Section 4.

Moreover, the connected components of the crystal graph plac_n corresponding to words of length k are in bijection with the connected components of $\mathbb{B}_n^{\otimes k}$. For example, compare the components of Fig. 1 with the ones on Fig. 8.

Likewise, the connected components of the quasi-crystal graph hypo_n corresponding to words with length k are also in bijection precisely with the connected components of $\mathbb{B}_n^{\ddot{\otimes}n}$. For instance, notice the components on Fig. 4 and the ones of Fig. 9.

As a consequence of Theorem 3.15, and following the previous discussion, we have the next result.

Corollary 3.16. *Let $\mathbb{B}_n$ be the standard crystal of type A_{n-1}. Then, every connected component of $\mathbb{B}_n^{\ddot{\otimes}k}$ satisfies the local quasi-crystal axioms of Definition 3.1. As a consequence, every seminormal quasi-crystal graph satisfying the local quasi-crystal axioms is isomorphic to a unique quasi-crystal graph of quasi-ribbon tableaux.*

Proof. The standard crystal $\mathbb{B}_n$ clearly satisfies the local quasi-crystal axioms. If follows from Theorem 3.15 that $\mathbb{B}_n^{\ddot{\otimes}k}$ satisfies the same axioms as well. Each component of $\mathbb{B}_n^{\ddot{\otimes}k}$ is in bijection with a component of hypo_n corresponding to a word w of length k, which is isomorphic, via the Krob–Thibon insertion, to a unique quasi-crystal component of quasi-ribbon tableaux. □

4. Type A_{n-1} Stembridge crystals and quasi-crystals

In what follows, let $\mathcal{C}$ be a crystal of type A_{n-1}.

Remark 4.1. We consider the weight lattice of $\mathcal{C}$ to be $\mathbb{Z}^n$ modulo $\mathbf{e_1}+\cdots+\mathbf{e_n}$, where $\{\mathbf{e_1},\ldots,\mathbf{e_n}\}$ denotes the canonical basis of $\mathbb{R}^n$. Therefore, we may fix a representative such that highest weights are always partitions, followed by a possible sequence of zeros. This implies that the entries of $\mathrm{wt}(x)$ are non-negative, for any $x \in \mathcal{C}$.

---

[1] Recall that with this tensor product convention, the vertices $x \otimes y$ and $x \ddot{\otimes} y$ are identified with the word yx.

We now state some auxiliary results. In what follows, $\mathrm{wt}_i(x)$ denotes the ith component of $\mathrm{wt}(x)$.

Lemma 4.2. *Let $\mathcal{C}$ be a seminormal crystal. Then,*

1. $\widetilde{\varepsilon}_i(x) \leq \mathrm{wt}_{i+1}(x)$,
2. $\widetilde{\varphi}_i(x) \leq \mathrm{wt}_i(x)$.

Furthermore, $\widetilde{\varepsilon}_i(x) = \mathrm{wt}_{i+1}(x)$ if and only if $\widetilde{\varphi}_i(x) = \mathrm{wt}_i(x)$.

Proof. We prove the first statement; the second is similar. Since $\mathcal{C}$ is seminormal, let $k_0 := \widetilde{\varepsilon}_i(x) = \max\{k : \widetilde{e}_i^{k}(x) \neq \perp\}$. If $k_0 > \mathrm{wt}_{i+1}(x)$, then, as $\mathrm{wt}(\widetilde{e}_i^{k_0}(x)) = \mathrm{wt}(x) + k_0\alpha_i$, we would have $\mathrm{wt}_{i+1}(\widetilde{e}_i^{k_0}(x)) = \mathrm{wt}_{i+1}(x) - k_0 < 0$, which is a contradiction by Remark 4.1.

Now suppose that $\widetilde{\varepsilon}_i(x) = \mathrm{wt}_{i+1}(x)$. Then, by **C2**, $\widetilde{\varphi}_i(x) = \widetilde{\varepsilon}_i(x) + \langle \mathrm{wt}(x), \alpha_i \rangle = \mathrm{wt}_{i+1}(x) + \mathrm{wt}_i(x) - \mathrm{wt}_{i+1}(x) = \mathrm{wt}_i(x)$. The other implication is proved analogously. □

We remark that in $\mathcal{C} = \mathsf{plac}_n$, $\widetilde{\varepsilon}_i(x) = \mathrm{wt}_{i+1}(x)$ and $\widetilde{\varphi}_i(x) = \mathrm{wt}_i(x)$ happen exactly when x has no i-inversions.

Next, we provide a construction to obtain quasi-crystal graphs from a connected crystal graph. Recently, Maas-Gariépy introduced an equivalent construction, by considering the induced subgraphs corresponding to fundamental quasi-symmetric functions [26].

Our construction may be described as follows. For every $i \in I$ and every vertex $x \in \mathcal{C}$, place an i-labelled loop on x if $\ddot{\varepsilon}_i(x) < \mathrm{wt}_{i+1}(x)$ (or equivalently, if $\ddot{\varphi}_i(x) < \mathrm{wt}_i(x)$). After this, remove all the i-labelled edges that have i-labelled loops on both ends (the existence of both loops is ensured by Lemma 4.4 below).

Definition 4.3. Let $(\mathcal{C}, \widetilde{e}_i, \widetilde{f}_i, \widetilde{\varepsilon}_i, \widetilde{\varphi}_i)$ be a connected Stembridge crystal. We construct a quasi-crystal $(\mathcal{Q}_{\mathcal{C}}, \ddot{e}_i, \ddot{f}_i, \ddot{\varepsilon}_i, \ddot{\varphi}_i)$ as follows: $\mathcal{Q}_{\mathcal{C}}$ has the same underlying set, index set I and weight function wt as $\mathcal{C}$, and we define

$$\ddot{\varepsilon}_i(x) := \begin{cases} \widetilde{\varepsilon}_i(x) & \text{if } \widetilde{\varepsilon}_i(x) = \mathrm{wt}_{i+1}(x) \\ +\infty & \text{otherwise,} \end{cases}$$

$$\ddot{e}_i(x) := \begin{cases} \widetilde{e}_i(x) & \text{if } \widetilde{\varepsilon}_i(x) = \mathrm{wt}_{i+1}(x) \\ \perp & \text{otherwise} \end{cases}$$

and set $\ddot{\varphi}_i(x) := \ddot{\varepsilon}_i(x) + \langle \mathrm{wt}(x), \alpha_i \rangle$, and $\ddot{f}_i(x) := y$ if and only if $\ddot{e}_i(y) = x$.

If $\widetilde{f}_i(x) \neq \perp$ and $\ddot{f}_i(x) = \perp$, we say that $\widetilde{f}_i$ is a *strict Kashiwara operator* on $x \in \mathcal{C}$. Otherwise, it is simply called a quasi-Kashiwara operator.

In what follows we consider $\mathcal{C}$ and $\mathcal{Q}_{\mathcal{C}}$ as in Definition 4.3.

Lemma 4.4. *Let $x, y \in \mathcal{C}$ be such that $\widetilde{e}_i(x) = y$. Then,*

1. *$\ddot{\varepsilon}_i(x) = +\infty$ if and only if $\ddot{\varepsilon}_i(y) = +\infty$.*
2. *For $|i - j| > 1$, $\ddot{\varepsilon}_j(x) = +\infty$ if and only if $\ddot{\varepsilon}_j(y) = +\infty$.*

Proof. By Definition 4.3 and Lemma 4.2, $\ddot{\varepsilon}_i(x) = +\infty$ implies that $\widetilde{\varepsilon}_i(x) < \mathrm{wt}_{i+1}(x)$, and consequently

$$\widetilde{\varepsilon}_i(x) - 1 < \mathrm{wt}_{i+1}(x) - 1.$$

Since $\widetilde{e}_i(x) = y$, by **C1** we have $\widetilde{\varepsilon}_i(y) = \widetilde{\varepsilon}_i(x) - 1$ and $\mathrm{wt}_{i+1}(y) = \mathrm{wt}_{i+1}(x) - 1$, and thus $\widetilde{\varepsilon}_i(y) < \mathrm{wt}_{i+1}(y)$ and $\ddot{\varepsilon}_i(y) = +\infty$. The proof of the reverse implication is analogous.

For the second statement, since $\ddot{e}_i(x) = y$, we have $\widetilde{e}_i(x) = y$ (and $\widetilde{\varepsilon}_i(x) = \mathrm{wt}_{i+1}(x)$), and by axiom **S1**, $\widetilde{\varepsilon}_j(x) = \widetilde{\varepsilon}_j(y)$. Suppose that $\ddot{\varepsilon}_j(x) = +\infty$. Then, $\widetilde{\varepsilon}_j(x) < \mathrm{wt}_{j+1}(x)$ and thus, $\widetilde{\varepsilon}_j(y) = \widetilde{\varepsilon}_j(x) < \mathrm{wt}_{i+1}(x)$. Since $|i - j| > 1$, we have $\mathrm{wt}_{j+1}(x) = \mathrm{wt}_{j+1}(y)$. Therefore, $\widetilde{\varepsilon}_j(y) < \mathrm{wt}_{j+1}(y)$ and $\ddot{\varepsilon}_j(y) = +\infty$. The other implication is proved similarly. □

Lemma 4.5. *Suppose that $\ddot{\varepsilon}_i(x) = +\infty$. Then, $\mathrm{wt}_i(x), \mathrm{wt}_{i+1}(x) > 0$.*

Proof. Since $\ddot{\varepsilon}_i(x) = +\infty$, we have $\widetilde{\varepsilon}_i(x) < \mathrm{wt}_{i+1}(x)$ and $\widetilde{\varphi}_i(x) < \mathrm{wt}_i(x)$, by Definition 4.3 and Lemma 4.2. Since $\mathcal{C}$ is seminormal, we have $\widetilde{\varepsilon}_i(x),\ \widetilde{\varphi}_i(x) \geq 0$, which implies that $\mathrm{wt}_i(x),\ \mathrm{wt}_{i+1}(x) > 0$. $\square$

Theorem 4.6. *The quasi-crystal graph $\mathcal{Q}_{\mathcal{C}}$ is seminormal and satisfies the axioms of Definition 3.1.*

Proof. For simplicity, we let $\mathcal{Q} := \mathcal{Q}_{\mathcal{C}}$. By construction, $\mathcal{Q}$ is seminormal.

Claim: $\mathcal{Q}$ satisfies LQ1. Let $i, i+1 \in I$. Suppose that $\ddot{\varepsilon}_i(x) = 0$. Since $\ddot{\varepsilon}_i(x) \neq +\infty$, by Definition 4.3, we have $\ddot{\varepsilon}_i(x) = \widetilde{\varepsilon}_i(x) = \mathrm{wt}_{i+1}(x) = 0$. By Lemma 4.2, we have $\widetilde{\varphi}_{i+1}(x) \leq \mathrm{wt}_{i+1}(x) = 0$. Therefore, $\widetilde{\varphi}_{i+1}(x) = 0 = \mathrm{wt}_{i+1}(x)$ and thus $\ddot{\varphi}_{i+1}(x) = 0$. The other implication is proved analogously.

Claim: $\mathcal{Q}$ satisfies LQ2. Suppose that $\ddot{e}_i(x) = y$. By Definition 4.3, we have $\widetilde{e}_i(x) = y$. To show that $\mathcal{Q}$ satisfies **LQ2 (1)**, suppose that $|i-j| > 1$. We claim that $\ddot{\varepsilon}_j(x) = +\infty$ if and only if $\ddot{\varepsilon}_j(y) = +\infty$. Indeed, since $\widetilde{e}_i(x) = y$, by **C1** we have $\mathrm{wt}_{j+1}(x) = \mathrm{wt}_{j+1}(y)$, and by axiom **S1**, we have $\widetilde{\varepsilon}_j(x) = \widetilde{\varepsilon}_j(y)$. Therefore, $\widetilde{\varepsilon}_j(x) < \mathrm{wt}_{j+1}(x)$ if and only if $\widetilde{\varepsilon}_j(y) < \mathrm{wt}_{j+1}(y)$. Thus, if $\ddot{\varepsilon}_j(x) = +\infty$, then $\ddot{\varepsilon}_j(y) = +\infty = \ddot{\varepsilon}_j(x)$. If $\ddot{\varepsilon}_j(x) \neq +\infty$, then $\ddot{\varepsilon}_j(y) \neq +\infty$, therefore $\ddot{\varepsilon}_j(x) = \widetilde{\varepsilon}_j(x)$ and $\ddot{\varepsilon}_j(y) = \widetilde{\varepsilon}_j(y)$, and thus $\ddot{\varepsilon}_j(x) = \ddot{\varepsilon}_j(y)$, which concludes the proof of **LQ2 (1)**.

We now prove that $\mathcal{Q}$ satisfies **LQ2 (2)** (the proof for **LQ2 (3)** is similar).

1. Suppose that $\ddot{\varepsilon}_{i+1}(x) = +\infty$ and that $\ddot{\varepsilon}_i(y) = 0$. Since $\mathcal{Q}$ satisfies **LQ1**, $\ddot{\varepsilon}_i(y) = 0$ implies that $\ddot{\varphi}_{i+1}(y) = 0$. In particular, $\ddot{\varphi}_{i+1}(y) \neq +\infty$, and by Definition 4.3, we have $\ddot{\varepsilon}_{i+1}(y) \neq +\infty$. Therefore, $\ddot{\varepsilon}_{i+1}(x) = +\infty \neq \ddot{\varepsilon}_{i+1}(y)$.
2. Now suppose that $\ddot{\varepsilon}_{i+1}(y) = 0$. We will show that $\ddot{\varepsilon}_{i+1}(x) = \ddot{\varepsilon}_{i+1}(y)$. By Definition 4.3 and Lemma 4.2, as $\ddot{\varepsilon}_{i+1}(y) = 0 \neq +\infty$, we have $\widetilde{\varepsilon}_{i+1}(y) = \mathrm{wt}_{i+2}(y) = 0$ and $\widetilde{\varphi}_{i+1}(y) = \mathrm{wt}_{i+1}(y)$. Since $\widetilde{e}_i(x) = y$, Remark 2.7 implies that $\widetilde{\varepsilon}_{i+1}(y) \geq \widetilde{\varepsilon}_{i+1}(x)$. Thus, since $\widetilde{\varepsilon}_{i+1}(y) = 0$, we have $\widetilde{\varepsilon}_{i+1}(x) \leq 0$, which implies, as $\mathcal{C}$ is seminormal, that $\widetilde{\varepsilon}_{i+1}(x) = 0$. Moreover, $\widetilde{e}_i(x) = y$ implies that $\mathrm{wt}_{i+2}(x) = \mathrm{wt}_{i+2}(y)$. Thus, we have $\widetilde{\varepsilon}_{i+1}(x) = \mathrm{wt}_{i+2}(y) = \mathrm{wt}_{i+2}(x) = 0$ and, consequently, $\ddot{\varepsilon}_{i+1}(x) = 0 = \ddot{\varepsilon}_{i+1}(y)$.
3. Suppose that $\ddot{\varepsilon}_{i+1}(x) \neq \ddot{\varepsilon}_{i+1}(y)$. We will show that $\ddot{\varepsilon}_{i+1}(x) = +\infty$ and $\ddot{\varepsilon}_i(y) = 0$. Since $\ddot{\varepsilon}_{i+1}(x) \neq \ddot{\varepsilon}_{i+1}(y)$, we claim that

   $$\ddot{\varepsilon}_{i+1}(x) = +\infty, \quad \ddot{\varepsilon}_{i+1}(y) \neq +\infty. \tag{4.1}$$

   If $\ddot{\varepsilon}_{i+1}(x) \neq +\infty$ and $\ddot{\varepsilon}_{i+1}(y) = +\infty$, we would have $\widetilde{\varepsilon}_{i+1}(x) = \mathrm{wt}_{i+2}(x)$ and $\widetilde{\varepsilon}_{i+1}(y) < \mathrm{wt}_{i+2}(y)$. Since $\widetilde{e}_i(x) = y$, we have $\mathrm{wt}(y) = \mathrm{wt}(x) + \alpha_i$, hence $\mathrm{wt}_{i+2}(x) = \mathrm{wt}_{i+2}(y)$. This would imply that

   $$\widetilde{\varepsilon}_{i+1}(y) < \mathrm{wt}_{i+2}(y) = \mathrm{wt}_{i+2}(x) = \widetilde{\varepsilon}_{i+1}(x),$$

   which contradicts Remark 2.7. If $\ddot{\varepsilon}_{i+1}(x),\ \ddot{\varepsilon}_{i+1}(y) \neq +\infty$, then we would have

   $$\widetilde{\varepsilon}_{i+1}(y) = \mathrm{wt}_{i+2}(y) = \mathrm{wt}_{i+2}(x) = \widetilde{\varepsilon}_{i+1}(x),$$

   and consequently, $\widetilde{\varepsilon}_{i+1}(x) = \widetilde{\varepsilon}_{i+1}(y)$, which implies $\ddot{\varepsilon}_{i+1}(x) = \ddot{\varepsilon}_{i+1}(y)$, since both are finite, contradicting the hypothesis that $\ddot{\varepsilon}_{i+1}(x) \neq \ddot{\varepsilon}_{i+1}(y)$. Therefore, Eq. (4.1) holds. In particular, as $\widetilde{\varepsilon}_{i+1}(x) < \mathrm{wt}_{i+2}(x) = \mathrm{wt}_{i+2}(y) = \widetilde{\varepsilon}_{i+1}(y)$, by axiom **S1** we have

   $$\widetilde{\varepsilon}_{i+1}(y) = \widetilde{\varepsilon}_{i+1}(x) + 1. \tag{4.2}$$

   Since $\ddot{\varepsilon}_{i+1}(x) = +\infty$, it remains to show that $\ddot{\varepsilon}_i(y) = 0$. So, suppose that $\ddot{\varepsilon}_i(y) = k > 0$. In particular, as $\ddot{\varepsilon}_i(y) \neq +\infty$ (because $\ddot{e}_i(x) = y$), we have $\widetilde{\varepsilon}_i(y) = \mathrm{wt}_{i+1}(y) = k$. Consider the connected component consisting of only i-labelled edges, containing y. Clearly, that component has a unique highest weight element, which is not y, since $\widetilde{\varepsilon}_i(y) > 0$. Let z be the highest weight element of that component. Then, there exists $z_1, \ldots, z_k = z \in \mathcal{Q}$ such that

   $$\begin{cases} \ddot{e}_i(y) = z_1 \text{ (and thus, } \widetilde{e}_i(y) = z_1), \\ \ddot{e}_i(z_l) = z_{l+1} \text{ (and thus, } \widetilde{e}_i(z_l) = z_{l+1}), \text{ for } l = 1, \ldots, k-1, \\ \ddot{\varepsilon}_i(z_k) = 0 \text{ (and thus, } \widetilde{\varepsilon}_i(z_k) = 0), \end{cases} \tag{4.3}$$

   as shown in the following diagram:

$$\begin{tikzcd}
 & & & z_k \arrow[d, "i"] \\
 & & & z_{k-1} \\
 & & & \vdots \\
 & & & z_1 \arrow[d, "i"] \\
\bullet \arrow[r, "i+1"] & \bullet \arrow[r, "i+1"] & \cdots \arrow[r, "i+1"] & y \arrow[d, "i"] \\
 & \bullet \arrow[r, "i+1"'] & \cdots \arrow[r, "i+1"'] & x
\end{tikzcd}$$

By **LQ1**, $\ddot{\varepsilon}_i(z_k) = 0$ implies that $\ddot{\varphi}_{i+1}(z_k) = 0$, and thus,

$$\widetilde{\varphi}_{i+1}(z_k) = 0. \tag{4.4}$$

By (4.1), we have $\ddot{\varepsilon}_{i+1}(y) \neq +\infty$ and hence,

$$\widetilde{\varphi}_{i+1}(y) = \mathrm{wt}_{i+1}(y) = k. \tag{4.5}$$

Thus, there exists $y_1, \ldots, y_k \in \mathcal{Q}$ such that

$$\begin{cases} \widetilde{f}_{i+1}(y) = y_1 \\ \widetilde{f}_{i+1}(y_l) = y_{l+1}, \ \text{for } l = 1, \ldots, k-1 \\ \widetilde{\varphi}_{i+1}(y_k) = 0, \end{cases} \tag{4.6}$$

as shown in the following diagram:

$$\begin{tikzcd}
 & & & z_k \arrow[d, "i"] \\
 & & & z_{k-1} \\
 & & & \vdots \\
 & & & z_1 \arrow[d, "i"] \\
\bullet \arrow[r, "i+1"] & \bullet \arrow[r, "i+1"] & \cdots \arrow[r, "i+1"] & y \arrow[d, "i"] \arrow[r, "i+1"] & y_1 & \cdots & y_{k-1} \arrow[r, "i+1"] & y_k \\
 & \bullet \arrow[r, "i+1"'] & \cdots \arrow[r, "i+1"'] & x
\end{tikzcd}$$

From Eqs. (4.2) and (4.5), and Proposition 2.6, we have

$$\widetilde{\varphi}_{i+1}(x) = \widetilde{\varphi}_{i+1}(y) = k. \tag{4.7}$$

Then, there exists $x_1, \dots, x_k \in \mathcal{Q}$ such that

$$\begin{cases} \widetilde{f}_{i+1}(x) = x_1, \\ \widetilde{f}_{i+1}(x_l) = x_{l+1}, \ \text{for } l = 1, \dots, k-1, \\ \widetilde{\varphi}_{i+1}(x_k) = 0. \end{cases} \tag{4.8}$$

We have $\widetilde{f}_i(y) = x$ and $\widetilde{f}_{i+1}(y) = y_1$. From (4.7), axiom **S2**$'$ implies that $\widetilde{f}_i\widetilde{f}_{i+1}(y) = \widetilde{f}_{i+1}\widetilde{f}_i(y)$ and thus

$$\begin{aligned} \widetilde{f}_i(y_1) &= \widetilde{f}_i\widetilde{f}_{i+1}(y) && \text{by (4.6)} \\ &= \widetilde{f}_{i+1}\widetilde{f}_i(y) \\ &= \widetilde{f}_{i+1}(x) = x_1 && \text{by (4.8)} \end{aligned}$$

Then, we have $\widetilde{f}_i(y_1) = x_1$, or equivalently, $y_1 = \widetilde{e}_i(x_1)$, as illustrated in the diagram below:

$$\begin{tikzcd}
 & & & z_k \arrow[d, "i"] & & & & \\
 & & & z_{k-1} & & & & \\
 & & & \vdots & & & & \\
 & & & z_1 \arrow[d, "i"] & & & & \\
\bullet \arrow[r, "i+1"] & \bullet \arrow[r, "i+1"] & \cdots \arrow[r, "i+1"] & y \arrow[r, "i+1"] \arrow[d, "i"] & y_1 \arrow[d, "i"] & \cdots & y_{k-1} \arrow[r, "i+1"] & y_k \\
 & \bullet \arrow[r, "i+1"'] & \cdots \arrow[r, "i+1"'] & x \arrow[r, "i+1"'] & x_1 & \cdots & x_{k-1} \arrow[r, "i+1"'] & x_k
\end{tikzcd}$$

Since $\widetilde{f}_{i+1}(x) = x_1$, we have $x = \widetilde{e}_{i+1}(x_1)$. By axiom **S1**, we either have $\widetilde{\varepsilon}_i(x) = \widetilde{\varepsilon}_i(x_1)$ or $\widetilde{\varepsilon}_i(x) = \widetilde{\varepsilon}_i(x_1) + 1$. We will show that both cases lead to contradictions.

Suppose that $\widetilde{\varepsilon}_i(x) = \widetilde{\varepsilon}_i(x_1)$. Then, since $\widetilde{e}_i(x) = y$ and $\widetilde{e}_i(x_1) = y_1$, **C1** implies that

$$k = \widetilde{\varepsilon}_i(y) = \widetilde{\varepsilon}_i(x) - 1 = \widetilde{\varepsilon}_i(x_1) - 1 = \widetilde{\varepsilon}_i(y_1).$$

Therefore, there exists $w_1, \dots, w_k$ such that

$$\begin{cases} \widetilde{e}_i(y_1) = w_1, \\ \widetilde{e}_i(w_l) = w_{l+1}, \ \text{for } l = 1, \dots, k-1, \\ \widetilde{\varepsilon}_i(w_k) = 0. \end{cases} \tag{4.9}$$

Thus, we have $\widetilde{e}_{i+1}(y_1) = y$ and $\widetilde{e}_i(y_1) = w_1$. Since $\widetilde{\varepsilon}_i(y) = \widetilde{\varepsilon}_i(y_1) > 0$, axiom **S2** implies that $\widetilde{e}_i\widetilde{e}_{i+1}(y_1) = \widetilde{e}_{i+1}\widetilde{e}_i(y_1)$ and thus

$$\begin{aligned} \widetilde{e}_{i+1}(w_1) &= \widetilde{e}_{i+1}\widetilde{e}_i(y_1) && \text{by (4.9)} \\ &= \widetilde{e}_i\widetilde{e}_{i+1}(y_1) \\ &= \widetilde{e}_i(y) && \text{by (4.6)} \\ &= z_1 && \text{by (4.3)} \end{aligned}$$

Applying this reasoning iteratively, we get $\widetilde{e}_{i+1}(w_l) = z_l$, for $l = 1, \dots, k$. In particular, we have $\widetilde{e}_{i+1}(w_k) = z_k$ and thus, $\widetilde{f}_{i+1}(z_k) = w_k$, as depicted in the following diagram:

$$\begin{tikzcd}
 & & & z_k \arrow[r, "i+1"] \arrow[d, "i"] & w_k \arrow[d, "i"] & & & \\
 & & & z_{k-1} \arrow[r, "i+1"] & w_{k-1} & & & \\
 & & & \vdots & \vdots & & & \\
 & & & z_1 \arrow[r, "i+1"] \arrow[d, "i"] & w_1 \arrow[d, "i"] & & & \\
\bullet \arrow[r, "i+1"] & \bullet \arrow[r, "i+1"] & \cdots \arrow[r, "i+1"] & y \arrow[r, "i+1"] \arrow[d, "i"] & y_1 \arrow[d, "i"] & \cdots & y_{k-1} \arrow[r, "i+1"] & y_k \\
 & \bullet \arrow[r, "i+1"'] & \cdots \arrow[r, "i+1"'] & x \arrow[r, "i+1"'] & x_1 & \cdots & x_{k-1} \arrow[r, "i+1"'] & x_k
\end{tikzcd}$$

This implies that $\widetilde{\varphi}_{i+1}(z_k) > 0$, which contradicts (4.4).
Now suppose that $\widetilde{\varepsilon}_i(x) = \widetilde{\varepsilon}_i(x_1) + 1$. Since we have $\widetilde{e}_{i+1}(y_1) = y$ and $\widetilde{e}_{i+1}(x_1) = x$, **C1** implies that

$$\widetilde{\varepsilon}_{i+1}(y_1) = \widetilde{\varepsilon}_{i+1}(y) + 1$$
$$\widetilde{\varepsilon}_{i+1}(x_1) = \widetilde{\varepsilon}_{i+1}(x) + 1$$

and thus, by (4.2), we get

$$\widetilde{\varepsilon}_{i+1}(y_1) = \widetilde{\varepsilon}_{i+1}(x_1) + 1.$$

Therefore, axiom **S3** implies that

$$\widetilde{e}_i\widetilde{e}_{i+1}^2\widetilde{e}_i(x_1) = \widetilde{e}_{i+1}\widetilde{e}_i^2\widetilde{e}_{i+1}(x_1),$$

and furthermore,

$$y = \widetilde{e}_i\widetilde{e}_{i+1}(x_1) \neq \widetilde{e}_{i+1}\widetilde{e}_i(x_1) = y.$$

which is again a contradiction.
Thus, the original assumption that $\ddot{\varepsilon}_i(y) > 0$ is false, and we have $\ddot{\varepsilon}_i(y) = 0$.

Claim: $\mathcal{Q}$ satisfies LQ3 and LQ3′. We will prove that $\mathcal{Q}$ satisfies axiom **LQ3**, the proof for **LQ3′** is similar. Let $x \in \mathcal{Q}$ and $i, j \in I$, such that $i \neq j$, and suppose that $\ddot{e}_i(x)$ and $\ddot{e}_j(x)$ are both defined. This implies that $\widetilde{e}_i(x)$ and $\widetilde{e}_j(x)$ are both defined as well and that

$$\widetilde{\varepsilon}_i(x) = \mathrm{wt}_{i+1}(x), \quad \widetilde{\varepsilon}_j(x) = \mathrm{wt}_{j+1}(x). \tag{4.10}$$

Case 1. Suppose that $|i-j| > 1$. Then, axiom **S1** implies that $\widetilde{\varepsilon}_i(\widetilde{e}_j(x)) = \widetilde{\varepsilon}_i(x)$, and therefore, by axiom **S2**,

$$\widetilde{e}_i\widetilde{e}_j(x) = \widetilde{e}_j\widetilde{e}_i(x). \tag{4.11}$$

Since $|i-j| > 1$, we have $\mathrm{wt}_{i+1}(x) = \mathrm{wt}_{i+1}(\widetilde{e}_j(x))$. Thus, (4.10) implies that

$$\widetilde{\varepsilon}_i(\widetilde{e}_j(x)) = \widetilde{\varepsilon}_i(x) = \mathrm{wt}_{i+1}(x) = \mathrm{wt}_{i+1}(\widetilde{e}_j(x)),$$

and consequently, $\ddot{\varepsilon}_i(\widetilde{e}_j(x)) = \ddot{\varepsilon}_i(\ddot{e}_j(x)) \neq +\infty$. Thus, $\ddot{e}_i\ddot{e}_j(x) = \widetilde{e}_i\widetilde{e}_j(x)$. Applying the same reasoning, we get $\ddot{e}_j\ddot{e}_i(x) = \widetilde{e}_j\widetilde{e}_i(x)$. Therefore, by (4.11), we have $\ddot{e}_i\ddot{e}_j(x) = \ddot{e}_j\ddot{e}_i(x)$.

Case 2. Suppose that $|i-j| = 1$, and without loss of generality, suppose that

$$\ddot{e}_i(x) = \widetilde{e}_i(x) = y, \quad \ddot{e}_{i+1}(x) = \widetilde{e}_{i+1}(x) = z.$$

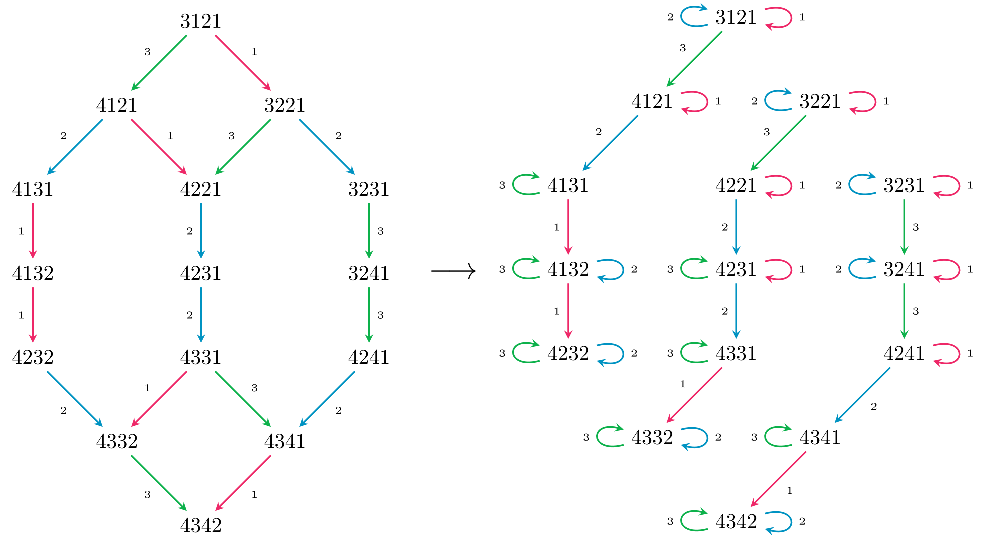


Fig. 10. A quasi-crystal graph (on the right) with three components having highest weights (2, 1, 1), (1, 2, 1) and (1, 1, 2), obtained using the construction of Definition 4.3 with a type A_3 connected crystal graph (on the left) with highest weight (2, 1, 1).

Since $\ddot{e}_{i+1}(x)$ is defined, we have $\ddot{\varepsilon}_{i+1}(x) \neq +\infty$. Therefore, since $\ddot{e}_i(x) = y$ and $\mathcal{Q}$ satisfies **LQ2**, we have

$$\ddot{\varepsilon}_{i+1}(x) = \ddot{\varepsilon}_{i+1}(y) \neq +\infty, \tag{4.12}$$

and, consequently, $\widetilde{\varepsilon}_{i+1}(x) = \widetilde{\varepsilon}_{i+1}(y)$. Thus, it follows from axiom **S2** that

$$\widetilde{e}_i\widetilde{e}_{i+1}(x) = \widetilde{e}_{i+1}\widetilde{e}_i(x). \tag{4.13}$$

From (4.12) and (4.13), we have $\ddot{e}_{i+1}\ddot{e}_i(x) = \widetilde{e}_{i+1}\widetilde{e}_i(x) = \widetilde{e}_i\widetilde{e}_{i+1}(x)$. Thus, it remains to show that $\ddot{e}_i\ddot{e}_{i+1}(x) = \widetilde{e}_i\widetilde{e}_{i+1}(x)$. We claim that $\ddot{\varphi}_i(z) \neq +\infty$. If $\ddot{\varphi}_i(z) = +\infty$ and $\ddot{\varphi}_{i+1}(x) > 0$, Propositions 3.6 (3.2) and axiom **LQ2** would imply that $\ddot{\varphi}_i(x) = +\infty$, which contradicts $\ddot{e}_i(x)$ being defined. If $\ddot{\varphi}_i(z) = +\infty$ and $\ddot{\varphi}_{i+1}(x) = 0$, then axiom **LQ1** would imply that $\ddot{\varepsilon}_i(x) = 0$, which also contradicts $\ddot{e}_i(x)$ being defined. Therefore, we have $\ddot{\varphi}_i(z) \neq +\infty$, and thus $\ddot{\varepsilon}_i(z) \neq +\infty$. Therefore, $\ddot{e}_i\ddot{e}_{i+1}(x) = \widetilde{e}_i\widetilde{e}_{i+1}(x)$. □

An example of the construction of Definition 4.3 is depicted in Fig. 10.

Recall that, given a standard tableau T filled with $\{1, \dots, n\}$, its *descent set* is given by the entries i such that $i+1$ is in a row of strictly greater index. If $\{i_1 < \cdots < i_k\}$ is the descent set of T, its *descent composition* is given by

$$\mathsf{DesComp}(T) = (i_1, i_2 - i_1, \dots, i_k - i_{k-1}, n - i_k).$$

It follows from the definition that i is in the descent set of a standard tableau T if and only if the word of T has an i-inversion.

Corollary 4.7. *Let $\mathcal{C}$ be a connected Stembridge crystal, of type A_{n-1}, having highest weight λ. Then the number of connected components in $\mathcal{Q}_{\mathcal{C}}$ is given by f_λ, the number of standard Young tableaux of shape λ.*

Proof. The character of $\mathcal{C}$ is the Schur function s_λ, where λ is the highest weight (see, for instance, [2]). Since the character of a quasi-crystal connected component is a fundamental quasi-symmetric function F_α, taking the characters of $\mathcal{Q}_{\mathcal{C}}$, one obtains a decomposition of s_λ as a sum of fundamental quasi-symmetric functions [26, Theorem 1]. The result then follows from the following decomposition [8],

$$s_\lambda = \sum_{T\in\mathsf{SYT}(\lambda)} F_{\mathsf{DesComp}(T)},$$

together with the fact that the fundamental quasi-symmetric functions F_α form a basis for the ring of quasi-symmetric functions. □

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