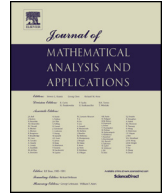




Contents lists available at [ScienceDirect](https://www.sciencedirect.com)

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa



Regular Articles

Self-improving boundedness of the maximal operator on quasi-Banach lattices over spaces of homogeneous type



Alina Shalukhina

Centro de Matemática e Aplicações, Departamento de Matemática, Faculdade de Ciências e Tecnologia, Universidade NOVA de Lisboa, Quinta da Torre, 2829-516 Caparica, Portugal

ARTICLE INFO

Article history:

Received 5 June 2024
 Available online 26 February 2025
 Submitted by M. Mastyló

Keywords:

Hardy–Littlewood maximal operator
 Quasi-Banach lattice
 Space of homogeneous type
 Self-improving property
 “Dyadic” maximal operator
 Variable Lebesgue space

ABSTRACT

We prove the self-improvement property of the Hardy–Littlewood maximal operator on quasi-Banach lattices with the Fatou property in the setting of spaces of homogeneous type. Our result is a generalization of the boundedness criterion obtained in 2010 by Lerner and Ombrosi for maximal operators on quasi-Banach function spaces over Euclidean spaces. The specialty of the proof for spaces of homogeneous type lies in using adjacent grids of Hytönen–Kairema dyadic cubes and studying the maximal operator alongside its “dyadic” version. Then we apply the obtained result to variable Lebesgue spaces over spaces of homogeneous type.

© 2025 The Author. Published by Elsevier Inc. This is an open access article under the CC BY license (<http://creativecommons.org/licenses/by/4.0/>).

1. Introduction

The primary interest in the study of the Hardy–Littlewood maximal operator, classically defined for a measurable function f on \mathbb{R}^n by

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy,$$

where the supremum of the integral means of f is taken over all balls B containing a point $x \in \mathbb{R}^n$, has always been related to finding necessary and sufficient conditions for the operator’s boundedness on various function spaces. Over the last two decades, characterization of spaces on which M is bounded became a subject of extensive research in the setting of variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$. This endeavor has recently culminated in Lerner’s criterion for the boundedness of M on $L^{p(\cdot)}(\mathbb{R}^n)$ forthcoming in [19]. Along with the effort to obtain such a characterization in terms of the exponent function, another natural question was raised: if M is bounded on a certain $L^{p(\cdot)}$ space, which perturbations of the exponent $p(\cdot)$ preserve the

E-mail address: a.shalukhina@campus.fct.unl.pt.

boundedness? Once we have the boundedness of M on one space, can it be automatically extended to a family of the closely related spaces?

A significant development in this direction was the 2005 result of Diening [10, Theorem 8.1] within which he proved, in particular, that M is bounded on a space $L^{p(\cdot)}(\mathbb{R}^n)$ with an exponent $p(\cdot)$ bounded away from one and infinity if and only if it is bounded on $L^{p(\cdot)/q}(\mathbb{R}^n)$ for some $q > 1$ (see also [11, Theorem 5.7.2]). This property received the name of “left-openness” in his work—the terminology borrowed from the theory of Muckenhoupt weights, whose classes are “left-open” in the sense that for a weight $w \in A_p$, there exists $q < p$ such that $w \in A_q$. Alternatively, this classical feature of the Muckenhoupt classes may be referred to as a self-improving property, so the boundedness result for the maximal operator adopts this name later as well.

Two years after Diening’s result, Lerner and Pérez [21, Corollary 1.3] proved the self-improving boundedness of M in the more general setting of quasi-Banach function spaces $X(\mathbb{R}^n)$. Thus this property, known for weighted Lebesgue spaces (as a consequence of the left-openness of the Muckenhoupt classes) and variable $L^{p(\cdot)}$ spaces (due to Diening), received a generalization and a unified proof. Expressed in the equivalent terms of the parameterized maximal function $M_r f = M(|f|^r)^{1/r}$ instead of “perturbations” of the space, the generalized result stated that M is bounded on X if and only if M_r is bounded on X for some $r > 1$.

Subsequently, in 2010, Lerner and Ombrosi [20, Theorem 1.2] complemented the self-improving property on quasi-Banach spaces X over \mathbb{R}^n by considering more general maximal operators $M_{\mathcal{B}}$ with respect to a basis \mathcal{B} and giving an additional “self-improvement” characterization in terms of $M_{\mathcal{B},r}$ with $r < 1$. It is exactly this result that we took as a reference point for our work—and transferred it, for the classical maximal operator, to a different topological setting. Except for switching from the abstract $M_{\mathcal{B}}$ to the classical M , our main Theorem 1.1 can be viewed as a generalization of the Lerner–Ombrosi result: on the topological level, we replace the Euclidean space \mathbb{R}^n by a space of homogeneous type (Ω, d, μ) , which is essentially a quasi-metric measure space with a doubling measure; on the functional level, we notice that the requirement on X to be a quasi-Banach function space is abundant for the proof—the framework of which we follow, though bringing in techniques specific for the new topology—and so assume only that X is a quasi-Banach lattice with the Fatou property (see [22] for differences between the above two concepts).

We begin in Section 2 by defining a quasi-Banach lattice and showing that any variable Lebesgue space with the exponent function bounded away from zero falls within its definition: this tangible example gives better perception of the abstract reasoning and is a natural setting for the later application of the main result. To make the connection between our generalization of the Lerner–Ombrosi theorem and the original self-improvement result for $L^{p(\cdot)}$ even more intuitive, we formulate Theorem 1.1 in terms of convexifications $X^{(r)}$ of a lattice X . These are logically the same “perturbations” of the space as $L^{rp(\cdot)}$ would be with respect to $L^{p(\cdot)}$, and we provide their precise definition in Subsection 2.2 following Maligranda [25, p. 99].

Having said that, we now present the main theorem.

Theorem 1.1. *Let $X(\Omega, d, \mu)$ be a quasi-Banach lattice over a space of homogeneous type (Ω, d, μ) . If $X(\Omega, d, \mu)$ has the Fatou property, the following statements are equivalent:*

- (1) M is bounded on $X(\Omega, d, \mu)$.
- (2) For all $s > 1$, M is bounded on $X^{(s)}(\Omega, d, \mu)$ and

$$\lim_{s \rightarrow 1^+} (s - 1) \|M\|_{X^{(s)} \rightarrow X^{(s)}} = 0.$$

- (3) There exists $r_0 \in (0, 1)$ such that if $r \in [r_0, 1)$, then M is bounded on $X^{(r)}(\Omega, d, \mu)$.

Technically, the above collection of equivalent statements splits into the two “trivial” implications (1) \Rightarrow (2) and (3) \Rightarrow (1) and the converse non-trivial implications (2) \Rightarrow (1) and (1) \Rightarrow (3) containing the self-

improvement property itself. The former are a simple consequence of Hölder’s inequality and extend the boundedness of M to “higher” convexifications of the lattice X ; the latter provide the extension to a range of “lower” convexifications and require a sort of reverse-Hölder bound for the maximal operator.

In our setting of spaces of homogeneous type—the review of which comprises the entire Section 3—such a bound is available for the so-called “dyadic” maximal function $M^{\mathcal{D}}w$ of a “dyadic” A_1 weight w . By the “dyadic” objects, introduced in Subsections 3.3 and 3.4, we always understand those defined through the adjacent grids of Hytönen–Kairema dyadic cubes (see [13, Section 4]) instead of balls. Then the Rubio de Francia iteration algorithm, which we apply after Lerner and Ombrosi, allows construction of appropriate “dyadic” A_1 weights for the use of the reverse-Hölder type bound (Corollary 3.9) based on the weak reverse Hölder inequality originally proved for the “dyadic” A_∞ weights by Anderson, Hytönen and Tapiola [1, Theorem 5.4]. Given also the equivalence between $M^{\mathcal{D}}$ and M , we are able to switch between the two operators when convenient and eventually prove the non-trivial implications using this trick.

Section 4 presents the proof of Theorem 1.1 in line with its division into separate implications—with Lemma 4.1 uniting the trivial implications and Theorems 4.2 and 4.4 corresponding to the “self-improvement” part. A great help in organizing our proof was Theorem 4.37 from the book by Cruz-Uribe and Fiorenza [7], which contains an adaptation of the Lerner–Ombrosi theorem to the special case of spaces $L^{p(\cdot)}(\mathbb{R}^n)$ with the “Banach” range of exponents $p(\cdot)$ not going below 1. Towards the close, in Subsection 4.4, we give an application of the main result to the variable Lebesgue spaces $L^{p(\cdot)}(\Omega, d, \mu)$ with $p(\cdot)$ bounded away from zero.

Let us also note that the reference paper by Lerner and Ombrosi [20] set in quasi-Banach spaces has recently given rise to a few further studies of the self-improvement property for maximal operators on Banach function spaces. Thus, e.g., a sharp version of the Lerner–Ombrosi theorem appeared in the work of Nieraeth [26, Theorem 2.34] for the maximal operator with respect to some general bases in the setting of Banach function spaces. Another similar self-improvement property of M on r -convex Banach function spaces was proved by Lorist and Nieraeth [23, Theorem 3.1], with the possibility to extend this result to Banach function spaces over spaces of homogeneous type [23, Remark 3.5].

We would like to underline that our main result is more general than [26, Theorem 2.34]. For one thing, we do not require that quasi-Banach lattices in our setting satisfy the saturation property and the triangle inequality with constant $C_\Delta = 1$, as it is required in [26]. Dealing with the quasi-Banach setting is more involved because it requires a version of the Aoki–Rolewicz theorem for infinite sums (see Theorem 2.1 below), whose validity was not sufficiently well explained in [20]. Furthermore, not only our essential Theorems 4.2 and 4.4, compared to [26, Theorem 2.34], establish the self-improvement property for the maximal operator $M^{\mathcal{D}}$ —that is, the maximal operator built with respect to the “nice” basis of sets \mathcal{D} satisfying the sharp A_1 self-improvement property, in terminology of [26] (which corresponds to Corollary 3.9 in our work)—but also transfer this property from $M^{\mathcal{D}}$ to the classical maximal operator M . The equivalence between $M^{\mathcal{D}}$ and M , which allows us to do this, and the properties of \mathcal{D} that make it a “nice” basis in the above sense are stemming from the specific construction which is the Hytönen–Kairema dyadic system.

2. Quasi-Banach lattices and their convexifications

2.1. Quasi-Banach lattices

We denote by $L^0(\Omega, \mu)$ the space of measurable complex-valued functions on a measure space (Ω, μ) . Let $X(\Omega, \mu) \subset L^0(\Omega, \mu)$ be a quasi-normed space—that is, a linear subspace of $L^0(\Omega, \mu)$ endowed with a quasi-norm $\|\cdot\|_X$, which differs from the usual norm by the weakened triangle inequality

$$\|f + g\|_X \leq C_\Delta(\|f\|_X + \|g\|_X)$$

holding with a constant $C_\Delta \geq 1$ independent of f and g . The space $X(\Omega, \mu)$ is called a quasi-normed lattice if it additionally satisfies the following *lattice property*:

$$\begin{aligned} \text{if } f \in X(\Omega, \mu), \text{ and } g \in L^0(\Omega, \mu) \text{ is such that } |g| \leq |f|, \\ \text{then } g \in X(\Omega, \mu) \text{ and } \|g\|_X \leq \|f\|_X. \end{aligned}$$

Quasi-normed lattices and their complete counterparts, quasi-Banach lattices, form the functional setting for our work. In the main result of this paper, a quasi-Banach lattice $X(\Omega, \mu)$, though taken over a specific measure space which is a space of homogeneous type, will also be required to have a property stronger than completeness—the *Fatou property* with constant $C_{\mathcal{F}} > 0$:

$$\begin{aligned} \text{if } 0 \leq f_n \uparrow f \text{ for a sequence } \{f_n\} \subset X(\Omega, \mu) \text{ and } \sup_{n \geq 0} \|f_n\|_X < \infty, \\ \text{then } f \in X(\Omega, \mu) \text{ and } \|f\|_X \leq C_{\mathcal{F}} \sup_{n \geq 0} \|f_n\|_X. \end{aligned}$$

The proof that the Fatou property of a quasi-normed lattice indeed implies its completeness is outlined in [22, Remark 2.1(ii)]; although a stronger form of the Fatou property is assumed there, the same argument applies unchanged within our definition. (For the case of normed lattices, see also [15, Ch. IV, § 3, Theorem 4].)

For quasi-Banach lattices with the Fatou property, we have a useful version of the Aoki–Rolewicz theorem for infinite sums.

Theorem 2.1. *Let a quasi-Banach lattice $X(\Omega, \mu)$ have the Fatou property with constant $C_{\mathcal{F}}$. Then for any nonnegative sequence $\{f_k\} \subset X(\Omega, \mu)$, there holds*

$$\left\| \sum_{k=0}^{\infty} f_k \right\|_X \leq 2^{1/\rho} C_{\mathcal{F}} \left(\sum_{k=0}^{\infty} \|f_k\|_X^\rho \right)^{1/\rho},$$

where the number $\rho \in (0, 1]$ is given by $2^{1/\rho-1} = C_\Delta$ and will hereafter be referred to as the *Aoki–Rolewicz exponent* of the lattice $X(\Omega, \mu)$.

Proof. For $n = 0, 1, \dots$, let us denote $F_n = \sum_{k=0}^n f_k$ and $F = \sum_{k=0}^{\infty} f_k$; then $F_n \uparrow F$. If $\sum_{k=0}^{\infty} \|f_k\|_X^\rho = \infty$, the statement is trivial. If

$$\sum_{k=0}^{\infty} \|f_k\|_X^\rho =: M < \infty,$$

then for all n , we have $\sum_{k=0}^n \|f_k\|_X^\rho \leq M$. By the Aoki–Rolewicz theorem for finite sums (see [18, p. 47]; cf. a weaker formulation in [14, p. 3] with $4^{1/\rho}$ in place of $2^{1/\rho}$), for every $n \geq 0$,

$$\|F_n\|_X \leq 2^{1/\rho} \left(\sum_{k=0}^n \|f_k\|_X^\rho \right)^{1/\rho} \leq 2^{1/\rho} M^{1/\rho} < \infty,$$

hence $\sup_{n \geq 0} \|F_n\|_X < \infty$. The Fatou property implies that $F \in X(\Omega, \mu)$ and

$$\|F\|_X \leq C_{\mathcal{F}} \sup_{n \geq 0} \|F_n\|_X \leq 2^{1/\rho} C_{\mathcal{F}} M^{1/\rho} = 2^{1/\rho} C_{\mathcal{F}} \left(\sum_{k=0}^{\infty} \|f_k\|_X^\rho \right)^{1/\rho},$$

which is the desired inequality.

2.2. Convexifications

Together with quasi-normed lattices $X(\Omega, \mu)$, we consider their r -convexifications, $r > 0$, defined by

$$X^{(r)}(\Omega, \mu) := \{f \in L^0(\Omega, \mu) : |f|^r \in X(\Omega, \mu)\}.$$

It is well-known that each of the convexifications is a quasi-normed lattice itself. This fact is mentioned without proof, e.g., in [25, p. 99]; being unable to provide a better reference, we prove it here.

Lemma 2.2. *If $X(\Omega, \mu)$ is a quasi-normed lattice with a quasi-norm $\|\cdot\|_X$, then $X^{(r)}(\Omega, \mu)$, for any $r > 0$, is a quasi-normed lattice with the quasi-norm*

$$\|f\|_{X^{(r)}} := \| |f|^r \|_X^{1/r}.$$

Proof. Fix $r > 0$, and let us check first that $X^{(r)}(\Omega, \mu)$ is a vector space. Take two functions $f, g \in X^{(r)}(\Omega, \mu)$. Since there holds the pointwise relation

$$|f + g|^r \leq (|f| + |g|)^r \leq \max\{1, 2^{r-1}\}(|f|^r + |g|^r), \tag{2.1}$$

and $|f|^r, |g|^r \in X(\Omega, \mu)$ by our choice, the function on the right-hand side of the above inequality belongs to $X(\Omega, \mu)$, which is a vector space. Then, by the lattice property, the smaller left-hand function $|f + g|^r \in X(\Omega, \mu)$, and thus $f + g \in X^{(r)}(\Omega, \mu)$. Trivially, we have $|\lambda f|^r = |\lambda|^r |f|^r \in X(\Omega, \mu)$, which implies that $\lambda f \in X^{(r)}(\Omega, \mu)$, for any $\lambda \in \mathbb{C}$. Thus, an r -convexification of a lattice is indeed a vector space.

Next, it follows from (2.1) and the quasi-triangle inequality for $X(\Omega, \mu)$, which holds with the “quasi-norm” constant $C_\Delta(X)$, that

$$\begin{aligned} \|f + g\|_{X^{(r)}} &= \| |f + g|^r \|_X^{1/r} \leq (\max\{1, 2^{r-1}\})^{1/r} \| |f|^r + |g|^r \|_X^{1/r} \\ &\leq \max\{1, 2^{1-1/r}\} C_\Delta(X)^{1/r} (\| |f|^r \|_X + \| |g|^r \|_X)^{1/r} \\ &\leq \max\{1, 2^{1-1/r}\} C_\Delta(X)^{1/r} \max\{1, 2^{1/r-1}\} (\| |f|^r \|_X^{1/r} + \| |g|^r \|_X^{1/r}) \\ &= C_\Delta(X^{(r)}) (\|f\|_{X^{(r)}} + \|g\|_{X^{(r)}}), \end{aligned}$$

where $C_\Delta(X^{(r)}) := 2^{|1-1/r|} C_\Delta(X)^{1/r}$. Hence, $\|\cdot\|_{X^{(r)}}$ satisfies the quasi-triangle inequality with the constant $C_\Delta(X^{(r)})$, about which we note, for later use, that

$$C_\Delta(X^{(r)}) \leq 2C_\Delta(X) \text{ whenever } r \geq 1. \tag{2.2}$$

Verifying the other axioms of the quasi-norm for $\|\cdot\|_{X^{(r)}}$ and establishing the lattice property in the space $X^{(r)}(\Omega, \mu)$ are straightforward.

Moreover, the action of “convexifying” a lattice preserves the Fatou property, and hence completeness.

Lemma 2.3. *If a quasi-Banach lattice $X(\Omega, \mu)$ has the Fatou property with constant $C_{\mathcal{F}}(X)$, then its r -convexification $X^{(r)}(\Omega, \mu)$, $r > 0$, is a quasi-Banach lattice satisfying the Fatou property with the constant*

$$C_{\mathcal{F}}(X^{(r)}) = C_{\mathcal{F}}(X)^{1/r}.$$

Proof. Let us take a sequence $\{f_n\}$ in $X^{(r)}(\Omega, \mu)$ such that $0 \leq f_n \uparrow f$ and

$$\sup_{n \geq 0} \|f_n\|_{X^{(r)}} = \sup_{n \geq 0} \|f_n^r\|_X^{1/r} < \infty.$$

Applying the Fatou property to the sequence $0 \leq f_n^r \uparrow f^r$ in $X(\Omega, \mu)$, we conclude that $f^r \in X(\Omega, \mu)$ —or equivalently, $f \in X^{(r)}(\Omega, \mu)$ —and

$$\|f\|_{X^{(r)}} = \|f^r\|_X^{1/r} \leq \left[C_{\mathcal{F}}(X) \sup_{n \geq 0} \|f_n^r\|_X \right]^{1/r} = C_{\mathcal{F}}(X)^{1/r} \sup_{n \geq 0} \|f_n\|_{X^{(r)}}.$$

Thus, $X^{(r)}(\Omega, \mu)$ satisfies the Fatou property with the constant $C_{\mathcal{F}}(X)^{1/r}$.

2.3. Example: variable Lebesgue spaces

For a function $p(\cdot) : \Omega \rightarrow (0, \infty]$ measurable on (Ω, μ) , called an exponent function, and an arbitrary $f \in L^0(\Omega, \mu)$, consider the modular functional associated with $p(\cdot)$ given by

$$m_{p(\cdot)}(f) := \int_{\Omega \setminus \Omega_\infty} |f(x)|^{p(x)} d\mu(x) + \operatorname{ess\,sup}_{x \in \Omega_\infty} |f(x)|,$$

where $\Omega_\infty = \{x \in \Omega : p(x) = \infty\}$. By customary definition, the variable Lebesgue space $L^{p(\cdot)}(\Omega, \mu)$ consists of all measurable functions f such that $m_{p(\cdot)}(f/\lambda) < \infty$ for some $\lambda > 0$ depending on f .

When an exponent function $p(\cdot)$ is essentially bounded away from zero, the space $L^{p(\cdot)}(\Omega, \mu)$ is a quasi-Banach space—for the setting of \mathbb{R}^n , this fact is mentioned without proof in [17, p. 940]. More generally, the following statement is true.

Theorem 2.4. *Given an exponent function $p(\cdot)$ such that*

$$p_- := \operatorname{ess\,inf}_{x \in \Omega} p(x) > 0,$$

the space $L^{p(\cdot)}(\Omega, \mu)$ is a quasi-Banach lattice, possessing the Fatou property, with respect to the Luxemburg–Nakano quasi-norm

$$\|f\|_{p(\cdot)} := \inf\{\lambda > 0 : m_{p(\cdot)}(f/\lambda) \leq 1\}. \tag{2.3}$$

Further, the “quasi-norm” constant is $C_\Delta = \max\{1, 2^{1/p_- - 1}\}$.

The proof of Theorem 2.4 employs the next elementary properties of the modular.

Lemma 2.5. *For any exponent function $p(\cdot)$, the following are true:*

- (i) $m_{p(\cdot)}$ is order preserving: if $|g| \leq |f|$, then $m_{p(\cdot)}(g) \leq m_{p(\cdot)}(f)$.
- (ii) For any $0 < \alpha < 1$,

$$m_{p(\cdot)}(\alpha f) \leq \alpha^{\min\{p_-, 1\}} m_{p(\cdot)}(f). \tag{2.4}$$

- (iii) If $p_- < 1$, then given $\alpha, \beta \geq 0$ such that $\alpha + \beta = 1$, there holds

$$m_{p(\cdot)}(\alpha f + \beta g) \leq \alpha^{p_-} m_{p(\cdot)}(f) + \beta^{p_-} m_{p(\cdot)}(g). \tag{2.5}$$

Proof. Properties (i) and (ii) follow simply from the definition of the modular. For (iii), observe first that if $p_- < 1$, then for any $x \in \Omega$ and each $y, z \geq 0$, we have

$$(\alpha y + \beta z)^{p(x)} \leq \alpha^{p_-} y^{p(x)} + \beta^{p_-} z^{p(x)}. \tag{2.6}$$

Indeed, if $p(x) \geq 1$, then by Jensen’s inequality,

$$(\alpha y + \beta z)^{p(x)} \leq \alpha y^{p(x)} + \beta z^{p(x)} \leq \alpha^{p_-} y^{p(x)} + \beta^{p_-} z^{p(x)};$$

otherwise, if $0 < p(x) < 1$, we use a simple inequality from [2, p. 121] to deduce

$$(\alpha y + \beta z)^{p(x)} \leq (\alpha y)^{p(x)} + (\beta z)^{p(x)} \leq \alpha^{p_-} y^{p(x)} + \beta^{p_-} z^{p(x)}.$$

Applying (2.6), we conclude that

$$\begin{aligned} m_{p(\cdot)}(\alpha f + \beta g) &\leq \alpha^{p_-} \int_{\Omega \setminus \Omega_\infty} |f(x)|^{p(x)} d\mu(x) + \beta^{p_-} \int_{\Omega \setminus \Omega_\infty} |g(x)|^{p(x)} d\mu(x) \\ &\quad + \alpha \operatorname{ess\,sup}_{x \in \Omega_\infty} |f(x)| + \beta \operatorname{ess\,sup}_{x \in \Omega_\infty} |g(x)| \\ &\leq \alpha^{p_-} m_{p(\cdot)}(f) + \beta^{p_-} m_{p(\cdot)}(g), \end{aligned}$$

which is exactly the desired inequality (2.5).

Proof of Theorem 2.4. We begin by checking that (2.3) indeed defines a quasi-norm on $L^{p(\cdot)}(\Omega, \mu)$, namely, that the functional $\|\cdot\|_{p(\cdot)}$ has the following properties:

- (a) $\|f\|_{p(\cdot)} = 0$ if and only if $f = 0$;
- (b) $\|\alpha f\|_{p(\cdot)} = |\alpha| \|f\|_{p(\cdot)}$ for all $\alpha \in \mathbb{C}$;
- (c) $\|f + g\|_{p(\cdot)} \leq C_\Delta (\|f\|_{p(\cdot)} + \|g\|_{p(\cdot)})$, where $C_\Delta = \max\{1, 2^{1/p_- - 1}\}$.

Clearly, $\|0\|_{p(\cdot)} = 0$. Suppose $\|f\|_{p(\cdot)} = 0$; then $m_{p(\cdot)}(f/\lambda) \leq 1$ for all $\lambda > 0$. We immediately have that $|f(x)| \leq \lambda$ for almost every $x \in \Omega_\infty$, hence $f = 0$ on Ω_∞ . At the same time, if $0 < \lambda < 1$, then

$$1 \geq \int_{\Omega \setminus \Omega_\infty} \left| \frac{f(x)}{\lambda} \right|^{p(x)} d\mu(x) \geq \lambda^{-p_-} \int_{\Omega \setminus \Omega_\infty} |f(x)|^{p(x)} d\mu(x),$$

which implies

$$\int_{\Omega \setminus \Omega_\infty} |f(x)|^{p(x)} d\mu(x) \leq \lim_{\lambda \rightarrow 0^+} \lambda^{p_-} = 0,$$

and therefore $f = 0$ on $\Omega \setminus \Omega_\infty$. Thus $f = 0$ and we have proved (a).

For (b), note that if $\alpha = 0$, this follows from (a). Otherwise, if $\alpha \neq 0$, we get

$$\begin{aligned} \|\alpha f\|_{p(\cdot)} &= \inf \left\{ \lambda > 0 : m_{p(\cdot)} \left(\frac{|\alpha|f}{\lambda} \right) \leq 1 \right\} \\ &= |\alpha| \inf \left\{ \frac{\lambda}{|\alpha|} > 0 : m_{p(\cdot)} \left(\frac{f}{\lambda/|\alpha|} \right) \leq 1 \right\} = |\alpha| \|f\|_{p(\cdot)}. \end{aligned}$$

As to (c), it is well known that $\|\cdot\|_{p(\cdot)}$ defines a norm on $L^{p(\cdot)}$ when $p_- \geq 1$, so the usual triangle inequality holds in this case (see, e.g., [7, Theorem 2.17] for the proof in \mathbb{R}^n). Therefore, it remains to establish the quasi-triangle inequality with the constant $C_\Delta = 2^{1/p_- - 1}$ when $0 < p_- < 1$.

Fix $\lambda_f > \|f\|_{p(\cdot)}$ and $\lambda_g > \|g\|_{p(\cdot)}$; then $m_{p(\cdot)}(f/\lambda_f) \leq 1$ and $m_{p(\cdot)}(g/\lambda_g) \leq 1$ due to the order preserving property. Let $\lambda = \lambda_f + \lambda_g$. Using inequality (2.5) and the fact that $t \mapsto t^{p_-}$ is concave and hence

$$y^{p_-} + z^{p_-} \leq 2^{1-p_-} (y+z)^{p_-} \quad \text{for each } y, z \geq 0,$$

we find that

$$\begin{aligned} m_{p(\cdot)}\left(\frac{f+g}{\lambda}\right) &= m_{p(\cdot)}\left(\frac{\lambda_f}{\lambda} \frac{f}{\lambda_f} + \frac{\lambda_g}{\lambda} \frac{g}{\lambda_g}\right) \\ &\leq \left(\frac{\lambda_f}{\lambda}\right)^{p_-} m_{p(\cdot)}(f/\lambda_f) + \left(\frac{\lambda_g}{\lambda}\right)^{p_-} m_{p(\cdot)}(g/\lambda_g) \\ &\leq \left(\frac{\lambda_f}{\lambda}\right)^{p_-} + \left(\frac{\lambda_g}{\lambda}\right)^{p_-} \leq 2^{1-p_-} \left(\frac{\lambda_f}{\lambda} + \frac{\lambda_g}{\lambda}\right)^{p_-} = 2^{1-p_-}. \end{aligned}$$

Then it follows by applying (2.4) with $\alpha = 2^{1-1/p_-}$ that

$$m_{p(\cdot)}\left(\frac{f+g}{2^{1/p_- - 1}\lambda}\right) \leq 2^{p_- - 1} m_{p(\cdot)}\left(\frac{f+g}{\lambda}\right) \leq 1,$$

and thus $\|f+g\|_{p(\cdot)} \leq 2^{1/p_- - 1}(\lambda_f + \lambda_g)$. Taking now infimum over all such λ_f and λ_g , we get the desired quasi-triangle inequality with $C_\Delta = 2^{1/p_- - 1}$. Since, moreover, the lattice property for $\|\cdot\|_{p(\cdot)}$ holds as a consequence of the order preserving property of the modular, we have proved that $L^{p(\cdot)}(\Omega, d, \mu)$ is a quasi-normed lattice with respect to $\|\cdot\|_{p(\cdot)}$.

To complete the proof, let us verify the Fatou property. Take a sequence $\{f_n\}$ in $L^{p(\cdot)}(\Omega, \mu)$ such that $0 \leq f_n \uparrow f$ and

$$\Lambda := \sup_{n \geq 0} \|f_n\|_{p(\cdot)} < \infty.$$

With this choice, $m_{p(\cdot)}(f_n/\Lambda) \leq 1$ for any $n \geq 0$. Using the monotone convergence theorem [12, Theorem 2.14], we get

$$\begin{aligned} m_{p(\cdot)}\left(\frac{f}{\Lambda}\right) &= \int_{\Omega \setminus \Omega_\infty} \left|\frac{f(x)}{\Lambda}\right|^{p(x)} d\mu(x) + \operatorname{ess\,sup}_{x \in \Omega_\infty} \left|\frac{f(x)}{\Lambda}\right| \\ &= \lim_{n \rightarrow \infty} \left(\int_{\Omega \setminus \Omega_\infty} \left|\frac{f_n(x)}{\Lambda}\right|^{p(x)} d\mu(x) + \operatorname{ess\,sup}_{x \in \Omega_\infty} \left|\frac{f_n(x)}{\Lambda}\right| \right) \\ &= \lim_{n \rightarrow \infty} m_{p(\cdot)}\left(\frac{f_n}{\Lambda}\right) \leq 1, \end{aligned}$$

which implies that $f \in L^{p(\cdot)}(\Omega, \mu)$ and $\|f\|_{p(\cdot)} \leq \sup_{n \geq 0} \|f_n\|_{p(\cdot)}$. Thus, the Fatou property holds with the constant $C_{\mathcal{F}} = 1$, and $L^{p(\cdot)}(\Omega, \mu)$ is a quasi-Banach lattice with the Luxemburg–Nakano quasi-norm.

Hence, any space $L^{p(\cdot)}(\Omega, \mu)$ with $p_- > 0$ is a lattice and can be convexified; in fact, its every convexification is a variable Lebesgue space itself—with the exponent function multiplied by the parameter of convexification. The following theorem extends [7, Proposition 2.18].

Theorem 2.6. *Given $s > 0$ and an exponent function $p(\cdot)$ with $p_- > 0$, the s -convexification of $L^{p(\cdot)}(\Omega, \mu)$ coincides with $L^{sp(\cdot)}(\Omega, \mu)$, and $\|\cdot\|_{(L^{p(\cdot)})^{(s)}}$ and $\|\cdot\|_{sp(\cdot)}$ are equivalent quasi-norms, such that*

$$2^{-(1/s)\max\{1/p_-, 1\}} \|f\|_{(L^{p(\cdot)})^{(s)}} \leq \|f\|_{sp(\cdot)} \leq 2^{\max\{1/sp_-, 1\}} \|f\|_{(L^{p(\cdot)})^{(s)}}$$

for all functions $f \in L^{sp(\cdot)}(\Omega, \mu)$.

Proof. Take an $f \in (L^{p(\cdot)})^{(s)}(\Omega, \mu)$. By the definition of the convexification, $|f|^s \in L^{p(\cdot)}(\Omega, \mu)$, and we fix an arbitrary $\lambda > \| |f|^s \|_{p(\cdot)}$. This choice of λ implies that $m_{p(\cdot)}(|f|^s/\lambda) \leq 1$ and thus

$$\int_{\Omega \setminus \Omega_\infty} \left(\frac{|f(x)|^s}{\lambda} \right)^{p(x)} d\mu(x) \leq 1, \quad \text{ess sup}_{x \in \Omega_\infty} \frac{|f(x)|^s}{\lambda} \leq 1,$$

or equivalently,

$$\int_{\Omega \setminus \Omega_\infty} \left(\frac{|f(x)|}{\lambda^{1/s}} \right)^{sp(x)} d\mu(x) \leq 1, \quad \text{ess sup}_{x \in \Omega_\infty} \frac{|f(x)|}{\lambda^{1/s}} \leq 1.$$

Summing the last two inequalities gives $m_{sp(\cdot)}(f/\lambda^{1/s}) \leq 2$, and so $f \in L^{sp(\cdot)}(\Omega, \mu)$. It follows then by inequality (2.4) that

$$m_{sp(\cdot)} \left(2^{-\max\{1/sp_-, 1\}} \frac{f}{\lambda^{1/s}} \right) \leq \frac{1}{2} m_{sp(\cdot)} \left(\frac{f}{\lambda^{1/s}} \right) \leq 1,$$

whence $\|f\|_{sp(\cdot)} \leq 2^{\max\{1/sp_-, 1\}} \lambda^{1/s}$. Taking the infimum over all such λ yields

$$\|f\|_{sp(\cdot)} \leq 2^{\max\{1/sp_-, 1\}} \|f\|_{(L^{p(\cdot)})^{(s)}}. \tag{2.7}$$

For the converse, take a function $f \in L^{sp(\cdot)}(\Omega, \mu)$ and fix any $\lambda > \|f\|_{sp(\cdot)}$ noting that $m_{sp(\cdot)}(f/\lambda) \leq 1$. Arguing as we did above, we consequently obtain that $m_{p(\cdot)}(|f|^s/\lambda^s) \leq 2$. Further application of inequality (2.4) gives us

$$m_{p(\cdot)} \left(2^{-\max\{1/p_-, 1\}} \frac{|f|^s}{\lambda^s} \right) \leq 1,$$

which implies that $f \in (L^{p(\cdot)})^{(s)}(\Omega, \mu)$ and $\| |f|^s \|_{p(\cdot)} \leq 2^{\max\{1/p_-, 1\}} \lambda^s$. By taking the infimum over all λ considered, we obtain

$$2^{-(1/s)\max\{1/p_-, 1\}} \|f\|_{(L^{p(\cdot)})^{(s)}} \leq \|f\|_{sp(\cdot)}. \tag{2.8}$$

Inequalities (2.7) and (2.8) are the two parts of the desired inequality.

Let us remark, however, that the modular $m_{p(\cdot)}$ which we use to define the variable Lebesgue space $L^{p(\cdot)}(\Omega, \mu)$ is not the only functional suitable for this purpose. As an alternative, one may take the “max”-modular

$$m_{p(\cdot)}^{\max}(f) := \max \left\{ \int_{\Omega \setminus \Omega_\infty} |f(x)|^{p(x)} d\mu(x), \text{ess sup}_{x \in \Omega_\infty} |f(x)| \right\}$$

and similarly declare $L^{p(\cdot)}(\Omega, \mu)$ to be the space of measurable functions f for which $m_{p(\cdot)}^{\max}(f/\lambda) < \infty$ at some $\lambda > 0$. Clearly, the new definition covers the same collection of functions as before. The analogue of the Luxemburg–Nakano quasi-norm based on the “max”-modular, i.e.

$$\|f\|_{p(\cdot)}^{\max} := \inf\{\lambda > 0 : m_{p(\cdot)}^{\max}(f/\lambda) \leq 1\},$$

is again a quasi-norm with the constant $C_{\Delta} = \max\{1, 2^{1/p_- - 1}\}$ whenever $p_- > 0$; in fact, the entire Theorem 2.4 remains valid for $\|\cdot\|_{p(\cdot)}^{\max}$ —one can easily trace that the proof goes identically line by line, with the corresponding changes from the sum to maximum where necessary, given that Lemma 2.5 still holds when $m_{p(\cdot)}$ is replaced by $m_{p(\cdot)}^{\max}$.

The two quasi-norms $\|\cdot\|_{p(\cdot)}$ and $\|\cdot\|_{p(\cdot)}^{\max}$ on $L^{p(\cdot)}(\Omega, \mu)$ are equivalent: the simple inequality $\max\{a, b\} \leq a + b \leq 2 \max\{a, b\}$ provides the estimate

$$\|f\|_{p(\cdot)}^{\max} \leq \|f\|_{p(\cdot)} \leq 2^{\max\{1/p_-, 1\}} \|f\|_{p(\cdot)}^{\max}.$$

A curious observation about the maximum quasi-norm is that $\|\cdot\|_{p(\cdot)}^{\max}$ is equal to the “convexified” quasi-norm based on it, while for the sum quasi-norm, we only had the weaker Theorem 2.6.

Lemma 2.7. *Given $s > 0$ and an exponent function $p(\cdot)$ with $p_- > 0$, for any $f \in L^{sp(\cdot)}(\Omega, \mu)$ the “convexified” quasi-norm*

$$\|f\|_{(L^{p(\cdot)})^{(s)}}^{\max} := (\| |f|^s \|_{p(\cdot)}^{\max})^{1/s}$$

is equal to the Luxemburg–Nakano quasi-norm $\|f\|_{sp(\cdot)}^{\max}$.

Proof. It is easy to see that since for any $\lambda > 0$,

$$\operatorname{ess\,sup}_{x \in \Omega_{\infty}} \frac{|f(x)|^s}{\lambda} \leq 1 \quad \text{if and only if} \quad \operatorname{ess\,sup}_{x \in \Omega_{\infty}} \frac{|f(x)|}{\lambda^{1/s}} \leq 1,$$

there holds the set equality

$$\left\{ \lambda > 0 : m_{p(\cdot)}^{\max} \left(\frac{|f|^s}{\lambda} \right) \leq 1 \right\} = \left\{ \lambda > 0 : m_{sp(\cdot)}^{\max} \left(\frac{f}{\lambda^{1/s}} \right) \leq 1 \right\}.$$

By passing to the infima of the two sets, we arrive at $\| |f|^s \|_{p(\cdot)}^{\max} = (\|f\|_{sp(\cdot)}^{\max})^s$, which yields the desired result.

In our experience, the maximum quasi-norm, though being less used than its “sum” counterpart, often brings more technical convenience to the proofs; so it is generally a good idea to check any $L^{p(\cdot)}$ -related results for both.

3. Review on spaces of homogeneous type

3.1. Quasi-metric spaces

First of all, spaces of homogeneous type are quasi-metric spaces, and thus have a topological structure weaker than metric spaces. By definition, a quasi-metric on a set Ω is a function $\rho : \Omega \times \Omega \rightarrow [0, \infty)$ such that

- (1) $\rho(x, y) = 0$ if and only if $x = y$,
- (2) $\rho(x, y) = \rho(y, x)$ for every $x, y \in \Omega$,
- (3) there exists a constant $A_0 > 0$ such that for every $x, y, z \in \Omega$,

$$\rho(x, y) \leq A_0(\rho(x, z) + \rho(z, y)).$$

A pair (Ω, ρ) is called a quasi-metric space. Note that if the set Ω has at least two distinct points, then necessarily the quasi-metric constant $A_0 \geq 1$. The case $A_0 = 1$, however, is the usual metric case.

Once we have a notion of “distance,” we can define a ball

$$B_\rho(x, r) := \{y \in \Omega : \rho(x, y) < r\}$$

centered at $x \in \Omega$ and of radius $r > 0$. A quasi-metric ρ naturally induces the topology τ_ρ on Ω , in which a set $G \subset \Omega$ is defined to be open if for each $x \in G$ there exists $\varepsilon > 0$ such that $B_\rho(x, \varepsilon) \subset G$.

An important fact about this canonical topology on quasi-metric spaces is that the same topology τ_ρ is unequivocally induced by any other quasi-metric ρ' equivalent to ρ in the sense that there exists a constant $c \geq 1$ such that for all $x, y \in \Omega$,

$$c^{-1}\rho(x, y) \leq \rho'(x, y) \leq c\rho(x, y).$$

Indeed, for any $x \in \Omega$ and $\varepsilon > 0$, such a relation implies that $B_{\rho'}(x, \varepsilon/c) \subset B_\rho(x, \varepsilon)$ and $B_\rho(x, \varepsilon/c) \subset B_{\rho'}(x, \varepsilon)$, and these inclusions of balls yield $\tau_\rho = \tau_{\rho'}$.

Quasi-metric spaces are also known for a certain misfortune with the balls: when $A_0 > 1$, some balls $B_\rho(x, r)$ may fail to be open. Hytönen and Kairema gave an elegant and simple example of such an occurrence in [13, p. 5]: by considering $\Omega = \{-1\} \cup [0, \infty)$ with the usual distance between all other pairs of points except $\rho(-1, 0) := 1/2$, one gets a quasi-metric space with $A_0 \geq 2$, in which the ball $B_\rho(-1, 1) = \{-1, 0\}$ does not contain any ball of the form $B_\rho(0, \varepsilon)$, and hence is not open. Another example can be found in [27, p. 4310].

Fortunately, Macías and Segovia [24, Theorem 2] proved that given any quasi-metric ρ , there exists an equivalent quasi-metric d such that d^α , for some $0 < \alpha < 1$, is a genuine metric. Since every ball $B_d(x, r)$ coincides then with a metric ball $B_{d^\alpha}(x, r^\alpha)$, it follows that $\tau_\rho = \tau_d = \tau_{d^\alpha}$ and every d -ball is open in this common topology. From now on, we will assume that an arbitrary quasi-metric ρ on Ω has been replaced with an equivalent well-behaved quasi-metric d and benefit from the fact that the associated balls $B(x, r) := B_d(x, r)$ are all open sets.

3.2. Setting for analysis

Now that the “ball-friendly” quasi-metric is chosen, we define the setting of spaces of homogeneous type following Coifman and Weiss [5,6], who introduced this notion in the 1970s.

Definition 3.1. A space of homogeneous type (Ω, d, μ) is a quasi-metric space (Ω, d) equipped with a non-negative Borel measure μ satisfying the doubling condition

$$0 < \mu(B(x, r)) \leq A\mu(B(x, r/2)) < \infty \tag{3.1}$$

with an absolute constant $A := A_\mu \geq 1$ for all balls $B(x, r)$.

By assuming that balls have positive, finite measure, we avoid trivial measures and ensure that μ is σ -finite.

The doubling condition (3.1) of a measure μ implies the following homogeneity property of the quasi-metric space (Ω, d) : there is a natural number

$$A_1 \leq A^{3 \log_2 A_0 + 5}$$

such that any ball $B(x, r)$ contains at most A_1 points x_i satisfying $d(x_i, x_j) \geq r/2$. This is the very first thing pointed out by Coifman and Weiss in their discussion of spaces of homogeneous type [5, p. 67]; in fact, their original—more general—definition of these spaces required the homogeneity condition instead of the existence of a doubling measure, which is reflected in the name of the spaces.

The homogeneity condition, in turn, implies the geometric doubling property of the quasi-metric d with the same constant A_1 —namely, any ball $B(x, r)$ can be covered by at most A_1 balls of radius $r/2$ (since we may always choose the smaller balls to be centered at the evenly spread points x_i). Thus, every space of homogeneous type is geometrically doubling; having said this, we gain access to an important tool of adjacent dyadic grids developed by Hytönen and Kairema for geometrically doubling quasi-metric spaces.

The usefulness of dyadic objects in harmonic analysis on the Euclidean space \mathbb{R}^n has long been known, and development of similar systems in a more general setting of quasi-metric spaces was only a matter of time. One of the first advancements in this area is due to Christ [4, Theorem 11] (see also [3, Ch. VI, Theorem 14]), who constructed a system of sets on (Ω, d, μ) satisfying many properties of the Euclidean dyadic cubes. His construction was further refined by Hytönen and Kairema [13, Theorem 2.2]; moreover, these authors observed that a number of results in harmonic analysis exploit several adjacent grids of dyadic cubes instead of just one fixed grid and designed an analogous system of dyadic grids in quasi-metric spaces with geometric doubling [13, Theorem 4.1]. We will use the version of this result from [1, Theorem 4.1].

Theorem 3.2. *Let (Ω, d) be a quasi-metric space with the quasi-metric constant A_0 and satisfy the geometric doubling condition with the constant A_1 . Suppose the parameter $\delta \in (0, 1)$ satisfies $96A_0^6\delta \leq 1$. Then there exist countable sets of points $\{z_\alpha^{k,t} : \alpha \in \mathcal{A}_k\}$, $k \in \mathbb{Z}$, $t = 1, 2, \dots, K = K(A_0, A_1, \delta)$, and a finite number of dyadic grids $\mathcal{D}^t := \{Q_\alpha^{k,t} : k \in \mathbb{Z}, \alpha \in \mathcal{A}_k\}$, such that*

(a) *for every $t \in \{1, 2, \dots, K\}$ and $k \in \mathbb{Z}$ one has*

- (i) $\Omega = \bigcup_{\alpha \in \mathcal{A}_k} Q_\alpha^{k,t}$ (disjoint union);
- (ii) if $Q, P \in \mathcal{D}^t$, then $Q \cap P \in \{\emptyset, Q, P\}$;
- (iii) if $Q_\alpha^{k,t} \in \mathcal{D}^t$, then

$$B(z_\alpha^{k,t}, c_1\delta^k) \subset Q_\alpha^{k,t} \subset B(z_\alpha^{k,t}, C_1\delta^k), \tag{3.2}$$

where $c_1 := (12A_0^4)^{-1}$ and $C_1 := 4A_0^2$;

- (iv) if $Q_\alpha^{k,t} \in \mathcal{D}^t$, there exist at least one $Q_\beta^{k+1,t} \in \mathcal{D}^t$, which is called a child of $Q_\alpha^{k,t}$, and exactly one $Q_\gamma^{k-1,t} \in \mathcal{D}^t$, which is called the parent of $Q_\alpha^{k,t}$, such that $Q_\beta^{k+1,t} \subset Q_\alpha^{k,t} \subset Q_\gamma^{k-1,t}$;

(b) *for every ball $B = B(x, r)$, there exists a cube $Q_B \in \bigcup_{t=1}^K \mathcal{D}^t$ such that $B \subset Q_B$ and $Q_B = Q_\alpha^{k-1,t}$ for some indices $\alpha \in \mathcal{A}_k$ and $t \in \{1, \dots, K\}$, where k is the unique integer satisfying $\delta^{k+1} < r \leq \delta^k$.*

For further work, fix a collection of the adjacent dyadic grids $\{\mathcal{D}^t : t = 1, 2, \dots, K\}$ defined in Theorem 3.2 and denote their union by

$$\mathcal{D} := \bigcup_{t=1}^K \mathcal{D}^t.$$

The sets $Q_\alpha^{k,t} \in \mathcal{D}$ are Borel sets [1, Remark 4.2] referred to as dyadic cubes with centers $z_\alpha^{k,t}$ and sidelengths $\ell(Q_\alpha^{k,t}) = \delta^k$. Of course, these are not cubes in the standard Euclidean sense—but the name was preserved given the similar properties of each individual dyadic system \mathcal{D}^t , listed in (a), to those of the classical dyadic grid in \mathbb{R}^n . In particular, inclusion (3.2) implies that for any cube $Q_\alpha^{k,t}$, there are the containing ball $B(z_\alpha^{k,t}, C_1\delta^k)$ and the contained ball $B(z_\alpha^{k,t}, c_1\delta^k)$, so we can trap a cube “between” two balls just like in \mathbb{R}^n . Conversely, part (b) guarantees that any quasi-metric ball is contained in a dyadic cube of a comparable size.

More specifically, in spaces of homogeneous type—where we have a doubling measure μ —dyadic properties (iii) and (b) yield the following relations between measures of cubes and their containing balls, and vice versa [13, Corollary 7.4].

Lemma 3.3. *There exists a constant $C = C(A_0, A, \delta) \geq 1$ such that for every dyadic cube $Q \in \mathcal{D}$ we have $\mu(B_Q) \leq C\mu(Q)$, where B_Q is the containing ball of Q as in (3.2). Conversely, given a ball $B = B(x, r)$, there exist a dyadic grid \mathcal{D}^t and a dyadic cube $Q_B \in \mathcal{D}^t$ such that $B \subset Q_B$ and $\mu(Q_B) \leq C\mu(B)$.*

This result is crucial for establishing equivalence between the classical maximal operator and its “dyadic” counterpart.

3.3. Maximal operators

Let f be a measurable function on (Ω, d, μ) , from now on denoted by $f \in L^0(\Omega, d, \mu)$. Then Mf , the Hardy–Littlewood maximal function of f , is defined for any $x \in \Omega$ by

$$Mf(x) := \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f(y)| d\mu(y),$$

where the supremum is taken over all quasi-metric balls $B \subset \Omega$ containing x . The maximal operator M is a sublinear operator acting by the rule $f \mapsto Mf$. Simply put, the function Mf renders the “maximum” mean value of f about the point x . By maximizing the means of other orders $r > 0$, we also define the parameterized maximal operator M_r as

$$M_r f(x) := M(|f|^r)(x)^{1/r} = \sup_{B \ni x} \left(\frac{1}{\mu(B)} \int_B |f(y)|^r d\mu(y) \right)^{1/r}. \tag{3.3}$$

Obviously, this formula yields the usual maximal operator M when $r = 1$.

As the following lemma shows, values of the parameterized maximal function increase with the parameter.

Lemma 3.4. *If $0 < r < s$ and $f \in L^0(\Omega, d, \mu)$, then for every $x \in \Omega$ there holds*

$$M_r f(x) \leq M_s f(x).$$

Proof. Fix an $x \in \Omega$ and a ball B containing x . By Hölder’s inequality with the exponent $s/r > 1$, we have

$$\begin{aligned} \frac{1}{\mu(B)} \int_B |f(y)|^r d\mu(y) &= \frac{1}{\mu(B)} \int_\Omega |f(y)|^r \chi_B(y) d\mu(y) \\ &\leq \frac{1}{\mu(B)} \left(\int_B |f(y)|^s d\mu(y) \right)^{r/s} \mu(B)^{1-r/s} \end{aligned}$$

$$= \left(\frac{1}{\mu(B)} \int_B |f(y)|^s d\mu(y) \right)^{r/s} \leq M_s f(x)^r.$$

This implies that

$$M_r f(x) = \sup_{B \ni x} \left(\frac{1}{\mu(B)} \int_B |f(y)|^r d\mu(y) \right)^{1/r} \leq M_s f(x).$$

Along with the classical maximal operator M , we will use the “dyadic” maximal operator $M^{\mathcal{D}}$ defined for $f \in L^0(\Omega, d, \mu)$ by

$$M^{\mathcal{D}} f(x) = \sup_{Q \ni x} \frac{1}{\mu(Q)} \int_Q |f(y)| d\mu(y), \quad x \in \Omega,$$

where the supremum is taken over all dyadic cubes $Q \in \mathcal{D}$ containing x . In style of Anderson, Hytönen and Tapiola [1], we put quotation marks around the word dyadic when we want to emphasize that the definition in question uses the collection \mathcal{D} of adjacent dyadic grids instead of a single grid.

Note that we can similarly define the “dyadic” parameterized maximal operator $M_r^{\mathcal{D}}$ by taking the supremum over cubes instead of balls in (3.3). Lemma 3.4 still remains true if we replace M_r and M_s by their “dyadic” versions $M_r^{\mathcal{D}}$ and $M_s^{\mathcal{D}}$.

Due to Lemma 3.3, we easily deduce the pointwise equivalence of the maximal functions Mf and $M^{\mathcal{D}}f$ (cf. [13, Proposition 7.9]).

Theorem 3.5. *Let $f \in L^0(\Omega, d, \mu)$. For all $x \in \Omega$, we have the pointwise estimates*

$$M^{\mathcal{D}} f(x) \leq CMf(x) \quad \text{and} \quad Mf(x) \leq CM^{\mathcal{D}} f(x)$$

with the same constant $C \geq 1$, independent of f , as in Lemma 3.3.

Proof. Assume that $x \in Q$, $Q \in \mathcal{D}$. Let B be the containing ball of Q as in (3.2). Then $\mu(B) \leq C\mu(Q)$ by Lemma 3.3, and therefore

$$\frac{1}{\mu(Q)} \int_Q |f(y)| d\mu(y) \leq \frac{C}{\mu(B)} \int_B |f(y)| d\mu(y) \leq CMf(x).$$

By taking the supremum over all dyadic cubes in \mathcal{D} containing x , we conclude from the above inequality that $M^{\mathcal{D}} f(x) \leq CMf(x)$.

For the reverse inequality, consider a ball $B \ni x$ and let $Q \in \mathcal{D}^t$, for some t , be the dyadic cube from Lemma 3.3 such that $B \subset Q$ and $\mu(Q) \leq C\mu(B)$. Then, just like in the argument above, but with the interchanged roles of B and Q , we have

$$\frac{1}{\mu(B)} \int_B |f(y)| d\mu(y) \leq \frac{C}{\mu(Q)} \int_Q |f(y)| d\mu(y) \leq CM^{\mathcal{D}} f(x),$$

which results in $Mf(x) \leq CM^{\mathcal{D}} f(x)$ after taking the supremum over all balls B containing x .

Both “dyadic” and classical maximal operators are countably subadditive, as stated in the next lemma.

Lemma 3.6. *If $\{f_k\} \subset L^0(\Omega, d, \mu)$ and $f = \sum_{k=0}^\infty f_k$, then for every $x \in \Omega$,*

$$M^{\mathcal{D}} f(x) \leq \sum_{k=0}^\infty M^{\mathcal{D}} f_k(x).$$

The same is true if we replace $M^{\mathcal{D}}$ by M .

Proof. The statement is a direct consequence of the countable additivity of the integral (see [12, Theorem 2.15]) resulting from the monotone convergence theorem—since for any fixed $x \in \Omega$, we have

$$\begin{aligned} M^{\mathcal{D}} f(x) &= \sup_{Q \ni x} \frac{1}{\mu(Q)} \int_Q |f(y)| d\mu(y) \leq \sup_{Q \ni x} \frac{1}{\mu(Q)} \int_Q \sum_{k=0}^\infty |f_k(y)| d\mu(y) \\ &= \sup_{Q \ni x} \sum_{k=0}^\infty \frac{1}{\mu(Q)} \int_Q |f_k(y)| d\mu(y) \leq \sum_{k=0}^\infty M^{\mathcal{D}} f_k(x), \end{aligned}$$

where all the suprema are taken over cubes $Q \in \mathcal{D}$ containing x . When replacing dyadic cubes by balls $B \ni x$, we obtain the same inequality for the operator M .

3.4. “Dyadic” A_1 weights

A nonnegative measurable function w on Ω is said to be a weight. The “dyadic” class $A_1^{\mathcal{D}}$ consists of the weights w such that

$$[w]_{A_1^{\mathcal{D}}} := \operatorname{ess\,sup}_{x \in \Omega} \frac{M^{\mathcal{D}} w(x)}{w(x)} < \infty.$$

For each $Q \in \mathcal{D}$, define the set of all its nearby cubes of that same generation,

$$\mathcal{N}_Q := \{Q' \in \mathcal{D} : Q' \cap Q \neq \emptyset, \ell(Q') = \ell(Q)\}.$$

According to [1, Definition 4.4], a generalized dyadic parent (gdp) of Q is any cube Q^* such that $\ell(Q^*) = \delta^{-2}\ell(Q)$ and for every $Q' \in \mathcal{N}_Q$ we have $Q' \subset Q^*$. Every cube has at least one gdp (see [1, Lemma 4.5]), and we set $S \geq 1$ to be such a constant that for all $Q \in \mathcal{D}$ and $Q' \in \mathcal{N}_Q$, there holds $\mu(Q^*) \leq S\mu(Q')$. This constant, introduced in [1, Section 5], is always finite—which we now prove by giving one of its possible values.

Theorem 3.7. *Given that $S = A(A_0/\delta^3)^{\log_2 A}$, there holds $\mu(Q^*) \leq S\mu(Q')$ for any gdp Q^* and all $Q' \in \mathcal{N}_Q$ related to an arbitrary cube $Q \in \mathcal{D}$.*

Proof. Note first that the doubling property (3.1) of the measure μ implies that for all $x \in \Omega$, $0 < r < R$ and $y \in B(x, R)$, one has

$$\frac{\mu(B(x, R))}{\mu(B(y, r))} \leq A \left(\frac{2A_0 R}{r} \right)^{\log_2 A} \tag{3.4}$$

(cf. [9, Lemma 2.3]). This follows by showing that for all $z \in B(x, R)$, the quasi-triangle inequality yields

$$d(y, z) \leq A_0(d(y, x) + d(x, z)) < 2A_0 R$$

and consequently $B(x, R) \subset B(y, 2A_0R)$, and then applying the well-known relation between the measures of concentric balls, see [13, Lemma 7.3], to conclude that

$$\mu(B(x, R)) \leq \mu(B(y, 2A_0R)) \leq A \left(\frac{2A_0R}{r} \right)^{\log_2 A} \mu(B(y, r)).$$

Let $\ell(Q) = \ell(Q') = \delta^k$; then $\ell(Q^*) = \delta^{k-2}$. According to Property (a)-(iii) from Theorem 3.2, there exist $x, y \in \Omega$ such that

$$B(y, c_1\delta^k) \subset Q' \subset Q^* \subset B(x, C_1\delta^{k-2}).$$

If we now apply inequality (3.4), recall the values of c_1 and C_1 , and make use of the definition of the parameter δ satisfying $96A_0^6\delta \leq 1$, we get that

$$\begin{aligned} \mu(Q^*) &\leq \mu(B(x, C_1\delta^{k-2})) \\ &\leq A \left(\frac{2A_0C_1\delta^{k-2}}{c_1\delta^k} \right)^{\log_2 A} \mu(B(y, c_1\delta^k)) \\ &\leq A \left(\frac{A_0}{\delta^3} \right)^{\log_2 A} \mu(Q'), \end{aligned}$$

and thus the claim follows.

The following result is an easy consequence of the weak reverse Hölder inequality from [1, Theorem 5.4]. We present it as given in the work by Karlovich [16, Lemma 6], and also take this occasion to correct a misleading typo in [16]: throughout Section 3 there, \mathcal{D} is meant to be the union of adjacent dyadic grids, just like in our text, instead of a single dyadic grid.

Theorem 3.8. *Let $w \in A_1^{\mathcal{D}}$ and $Q \in \mathcal{D}$. Then for every η satisfying*

$$0 < \eta \leq \frac{1}{2S^2K[w]_{A_1^{\mathcal{D}}}}, \quad (3.5)$$

one has

$$\left(\frac{1}{2\mu(Q)} \int_Q w^{1+\eta}(y) d\mu(y) \right)^{\frac{1}{1+\eta}} \leq S[w]_{A_1^{\mathcal{D}}} \frac{1}{\mu(Q)} \int_Q w(y) d\mu(y). \quad (3.6)$$

As a corollary to this theorem, we have a pointwise estimate for the “dyadic” parameterized maximal functions of $A_1^{\mathcal{D}}$ weights.

Corollary 3.9. *If $w \in A_1^{\mathcal{D}}$, then for every η in the range (3.5) and for a. e. $x \in \Omega$,*

$$M_{1+\eta}^{\mathcal{D}} w(x) \leq 2S[w]_{A_1^{\mathcal{D}}}^2 w(x).$$

Proof. Take an $x \in \Omega$ and fix a cube $Q \ni x$. Since $w \in A_1^{\mathcal{D}}$, it follows from inequality (3.6) and the definition of the $A_1^{\mathcal{D}}$ weight that for all η satisfying (3.5),

$$\begin{aligned} \left(\frac{1}{\mu(Q)} \int_Q w^{1+\eta}(y) d\mu(y) \right)^{\frac{1}{1+\eta}} &\leq 2^{\frac{1}{1+\eta}} S[w]_{A_1^p} \frac{1}{\mu(Q)} \int_Q w(y) d\mu(y) \\ &\leq 2S[w]_{A_1^p} M^{\mathcal{D}} w(x) \\ &\leq 2S[w]_{A_1^p}^2 w(x), \end{aligned}$$

where the last inequality holds for almost every x . Then, if we take the supremum over all cubes Q containing x on the left-hand side, we get the desired inequality.

4. Proof of the main theorem

4.1. Boundedness on “higher” convexifications

Consider first a quasi-metric measure space (Ω, d, μ) without the requirement on μ to be doubling. We easily prove that the boundedness of M on a quasi-normed lattice over (Ω, d, μ) entails its boundedness on all of the “higher” convexifications of the lattice.

Lemma 4.1. *Let $X(\Omega, d, \mu)$ be a quasi-normed lattice over a quasi-metric measure space (Ω, d, μ) . If M is bounded on $X^{(r)}(\Omega, d, \mu)$ for some $r > 0$, then M is bounded on $X^{(s)}(\Omega, d, \mu)$ for all $s > r$ and*

$$\|M\|_{X^{(s)} \rightarrow X^{(s)}} \leq \|M\|_{X^{(r)} \rightarrow X^{(r)}}^{r/s}. \tag{4.1}$$

Proof. Fix an $s > r$ and take an $f \in X^{(s)}(\Omega, d, \mu)$. Note that since $X(\Omega, d, \mu)$ is a quasi-normed lattice, so are $X^{(r)}(\Omega, d, \mu)$ and $X^{(s)}(\Omega, d, \mu)$ by Lemma 2.2. Then the pointwise inequality from Lemma 3.4 and the boundedness of M on $X^{(r)}(\Omega, d, \mu)$ imply that

$$\begin{aligned} \|Mf\|_{X^{(s)}} &= \|(Mf)^s\|_X^{1/s} = \|(Mf)^{s/r}\|_{X^{(r)}}^{r/s} = \|M_{r/s}(|f|^{s/r})\|_{X^{(r)}}^{r/s} \\ &\leq \|M(|f|^{s/r})\|_{X^{(r)}}^{r/s} \leq \|M\|_{X^{(r)} \rightarrow X^{(r)}}^{r/s} \| |f|^{s/r} \|_{X^{(r)}}^{r/s} \\ &= \|M\|_{X^{(r)} \rightarrow X^{(r)}}^{r/s} \|f\|_{X^{(s)}}. \end{aligned}$$

Thus, M is bounded on $X^{(s)}(\Omega, d, \mu)$ and (4.1) is satisfied.

4.2. Self-improving property

Extending the boundedness of M from a quasi-normed lattice to a range of its “lower” convexifications, however, is not at all trivial. We were able to construct such an extension for a more specific class of lattices—quasi-Banach lattices with the Fatou property. In contrast to Lemma 4.1, the assumption that the lattices are taken over spaces of homogeneous type is crucial for the converse result.

As we noted in the Introduction, it is this converse result that is generally called the *self-improving property* of the boundedness of the maximal operator. We establish it in two parts—first, “pushing” the parameter of convexification from $r = 1$ down (Theorem 4.2), and then adapting the designed argument to pass from convexifications of order $s > 1$ right down to $s = 1$ (Theorem 4.4).

Theorem 4.2. *Let $X(\Omega, d, \mu)$ be a quasi-Banach lattice with the Fatou property. Suppose M is bounded on $X(\Omega, d, \mu)$. Then there exists $r_0 \in (0, 1)$ such that if $r \in [r_0, 1)$, then M is bounded on $X^{(r)}(\Omega, d, \mu)$.*

Proof. Instead of proving the theorem directly, we show that the same statement is true for the “dyadic” maximal operator $M^{\mathcal{D}}$ and use the equivalence of the maximal functions Mf and $M^{\mathcal{D}}f$ to pass from M to $M^{\mathcal{D}}$ and backwards.

Since M is bounded on $X(\Omega, d, \mu)$, it follows from Theorem 3.5 that $M^{\mathcal{D}}$ is also bounded on $X(\Omega, d, \mu)$ and $\|M^{\mathcal{D}}\|_{X \rightarrow X} \leq C\|M\|_{X \rightarrow X}$. To show that there is a number $r_0 \in (0, 1)$ such that $M^{\mathcal{D}}$ is then bounded on any $X^{(r)}(\Omega, d, \mu)$ with $r_0 \leq r < 1$, we first construct an $A_1^{\mathcal{D}}$ weight applying the Rubio de Francia iteration algorithm (see, e.g., [8] for a nice introduction to techniques based on it).

Fix an ε such that $0 < \varepsilon\|M^{\mathcal{D}}\|_{X \rightarrow X} < 2^{-1/\rho}$, where ρ is the Aoki–Rolewicz exponent of the lattice $X(\Omega, d, \mu)$. Given $h \in X(\Omega, d, \mu)$, define

$$\mathcal{R}_\varepsilon^{\mathcal{D}}h(x) := \sum_{k=0}^{\infty} \varepsilon^k (M^{\mathcal{D}})^k h(x), \quad x \in \Omega,$$

where $(M^{\mathcal{D}})^k$ is the operator $M^{\mathcal{D}}$ iterated k times and $(M^{\mathcal{D}})^0 h = |h|$. This “dyadic” Rubio de Francia operator has the following properties:

- (a) trivially, $|h(x)| \leq \mathcal{R}_\varepsilon^{\mathcal{D}}h(x)$ for all $x \in \Omega$;
- (b) $\mathcal{R}_\varepsilon^{\mathcal{D}}$ is bounded on $X(\Omega, d, \mu)$ and $\|\mathcal{R}_\varepsilon^{\mathcal{D}}\|_{X \rightarrow X} \leq 4^{1/\rho}C_{\mathcal{F}}$ (like before, $C_{\mathcal{F}}$ denotes the constant in the Fatou property for the lattice $X(\Omega, d, \mu)$);
- (c) $\mathcal{R}_\varepsilon^{\mathcal{D}}h \in A_1^{\mathcal{D}}$ and $[\mathcal{R}_\varepsilon^{\mathcal{D}}h]_{A_1^{\mathcal{D}}} \leq 1/\varepsilon$.

Property (b) follows from Theorem 2.1, according to which

$$\begin{aligned} \|\mathcal{R}_\varepsilon^{\mathcal{D}}h\|_X &\leq 2^{1/\rho}C_{\mathcal{F}} \left(\sum_{k=0}^{\infty} \varepsilon^{k\rho} \|(M^{\mathcal{D}})^k h\|_X^\rho \right)^{1/\rho} \\ &\leq 2^{1/\rho}C_{\mathcal{F}} \left(\sum_{k=0}^{\infty} (\varepsilon\|M^{\mathcal{D}}\|_{X \rightarrow X})^{k\rho} \right)^{1/\rho} \|h\|_X \\ &\leq 2^{1/\rho}C_{\mathcal{F}} \left(\sum_{k=0}^{\infty} (2^{-1/\rho})^{k\rho} \right)^{1/\rho} \|h\|_X \\ &= 4^{1/\rho}C_{\mathcal{F}}\|h\|_X. \end{aligned}$$

Property (c) is a consequence of the “countable subadditivity” (see Lemma 3.6) and homogeneity of the maximal operator, since for any $x \in \Omega$ we have

$$M^{\mathcal{D}}(\mathcal{R}_\varepsilon^{\mathcal{D}}h)(x) \leq \sum_{k=0}^{\infty} \varepsilon^k (M^{\mathcal{D}})^{k+1}h(x) \leq \frac{1}{\varepsilon}\mathcal{R}_\varepsilon^{\mathcal{D}}h(x).$$

Denote $\eta_0 := \varepsilon/(2S^2K)$. By Property (c),

$$\eta_0 \leq \frac{1}{2S^2K[\mathcal{R}_\varepsilon^{\mathcal{D}}h]_{A_1^{\mathcal{D}}}},$$

therefore, it follows from Corollary 3.9 and again (c) that for every $0 < \eta \leq \eta_0$ and almost every $x \in \Omega$, there holds

$$M_{1+\eta}^{\mathcal{D}}(\mathcal{R}_\varepsilon^{\mathcal{D}}h)(x) \leq \frac{2S}{\varepsilon^2}\mathcal{R}_\varepsilon^{\mathcal{D}}h(x). \tag{4.2}$$

Declare $r_0 = 1/(1 + \eta_0)$. Then fix $r \in [r_0, 1)$ and determine the unique number $\eta \in (0, \eta_0]$ such that $r = 1/(1 + \eta)$. Then by Properties (a) and (b) and inequality (4.2), for any $f \in X^{(r)}(\Omega, d, \mu)$ we have

$$\begin{aligned} \|M^{\mathcal{D}} f\|_{X^{(r)}} &= \|(M^{\mathcal{D}} f)^r\|_X^{1/r} = \|M_{1/r}^{\mathcal{D}}(|f|^r)\|_X^{1/r} \\ &= \|M_{1+\eta}^{\mathcal{D}}(|f|^r)\|_X^{1/r} \leq \|M_{1+\eta}^{\mathcal{D}}(\mathcal{R}_\varepsilon^{\mathcal{D}}(|f|^r))\|_X^{1/r} \\ &\leq \left(\frac{2S}{\varepsilon^2}\right)^{1/r} \|\mathcal{R}_\varepsilon^{\mathcal{D}}(|f|^r)\|_X^{1/r} \leq \left(\frac{2S}{\varepsilon^2}\right)^{1/r} (4^{1/\rho} C_{\mathcal{F}})^{1/r} \| |f|^r \|_X^{1/r} \\ &= \left(\frac{2S \cdot 4^{1/\rho} C_{\mathcal{F}}}{\varepsilon^2}\right)^{1/r} \|f\|_{X^{(r)}}, \end{aligned}$$

which means that $M^{\mathcal{D}}$ is bounded on $X^{(r)}(\Omega, d, \mu)$. Once again, Theorem 3.5 guarantees that in this case M is also bounded on all $X^{(r)}(\Omega, d, \mu)$, $r_0 \leq r < 1$, and $\|M\|_{X^{(r)} \rightarrow X^{(r)}} \leq C \|M^{\mathcal{D}}\|_{X^{(r)} \rightarrow X^{(r)}}$.

Remark 4.3. The construction in the proof implies that

$$r_0 > \left(1 + \frac{C}{2^{1/\rho+1} S^2 K \|M\|_{X \rightarrow X}}\right)^{-1}.$$

Intuitively, this lower estimate for the parameter of convexification r_0 down to which the boundedness of M can be extended—depending on the quasi-norm of the maximal operator—admits the following interpretation: a smaller value of $\|M\|_{X \rightarrow X}$ allows to achieve a lower (that is, better) self-improvement “threshold” r_0 .

The second part of the self-improving property uses the argument from the above proof and follows almost as its corollary.

Theorem 4.4. *Let $X(\Omega, d, \mu)$ be a quasi-Banach lattice with the Fatou property. Suppose that there exists $s_0 > 1$ such that M is bounded on $X^{(s)}(\Omega, d, \mu)$ for every $1 < s < s_0$, and*

$$\lim_{s \rightarrow 1^+} (s - 1) \|M\|_{X^{(s)} \rightarrow X^{(s)}} = 0. \tag{4.3}$$

Then M is bounded on $X(\Omega, d, \mu)$.

Proof. It is immediate from the equivalence of the classical and “dyadic” maximal functions, given by Theorem 3.5, that not only M but also $M^{\mathcal{D}}$ is bounded on each $X^{(s)}(\Omega, d, \mu)$, $1 < s < s_0$, and limit (4.3) holds true if we replace M by $M^{\mathcal{D}}$.

Therefore, we can find an $s \in (1, s_0)$ such that

$$4C_{\Delta}(X) \cdot 2S^2 K (s - 1) \|M^{\mathcal{D}}\|_{X^{(s)} \rightarrow X^{(s)}} < 1$$

and fix $\varepsilon = 2S^2 K (s - 1)$. Denote by ρ_s the Aoki–Rolewicz exponent of $X^{(s)}(\Omega, d, \mu)$; note that $2^{1/\rho_s} = 2C_{\Delta}(X^{(s)}) \leq 4C_{\Delta}(X)$ due to inequality (2.2). Then

$$0 < \varepsilon \|M^{\mathcal{D}}\|_{X^{(s)} \rightarrow X^{(s)}} < \frac{1}{4C_{\Delta}(X)} \leq 2^{-1/\rho_s}.$$

After this choice of ε , we can repeat the argument from the proof of Theorem 4.2 almost verbatim—it suffices to apply it to $X = X^{(s)}$ and $r = r_0 = 1/s$ with the corresponding change of constants $\rho = \rho_s$ and

$C_{\mathcal{F}} = C_{\mathcal{F}}(X^{(s)})$. Following this “re-designation,” $X^{(r)}$ becomes $(X^{(s)})^{(1/s)} = X^{(1)} = X$. By Lemma 2.3, $C_{\mathcal{F}}(X^{(s)}) = C_{\mathcal{F}}(X)^{1/s}$, and we conclude from the earlier argument that for all $f \in X(\Omega, d, \mu)$,

$$\|M^{\mathcal{D}}f\|_X \leq C_{\mathcal{F}}(X) \left(\frac{2S \cdot 4^{1/\rho_s}}{\varepsilon^2} \right)^s \|f\|_X,$$

hence the “dyadic” operator $M^{\mathcal{D}}$ is bounded on $X(\Omega, d, \mu)$. By equivalence, M is bounded on $X(\Omega, d, \mu)$, too.

4.3. Proof of Theorem 1.1

By bringing together the results of Lemma 4.1 and Theorems 4.2 and 4.4, we obtain the main Theorem 1.1.

Indeed, implications (1) \Rightarrow (2) and (3) \Rightarrow (1) essentially give the extension of the boundedness of M to the “higher” convexifications of a lattice, and are thus ensured by Lemma 4.1. Implication (1) \Rightarrow (3) is exactly Theorem 4.2. Finally, implication (2) \Rightarrow (1) is a corollary of Theorem 4.4, since Condition (2) taken as a premise can be weakened by assuming that there exists $s_0 > 1$ such that M is bounded on $X^{(s)}(\Omega, d, \mu)$ if $1 < s < s_0$ and limit (4.3) holds. \square

4.4. Application to variable Lebesgue spaces

Finally, we give an application of Theorem 1.1 to the spaces $L^{p(\cdot)}(\Omega, d, \mu)$ with $p_- > 0$, which by Theorem 2.4 are quasi-Banach lattices with respect to the quasi-norm $\|\cdot\|_{p(\cdot)}$ and have the Fatou property. To translate the applied result from the abstract language of convexifications to the clear language of variable Lebesgue spaces with different exponent functions, we use Theorem 2.6 and therefore obtain the following special-case version of the main theorem.

Corollary 4.5. *Given a variable Lebesgue space $L^{p(\cdot)}(\Omega, d, \mu)$ with $p_- > 0$, the following are equivalent:*

- (1) M is bounded on $L^{p(\cdot)}(\Omega, d, \mu)$.
- (2) For all $s > 1$, M is bounded on $L^{sp(\cdot)}(\Omega, d, \mu)$ and

$$\lim_{s \rightarrow 1^+} (s - 1) \|M\|_{L^{sp(\cdot)} \rightarrow L^{sp(\cdot)}} = 0.$$

- (3) There exists $r_0 \in (0, 1)$ such that if $r \in [r_0, 1)$, then M is bounded on $L^{rp(\cdot)}(\Omega, d, \mu)$.

Proof. Once we set $X(\Omega, d, \mu) = L^{p(\cdot)}(\Omega, d, \mu)$ in the statement of Theorem 1.1, the desired result follows immediately if the boundedness of M on convexifications $(L^{p(\cdot)})^{(s)}(\Omega, d, \mu) = L^{sp(\cdot)}(\Omega, d, \mu)$, $s > 0$, is understood with respect to the “convexified” operator quasi-norm defined by

$$\|M\|_{(L^{p(\cdot)})^{(s)} \rightarrow (L^{p(\cdot)})^{(s)}} = \sup_{\substack{f \in L^{sp(\cdot)} \\ f \neq 0}} \frac{\|Mf\|_{(L^{p(\cdot)})^{(s)}}}{\|f\|_{(L^{p(\cdot)})^{(s)}}}.$$

However, the equivalence of the “convexified” quasi-norm $\|\cdot\|_{(L^{p(\cdot)})^{(s)}}$ and the Luxemburg-Nakano quasi-norm $\|\cdot\|_{sp(\cdot)}$, as established in Theorem 2.6, implies that

$$c(p_-, s)^{-1} \|M\|_{L^{sp(\cdot)} \rightarrow L^{sp(\cdot)}} \leq \|M\|_{(L^{p(\cdot)})^{(s)} \rightarrow (L^{p(\cdot)})^{(s)}} \leq c(p_-, s) \|M\|_{L^{sp(\cdot)} \rightarrow L^{sp(\cdot)}},$$

where

$$c(p_-, s) = 2^{\max\{1/sp_-, 1\} + (1/s) \max\{1/p_-, 1\}}.$$

Due to this relation, the boundedness of M in the sense of the “convexified” quasi-norm is equivalent to that defined through the usual quasi-norm of the variable Lebesgue space, and hence the claim of the corollary follows.

As we noted at the end of Subsection 2.3, one may alternatively consider the maximum quasi-norm $\|\cdot\|_{p(\cdot)}^{\max}$ on the spaces $L^{p(\cdot)}(\Omega, d, \mu)$ satisfying $p_- > 0$, with respect to which these spaces are again quasi-Banach lattices in possession of the Fatou property. Thus, we apply the main Theorem 1.1 to a variable Lebesgue space with the maximum quasi-norm and immediately get a version of Corollary 4.5 for the $\|\cdot\|_{p(\cdot)}^{\max}$ implied—in this case, the translation from the language of convexifications is automatic due to the equality of $\|\cdot\|_{p(\cdot)}^{\max}$ and the “convexified” quasi-norm, established in Lemma 2.7.

The above two applications of the main result to variable Lebesgue spaces with different underlying quasi-norms complete this work.

Acknowledgments

This work is funded by national funds through the FCT – Fundação para a Ciência e a Tecnologia, I.P., under the scope of the projects UIDB/00297/2020 (<https://doi.org/10.54499/UIDB/00297/2020>) and UIDP/00297/2020 (<https://doi.org/10.54499/UIDP/00297/2020>) (Center for Mathematics and Applications) and under the scope of the PhD scholarship UI/BD/154284/2022.

I would like to thank Dr. Oleksiy Karlovykh, who supervised this work as a part of my PhD research, for encouraging me to address the self-improvement problem and our insightful conversations all the way through.

References

- [1] T.C. Anderson, T. Hytönen, O. Tapiola, Weak A_∞ weights and weak reverse Hölder property in a space of homogeneous type, *J. Geom. Anal.* 27 (2017) 95–119, <https://doi.org/10.1007/s12220-015-9675-6>.
- [2] R.E. Castillo, H. Rafeiro, *An Introductory Course in Lebesgue Spaces*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, Springer, Cham, 2016.
- [3] M. Christ, *Lectures on Singular Integral Operators*, CBMS Regional Conference Series in Mathematics, vol. 77, American Mathematical Society, Providence, RI, 1990. Published for the Conference Board of the Mathematical Sciences, Washington, DC.
- [4] M. Christ, A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral, *Colloq. Math.* 60/61 (1990) 601–628, <https://doi.org/10.4064/cm-60-61-2-601-628>.
- [5] R.R. Coifman, G. Weiss, Analyse harmonique non-commutative sur certains espaces homogènes, *Lecture Notes in Mathematics*, vol. 242, Springer-Verlag, Berlin-New York, 1971, Étude de certaines intégrales singulières.
- [6] R.R. Coifman, G. Weiss, Extensions of Hardy spaces and their use in analysis, *Bull. Am. Math. Soc.* 83 (1977) 569–645, <https://doi.org/10.1090/S0002-9904-1977-14325-5>.
- [7] D.V. Cruz-Uribe, A. Fiorenza, *Variable Lebesgue Spaces, Applied and Numerical Harmonic Analysis*, Birkhäuser/Springer, Heidelberg, 2013, Foundations and Harmonic Analysis.
- [8] D.V. Cruz-Uribe, J.M. Martell, C. Pérez, *Weights, Extrapolation and the Theory of Rubio de Francia*, *Operator Theory: Advances and Applications*, vol. 215, Birkhäuser/Springer Basel AG, Basel, 2011.
- [9] D.V. Cruz-Uribe, P. Shukla, The boundedness of fractional maximal operators on variable Lebesgue spaces over spaces of homogeneous type, *Stud. Math.* 242 (2018) 109–139, <https://doi.org/10.4064/sm8556-6-2017>.
- [10] L. Diening, Maximal function on Musielak-Orlicz spaces and generalized Lebesgue spaces, *Bull. Sci. Math.* 129 (2005) 657–700, <https://doi.org/10.1016/j.bulsci.2003.10.003>.
- [11] L. Diening, P. Harjulehto, P. Hästö, M. Růžička, *Lebesgue and Sobolev Spaces with Variable Exponents*, *Lecture Notes in Mathematics*, vol. 2017, Springer, Heidelberg, 2011.
- [12] G.B. Folland, *Real Analysis: Modern Techniques and Their Applications*, Wiley, New York, 1999.
- [13] T. Hytönen, A. Kairema, Systems of dyadic cubes in a doubling metric space, *Colloq. Math.* 126 (2012) 1–33, <https://doi.org/10.4064/cm126-1-1>.
- [14] N.J. Kalton, N.T. Peck, J.W. Roberts, *An F -Space Sampler*, *London Mathematical Society Lecture Note Series*, vol. 89, Cambridge University Press, Cambridge, 1984.
- [15] L.V. Kantorovich, G.P. Akilov, *Functional Analysis*, second ed., Pergamon Press, Oxford-Elmsford, N.Y., 1982. Translated from the Russian by Howard L. Silcock.

- [16] A. Karlovich, Hardy-Littlewood maximal operator on the associate space of a Banach function space, *Real Anal. Exch.* 44 (2019) 119–140, <https://doi.org/10.14321/realanalexch.44.1.0119>.
- [17] H. Kempka, J. Vybíral, Lorentz spaces with variable exponents, *Math. Nachr.* 287 (2014) 938–954, <https://doi.org/10.1002/mana.201200278>.
- [18] H. König, *Eigenvalue Distribution of Compact Operators, Operator Theory: Advances and Applications*, vol. 16, Birkhäuser Verlag, Basel, 1986.
- [19] A.K. Lerner, A boundedness criterion for the maximal operator on variable Lebesgue spaces, *J. Anal. Math.* (2024), [accepted](#).
- [20] A.K. Lerner, S. Ombrosi, A boundedness criterion for general maximal operators, *Publ. Mat.* 54 (2010) 53–71, https://doi.org/10.5565/PUBLMAT_54110_03.
- [21] A.K. Lerner, C. Pérez, A new characterization of the Muckenhoupt A_p weights through an extension of the Lorentz-Shimogaki theorem, *Indiana Univ. Math. J.* 56 (2007) 2697–2722, <https://doi.org/10.1512/iumj.2007.56.3112>.
- [22] E. Lorist, Z. Nieraeth, Banach function spaces done right, *Indag. Math. (N.S.)* 35 (2024) 247–268, <https://doi.org/10.1016/j.indag.2023.11.004>.
- [23] E. Lorist, Z. Nieraeth, Extrapolation of compactness on banach function spaces, *J. Fourier Anal. Appl.* 30 (2024) 30, <https://doi.org/10.1007/s00041-024-10087-x>.
- [24] R.A. Macías, C. Segovia, Lipschitz functions on spaces of homogeneous type, *Adv. Math.* 33 (1979) 257–270, [https://doi.org/10.1016/0001-8708\(79\)90012-4](https://doi.org/10.1016/0001-8708(79)90012-4).
- [25] L. Maligranda, Type, cotype and convexity properties of quasi-Banach spaces, in: *Banach and Function Spaces*, Yokohama Publ., Yokohama, 2004, pp. 83–120.
- [26] Z. Nieraeth, Extrapolation in general quasi-Banach function spaces, *J. Funct. Anal.* 285 (2023) 110130, <https://doi.org/10.1016/j.jfa.2023.110130>, 109.
- [27] M. Paluszyński, K. Stempak, On quasi-metric and metric spaces, *Proc. Am. Math. Soc.* 137 (2009) 4307–4312, <https://doi.org/10.1090/S0002-9939-09-10058-8>.