DEPARTMENT OF
MATHEMATICS

BEATRIZ DE ALMEIDA CURIOSO
Master in Mathematics and Applications

MULTIPLE STATE MODELS IN DISABILITY INSURANCE

ESTIMATION AND GRADUATION OF TRANSITION INTENSITIES CONSIDERING RISK FACTORS

MASTER IN ACTUARIAL MATHEMATICS

NOVA University Lisbon
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Estimation and Graduation of Transition Intensities Considering Risk Factors

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Aos meus pais, António e Clarinda.
Aos meus irmãos e melhores amigos, Isabel e Guilherme.
Às minhas avós, Adélia e Deolinda,
e ao resto da minha família.
Ao Carlos.

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Abstract

Disability Insurance and, in particular, Long-Term Care (LTC) Insurance have become increasingly important research areas due to the accelerated population ageing, and consequent increase of the elderly and dependent population, observed in the developed countries. In Portugal, although the same social and economic problems are present, there are no private insurance companies offering LTC Insurance products yet, mostly due to the complexities involved with modelling the underlying risks of this type of insurance, as well as to the lack of data that allows for an adequate risk measurement.

The main purpose of this Master's Thesis is to present a thorough study of Multiple State Models, while focusing on the estimation and graduation of the corresponding transition intensities, as well as to develop a practical application of the studied methods that highlights the importance of the use of these models in Disability Insurance.

In the first part, we carry out a literature review for the definition and properties of continuous-time Markov chains, as well as the formulation of Multiple State Models through the use of these stochastic processes. Moreover, based on the Transition Intensities Approach, we study and develop methods for the estimation and graduation of the transition intensities, adapting the results to our own Disability multiple state model.

In the second part of this work, we present an original practical application of the studied methods, which incorporates observable risk factors, such as demographic characteristics, in the estimation and graduation of the transition intensities of the Disability multiple state model. Data simulation was performed to obtain a dataset of individuals with the demographic characteristics of the Portuguese population and their trajectories through the multiple state model.

The proposed process allows us to obtain a form for the transition intensities, as a function of age, specific for the different risk profiles in our population. The work developed in this thesis can thus be used as a starting point to build a pricing structure adapted to the Portuguese population.

Keywords: Disability Insurance, Long-Term Care, Continuous-time Markov chains, Multiple State Models, Transition Intensities Approach, Graduation methods, Data simulation

Resumo

Os Seguros de Invalidez e, em particular, os Seguros de Cuidados Continuados têm-se tornado áreas de investigação cada vez mais importantes devido ao acelerado envelhecimento populacional, e consequente aumento da população idosa e dependente, observado nos países desenvolvidos. Em Portugal, embora este problema social e económico esteja presente, ainda não existem seguradoras privadas que ofereçam este tipo de produtos, devido sobretudo às complexidades envolvidas na modelação dos riscos subjacentes, bem como à falta de dados necessários para uma mensuração adequada do risco.

O principal objetivo desta dissertação é apresentar um estudo aprofundado dos Modelos de Estados Múltiplos, focando-se na estimação e graduação das correspondentes intensidades de transição, bem como desenvolver uma aplicação prática dos métodos estudados que destaque a importância da utilização destes modelos no Seguro de Invalidez.

Na primeira parte, realizamos uma revisão de literatura das cadeias de Markov a tempo contínuo, bem como a formulação de Modelos de Estados Múltiplos através destes processos estocásticos. Além disso, com base na Abordagem das Intensidades de Transição, desenvolvemos métodos para a estimação e graduação das intensidades de transição, adaptando os resultados ao nosso próprio modelo de estados múltiplos.

Na segunda parte deste trabalho, apresentamos uma aplicação prática original dos métodos estudados, que incorpora fatores de risco observáveis, como as características demográficas, na estimação e graduação das intensidades de transição do modelo de estados múltiplos. O processo de simulação foi utilizado para obtermos um conjunto de dados de indivíduos com as características demográficas da população Portuguesa e as suas trajetórias através do modelo de estados múltiplos.

O processo proposto permite-nos obter uma forma para as intensidades de transição, em função da idade, específica para os diferentes perfis de risco da nossa população. O trabalho desenvolvido nesta tese pode assim ser utilizado como ponto de partida para a construção de uma estrutura tarifária adaptada à população portuguesa.

Palavras-chave: Seguro de Invalidez, Cuidados Continuados, Cadeias de Markov a tempo contínuo, Modelos de Estados Múltiplos, Abordagem de Intensidades de Transição, Métodos de graduação, Simulação de dados

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1 Introduction

Generally speaking, Health Insurance designates the range of insurance products that provide benefits in the case of an event that causes an alteration of the insured's health conditions, and results in loss of income or expenses related to health care (or both), see [25]. Since there are various possible changes (with different degrees of severity) to the health conditions of individuals and, consequently, different impacts on their life (whether financial, or in terms of quality of life), Health Insurance is a crucial tool for managing the impact that such event might represent.

Although there are several types of Health Insurance products (for example, Accident, Sickness or Critical Illness Insurance), in this work we shall focus on Disability Insurance. This type of insurance provides benefits to individuals in case they became sick or disabled (temporarily or permanently) and are not able to perform their usual professional activity.

Concerning the protection against the loss of income, the benefits of Disability Insurance can consist of a single payment (also named lump sum) or of fixed-amount periodical payments (usually called disability annuities). Another possible feature is the waiver-of-premium benefit, in which the insured is not required to pay the premiums during the periods of disability.

Regarding the medical expenses, (short-term) sickness insurance usually provides the return of charges with hospitalisation, surgery or medical appointments for disability spells of short duration. However, there are cases when individuals require financial and medical support for longer periods (or even until the end of their life) due to chronic conditions or the effects of ageing on their health. The insurance product to be used in these circumstances is called Long-Term Care (LTC) Insurance.

In addition to the income protection and reimbursement of medical expenses, the benefits of LTC Insurance can also include continuous nursing or medical care services, as the ones provided by the National Network of Integrated Continuous Care (in Portuguese, Rede Nacional de Cuidados Continuados Integrados, or RNCCI), required due to chronic conditions. Moreover, it is also common to provide different benefits depending on the degree or severity of the disability itself.

Therefore, one of the main features of LTC Insurance is that it can be viewed as the generalisation of Disability Insurance, with different possible status or degrees of disability. The Barthel Index, see [21], is one of the indicators used to classify individuals based on their level of functional dependence. It is a scale that measures the ability to perform, without assistance, Activities of Daily Living (ADL), such as feeding, bathing, mobility, among others. A classification or status is assigned based on the number of activities that the individuals cannot complete, which expresses their disability severity.

From an Actuarial perspective, the main concern when dealing with Health Insurance (as it is for the other types of insurance), is the modelling of the underlying risks. However, we are presented with a higher complexity in this case, not only due to the wide range of possible status of an individual's heath, but also because of the various benefits that can be provided by this type of insurance. Therefore, robust models are required to account for the risk modelling of the different events and underlying benefits, as well as for the respective premium and reserve calculations.

The actuarial models used for the study of Health Insurance (including Disability and LTC Insurance) are Multiple State Models. There exists a well-established mathematical study of the theory of Markov chains and their relation with Multiple State Models, see [10]. Furthermore, the application of Multiple State Models in Health Insurance, and in particular Disability and LTC, has also been thoroughly studied, see [4, 10, 25]. Regarding the methodology for the formulation of these models, the mentioned studies follow the Transition Intensities Approach, proposed in [30], in which the transition intensities are considered the fundamental quantities (or parameters) of Multiple State Models.

In this work, we aim to model the health status of individuals throughout their life for Health, Disability or LTC Insurance purposes. Therefore, our first goal is to study the implementation of Multiple State Models, using continuous-time Markov chains, while focusing on the estimation of the models' parameters (the transition intensities). Furthermore, we shall study a framework for the graduation of the transition intensities based on Generalised Linear Models (GLMs), see [10, 28], and adapt the results to our own Disability multiple state model, with a practical application in mind.

From a social and economic perspective, Disability Insurance and, in particular, LTC Insurance have recently become areas of interest in the developed countries due to the observed demographic trends. The accelerated population ageing, caused by the increase in life expectancy and the decline in fertility rates, is expected to intensify in the following years, see [23]. With the increase in the elderly (and consequently, dependent) population, various countries (namely, Germany, United Kingdom, France, Switzerland, among others, see [10, p. 188]) have not only developed public networks for continuous care, but also LTC products being provided by private insurance companies.

However, as mentioned, LTC Insurance products are complex to implement due to their extensive coverage, both in types of benefits and in the duration of the policies, which result in high premiums that are not accessible to the general population. Additionally, since this area and products are rather recent, there is a lack of data concerning disability events, which is required to obtain accurate models for premium and reserve calculations.

In Portugal, population ageing is a trend that is also verified, see [14], and there is a public network in place (the RNCCI) which provides continuous health care and social support to individuals in a situation of dependence. Although there are no private companies in Portugal providing LTC Insurance products, mostly due to the high risk involved and the lack of data, see [6], some research work has been developed in this area.

In [24], using 2015 data from the RNCCI database, the author develops the simulation and calibration of a multiple state model in discrete time, for the calculation of the associated costs and premiums for a LTC product in Portugal. In [5, 6], using the same dataset and relaxing the assumption of a discrete model, the authors present a method to estimate and calibrate transition intensities for non-homogeneous continuous-time Markov chains, and present a general multiple state model using official Portuguese data. Thus, a consistent framework has been developed, which can be used as a starting point for insurers to develop LTC Insurance products in Portugal.

In this work, we aim to develop further statistical methods for the formulation of Disability and LTC Insurance products in Portugal, by incorporating the demographic characteristics of the population in the framework. Therefore, our second goal is to present an original actuarial application of the studied graduation methods of the transition intensities in our Disability multiple state model, while extending the explanatory variables to other risk factors than solely the age of the individuals. Since the required data to apply the studied graduation techniques is not available, we resort to data simulation to produce a dataset of a population with the characteristics of the Portuguese one.

The proposed process differentiates individuals based on their risk profile when determining their transition intensities and, consequently, the probabilities that are used in the calculation of premiums and reserves for Disability or LTC Insurance. Therefore, the work developed in this thesis can be used as a starting point to build a pricing structure adapted to the risk profile of the individuals in the Portuguese population.

Regarding the structure of this work, in Chapter 2, we present the main definitions and notations regarding continuous-time Markov chains as well as their underlying properties. Then, we study the formulation of Multiple State Models through the use of Markov chains and present some examples that will be used throughout this work to illustrate the stated results.

In Chapter 3, we perform a study of Multiple State Models based on the Transition Intensities Approach, while focusing on the estimation of the transition intensities. Moreover, we present a parametric method to graduate the crude estimates of the transition intensities based on the Gompertz-Makeham law and GLMs.

Finally, in Chapter 4, we develop a practical application of the methods presented in the previous Chapter to estimate and graduate the transition intensities of the chosen multiple state model. To obtain the required data, we simulate a dataset of individuals with the demographic characteristics of the Portuguese population and their trajectories. We then present a form for the transition intensities as a function of the age of the individuals, in which the parameters depend on the risk profile of the selected group of individuals.

2

Continuous-time Markov Chains and Multiple State Models

In this Chapter, we present the main definitions, notations and properties that will be used throughout this work, regarding Markov chains. In particular, we define transition probabilities and transition intensities of Markov chains, and demonstrate some important results that establish the relations between the two. Finally, we study the formulation of Multiple State Models through the use of Markov chains and their respective properties.

In [10], Haberman and Pitacco develop a thorough mathematical study of the theory of Markov processes and their use in Multiple State Models. Thus, most of the results and proofs presented in this Chapter follow that book. For the definition and properties of Multiple State Models, we studied the analysis in [4] and [25] and our research also includes the works in [29] and [33].

2.1 Continuous-time Markov Chains

Definition 2.1.1. *A stochastic process is a family $\{X_t,\ t \in T\}$ of random variables, where T is a set of indexes which usually symbolises time, X_t represents the state of the process at time t, and S is the set where $\{X_t\}$ takes values in (designated as the state space).*

In other words, a stochastic process can be seen as a model of the evolution of a random phenomenon through time.

Definition 2.1.2. *Let $\{X_t,\ t \in T\}$ be a stochastic process.*

If T is a countable set (usually $T = \mathbb{Z}$ or $T = \mathbb{N}_0$), $\{X_t\}$ is commonly denoted as a discrete-time process and is represented by $\{X_n,\ n \in T\}$.

On the other hand, if T is uncountable ($T = \mathbb{R}$ or more commonly $T = [0, +\infty)$), $\{X_t\}$ is usually referred to as a continuous-time process and is denoted by $\{X(t),\ t \in T\}$.

In this work, we aim to model the health status of individuals throughout their life for Health, Disability or Long-Term Care (LTC) Insurance purposes. Thus, we will be focusing on continuous-time stochastic processes that illustrate the state of an individual throughout time as they age.

Therefore, let us consider $\{X(x),\ x \in T\}$, where $X(x)$ represents the state an individual is in at age x, with $T = [0, +\infty)$ and S a finite countable state space.

To model the state of the individual throughout time, one can use conditional probabilities of the form

$$P[X(x+t) = j|X(x) = i, X(u) = x(u) \text{ for } 0 \leq u < x], \tag{2.1}$$

for $x, t \geq 0,\ i, j, x(u) \in S\ (0 \leq u < x)$, which represent the probability that the individual will be in state j at age $x + t$ given their previous values at ages $0 \leq u \leq x$.

If the future value of the stochastic process (i.e. the future state of the individual) does not depend on their past values and only on their present state, we say that the process verifies the memorylessness or the Markov property and is called a Markov chain.

Definition 2.1.3. *A continuous-time stochastic process $\{X(x),\ x \in T\}$ is said to be a Markov chain if it satisfies the property*

$$P[X(x+t) = j|X(x) = i, X(u) = x(u) \text{ for } 0 \leq u < x] = P[X(x+t) = j|X(x) = i] \tag{2.2}$$

for $x, t \geq 0,\ i, j, x(u) \in S\ (0 \leq u < x)$.

Moreover, Markov chains can be classified according to their dependence on time.

Definition 2.1.4. *A Markov chain $\{X(x),\ x \in T\}$ is time-homogeneous if*

$$P[X(x+t) = j|X(x) = i] = P[X(t) = j|X(0) = i], \tag{2.3}$$

i.e. the conditional probabilities are independent of the age x and depend only on the difference between moments t.

Otherwise, the Markov chain is said to be time-inhomogeneous or non-homogeneous.

Using these conditional probabilities, we can define the transition probabilities between the states of the model and build a representation in matrix form since the state space is finite countable.

For the remainder of this Chapter, let us consider a continuous-time Markov chain $\{X(x),\ x \in T\}$, with $T = [0, +\infty)$ and finite countable state space S.

2.1.1 Transition Probabilities

Definition 2.1.5. *Let $i, j \in S$ and $x, t \geq 0$.*

The transition probability between states i and j, for an aged x individual, denoted by ${}_tp_x^{ij}$, is the probability that a life aged x in state i will be in state j at age $x + t$:

$${}_tp_x^{ij} \equiv P_{ij}(x,t) = P[X(x+t) = j|X(x) = i]. \tag{2.4}$$

In particular, for $t = 0$, ${}_0p_x^{ij} = \delta_{ij}$, *where* δ_{ij} *denotes the Kronecker delta, meaning that*

$$\begin{cases} {}_0p_x^{ii} = 1, \\ {}_0p_x^{ij} = 0,\ i \neq j. \end{cases} \tag{2.5}$$

Remark 2.1.1. *For homogeneous Markov chains, this probability can be simplified to*

$$P_{ij}(x,t) = P[X(x+t) = j | X(x) = i] = P[X(t) = j | X(0) = i] \equiv P_{ij}(t),\ \forall x \geq 0, \tag{2.6}$$

meaning that the transition probabilities depend only on the states i, j *and the time* t *of the process.*

On the other hand, for non-homogeneous Markov chains, the transition probabilities also depend on the age x *of the individual.*

These probabilities can be represented in matrix form.

Definition 2.1.6. *Let* $x, t \geq 0$, $n \in \mathbb{N}$ *and* $S = \{1, \dots, n\}$.

The transition probability matrix is given by $\boldsymbol{P}(x,t) = [P_{ij}(x,t)]_{i,j \in S} = [{}_tp_x^{ij}]_{i,j \in S}$, *i.e.*

$${}_t\boldsymbol{P}_x \equiv \boldsymbol{P}(x,t) = \begin{bmatrix} {}_tp_x^{11} & {}_tp_x^{12} & \dots & {}_tp_x^{1n} \\ {}_tp_x^{21} & {}_tp_x^{22} & \dots & {}_tp_x^{2n} \\ \vdots & \vdots & \ddots & \vdots \\ {}_tp_x^{n1} & {}_tp_x^{n2} & \dots & {}_tp_x^{nn} \end{bmatrix}. \tag{2.7}$$

Remark 2.1.2. *In the homogeneous case,* $\boldsymbol{P}(x,t) \equiv \boldsymbol{P}(t) = [P_{ij}(t)]_{i,j \in S}$, $\forall x \geq 0$, *meaning that the probability matrix is the same for all individuals regardless of their age* x.

We observe that the transition probability matrix is a stochastic matrix since:

- For all $i, j \in S$, ${}_tp_x^{ij} \in [0,1]$, and
- For all $i \in S$, $\sum_{j \in S} {}_tp_x^{ij} = 1$, i.e. all the rows of matrix (2.7) sum to one.

Additionally, transition probabilities also verify the following property.

Theorem 2.1.1 (Chapman-Kolmogorov Equations)**.** *For* $i, j \in S$ *and* $x, t, r \geq 0$,

$${}_{t+r}p_x^{ij} = \sum_{k \in S} {}_tp_x^{ik}\ {}_rp_{x+t}^{kj}. \tag{2.8}$$

In matrix form, we have that $\boldsymbol{P}(x, t+r) = \boldsymbol{P}(x,t)\boldsymbol{P}(x+t, r)$.

Proof. We mainly follow the proof in [10, p. 16].

By the law of total probability, since S is a finite or countable set, we have that

$$P[X(x+t+r) = j] = \sum_{k \in S} P[X(x+t+r) = j, X(x+t) = k],$$

and so

$$\begin{aligned}
{}_{t+r}p_x^{ij} &= P[X(x+t+r)=j|X(x)=i] = \\
&= \sum_{k\in S} P[X(x+t+r)=j, X(x+t)=k|X(x)=i] = \\
&= \sum_{k\in S} P[X(x+t)=k|X(x)=i]P[X(x+t+r)=j|X(x+t)=k, X(x)=i],
\end{aligned}$$

where in the last step we use $P(A\cap B|C) = P(A|B\cap C)P(B|C)$.
Finally, since the Markov property is verified, we conclude

$$\begin{aligned}
{}_{t+r}p_x^{ij} &= \sum_{k\in S} P[X(x+t)=k|X(x)=i]P[X(x+t+r)=j|X(x+t)=k, X(x)=i] = \\
&= \sum_{k\in S} P[X(x+t)=k|X(x)=i]P[X(x+t+r)=j|X(x+t)=k] = \\
&= \sum_{k\in S} {}_tp_x^{ik}\, {}_rp_{x+t}^{kj}.
\end{aligned}$$

□

Remark 2.1.3. *Since we did not require property (2.3) to prove the result, the Chapman-Kolmogorov Equations are verified for both homogeneous and non-homogeneous Markov chains.*

For homogeneous Markov chains, this system of equations can be simplified to

$$P_{ij}(t+r) = \sum_{k\in S} P_{ik}(t)P_{kj}(r),\ \forall x \geq 0, \tag{2.9}$$

and so, in matrix form, we have $\boldsymbol{P}(t+r) = \boldsymbol{P}(t)\boldsymbol{P}(r)$ *and consequently* $\boldsymbol{P}(t) = (\boldsymbol{P}(1))^t$.

Since we are considering processes in continuous time, we can think of the rate at which a Markov chain transitions between states in a given instant, that is, the variation of the transition probabilities in an infinitesimal interval of time. Hence, we can define the transition intensities between the states of the model, as well as the representation in matrix form.

2.1.2 Transition Intensities

Definition 2.1.7. *Let* $i, j \in S$ *and* $x \geq 0$.

The transition intensity (or force of transition) between states i *and* j*, for an aged* x *individual, denoted by* μ_x^{ij}*, is defined by:*

$$\mu_x^{ij} \equiv \mu_{ij}(x) = \lim_{h\to 0^+} \frac{{}_hp_x^{ij}}{h},\ i \neq j, \tag{2.10}$$

or, equivalently,

$${}_hp_x^{ij} = h\mu_x^{ij} + o(h), \tag{2.11}$$

where $\lim_{h\to 0^+} \dfrac{o(h)}{h} = 0$.

This means that the transition intensity μ_x^{ij} can be seen as the derivative of the transition probability ${}_tp_x^{ij}$ at time $t = 0$, for each age $x \geq 0$. This definition provides an approach for obtaining transition intensities, given the transition probabilities.

Remark 2.1.4. *In homogeneous Markov chains, the transition intensities are constant for all ages and depend only on the states i, j, since*

$$\mu_x^{ij} = \lim_{h\to 0^+} \frac{P_{ij}(x,h)}{h} = \lim_{h\to 0^+} \frac{P_{ij}(h)}{h} \equiv \mu_{ij},\ \forall x \geq 0, i \neq j. \tag{2.12}$$

The transition intensities can also be represented in matrix form.

Definition 2.1.8. *Let $x \geq 0$, $n \in \mathbb{N}$ and $S = \{1, \ldots, n\}$.*

The transition intensity matrix is given by $\boldsymbol{Q}(x) = [\mu_x^{ij}]_{i,j\in S}$, i.e.

$$\boldsymbol{Q}(x) = \begin{bmatrix} \mu_x^{11} & \mu_x^{12} & \cdots & \mu_x^{1n} \\ \mu_x^{21} & \mu_x^{22} & \cdots & \mu_x^{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_x^{n1} & \mu_x^{n2} & \cdots & \mu_x^{nn} \end{bmatrix}. \tag{2.13}$$

Note that transition intensities satisfy the following properties:

- For all $i, j \in S$, $\mu_x^{ij} \geq 0$, and
- For any $i \in S$,

$$\mu_x^{ii} = -\sum_{j\in S, j\neq i} \mu_x^{ij}, \text{ or equivalently, } \sum_{j\in S} \mu_x^{ij} = 0, \tag{2.14}$$

  i.e. all the rows of matrix (2.13) sum to zero.

Remark 2.1.5. *In the homogeneous case, $\boldsymbol{Q}(x) \equiv \boldsymbol{Q} = [\mu_{ij}]_{i,j\in S}$ and the transition matrix is constant and the same for all individuals regardless of their age x.*

We have already seen that, by definition, we are able to obtain the transition intensities given the transition probabilities. The next result provides a method to calculate the transition probabilities given the intensities.

2.1.3 Kolmogorov Forward Equations

The following result is of extreme importance since it establishes a relation between the transition probabilities and the transition intensities.

Theorem 2.1.2 (Kolmogorov Forward Equations)**.** *For $i, j \in S$ (not necessarily different) and $x, t \geq 0$,*

$$\frac{d}{dt}{}_tp_x^{ij} = \sum_{\substack{k\in S\\k\neq j}} ({}_tp_x^{ik}\mu_{x+t}^{kj} - {}_tp_x^{ij}\mu_{x+t}^{jk}) = \sum_{\substack{k\in S\\k\neq j}} {}_tp_x^{ik}\mu_{x+t}^{kj} - {}_tp_x^{ij}\sum_{\substack{k\in S\\k\neq j}} \mu_{x+t}^{jk} \tag{2.15}$$

or, equivalently, in matrix form, $\dfrac{\partial}{\partial t}\boldsymbol{P}(x,t) = \boldsymbol{P}(x,t)\boldsymbol{Q}(t)$.

Proof. We mainly follow the proof in [10, p. 18].

By definition,

$$\frac{d}{dt}{}_tp_x^{ij} = \lim_{h\to 0^+} \frac{{}_{t+h}p_x^{ij} - {}_tp_x^{ij}}{h},$$

and by the Chapman-Kolmogorov Equations, we have that

$${}_{t+h}p_x^{ij} = \sum_{k\in S} {}_tp_x^{ik}\, {}_hp_{x+t}^{kj}.$$

Therefore,

$$\begin{aligned}
\frac{d}{dt}{}_tp_x^{ij} &= \lim_{h\to 0^+} \frac{\sum_{k\in S} {}_tp_x^{ik}\, {}_hp_{x+t}^{kj} - {}_tp_x^{ij}}{h} = \\
&= \lim_{h\to 0^+} \frac{\sum_{k\in S, k\neq j} {}_tp_x^{ik}\, {}_hp_{x+t}^{kj} + {}_tp_x^{ij}\, {}_hp_{x+t}^{jj} - {}_tp_x^{ij}}{h} = \\
&= \sum_{\substack{k\in S\\ k\neq j}} \lim_{h\to 0^+} \frac{{}_tp_x^{ik}\, {}_hp_{x+t}^{kj}}{h} + \lim_{h\to 0^+} \frac{{}_tp_x^{ij}({}_hp_{x+t}^{jj} - 1)}{h} = \\
&= \sum_{\substack{k\in S\\ k\neq j}} {}_tp_x^{ik} \lim_{h\to 0^+} \frac{{}_hp_{x+t}^{kj}}{h} - {}_tp_x^{ij} \lim_{h\to 0^+} \frac{(1 - {}_hp_{x+t}^{jj})}{h} = \\
&= \sum_{\substack{k\in S\\ k\neq j}} {}_tp_x^{ik} \lim_{h\to 0^+} \frac{{}_hp_{x+t}^{kj}}{h} - {}_tp_x^{ij} \lim_{h\to 0^+} \frac{\sum_{k\in S, k\neq j} {}_hp_{x+t}^{jk}}{h} = \\
&= \sum_{\substack{k\in S\\ k\neq j}} {}_tp_x^{ik} \lim_{h\to 0^+} \frac{{}_hp_{x+t}^{kj}}{h} - {}_tp_x^{ij} \sum_{\substack{k\in S\\ k\neq j}} \lim_{h\to 0^+} \frac{{}_hp_{x+t}^{jk}}{h} = \sum_{\substack{k\in S\\ k\neq j}} {}_tp_x^{ik} \mu_{x+t}^{kj} - {}_tp_x^{ij} \sum_{\substack{k\in S\\ k\neq j}} \mu_{x+t}^{jk}.
\end{aligned}$$

□

Remark 2.1.6. *Since we did not require property (2.3) to prove the result, the Kolmogorov Forward Equations are verified for both homogeneous and non-homogeneous Markov chains.*

For homogeneous Markov chains, the system of equations can be simplified and will have only one variable: for $i, j \in S$ (not necessarily different) and $t \geq 0$,

$$\frac{d}{dt}P_{ij}(t) = \sum_{\substack{k\in S\\ k\neq j}} P_{ik}(t)\mu_{kj} - P_{ij}(t) \sum_{\substack{k\in S\\ k\neq j}} \mu_{jk} = \sum_{\substack{k\in S\\ k\neq j}} P_{ik}(t)\mu_{kj} - P_{ij}(t)\mu_{jj} \tag{2.16}$$

or, equivalently, in matrix form, $\frac{\partial}{\partial t}\boldsymbol{P}(t) = \boldsymbol{P}(t)\boldsymbol{Q}(t)$.

Therefore, the Kolmogorov Forward Equations provide an approach to calculate the transition probabilities given the intensities.

To numerically compute the solution of these differential equations, the following equivalent expression is used: for $i, j \in S$ and $x, t, h \geq 0$,

$$_{t+h}p_x^{ij} = {}_tp_x^{ij} - h \sum_{\substack{k \in S \\ k \neq j}} ({}_tp_x^{ij} \mu_{x+t}^{jk} - {}_tp_x^{ik} \mu_{x+t}^{kj}) + o(h). \tag{2.17}$$

Similarly to the transition probabilities between states, we can also define the probability that the process stays in a given state throughout a given time period.

2.1.4 Sojourn Probabilities

Definition 2.1.9. *Let $i \in S$ and $x, t \geq 0$.*

The sojourn probability (or occupancy probability) for state i, for an aged x individual, denoted by ${}_tp_x^{\overline{ii}}$, corresponds to the probability that a life aged x in state i stays in state i throughout the period from age x to age $x + t$:

$$_tp_x^{\overline{ii}} \equiv P_{\overline{ii}}(x, t) = P[X(x + s) = i, \forall s \in [0, t] | X(x) = i]. \tag{2.18}$$

Note that we have ${}_0p_x^{\overline{ii}} = {}_0p_x^{ii} = 1$.

Remark 2.1.7. *For homogeneous Markov chains, this probability can be simplified to*

$$\begin{aligned} P_{\overline{ii}}(x, t) &= P[X(x + s) = i, \forall s \in [0, t] | X(x) = i] = \\ &= P[X(s) = i, \forall s \in [0, t] | X(0) = i] \equiv P_{\overline{ii}}(t), \ \forall x \geq 0, \end{aligned} \tag{2.19}$$

and the sojourn probabilities do not depend of the age x of the individual, as it happens with the transition probabilities.

The sojourn probabilities satisfy a similar property to the Chapman-Kolmogorov Equations.

Theorem 2.1.3. *For $i \in S$ and $x, t, r \geq 0$,*

$$_{t+r}p_x^{\overline{ii}} = {}_tp_x^{\overline{ii}} \, {}_rp_{x+t}^{\overline{ii}}. \tag{2.20}$$

Proof. We mainly follow the proof in [10, p. 16].

We have

$$\begin{aligned} {}_{t+r}p_x^{\overline{ii}} &= P[X(x + s) = i, \forall s \in [0, t + r] | X(x) = i] = \\ &= P[(X(x + s) = i, \forall s \in [0, t]) \wedge (X(x + s) = i, \forall s \in [t, t + r]) | X(x) = i] = \\ &= P[X(x + s) = i, \forall s \in [0, t] | X(x) = i] \\ &\quad \times P[X(x + s) = i, \forall s \in [t, t + r] | (X(x + s) = i, \forall s \in [0, t]) \wedge (X(x) = i)] = \\ &= P[X(x + s) = i, \forall s \in [0, t] | X(x) = i] \\ &\quad \times P[X(x + s) = i, \forall s \in [t, t + r] | X(x + s) = i, \forall s \in [0, t]], \end{aligned}$$

where, again, we use $P(A \cap B|C) = P(A|B \cap C)P(B|C)$.
Finally, since the Markov property is verified, we conclude

$$\begin{aligned}
{}_{t+r}p_x^{\overline{ii}} &= P[X(x+s) = i, \forall s \in [0,t] | X(x) = i] \\
&\quad \times P[X(x+s) = i, \forall s \in [t, t+r] | X(x+s) = i, \forall s \in [0,t]] = \\
&= P[X(x+s) = i, \forall s \in [0,t] | X(x) = i] P[X(x+s) = i, \forall s \in [t, t+r] | X(x+t) = i] = \\
&= {}_t p_x^{\overline{ii}}\, {}_r p_{x+t}^{\overline{ii}}.
\end{aligned}$$

□

Remark 2.1.8. *Since we did not require property (2.3) to prove the result, these equations are verified for both homogeneous and non-homogeneous Markov chains.*

For homogeneous Markov chains, this system of equations can be simplified to

$$P_{\overline{ii}}(t+r) = P_{\overline{ii}}(t) P_{\overline{ii}}(r),\ \forall x \geq 0. \tag{2.21}$$

With a similar reasoning to the Kolmogorov Forward Equations, we can obtain an expression for the derivative of sojourn probabilities.

Theorem 2.1.4. *For $i \in S$ and $x, t \geq 0$,*

$$\frac{d}{dt} {}_t p_x^{\overline{ii}} = -{}_t p_x^{\overline{ii}} \sum_{\substack{k \in S \\ k \neq i}} \mu_{x+t}^{ik}. \tag{2.22}$$

Proof. We mainly follow the proof in [10, p. 19].

Following the definition of derivative and (2.20), we have

$$\frac{d}{dt} {}_t p_x^{\overline{ii}} = \lim_{h \to 0^+} \frac{{}_{t+h} p_x^{\overline{ii}} - {}_t p_x^{\overline{ii}}}{h} = \lim_{h \to 0^+} \frac{{}_t p_x^{\overline{ii}}\, {}_h p_{x+t}^{\overline{ii}} - {}_t p_x^{\overline{ii}}}{h} = -{}_t p_x^{\overline{ii}} \lim_{h \to 0^+} \frac{1 - {}_h p_{x+t}^{\overline{ii}}}{h}.$$

The probability of two or more transitions in the interval $[x+t, x+t+h)$ is $o(h)$, where $\lim_{h \to 0} \frac{o(h)}{h} = 0$, and so ${}_h p_{x+t}^{\overline{ii}} = {}_h p_{x+t}^{ii} + o(h)$. Therefore,

$$\begin{aligned}
\frac{d}{dt} {}_t p_x^{\overline{ii}} &= -{}_t p_x^{\overline{ii}} \lim_{h \to 0^+} \frac{1 - {}_h p_{x+t}^{\overline{ii}}}{h} = -{}_t p_x^{\overline{ii}} \lim_{h \to 0^+} \frac{1 - {}_h p_{x+t}^{ii} - o(h)}{h} = \\
&= -{}_t p_x^{\overline{ii}} \lim_{h \to 0^+} \frac{\displaystyle\sum_{k \in S, k \neq i} {}_h p_{x+t}^{ik} - o(h)}{h} = \\
&= -{}_t p_x^{\overline{ii}} \left(\sum_{\substack{k \in S \\ k \neq i}} \lim_{h \to 0^+} \frac{{}_h p_{x+t}^{ik}}{h} - \lim_{h \to 0^+} \frac{o(h)}{h} \right) = -{}_t p_x^{\overline{ii}} \sum_{\substack{k \in S \\ k \neq i}} \mu_{x+t}^{ik}.
\end{aligned}$$

□

Remark 2.1.9. *Since we did not require property (2.3) to prove the result, these equations are verified for both homogeneous and non-homogeneous Markov chains.*

For homogeneous Markov chains, the system of equations can be simplified and will have only one variable: for $i \in S$ and $t \geq 0$,

$$\frac{d}{dt} P_{\overline{ii}}(t) = -P_{\overline{ii}}(t) \sum_{\substack{k \in S \\ k \neq i}} \mu_{ik}. \tag{2.23}$$

Therefore, by solving the differential equation, we also have an approach to calculate the sojourn probabilities given the transition intensities.

Theorem 2.1.5. *For $i \in S$ and $x, t \geq 0$,*

$$_tp_x^{\overline{ii}} = \exp\left\{ -\int_0^t \sum_{\substack{k \in S \\ k \neq j}} \mu_{x+s}^{ik} \, ds \right\}. \tag{2.24}$$

Proof. We mainly follow the proof in [4, p. 296].

We have

$$\frac{d}{dt} {}_tp_x^{\overline{ii}} = -{}_tp_x^{\overline{ii}} \sum_{\substack{k \in S \\ k \neq i}} \mu_{x+t}^{ik} \Leftrightarrow \frac{\frac{d}{dt} {}_tp_x^{\overline{ii}}}{{}_tp_x^{\overline{ii}}} = -\sum_{\substack{k \in S \\ k \neq i}} \mu_{x+t}^{ik} \Leftrightarrow \frac{d}{dt} \ln\left({}_tp_x^{\overline{ii}}\right) = -\sum_{\substack{k \in S \\ k \neq i}} \mu_{x+t}^{ik} \Leftrightarrow$$

$$\Leftrightarrow \ln\left({}_tp_x^{\overline{ii}}\right) - \ln\left({}_0p_x^{\overline{ii}}\right) = -\int_0^t \sum_{\substack{k \in S \\ k \neq i}} \mu_{x+t}^{ik} \Leftrightarrow {}_tp_x^{\overline{ii}} = \exp\left\{ -\int_0^t \sum_{\substack{k \in S \\ k \neq j}} \mu_{x+s}^{ik} \, ds \right\},$$

since $\ln\left({}_0p_x^{\overline{ii}}\right) = \ln(1) = 0$. □

Remark 2.1.10. *For homogeneous Markov chains, we have the simplified solution: for $i \in S$ and $t \geq 0$,*

$$P_{\overline{ii}}(t) = \exp\left\{ -\int_0^t \sum_{\substack{k \in S \\ k \neq j}} \mu_{ik} \, ds \right\}. \tag{2.25}$$

We have now defined all the elements that are required to describe Markov chains as well as their underlying properties. Continuous-time stochastic processes (and, in particular, Markov chains) can be used as the technical instruments behind various mathematical and actuarial applications, namely Multiple State Models.

2.2 Multiple State Models

Let us consider a general multiple state model, with $T = [0, +\infty)$ and a finite countable state space $S = \{1, \ldots, n\}$, with $n \in \mathbb{N}$. Each state represents a different condition for an individual, instantaneous transitions are possible between selected pair of states and the model describes the random transitions of the individual throughout time between the possible conditions.

Thus, for each $x \geq 0$, the random variable $X(x)$ takes one of the values $\{1, \ldots, n\}$, which represents the state in which the individual is in at a given age x (i.e. $X(x) = i$ means that the individual is in state $i \in S$ at age x). The set $\{X(x) : x \geq 0\}$ is then a continuous-time stochastic process and we define the transition probabilities and intensities of the multiple state model by (2.4) and (2.10), respectively.

As illustrated in [4, p. 286], some examples of commonly used Multiple State Models for actuarial purposes are:

- Alive-Dead model, a two-state model with only one possible transition from the Alive state to the Dead state (the individual cannot leave the Dead state once they enter it);
- Permanent Disability model, a three-state model (Healthy, Sick/Disabled, Dead) in which the individual cannot return to the Healthy state once they leave and can transition to the Dead state from both the other two states;
- Sickness-Death model, similar to the Permanent Disability model but with the possibility of recovery from the Sick/Disabled state to the Healthy state.

From a mathematical point of view, some assumptions must be established in order for the models to be well defined.

Theorem 2.2.1 ([4, p. 290]). *Regarding Multiple State Models, we assume that:*

1. *The process $\{X(x) x \geq 0\}$ is a continuous-time Markov chain, which means that the probability of the future transitions is completely determined by the current state of the process.*

   In other words, for any states $i, j \in S$ and any times $x, x+t \geq 0$, the conditional probability ${}_tp_x^{ij} = P[X(x+t) = j | X(x) = i]$ does not depend on any previous values of the process before time x;

2. *For any interval of time $h > 0$,*

$$P(2 \text{ or more transitions within a time period of length } h) = o(h), \tag{2.26}$$

   meaning that, for a small interval of time h, the probability of more than one transition in that interval is so small that it can be ignored;

3. *For all states $i, j \in S$ and all ages $x \geq 0$, ${}_tp_x^{ij}$ is a differentiable function of t. Additionally, μ_x^{ij} is a bounded and integrable function of x.*

   This assumption ensures the necessary conditions for the existence and computation of the transition intensities and probabilities.

Next, we present some examples of Multiple State Models that will be used throughout this work to illustrate the stated results. We start by describing the states and transitions and then present the correspondent probability matrix, intensity matrix and Kolmogorov Forward Equations.

2.2.1 Sickness-Death Model

The first Multiple State Model is commonly used for Disability Insurance purposes, for example in policies for which the benefits are payable only during periods of disability or sickness, see [4, p. 289], and is known as the standard Sickness-Death model.

Let us consider the three-state continuous-time Markov model, with $T = [0, +\infty)$ and $S = \{1, 2, 3\}$, as represented in Figure 2.1.

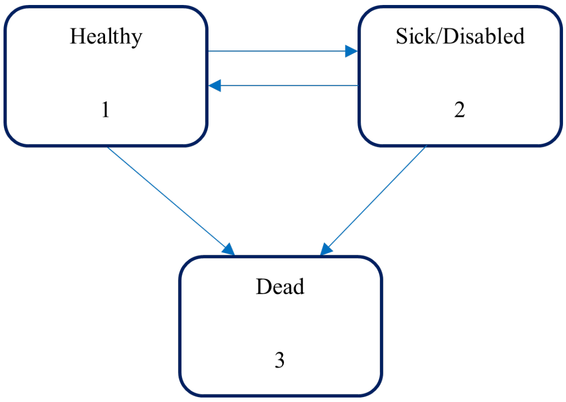


Figure 2.1: Sickness-Death model.

The states of this model and the existing transitions between the states are the following:

- State 1 ≡ Healthy. It is possible to transition from this state to any of the other two states, i.e. the individual might acquire a disability, or die while healthy ($1 \longrightarrow 2$, $1 \longrightarrow 3$, respectively);
- State 2 ≡ Sick/Disabled. From this state it is also possible to transition to any of the other two states, i.e. the individual might make a recovery to the Healthy state, or die while having a disability ($2 \rightarrow 1$, $2 \rightarrow 3$, respectively);
- State 3 ≡ Dead. As mentioned, the individual's death might occur from any of the other two states. Moreover, this is an absorbing state since there are no transitions from it.

Based on the description above, we conclude that the available transition intensities in this multiple state model are $\mu_x^{12}, \mu_x^{13}, \mu_x^{21}, \mu_x^{23}$, and so, for $x \geq 0$, $\mu_x^{3j} = 0$, $\forall j \in S$. Therefore, the transition intensity matrix of this model is given by

$$Q(x) = \begin{bmatrix} \mu_x^{11} & \mu_x^{12} & \mu_x^{13} \\ \mu_x^{21} & \mu_x^{22} & \mu_x^{23} \\ \mu_x^{31} & \mu_x^{32} & \mu_x^{33} \end{bmatrix} = \begin{bmatrix} -\left(\mu_x^{12} + \mu_x^{13}\right) & \mu_x^{12} & \mu_x^{13} \\ \mu_x^{21} & -\left(\mu_x^{21} + \mu_x^{23}\right) & \mu_x^{23} \\ 0 & 0 & 0 \end{bmatrix} \tag{2.27}$$

using the fact that $\mu_x^{ii} = -\sum_{j \in S, j \neq i} \mu_x^{ij}, \forall i \in S$.

Moreover, we also deduce that ${}_tp_x^{31} = {}_tp_x^{32} = 0$, and so ${}_tp_x^{33} = 1$ since $\sum_{j \in S} {}_tp_x^{3j} = 1$. Therefore, for $x, t \geq 0$, the transition probability matrix is given by

$$P(x,t) = \begin{bmatrix} {}_tp_x^{11} & {}_tp_x^{12} & {}_tp_x^{13} \\ {}_tp_x^{21} & {}_tp_x^{22} & {}_tp_x^{23} \\ {}_tp_x^{31} & {}_tp_x^{32} & {}_tp_x^{33} \end{bmatrix} = \begin{bmatrix} {}_tp_x^{11} & {}_tp_x^{12} & {}_tp_x^{13} \\ {}_tp_x^{21} & {}_tp_x^{22} & {}_tp_x^{23} \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.28}$$

Finally, the Kolmogorov Forward Equations of this multiple state model, for $x, t \geq 0$, are

$$\frac{d}{dt}{}_tp_x^{11} = {}_tp_x^{12}\mu_x^{21} - {}_tp_x^{11}(\mu_{x+t}^{12} + \mu_{x+t}^{13}) \tag{2.29a}$$
$$\frac{d}{dt}{}_tp_x^{12} = {}_tp_x^{11}\mu_{x+t}^{12} - {}_tp_x^{12}(\mu_{x+t}^{21} + \mu_{x+t}^{23}) \tag{2.29b}$$
$$\frac{d}{dt}{}_tp_x^{13} = {}_tp_x^{11}\mu_{x+t}^{13} + {}_tp_x^{12}\mu_{x+t}^{23} \tag{2.29c}$$
$$\frac{d}{dt}{}_tp_x^{21} = {}_tp_x^{22}\mu_{x+t}^{21} - {}_tp_x^{21}(\mu_{x+t}^{12} + \mu_{x+t}^{13}) \tag{2.29d}$$
$$\frac{d}{dt}{}_tp_x^{22} = {}_tp_x^{21}\mu_{x+t}^{12} - {}_tp_x^{22}(\mu_{x+t}^{21} + \mu_{x+t}^{23}) \tag{2.29e}$$
$$\frac{d}{dt}{}_tp_x^{23} = {}_tp_x^{21}\mu_{x+t}^{13} - {}_tp_x^{22}\mu_{x+t}^{23} \tag{2.29f}$$

2.2.2 Disability Model with Two Disability States

The Sickness-Death model can be generalised for Long-Term Care (LTC) Insurance purposes, by considering more than one disability state to account for different degrees of disability. The Barthel Index, see [21], is a scale that measures the ability of individuals to complete activities of daily living (ADL), based on the level of assistance required. This index can then be used to build a classification based on their degree of dependence or disability, and a corresponding Multiple State Model.

In [6], a LTC model with three dependence states based on a reduced Barthel Index is considered, allowing for transitions between all dependence states. In this work, we shall consider instead two Disability states and assume that a full recovery from a severe disability is not possible. Although the chosen model has only two Disability states, the methodology and results presented can be generalised to Disability models with more states.

Let us consider the four-state continuous-time Markov model, with $T = [0, +\infty)$ and $S = \{1, 2, 3, 4\}$, as represented in Figure 2.2.

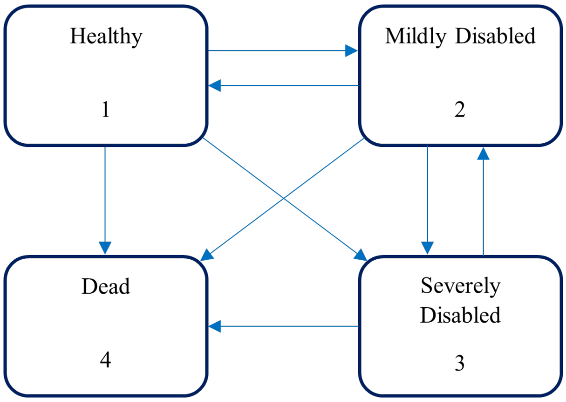


Figure 2.2: Disability model with two Disability states.

The states of this model and the existing transitions between the states are the following:

- State 1 ≡ Healthy. It is possible to transition from this state to any of the remaining states, i.e. the individual might acquire a mild disability, acquire a severe disability or die while Healthy ($1 \longrightarrow 2$, $1 \longrightarrow 3$, $1 \longrightarrow 4$, respectively);

- State 2 ≡ Mildly Disabled. From this state it is also possible to transition to any of the remaining states, i.e. the individual might make a recovery to the Healthy state, acquire a severe disability or die while having a mild disability ($2 \longrightarrow 1$, $2 \longrightarrow 3$, $2 \longrightarrow 4$, respectively);

- State 3 ≡ Severely Disabled. It is not possible for the individual to make a full recovery if they have a severe disability, i.e. to transition from this state directly to the Healthy state. The individual might only make a recovery to a mild disability or die while having a severe disability ($3 \longrightarrow 2$, $3 \longrightarrow 4$, respectively);

- State 4 ≡ Dead. As mentioned, the individual's death might occur from any of the remaining states. Moreover, this is an absorbing state since there are no transitions from it.

Based on the description above, we conclude that, for $x \geq 0$, $\mu_x^{31} = 0$ and $\mu_x^{4j} = 0$, $\forall j \in S$, and thus the transition intensity matrix of this model is given by

$$
\begin{aligned}
Q(x) &= \begin{bmatrix} \mu_x^{11} & \mu_x^{12} & \mu_x^{13} & \mu_x^{14} \\ \mu_x^{21} & \mu_x^{22} & \mu_x^{23} & \mu_x^{24} \\ \mu_x^{31} & \mu_x^{32} & \mu_x^{33} & \mu_x^{34} \\ \mu_x^{31} & \mu_x^{32} & \mu_x^{33} & \mu_x^{44} \end{bmatrix} = \\
&= \begin{bmatrix} -\left(\mu_x^{12} + \mu_x^{13} + \mu_x^{14}\right) & \mu_x^{12} & \mu_x^{13} & \mu_x^{14} \\ \mu_x^{21} & -\left(\mu_x^{21} + \mu_x^{23} + \mu_x^{24}\right) & \mu_x^{23} & \mu_x^{24} \\ 0 & \mu_x^{32} & -\left(\mu_x^{32} + \mu_x^{34}\right) & \mu_x^{34} \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}
\tag{2.30}
$$

using the fact that $\mu_x^{ii} = -\sum_{j \in S, j \neq i} \mu_x^{ij}, \ \forall i \in S$.

Moreover, we also deduce that ${}_tp_x^{4j} = 0, \ \forall j \in S \backslash \{4\}$, and so ${}_tp_x^{44} = 1$ since $\sum_{j \in S} {}_tp_x^{4j} = 1$. Therefore, for $x, t \geq 0$, the transition probability matrix is given by

$$P(x,t) = \begin{bmatrix} {}_tp_x^{11} & {}_tp_x^{12} & {}_tp_x^{13} & {}_tp_x^{14} \\ {}_tp_x^{21} & {}_tp_x^{22} & {}_tp_x^{23} & {}_tp_x^{24} \\ {}_tp_x^{31} & {}_tp_x^{32} & {}_tp_x^{33} & {}_tp_x^{34} \\ {}_tp_x^{31} & {}_tp_x^{32} & {}_tp_x^{33} & {}_tp_x^{44} \end{bmatrix} = \begin{bmatrix} {}_tp_x^{11} & {}_tp_x^{12} & {}_tp_x^{13} & {}_tp_x^{14} \\ {}_tp_x^{21} & {}_tp_x^{22} & {}_tp_x^{23} & {}_tp_x^{24} \\ {}_tp_x^{31} & {}_tp_x^{32} & {}_tp_x^{33} & {}_tp_x^{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.31}$$

Note that, even though the transition intensity $\mu_x^{31} = 0$, the transition probability ${}_tp_x^{31}$ is positive since it is possible for the individual to transition from the Severely Disabled state to the Healthy state for $t > 0$ (in more than one step).

Finally, the Kolmogorov Forward Equations of this multiple state model, for $x, t \geq 0$, are

$$\frac{d}{dt}{}_tp_x^{11} = {}_tp_x^{12}\mu_x^{21} - {}_tp_x^{11}(\mu_{x+t}^{12} + \mu_{x+t}^{13} + \mu_{x+t}^{14}) \tag{2.32a}$$

$$\frac{d}{dt}{}_tp_x^{12} = {}_tp_x^{11}\mu_{x+t}^{12} + {}_tp_x^{13}\mu_{x+t}^{32} - {}_tp_x^{12}(\mu_{x+t}^{21} + \mu_{x+t}^{23} + \mu_{x+t}^{24}) \tag{2.32b}$$

$$\frac{d}{dt}{}_tp_x^{13} = {}_tp_x^{11}\mu_{x+t}^{13} + {}_tp_x^{12}\mu_{x+t}^{23} - {}_tp_x^{13}(\mu_{x+t}^{32} + \mu_{x+t}^{34}) \tag{2.32c}$$

$$\frac{d}{dt}{}_tp_x^{14} = {}_tp_x^{11}\mu_{x+t}^{14} + {}_tp_x^{12}\mu_{x+t}^{24} + {}_tp_x^{13}\mu_{x+t}^{34} \tag{2.32d}$$

$$\frac{d}{dt}{}_tp_x^{21} = {}_tp_x^{22}\mu_{x+t}^{21} - {}_tp_x^{21}(\mu_{x+t}^{12} + \mu_{x+t}^{13} + \mu_{x+t}^{14}) \tag{2.32e}$$

$$\frac{d}{dt}{}_tp_x^{22} = {}_tp_x^{21}\mu_{x+t}^{12} + {}_tp_x^{23}\mu_{x+t}^{32} - {}_tp_x^{22}(\mu_{x+t}^{21} + \mu_{x+t}^{23} + \mu_{x+t}^{24}) \tag{2.32f}$$

$$\frac{d}{dt}{}_tp_x^{23} = {}_tp_x^{21}\mu_{x+t}^{13} + {}_tp_x^{22}\mu_{x+t}^{23} - {}_tp_x^{23}(\mu_{x+t}^{32} + \mu_{x+t}^{34}) \tag{2.32g}$$

$$\frac{d}{dt}{}_tp_x^{24} = {}_tp_x^{21}\mu_{x+t}^{14} + {}_tp_x^{22}\mu_{x+t}^{24} + {}_tp_x^{23}\mu_{x+t}^{34} \tag{2.32h}$$

$$\frac{d}{dt}{}_tp_x^{31} = {}_tp_x^{32}\mu_{x+t}^{21} - {}_tp_x^{31}(\mu_{x+t}^{12} + \mu_{x+t}^{13} + \mu_{x+t}^{14}) \tag{2.32i}$$

$$\frac{d}{dt}{}_tp_x^{32} = {}_tp_x^{31}\mu_{x+t}^{12} + {}_tp_x^{33}\mu_{x+t}^{32} - {}_tp_x^{32}(\mu_{x+t}^{21} + \mu_{x+t}^{23} + \mu_{x+t}^{24}) \tag{2.32j}$$

$$\frac{d}{dt}{}_tp_x^{33} = {}_tp_x^{31}\mu_{x+t}^{13} + {}_tp_x^{32}\mu_{x+t}^{23} - {}_tp_x^{33}(\mu_{x+t}^{32} + \mu_{x+t}^{34}) \tag{2.32k}$$

$$\frac{d}{dt}{}_tp_x^{34} = {}_tp_x^{31}\mu_{x+t}^{14} + {}_tp_x^{32}\mu_{x+t}^{24} + {}_tp_x^{33}\mu_{x+t}^{34} \tag{2.32l}$$

One of the main goals of this work is to study the implementation of Multiple State Models, using continuous-time Markov chains, while focusing on the estimation and graduation of the transition intensities. There are several methodologies for the formulation of these models, but we shall consider the one that regards the transition intensities as the models' fundamental quantities, the Transition Intensities Approach.

3

Estimation and Graduation of Transition Intensities

In this Chapter, we study the formulation of Multiple State Models based on the Transition Intensities Approach, which considers the transition intensities as the models' parameters. Moreover, considering the assumption that the transition intensities are constant between ages, we analyse examples of Multiple State Models and present methods for obtaining crude estimates of the transition intensities as well as a graduation of these estimates based on Generalised Linear Models.

The Transition Intensities Approach was introduced in [30], where the author also presents methods for the parameters' estimation, and so we follow this article in our review. Regarding the estimation and graduation methods, our research includes the works in [4] and [10], as well as [19] and [28] for the study of Generalised Linear Models and their application in graduation techniques.

3.1 Transition Intensities Approach

In [9], Haberman presented an approach for the study of Multiple State Models, in which the model consists of increment-decrement life tables and is specified by Flow, Orientation and Integration equations (the FOI-approach). Using these three sets of equations, it is possible to estimate the transition probabilities between the states of the model.

Waters, in [30], proposed the Transition Intensities Approach (or TI-approach) as an alternative to the FOI-approach, in which the transition intensities are considered the fundamental quantities (or the parameters) of Multiple State Models. The suggested approach has some theoretical and practical advantages, namely in terms of the underlying nature of the models and assumptions for computational aid, which are useful especially when dealing with actuarial applications.

When specifying Multiple State Models, the TI-approach emphasises their stochastic nature since, given the transition intensities, one can completely determine the underlying continuous-time Markov chain. This setting is not always clear in the FOI-approach since the model is algebraically specified by the mentioned three sets of equations and only afterwards one can determine the corresponding Markov chain.

Since the estimation of the transition intensities of a given model is based on observed real data, and it is extremely difficult to derive transition probabilities analytically or could constitute a high computational cost, there are some natural assumptions that can be made to simplify computations. The main logical hypothesis to consider is that the transition intensities are piece-wise constant functions of age, meaning that the transition intensities are the same between exact ages ([30, p. 370]).

The analogous hypothesis in the FOI-approach would affect the underlying integration equations and consequently the specification of the model, which could lead to unreasonable results (as demonstrated in [12]). On the other hand, this assumption is not necessary to the specification of Multiple State Models in the TI-approach, merely serving for computational aid purposes. Therefore, this assumption constitutes one of the main advantages of the TI-approach when compared to the FOI-approach and we shall use it for the estimation of the transition intensities.

Based on the TI-approach, the complete process for determining transition probabilities for actuarial practice is as follows ([25, p. 136]):

1. Collection and processing of data regarding the transitions between states;

2. Estimation of the transition intensities, with the assumptions that these are constant between exact ages;

3. Graduation of the crude estimates, to obtain more realistic smoothed transition intensities;

4. Derivation of the transition probabilities, using the Kolmogorov Forward Equations;

5. Calculation of actuarial values for practical applications, namely premiums and reserves.

In the following sections, we shall present methods to obtain and graduate estimates for the transition intensities (steps 2 and 3). The chosen estimation approach will be demonstrated for the particular model introduced in 2.2.1 and then the analogous result developed for the model presented in 2.2.2, showing that this methodology can be adapted to other Multiple State Models as appropriate.

3.2 Estimation of Transition Intensities

In this work, we shall follow a maximum likelihood approach to obtain the crude estimates of the transition intensities. Moreover, as mentioned, we assume that the transition intensities are piece-wise constant functions between exact ages. This technique was introduced in [30] and also studied in [4] and [10].

Let us consider the Sickness-Death multiple state model, presented in 2.2.1, and a population of individuals under observation during a certain period of consecutive years.

For each individual, we collect the data referring to the transitions between states during the observation period, including the times between successive transitions (or waiting times in each state) and the number of transitions of each type. Hence, each individual symbolises an independent realisation of the underlying continuous-time stochastic process $\{X(x),\ x \geq 0\}$, with x being the individual's age.

Moreover, without loss of generality, we choose an exact age interval, say $(x, x+1)$, over which we can reasonably assume that

$$\mu_{x+s}^{ij} = \mu_x^{ij},\ \forall s \in [0,1),\ i,j \in S, \tag{3.1}$$

i.e. the transition intensities are constants. We also assume that there are N individuals under observation with age within the considered range $(x, x+1)$.

First, we shall determine the likelihood function for each individual under observation (as a function of the transition intensities). Let us present an example which illustrates the result to follow.

Example 3.2.1. *Let us suppose an individual is under observation during the period $t \in [0,1]$, where t is measured in years from age x, and that the sample path of the individual is the following: starts in State 1, transitions to State 2 at $t = 0.2$, then returns to State 1 at $t = 0.5$, transitions again to State 2 at $t = 0.9$ and remains in that state until the end of the observation period.*

For a clearer understanding, we partition the observation period in appropriate intervals, which are delimited by a transition or by the limits of the observation period. Then, we determine the contribution to the likelihood function of each interval:

- *For $t \in (0, 0.2]$, the individual remains in State 1 until the end of the interval, and then transitions to State 2. Thus, the probability function in this interval is a combination of the density function associated with remaining and then leaving State 1 at $t = 0.2$, given by*

$$_{0.2}p_x^{\overline{11}} \left(\mu_x^{12} + \mu_x^{13}\right),$$

  and the probability that the individual transitions to State 2 and not State 3 at $t = 0.2$, given by

$$\frac{\mu_x^{12}}{\mu_x^{12} + \mu_x^{13}}.$$

  Therefore, the contribution to the complete likelihood function of this interval is given by

$$_{0.2}p_x^{\overline{11}} \left(\mu_x^{12} + \mu_x^{13}\right) \frac{\mu_x^{12}}{\mu_x^{12} + \mu_x^{13}} = {_{0.2}p_x^{\overline{11}}} \mu_x^{12}.$$

- *For $t \in (0.2, 0.5]$, the reasoning is similar but, in this case, the individual remains in State 2 for a duration of $t = 0.5 - 0.2 = 0.3$, and then transitions back to State 1. Therefore, the contribution to the likelihood function of this interval is now a combination of the density* *function associated with remaining and then leaving State 2 at $t = 0.3$, and the probability that the individual transitions to State 1 (and not State 3) at $t = 0.3$, namely*

$$ {}_{0.3}p_x^{\overline{22}} \mu_x^{21}. $$

- *For $t \in (0.5, 0.9]$, the behaviour of the individual is similar to the first interval, except that they remain in State 1 for a duration of $t = 0.9 - 0.5 = 0.4$. Thus, the contribution to the likelihood function of this interval is given by*

$$ {}_{0.4}p_x^{\overline{11}} \mu_x^{12}. $$

- *For $t \in (0.9, 1]$, the individual remains in State 2 for a duration of $t = 1 - 0.9 = 0.1$ until the end of the observation period, and therefore the contribution to the likelihood function of this interval is simply given by*

$$ {}_{0.1}p_x^{\overline{22}}. $$

Finally, the likelihood function of this individual is given by

$$ {}_{0.2}p_x^{\overline{11}} \mu_x^{12} {}_{0.3}p_x^{\overline{22}} \mu_x^{21} {}_{0.4}p_x^{\overline{11}} \mu_x^{12} {}_{0.1}p_x^{\overline{22}} = {}_{0.2}p_x^{\overline{11}} {}_{0.3}p_x^{\overline{22}} {}_{0.4}p_x^{\overline{11}} {}_{0.1}p_x^{\overline{22}} (\mu_x^{12})^2 \mu_x^{21}. $$

Following the (2.24) for the calculation of sojourn probabilities, and considering assumption (3.1), we know that for $t \in [0, 1]$ we have

$$ \begin{aligned} {}_tp_x^{\overline{11}} &= \exp\left\{ -\int_0^t \left(\mu_{x+s}^{12} + \mu_{x+s}^{13}\right) ds \right\} = \\ &= \exp\left\{ -\int_0^t \left(\mu_x^{12} + \mu_x^{13}\right) ds \right\} = \exp\left\{ -\left(\mu_x^{12} + \mu_x^{13}\right) t \right\}, \end{aligned} \tag{3.2} $$

and

$$ \begin{aligned} {}_tp_x^{\overline{22}} &= \exp\left\{ -\int_0^t \left(\mu_{x+s}^{21} + \mu_{x+s}^{23}\right) ds \right\} = \\ &= \exp\left\{ -\int_0^t \left(\mu_x^{21} + \mu_x^{23}\right) ds \right\} = \exp\left\{ -\left(\mu_x^{21} + \mu_x^{23}\right) t \right\}, \end{aligned} \tag{3.3} $$

and thus the likelihood function of this individual can be rewritten as

$$ \begin{aligned} & e^{-0.2\left(\mu_x^{12}+\mu_x^{13}\right)} e^{-0.3\left(\mu_x^{21}+\mu_x^{23}\right)} e^{-0.4\left(\mu_x^{12}+\mu_x^{13}\right)} e^{-0.1\left(\mu_x^{21}+\mu_x^{23}\right)} (\mu_x^{12})^2 \mu_x^{21} = \\ = & e^{-0.6\left(\mu_x^{12}+\mu_x^{13}\right)} e^{-0.4\left(\mu_x^{21}+\mu_x^{23}\right)} (\mu_x^{12})^2 \mu_x^{21}. \end{aligned} $$

We note that the exact instants in which the individual transitions between states are not explicit in the likelihood function. Instead, the likelihood solely depends on the total waiting times in each state ($0.2 + 0.4 = 0.6$ in State 1 and $0.3 + 0.1 = 0.4$ in State 2), and the number of transitions of each type (2 transitions from State 1 to State 2 and 1 transition from State 2 to State 1), which is precisely the data collected for each individual.

Following the example, we note that for a single individual, each stay of duration $t \in [0,1]$ in a given state, followed by a transition to a different state, contributes with a factor of the form

$$ {}_t p_x^{\overline{ii}} \mu_x^{ij} $$

to the likelihood of the sample path. Moreover, by (3.2) and (3.3), each sojourn probability in the healthy or sick states (States 1 and 2, respectively) can be rewritten as

$$ \exp\left\{-\left(\mu_x^{12}+\mu_x^{13}\right)t\right\} \text{ and } \exp\left\{-\left(\mu_x^{21}+\mu_x^{23}\right)t\right\}, $$

respectively. Thus, defining

- H_k - waiting time of individual k in the Healthy state;
- S_k - waiting time of individual k in the Sick state;
- HS_k - number of transitions of individual k from Healthy to Sick;
- HD_k - number of transitions of individual k from Healthy to Dead;
- SH_k - number of transitions of individual k from Sick to Healthy;
- SD_k - number of transitions of individual k from Sick to Dead;

we conclude that the likelihood function for individual k, as a function of the transition intensities, is given by

$$ \mathcal{L}_k\left(\mu_x^{12},\mu_x^{13},\mu_x^{21},\mu_x^{23}\right) = e^{-H_k\left(\mu_x^{12}+\mu_x^{13}\right)} e^{-S_k\left(\mu_x^{21}+\mu_x^{23}\right)} (\mu_x^{12})^{HS_k} (\mu_x^{13})^{HD_k} (\mu_x^{21})^{SH_k} (\mu_x^{23})^{SD_k}. \quad (3.4) $$

After determining the likelihood function for each individual, we want to determine the likelihood function given the whole population under observation.

Since the sample paths of the individuals are mutually independent, the likelihood function of the observed population is the product of the likelihoods of all the individuals. Thus, defining $H = \sum_{k=1}^{N} H_k$ (and so on), the likelihood function for the transition intensities (or parameters) is given by

$$ \begin{aligned} \mathcal{L}\left(\mu_x^{12},\mu_x^{13},\mu_x^{21},\mu_x^{23}\right) &= \prod_{k=1}^{N} \mathcal{L}_k\left(\mu_x^{12},\mu_x^{13},\mu_x^{21},\mu_x^{23}\right) = \\ &= e^{-H\left(\mu_x^{12}+\mu_x^{13}\right)} e^{-S\left(\mu_x^{21}+\mu_x^{23}\right)} (\mu_x^{12})^{HS} (\mu_x^{21})^{SH} (\mu_x^{13})^{HD} (\mu_x^{23})^{SD} = \\ &= e^{-H\mu_x^{12}} (\mu_x^{12})^{HS} e^{-H\mu_x^{13}} (\mu_x^{13})^{HD} e^{-S\mu_x^{21}} (\mu_x^{21})^{SH} e^{-S\mu_x^{23}} (\mu_x^{23})^{SD}. \end{aligned} \quad (3.5) $$

Then, factorising the likelihood function in functions of each parameter to be estimated, we obtain elements of the form

$$ \mathcal{L}\left(\mu_x^{ij}\right) = e^{-\text{waiting time}\times\mu_x^{ij}} (\mu_x^{ij})^{\text{number of transitions}}. \quad (3.6) $$

For example, for the estimation of the transition intensity between the Healthy and Sick states, we have

$$\mathcal{L}\left(\mu_x^{12}\right) = e^{-H\mu_x^{12}}(\mu_x^{12})^{HS} \quad \Rightarrow \quad \ell\left(\mu_x^{12}\right) = \log \mathcal{L}\left(\mu_x^{12}\right) = -H\mu_x^{12} + HS\log(\mu_x^{12}) \tag{3.7}$$

and so the maximum likelihood estimator for this parameter is given by

$$\frac{d\ell}{d\mu_x^{12}} = 0 \Leftrightarrow -H + \frac{HS}{\mu_x^{12}} = 0 \Leftrightarrow \hat{\mu}_x^{12} = \frac{HS}{H} \tag{3.8}$$

Following the same reasoning, we conclude that the maximum likelihood estimators for the four transition intensities are given by

$$\hat{\mu}_x^{12} = \frac{HS}{H}, \quad \hat{\mu}_x^{13} = \frac{HD}{H}, \quad \hat{\mu}_x^{21} = \frac{SH}{S}, \quad \hat{\mu}_x^{23} = \frac{SD}{S}. \tag{3.9}$$

Note that, as usual, the obtained estimators are random variables, since they are a ratio of the number of transitions and the total waiting time (also known as the central exposure to risk), both random variables.

Following the same methodology, we shall develop the analogous result for the Disability model with two Disability states, presented in 2.2.2. Considering the same set of assumptions and defining

- H_k - waiting time of individual k in the Healthy state;
- $D1_k$ - waiting time of individual k in the Mildly Disabled state;
- $D2_k$ - waiting time of individual k in the Severely Disabled state;
- $HD1_k$ - number of transitions of individual k from Healthy to Mildly Disabled;
- $HD2_k$ - number of transitions of individual k from Healthy to Severely Disabled;
- HD_k - number of transitions of individual k from Healthy to Dead;
- $D1H_k$ - number of transitions of individual k from Mildly Disabled to health;
- $D1D2_k$ - number of transitions of individual k from Mildly Disabled to Severely Disabled;
- $D1D_k$ - number of transitions of individual k from Mildly Disabled to Dead;
- $D2D1_k$ - number of transitions of individual k from Severely Disabled to Mildly Disabled;
- $D2D_k$ - number of transitions of individual k from Severely Disabled to Dead;

as well as $H = \sum_{k=1}^{N} H_k$ (and so on), by following the same reasoning, we get that the likelihood function for the population under observation as a function of the transition intensities (or parameters) $\boldsymbol{\mu} = \left(\mu_x^{12}, \mu_x^{13}, \mu_x^{14}, \mu_x^{21}, \mu_x^{23}, \mu_x^{24}, \mu_x^{32}, \mu_x^{34}\right)$ is given by

$$\begin{aligned}\mathcal{L}(\boldsymbol{\mu}) =& e^{-H\mu_x^{12}}(\mu_x^{12})^{HD1}e^{-H\mu_x^{13}}(\mu_x^{13})^{HD2}e^{-H\mu_x^{14}}(\mu_x^{14})^{HD}e^{-D1\mu_x^{21}}(\mu_x^{21})^{D1H}\\ &\times e^{-D1\mu_x^{23}}(\mu_x^{23})^{D1D2}e^{-D1\mu_x^{24}}(\mu_x^{24})^{D1D}e^{-D2\mu_x^{32}}(\mu_x^{32})^{D2D1}e^{-D2\mu_x^{34}}(\mu_x^{34})^{D2D}.\end{aligned} \tag{3.10}$$

The likelihood function can also be factorised in elements of the form (3.6) for each parameter to be estimated, and thus we conclude that the maximum likelihood estimators for the available transition intensities are given by

$$\begin{aligned}\hat{\mu}_x^{12} = \frac{HD1}{H}, \quad \hat{\mu}_x^{13} = \frac{HD2}{H}, \quad \hat{\mu}_x^{14} = \frac{HD}{H}, \quad \hat{\mu}_x^{21} = \frac{D1H}{D1},\\ \hat{\mu}_x^{23} = \frac{D1D2}{D1}, \quad \hat{\mu}_x^{24} = \frac{D1D}{D1}, \quad \hat{\mu}_x^{32} = \frac{D2D1}{D2}, \quad \hat{\mu}_x^{34} = \frac{D2D}{D2}.\end{aligned} \tag{3.11}$$

Following the approach described above for each exact age interval in the observation period, we obtain crude estimates of the transition intensities for each year of age, which builds a piece-wise constant function between ages (as per our initial assumption). However, this methodology and hypothesis could lead to the loss of some expected properties, namely the smoothness of the transition intensity function between ages. Therefore, a graduation of these raw estimates can be used to obtain more realistic values.

3.3 Graduation of Transition Intensities

The crude estimates of the transition intensities obtained previously may present some random fluctuations, that are not desirable, which are a result of the hypothesis we have assumed to calculate the estimates. By assuming that the transition intensities are constant functions between exact ages, and calculating an estimator separately for each interval, we are also assuming that the transition intensities for different ages are mutually independent. However, it is commonly expected that the transition intensity for a given age has a relation to the ones in its neighbourhood and that the transition intensity function would have a smooth evolution across all ages ([10, p. 145]).

As a result, some techniques for the adjustment of the crude estimates are applied in order to obtain more realistic and regular values for the population under observation. This procedure is called graduation and ensures the expected degree of smoothness in the transition intensities estimates as well as the required mathematical properties for the calculation of the transition probabilities and actuarial values.

3.3.1 Graduation Methods

There are several kinds of graduation methods (see [1] for a thorough study of the matter), which can be categorised in graphic, parametric and non-parametric methods. In particular, parametric methods are based on the assumption that the values we wish to graduate can be written as a mathematical function (for example, a polynomial), in which the parameters are determined by estimation methods, such as maximum likelihood, minimum chi-square or weighted least squares.

In this work, we shall follow a parametric approach based on Generalised Linear Models, in particular Poisson regression models, with predictors of the Gompertz-Makeham type, and so we start with a brief introduction of the required concepts.

3.3.1.1 Gompertz-Makeham Law

The Gompertz-Makeham mortality law is one of the main expressions used to model death rates as age-dependent functions. It was initially proposed by Gompertz, [8], who suggested that mortality rates increase exponentially with age and can be written in the form

$$\mu_x = \alpha e^{\beta x}, \ \alpha > 0, \ \beta > 0, \tag{3.12}$$

or, equivalently, setting $B = \alpha$ and $\beta = \ln(c)$,

$$\mu_x = Bc^x, \ B > 0, c > 1. \tag{3.13}$$

The Gompertz function was later generalised by Makeham, [22], who introduced an age-independent component to the expression. The Makeham term increases the mortality rate by a constant that is equal for all ages and thus the Gompertz-Makeham law can be written as

$$\mu_x = A + Bc^x, \ A \geq 0, \ B > 0, \ c > 1. \tag{3.14}$$

Although the Gompertz-Makeham law is commonly used to describe mortality rates, it can also be applied to model other age-dependant transitions in Multiple State Models, for example Long-Term Care or Disability Insurance ([10, p. 100]).

Furthermore, it was one of the first expressions used as the base for the graduation of mortality rates estimates. As described in [7], the CMIB (Continuous Mortality Investigation Bureau) of the Institute and Faculty of Actuaries (IFoA) uses a generalisation of the Gompertz-Makeham law to graduate the force of mortality for the construction of life-tables.

Let r and s be non-negative integers (not both zero), $\theta = (\alpha_0, \ldots, \alpha_{r-1}; \beta_0, \ldots, \beta_{s-1})$ a vector of parameters, and

$$GM_\theta^{r,s}(x) = \sum_{i=0}^{r-1} \alpha_i x^i + \exp\left(\sum_{j=0}^{s-1} \beta_j x^j\right), \tag{3.15}$$

with the assumption that, if $r = 0$ or $s = 0$, the first or the second group of terms are set to 0, respectively.

This family of parametric functions are known as the Gompertz-Makeham formula of type (r, s), or $GM(r, s)$ formula. We notice that the formula $GM(0, 2)$ corresponds to the Gompertz function (3.12), setting $\alpha = e^{\beta_0}$ and $\beta = \beta_1$, and $GM(1,2)$ is the Gompertz-Makeham law (3.14), setting $A = \alpha_0$, $B = e^{\beta_0}$ and $c = e^{\beta_1}$.

For the graduation of mortality indicators, the CMIB methodology then claims that, given a vector of parameters (or coefficients) θ, estimated from the observed data, the model $\mu_x = GM_\theta^{r,s}$ provides an appropriate fit, see [2]. We note that this approach is also applicable to the graduation of other transition intensities in Multiple State Models, and more importantly, can be incorporated in graduation using Generalised Linear Models, see [2], as we shall see in the graduation process used in this work.

3.3.1.2 Generalised Linear Models

Generalised Linear Models (GLMs) are an extension of classical linear models for response variables that are not necessarily Normally distributed, and for non-linear transformations. GLMs have various applications in actuarial work, namely in Multiple State Models used for Health Insurance, in the modelling of Life Insurance lapse rates (where lapse is the premature termination of a policy), and for premium rating and claims reserving in Non-life Insurance, see [11].

To describe classical linear models, let us consider a vector $\mathbf{y} = (y_i)$, $i = 1, 2, \ldots, n$, of observations (or response variables), assumed to be the realisation of $\mathbf{Y} = (Y_i)$, a vector of independent and Normally distributed random variables. Moreover, let us also consider vectors $\mathbf{x}_j = (x_{ij})$, $j = 1, 2, \ldots, p$, of covariates (or explanatory variables), each containing n known observations, that model the response variables.

In order to estimate the expected value of $\mathbf{Y}$ given the observations $\mathbf{X}$, $\mathbf{m} = E(\mathbf{Y}|\mathbf{X}) \equiv E(\mathbf{Y})$, linear models assume there exists a linear combination of the covariates $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p$, such that

$$\mathbf{m} = \sum_{j=1}^{p} \beta_j \mathbf{x}_j = \boldsymbol{X\beta}. \tag{3.16}$$

For the expected value of each response variable, letting i be the index for the observations, and considering x_{ij} the value of the jth covariate for observation i, we have that

$$m_i = E(Y_i) = \sum_{j=1}^{p} \beta_j x_{ij},\ i = 1, 2, \ldots, n. \tag{3.17}$$

The β_j, $j = 1, 2, \ldots, p$, are unknown parameters to be estimated by maximum likelihood based on the observations $\mathbf{y}$.

Then, classical linear models may be generalised in two ways. The first is to consider not only Normally distributed response variables, but also distributions of the Exponential Family of distributions. In addition to the Normal distribution, the Exponential Family includes several other distributions, such as the Poisson, Binomial, Gamma, Inverse Gaussian and, of course, the Exponential distribution.

Secondly, a non-linear transformation can be applied to the means of the independent response variables we wish to estimate, **m**, to link it to the combination of the explanatory variables, also called the linear predictor

$$\boldsymbol{\eta} = X\boldsymbol{\beta} \tag{3.18}$$

This transformation is called the link function and is required to be monotonic and differentiable, so that

$$\boldsymbol{\eta} = g(\mathbf{m}) \Rightarrow \mathbf{m} = g^{-1}(\boldsymbol{\eta}) = g^{-1}(\boldsymbol{X} \cdot \boldsymbol{\beta}) \tag{3.19}$$

and we are able to obtain estimates for the values of **m** by estimating the βs. The most commonly used link function for each distribution (also known as the canonical link function) are the Identity function for the Normal distribution, the logarithmic function for the Poisson distribution and the logit function for the Binomial distribution.

The steps involved in constructing a GLM, given a response variable **Y**, are the following ([19, p. 65]):

1. Choose a distribution for the response variable **Y** (member of the Exponential Family of distributions);
2. Choose a link function g (if appropriate, the canonical link function for the chosen distribution);
3. Choose explanatory variables **x** in terms of which $g(\mathbf{m})$ is to be modelled;
4. Collect observations on the response **Y** and corresponding values on the explanatory variables **x**;
5. Fit the model by estimating β (using, for example, maximum likelihood estimation);
6. Given the estimate of β, generate predictions (or fitted values) of **Y** for different settings of **x** and test the goodness of fit.

To graduate the obtained raw estimates of the transition intensities, we shall use a Poisson regression, i.e. a GLM with a Poisson distributed response variable, with the logarithm as the link function, following the process described next.

3.3.2 Graduation Process

As mentioned, we shall follow a parametric graduation method based on GLMs, which was first proposed in [27] for the graduation of the force of mortality. In [28], it was then applied to the graduation of the transition intensities in the standard Sickness-Death model, noting that it can be generalised to other Multiple State Models if the required data is available. We note that, although in [28] the explanatory variables considered are the age at the onset of sickness and, for the transitions from the Sick state, the sickness duration, we might wish to take other observable risk factors into account.

Regarding the modelling of mortality indicators, some methods have been proposed to include the heterogeneity of a population and the effect of different observable variables in their mortality. One approach is considering a Poisson regression model inspired by the Cox proportional hazards model ([3]), in which a multiplicative factor includes the effect of the various explanatory variables in the force of mortality ([2]). Furthermore, in [26], a graduation of the force of mortality is performed taking into account more than one of the possible explanatory variables available in the data (gender, smoking habit, medical status and age).

In this work, we shall generalise the approach presented in [28] to the multiple state model introduced in 2.2.2, extending the explanatory variables to other demographic factors besides solely the individuals' age (similar to what was done in [26]).

Firstly, for the graduation of each transition intensity, the data is organised in a grid of units or cells denoted by $u \equiv (x_1, x_2, \ldots)$, where $x_1, x_2, \ldots$ are the covariates to be considered in the GLM. For example, if we assume the explanatory variables are:

- $x_1 \equiv$ age, divided in 10 age groups;
- $x_2 \equiv$ sex, with values Male or Female;
- $x_3 \equiv$ smoking habits, with values Smoker or Non-smoker;

we obtain a grid of $10 \times 2 \times 2 = 40$ cross-classified units u.

Then, the observed data consists of a set of ordered pairs (i_u, e_u), where i_u is the number of transitions accruing from the central exposures (or waiting times) e_u, for each unit u. The covariates included in the definition of the units u will be different for each intensity, as well as the definition of i_u and e_u:

- For μ_u^{12}, μ_u^{13} and μ_u^{14}, i_u denotes the observed number of transitions from the Healthy state to the Mildly Disabled, Severely Disabled and Dead states, respectively, accruing from corresponding exposures e_u, which, in this case, is the total waiting time in the Healthy state;

- For μ_u^{21}, μ_u^{23} and μ_u^{24}, i_u denotes the observed number of transitions from the Mildly Disabled state to the Healthy, Severely Disabled and Dead states, respectively, accruing from corresponding exposures e_u, which, in this case, is the total waiting time in the Mildly Disabled state;

- For μ_u^{32} and μ_u^{34}, i_u denotes the observed number of transitions from the Severely Disabled state to the Mildly Disabled and Dead states, respectively, accruing from corresponding exposures e_u, which, in this case, is the total waiting time in the Severely Disabled state.

Without loss of generality, let μ be one of the transition intensities available in the Disability model with two Disability states, presented in 2.2.2. Based on the GLM framework, we assume that there exists a differentiable and injective (and thus invertible) function g that links the intensities μ_u to a linear predictor

$$\eta_u = \sum_j h_j(u)\beta_j \tag{3.20}$$

where functions h_j define the known covariate structure of the current model and coefficients β_j are the unknown regression parameters. In other words,

$$g(\mu_u) = \eta_u \Rightarrow \mu_u = g^{-1}(\eta_u). \tag{3.21}$$

The next step for the graduation process is to obtain the estimates for the β_j, for which we shall follow a maximum likelihood approach.

First, we require a distribution for the response variables of the GLM, which, in this case, are the variables i_u corresponding to the number of observed transitions for each unit u. Let us note that a Poisson distributed random variable, $X \sim \text{Poi}(\lambda)$, $\lambda > 0$, has probability function

$$f(k) = P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!},\ k = 0, 1, 2, \dots,$$

expected value and variance given by $E(X) = Var(X) = \lambda$, and log-likelihood function

$$\begin{aligned}\ell(\lambda) = \log \mathcal{L}(\lambda) = \log \prod_i f(k_i|\lambda) = \sum_i \log\left(\frac{e^{-\lambda}\lambda^{k_i}}{k_i!}\right) = \\ = \sum_i \big(-\lambda + k_i \log(\lambda) - \log(k_i!)\big).\end{aligned}$$

Assuming that the transition intensities are constant between exact ages, we have determined that the likelihood function for the transition intensities is of the form (3.6), which we notice is identical to the probability function a Poisson distribution.

Therefore, we shall assume that the response variables i_u are independent and follow a Poisson distribution, in which the expected number of transitions is given by the product of the exposures to risk by the respective transition intensity. In other words,

$$i_u \sim \text{Poi}(e_u \mu_u),\ \text{independently for all } u \tag{3.22}$$

and the mean and variance are equal and respectively given by

$$m_u = E(i_u) = e_u \mu_u \text{ and } Var(i_u) = m_u. \tag{3.23}$$

However, the same transition between two states can be observed more than once for each individual (except transitions to the Dead state), meaning that the variance of the observed number of transitions might be higher than the mean (in other words, overdispersion could be verified).

To take the possibility of overdispersion into account, we shall further consider that i_u are independent overdispersed Poisson variables,

$$i_u \sim \text{Poi}(e_u\mu_u), \text{ independently for all } u \tag{3.24}$$

with mean and variance respectively given by

$$m_u = E(i_u) = e_u\mu_u \text{ and } Var(i_u) = \phi m_u, \tag{3.25}$$

where $\phi > 1$ is the scale or dispersion parameter. Note that, following from (3.20) and (3.21), for the expected value we obtain

$$m_u = e_u\mu_u = e_u g^{-1}(\eta_u) = e_u g^{-1}\left(\sum_j h_j(u)\beta_j\right), \tag{3.26}$$

which establishes a relation between m_u and the unknown β_j we intend to estimate.

Then, considering $\mathbf{m} = (m_u)$ and $\mathbf{i} = (i_u)$, the log-likelihood function associated with the Poisson distributed response variables is given by

$$\ell(\mathbf{i};\mathbf{m}) = \log \mathcal{L}(\mathbf{i};\mathbf{m}) = \sum_u \left(-m_u + i_u \log(m_u) - \log(i_u!)\right) \tag{3.27}$$

which can be expressed in terms of the β_j using (3.26). Finally, the estimates $\hat{\beta}_j$ are obtained by maximizing the log-likelihood function (3.27).

To conclude the graduation process, we now choose the function g which links the intensities to the linear predictor containing the estimated β_j. We shall use the log-link function, since it is the canonical link associated with the Poisson distributed response variables i_u.

First, using the logarithm in (3.21), we have that

$$\log(\mu_u) = \eta_u \Leftrightarrow \mu_u = \exp(\eta_u) \Leftrightarrow \mu_u = \exp\left(\sum_j h_j(u)\hat{\beta}_j\right) \tag{3.28}$$

replacing the linear predictor by its expression in (3.20). Moreover, by the properties of the logarithmic function,

$$\log(m_u) = \log(e_u\mu_u) = \log(e_u) + \log(\mu_u) = \log(e_u) + \sum_j h_j(u)\hat{\beta}_j \tag{3.29}$$

and we observe that the terms $\log(e_u)$ act in our linear model as an additional variable with known regression coefficient equal to 1, i.e. as an offset. Therefore, the graduated transition intensities are given by

$$m_u = e_u\hat{\mu}_u \Leftrightarrow \hat{\mu}_u = \frac{m_u}{e_u} \Leftrightarrow \hat{\mu}_u = \exp\left(\sum_j h_j(u)\hat{\beta}_j\right) \tag{3.30}$$

In order to test the overall goodness of fit of the current model, we calculate its deviance according to the likelihood ratio principle, where we compare the likelihood of the current model, c, with the one of the saturated model, s. Note that the saturated model is the model where the fitted values have a perfect fit, i.e. are the actual observed values.

First, we have that the value of the log-likelihood function (3.27) under the current model (predictor) and the saturated model structure are, respectively,

$$\ell_c = \log \mathcal{L}_c \equiv \ell(\mathbf{i};\mathbf{m}) = \sum_u \big(-\hat{m}_u + i_u \log(\hat{m}_u) - \log(i_u!)\big) \tag{3.31}$$

and

$$\ell_s = \log \mathcal{L}_s \equiv \ell(\mathbf{i};\mathbf{i}) = \sum_u \big(-i_u + i_u \log(i_u) - \log(i_u!)\big) \tag{3.32}$$

where $\hat{m}_u$ denote the resulting fitted values, given by

$$\hat{m}_u = e_u \exp\left(\sum_j h_j(u)\hat{\beta}_j\right) \tag{3.33}$$

Therefore, the model deviance is given by

$$\begin{aligned} D(c,s) &= -2\log\left(\frac{\mathcal{L}_c}{\mathcal{L}_s}\right) = -2(\log \mathcal{L}_c - \log \mathcal{L}_s) = -2(\ell_c - \ell_s) = 2(\ell_s - \ell_c) = \\ &= 2\sum_u \{-(i_u - \hat{m}_u) + i_u \log(i_u/\hat{m}_u)\} \end{aligned} \tag{3.34}$$

Finally, denoting by d_u the individual components of the model deviance as determined above, i.e. $D(c,s) = \sum_u d_u$, we have that the deviance residuals are given by

$$r_u = \operatorname{sign}(i_u - \hat{m}_u)\sqrt{d_u}, \tag{3.35}$$

where the sign function returns -1, $+1$ or 0 according to the sign (positive or negative) of the corresponding expression.

We note that graduation methods based on GLMs are of extreme importance, not only because of the various actuarial application of GLMs that we have mentioned, but also due to the versatility of this approach, since it can be applied to the available transition intensities in Multiple State Models and not only to mortality indicators ([28]).

In the following Chapter, we shall develop an application of the studied methods, to estimate and graduate the transition intensities of the multiple state model presented in 2.2.2, for a simulated population that mimics the Portuguese one.

4

Practical Application

In this Chapter, we develop a practical application of the results presented in Chapter 3 regarding the estimation and graduation of the transition intensities. We consider the multiple state model introduced in Section 2.2.2, represented in Figure 4.1.

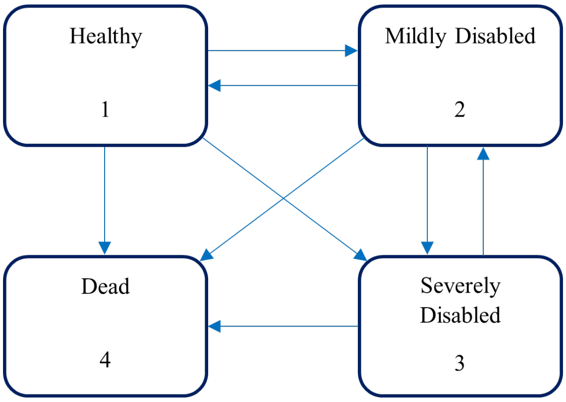


Figure 4.1: Disability model with two Disability states.

We start by analysing the risk factors (or covariates) that the observed transition intensities would depend on when dealing with real data. Since the required data to apply the mentioned graduation techniques is not available, we resort to data simulation to produce a dataset of a population and their trajectories through the states of the chosen multiple state model for a given observation period. Finally, we apply the studied GLM framework to graduate the estimates of the transition intensities, that will now depend not only on the individuals' age but also on other risk factors. The R software (version 4.3.2) is used to perform the simulation and calculations.

4.1 Data Simulation

In Portugal, there is currently a lack of data referring to Disability and Long-Term Care (LTC) Insurance, see [6]. Therefore, it is challenging to obtain the necessary data to estimate and graduate transition intensities following the TI-approach (namely, number of transitions of each type and the corresponding central exposures).

For the purpose of this work, we shall resort to data simulation, i.e. the process of generating a dataset that mimics the characteristics and properties expected in real data. We start by simulating a dataset with the demographic characteristics of the Portuguese population based on the latest health statistics, see [13, 15], and census, see [16, 17, 18]. Then, using the calibrated transitions intensities obtained in [6] for a similar multiple state model, we use the transition intensity matrix of our model to simulate trajectories for the individuals in the generated population in a given observation period.

4.1.1 Population and Risk Factors

In a real-world scenario, the risk or probability of an individual transitioning between the states of the model can depend on additional factors besides solely their age, as referred in [31]. Based on the available data for the Portuguese population, the risk factors (or explanatory variables) that we shall consider in the study of our multiple state model and the underlying transition intensities are:

- Age, which we categorise into age groups;
- Sex (Male or Female);
- Region, namely the population size (number of individuals) of the region of residence;
- Body Mass Index (BMI), a high-level indicator of an individual's health;
- Smoking habits;
- Exercise habits.

We shall simulate a population of 100,000 individuals, in which the explanatory variables for each individual are sampled according to the available statistical data on the Portuguese population. Moreover, for the purposes of the simulation of the variables, we consider them to be mutually independent.

The distribution of the resident population in Portugal, in 2021, according to age group and sex as well as the distribution of the resident population in Portugal, by population size of region is available in the data obtained in the Population and Housing Census (in Portuguese, Censos) conducted by INE (Instituto Nacional de Estatística, Portuguese for National Institute for Statistics) in 2021, see [16, 17, 18].

Additionally, a National Health Inquiry was conducted by INE in 2019 and presents the distribution of the resident population in Portugal, in 2019, according to each remaining covariates (body mass index, smoking and exercise habits) by age group and sex, see [15].

4.1.1.1 Age and Sex

Age and sex are considered the main factors when studying both mortality and health risks, due to the impact of ageing in the human condition as well as the differences observed between Males and Females, see [32]. In [16, 17], we can observe the distribution of Male and Female resident population in Portugal, in 2021, divided by the following age groups: 0-4, 5-9, 10-14, 15-19, 20-24, 25-29, 30-34, 35-39, 40-44, 45-49, 50-54, 55-59, 60-64, 65-69, 70-74, and over 75 years old.

To simulate the age of the individuals in our population (noting that we shall only consider exact ages), we consider a discrete uniform distribution for each five-year age group and generate a random sample in which each age had a 20% probability. For the "over 75 years old" age group, we consider a discrete triangular distribution with lower limit and mode equal to 75 and upper limit equal to 100 to generate ages up to 100 years old with decreasing probability. Finally, we randomly select the sex of an individual based on their age according to the proportions available in the data for each age group.

In Figure 4.2, we present the plot of the distribution of the generated population by age groups and sex, in the form of a age-sex pyramid. We note that there are more Female than Male individuals in most of the age groups and that it is an ageing population, with the majority of the individuals aged between 40 and 75 years old.

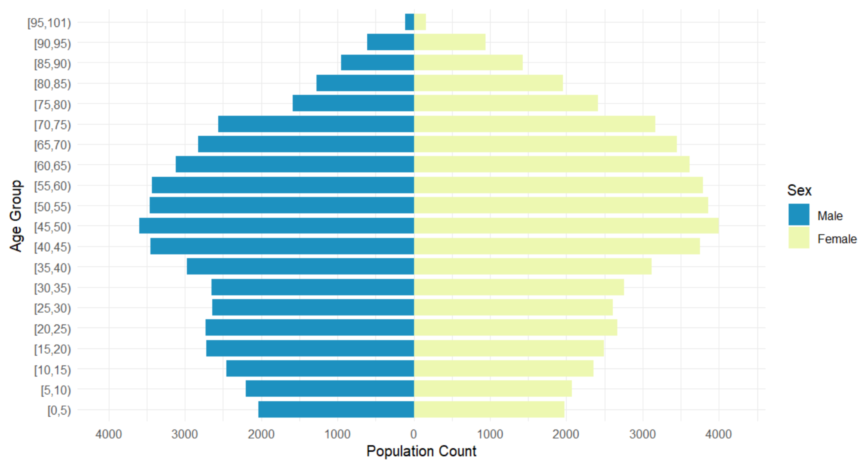


Figure 4.2: Age-sex pyramid of the simulated population.
Source: Author preparation.

4.1.1.2 Region

The type of region in which an individual lives, particularly in terms of population density or population size, can also affect their health. Areas with a larger number of residents are usually urbanised and present higher levels of pollution, resulting in increasing health risks for the inhabitants, see [31]. In [18], is presented the distribution of the resident population in Portugal, in 2021, according to the population size of the region of residence, in the following classes (of number of individuals): 0-1,999, 2,000-4,999, 5,000-9,999, 10,000-19,999, 20,000-99,999, and 100,000 or more individuals.

To simulate the population size of the region of the individuals in our population, we generate a random sample in which the probabilities are given by the proportion of each class of population size in the available data.

In Figure 4.3, we present the pie chart with the distribution of the generated population by the population size of the region of residence. The individuals in the simulated population most frequently live in regions of smaller population size, with 0-1,999 individuals, although there are more than a quarter of the individuals living in highly populated regions, with 20,000-99,999 and 100,000 or more individuals.

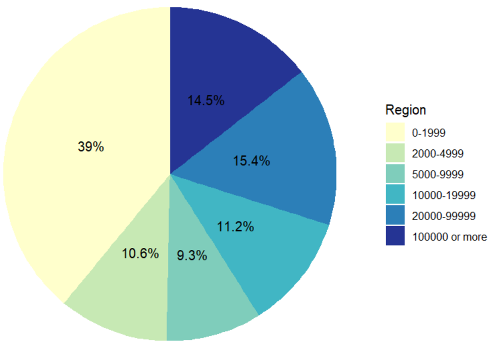


Figure 4.3: Distribution of the simulated population by the population size of the region.
Source: Author preparation.

4.1.1.3 Body Mass Index

The body mass index (or BMI) is used to categorise an individual's weight based on their height and it is calculated by the body mass (in kilograms) divided by the square of the body height (in metres). It is commonly used as a high-level indicator of an individual's health, since values outside of the expected normal range are associated with higher mortality and health risks, see [31]. In [15], we can observe the distribution of the resident population in Portugal, in 2019, over 18 years old, divided by sex and age groups (18-24, 25-34, 35-44, 45-54, 55-64, 65-74,75-84, and over 85 years old), according to the following classes of BMI:

- Underweight (IMC < 18.5 kg/m^2);
- Normal weight (IMC ≥ 18.5 kg/m^2 e IMC < 25 kg/m^2);
- Overweight I (IMC ≥ 25 kg/m^2 e IMC < 27 kg/m^2);
- Overweight II (IMC ≥ 27 kg/m^2 e IMC < 30 kg/m^2);
- Obese (IMC ≥ 30 kg/m^2).

For the simulation, we randomly select the class of BMI for each individual in our population based on their age and sex, using as probabilities the proportions in the available data. For the individuals under 18 years old, we consider the same proportions as for the 18-24 years old individuals.

In Figure 4.4, we present the plot of the distribution of the generated population by the classes of BMI, divided by age groups and sex, in the form of a stacked bar plot with the corresponding proportions. We observe that the majority of the individuals are overweight or obese in most of the age groups, with higher proportion in the Male individuals than in the Female individuals.

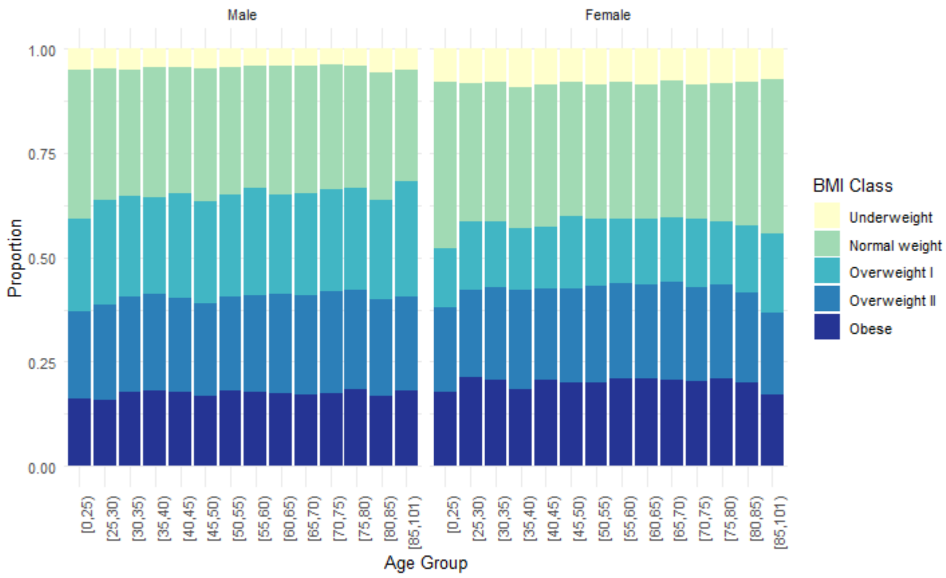


Figure 4.4: Distribution of the simulated population by the classes of BMI.
Source: Author preparation.

4.1.1.4 Smoking Habits

The smoking habits of an individual are a crucial factor when modelling their health risks, since smoking is one of the main causes of cardiovascular and respiratory diseases as well as of increased risk of death caused by smoking-related diseases, see [31]. In [15], there is the distribution of the resident population in Portugal, in 2019, over 15 years old, according to their smoking habits (Smoker and Non-smoker), divided by sex and age groups (15-24, 25-34, 35-44, 45-54, 55-64, 65-74,75-84, and over 85 years old).

For the simulation, we randomly select the smoking habits for each individual in our population based on their age and sex, using as probabilities the proportions in the available data. Note that all individuals under 15 years old are considered as Non-smokers.

In Figure 4.5, we present the bar plot of the distribution of the generated population by smoking habits, divided by age groups and sex. We note that most of the smoking individuals are aged between 15 and 50 years, with more Male individuals than Female individuals, and that the great majority of the individuals are non-smokers.

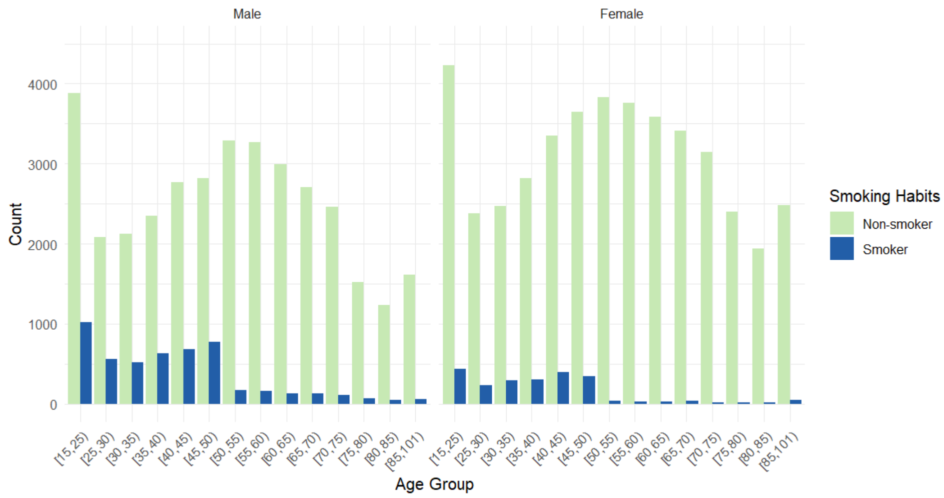


Figure 4.5: Distribution of the simulated population by the smoking habits.
Source: Author preparation.

4.1.1.5 Exercise Habits

Finally, exercise habits are also relevant when dealing with health risks, since the regular practice of physical activity presents major health benefits and improved quality of life, see [31]. In [15], we can observe the distribution of the resident population in Portugal, in 2019, over 15 years old, who practices physical activity at least 1 day per week, divided by sex and age groups (15-24, 25-34, 35-44, 45-54, 55-64, and over 65 years old), according to the time spent doing physical activity in a normal week:

- Less than 2 hours;
- 2 to less than 3 hours;
- 3 to less than 5 hours;
- 5 or more hours.

For the simulation, we randomly select the exercise habits for each individual in our population based on their age and sex, using as probabilities the proportions in the available data. For the individuals under 15 years old, we consider the same proportions as for the 15-24 years old individuals.

In Figure 4.6, we present the plot of the distribution of the generated population by exercise habits, divided by age groups and sex, in the form of a stacked bar plot with the corresponding proportions. We observe that a more frequent practice of exercise (5 or more hours per week) is more common in the Male individuals than in the Female individuals, although the majority of the population only practices physical activity less than 2 hours per week.

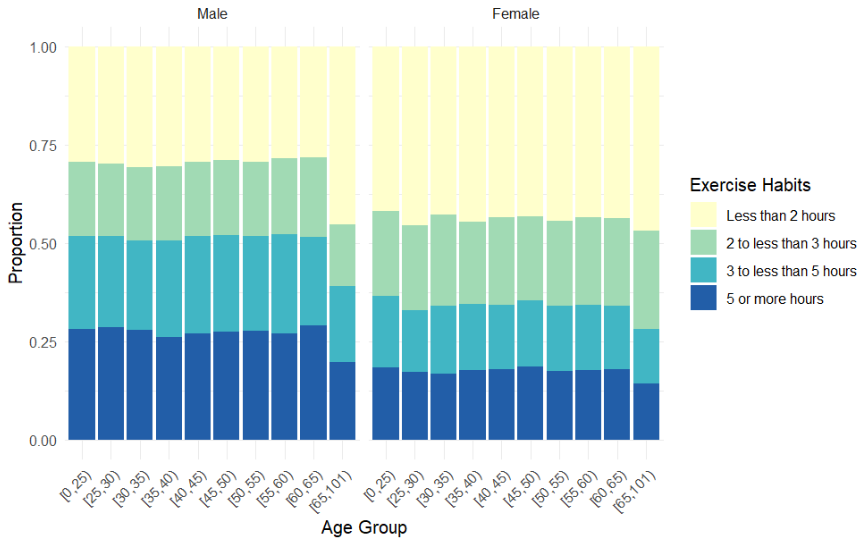


Figure 4.6: Distribution of the simulated population by the exercise habits.
Source: Author preparation.

4.1.2 Trajectories

As mentioned, we shall consider the Disability model with two Disability states presented in 2.2.2. To simulate the trajectories of the individuals in our population (based on their demographic characteristics or, in other words, their risk profile) through the multiple state model, we first require to generate the following elements:

- Initial State, i.e. the state in which the individual is in at the start of the observation period and from where their trajectory will start;
- Exposure, i.e. the duration of the period in which the individual is under observation;
- Transition Intensities, namely the ones included in the transition intensity matrix (2.31) corresponding to our model.

Firstly, we assume the initial state of the trajectories can be any state of the model except the Dead state (since it is an absorbing state and of no relevance to our study). In [13], INE published the Health Statistics referring to the year of 2022, which includes the distribution of the resident population in Portugal, in 2022, over 16 years old, divided by sex and age groups (16-64, and over 65 years old), according to the limitation in the realisation of tasks:

- Not limited, which corresponds to the Healthy state;
- Limited but not severely, which corresponds to the Mildly Disabled state;
- Severely limited, which corresponds to the Severely Disabled state.

For the simulation, we randomly select the initial state for each individual in our population based on their age and sex, using as probabilities the proportions in the available data. For the individuals under 16 years old, we consider the same proportions as for the 16-64 years old individuals.

In Figure 4.7, we present the plot of the distribution of the generated population by initial state, divided by age groups and sex, in the form of a stacked bar plot with the corresponding proportions. We note that around 3/4 of the population under 65 years old starts in the Healthy state, while the proportion of the Mildly Disabled and Severely Disabled states is higher for the individuals over 65 years old, especially for the Female individuals.

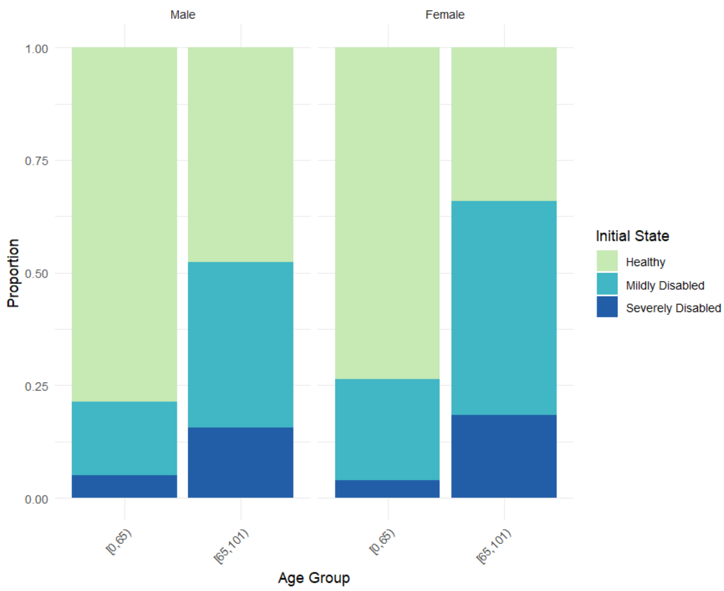


Figure 4.7: Distribution of the simulated population by the initial state.
Source: Author preparation.

In Table 4.1, we present a sample of the simulated population, including the demographic characteristics and the initial state.

Table 4.1: Sample of the simulated dataset.

Age	Sex	Region	BMI	Smoking	Exercise	Initial State
54	Female	2,000-4,999	Underweight	Non-smoker	2 to less than 3 hours	Mildly Disabled
14	Female	5,000-9,999	Obese	Non-smoker	5 or more hours	Mildly Disabled
38	Female	0-1,999	Underweight	Non-smoker	5 or more hours	Severely Disabled
26	Male	10,000-19,999	Normal weight	Non-smoker	Less than 2 hours	Healthy
48	Female	0-1,999	Normal weight	Non-smoker	Less than 2 hours	Healthy
62	Male	0-1,999	Overweight I	Non-smoker	Less than 2 hours	Healthy
37	Female	10,000-19,999	Overweight I	Non-smoker	5 or more hours	Healthy
80	Female	2,000-4,999	Overweight II	Non-smoker	5 or more hours	Severely Disabled
28	Female	100,000 or more	Obese	Non-smoker	2 to less than 3 hours	Healthy
53	Male	2,000-4,999	Normal weight	Non-smoker	2 to less than 3 hours	Healthy
70	Female	20,000-99,999	Obese	Non-smoker	2 to less than 3 hours	Mildly Disabled
40	Male	20,000-99,999	Overweight I	Smoker	5 or more hours	Healthy

Then, we shall simulate the exposure of the individuals in our population. Exposure (or exposure to risk) is the duration of the observation period, in which transitions can be observed for an individual.

We assume that the individuals in our population are under observation for at least 6 months and at most 12 months, and that the exposure is independent from their demographic characteristics. Thus, measuring the exposure in months, we consider a continuous uniform distribution $\mathcal{U}[6,12]$ and generate a random exposure for each individual.

Regarding the transition intensities required in the transition intensity matrix (2.31), following [6] and [10], firstly we shall assume that we have intensities of the Gompertz-Makeham type, i.e. of the form

$$\mu_x^{ij} = \gamma_{ij} + 10^{\alpha_{ij}x+\beta_{ij}},\ \alpha_{ij}, \beta_{ij}, \gamma_{ij} \in \mathbb{R}, \tag{4.1}$$

or equivalently, setting $\tilde{\alpha}_{ij} = \ln(10) \cdot \alpha_{ij}$ and $\tilde{\beta}ij = \ln(10) \cdot \beta_{ij}$,

$$\mu_x^{ij} = \gamma_{ij} + e^{\tilde{\alpha}_{ij}x+\tilde{\beta}_{ij}},\ \tilde{\alpha}_{ij}, \tilde{\beta}_{ij}, \gamma_{ij} \in \mathbb{R}, \tag{4.2}$$

in which the transition intensities exponentially increase with the age of the individual. This is a reasonable assumption that is commonly used in other countries for Disability Insurance, see [10].

As a starting point, we shall consider the calibrated transition intensities parameters obtained in [6] for a multiple state model with three dependence states. These were determined by calibrating the parameters used the Danish model for Disability Insurance ([10, p. 100]), using data from 2015 of the RNCCI (Rede Nacional de Cuidados Continuados Integrados, Portuguese for National Network of Integrated Continuous Care). Therefore, the initial transition intensities for the individuals of our population are of the form (4.1), with the parameters for each transition presented in Table 4.2.

Table 4.2: Initial parameters for the Gompertz-Makeham type transition intensities.

State	i	j	γ_{ij}	α_{ij}	β_{ij}
Healthy	1	2	0.00040	0.060	-5.46
	1	3	0.00044	0.052	-5.46
	1	4	0.00050	0.038	-4.12
Mildly Disabled	2	1	0.00040	0.060	-5.46
	2	3	0.00043	0.054	-5.46
	2	4	0.00050	0.037	-4.12
Severely Disabled	3	2	0.00043	0.054	-5.46
	3	4	0.00042	0.054	-4.12

Then, we want to adjust the transition intensities (in this case, by adjusting the Gompertz-Makeham parameters) based on the risk profile of the individuals. A brief statistical analysis demonstrates that the most frequent characteristics (excluding the age) present in our simulated population are the ones in Table 4.3.

Table 4.3: Most frequent demographic characteristics of the simulated population.

Sex	Region	BMI	Smoking	Exercise
Female	0-1,999	Normal weight	Non-smoker	Less than 2 hours

For the individuals with these demographic characteristics, the parameters will remain the ones in Table 4.2, and we shall adjust the parameters for the remaining characteristics depending on if we want to increase or decrease the risk and consequently the transition intensity.

We shall consider a model of multiplicative adjustment factors, i.e. for a transition intensity of the form (4.1), the adjusted parameters are given by

$$\alpha^*_{ij} = a \cdot \alpha_{ij},\ \beta^*_{ij} = b \cdot \beta_{ij},\ \gamma^*_{ij} = c \cdot \gamma_{ij},\ a, b, c \in \mathbb{R}. \tag{4.3}$$

Note that, for the individuals of the population with the characteristics in Table 4.3, the multiplicative adjustment factors are equal to 1, and that the adjustment factors are cumulative, meaning that if a member has more than one characteristic different from the ones in Table 4.3, the final adjustment factor will be the product of all the adjustments.

For example, regarding the intensity for the transition from the Healthy state to the Mildly Disabled state, μ_x^{12}, the adjustment factors according to each explanatory variable that we shall consider are obtained by the following reasoning and presented in Table 4.4:

- Regarding the Sex, we want to aggravate the risk (and thus increase the intensity) for the Male individuals when comparing to Female individuals, particularly for advanced ages;

- Regarding the Region, we wish to increase the risk with the population size, with higher intensities observed for larger populations;

- Regarding the BMI, we want to aggravate the risk for values outside of the "Normal weight" range, with the intensity increasing with the deviation from the normal values;

- Regarding the smoking habits, we wish to increase the intensity for the Smoker individuals when comparing to Non-smoker individuals;

- Regarding the exercise habits, we want to decrease the risk with the increase in frequency of physical activity, with more frequent exercise resulting in a greater reduction in the intensity.

Table 4.4: Adjustment factors according to each variable for the transition intensity μ_x^{12}.

Variable	Class	a	b	c
Sex	Male	1.05	0.9998	1.3
Region	2,000-4,999	1.005	0.9999	1.01
	5,000-9,999	1.001	0.9998	1.02
	10,000-19,999	1.0015	0.9997	1.03
	20,000-99,999	1.002	0.9996	1.04
	100,000 or more	1.004	0.9995	1.05
BMI	Underweight	1.01	0.9999	1.2
	Overweight I	1.01	0.9999	1.2
	Overweight II	1.03	0.9995	1.25
	Obese	1.05	0.999	1.3
Smoking	Smoker	1.055	0.997	1.3
Exercise	2 to less than 3 hours	0.995	0.9997	0.87
	3 to less than 5 hours	0.985	0.9995	0.83
	5 or more hours	0.965	0.999	0.77

We then follow a similar process for the remaining transition intensities.

Finally, since we are simulating the values of the transition intensities and not dealing with observed data (which usually presents a certain level of variability), we should insert a random error to our generated values.

We shall consider a Normal distributed error term, with 0 mean and an appropriate standard deviation. Since we have assumed that the transition intensities are of the Gompertz-Makeham type (increasing exponentially with the age), there is a wide range of possible values depending on the age of the individual. Thus, it might not be reasonable to assume a constant standard deviation and we shall instead consider a random error proportional to the value of the resulting transition intensity. Therefore, the final value of the generated transition intensities is given by

$$\mu_{x\,\mathbf{final}}^{ij} = \mu_x^{ij} + \varepsilon_x^{ij}, \tag{4.4}$$

where $\varepsilon_x^{ij} \sim \mathcal{N}(0, 0.05 \cdot \mu_x^{ij})$.

Finally, we shall simulate the trajectory of each individual in our population, i.e. the sequence of states and the times spent in each state observed over their observation period. Since we are only considering exact ages in our population, we assume that the value of the transition intensities is constant between ages. Thus, we shall consider that the holding times (or sojourn times) in each state and the sequence of states can be obtained from the transition intensities, as seen in [29, p. 165].

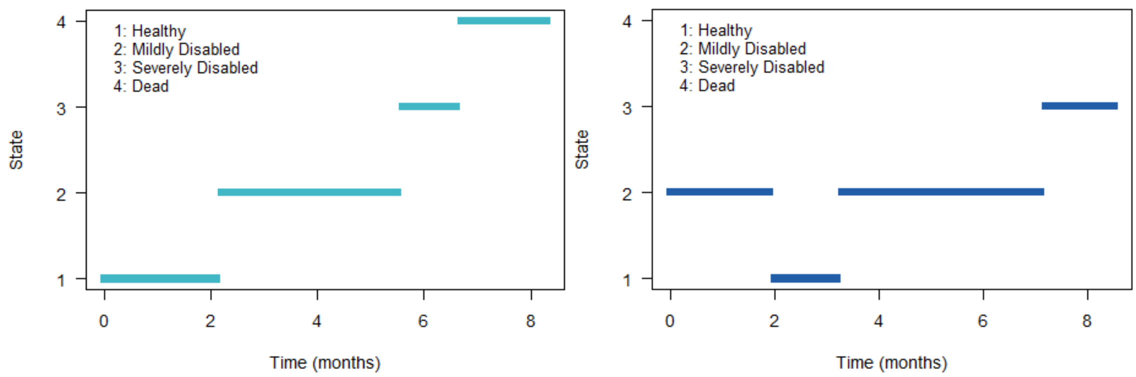
Let us consider an individual, with simulated initial state and exposure, as well as transition intensity matrix (2.31), with the transition intensities depending on their risk profile. Starting from the initial state, the individual will remain in that state for a certain amount of time before they transition to the next state. The sojourn time in each state i is Exponentially distributed, with rate equal to the total intensity of leaving state i, which we shall denote by μ_x^i, with $\mu_x^i = \sum_{j \neq i} \mu_x^{ij}$.

If the generated sojourn time is greater than the observation period, the trajectory ends with the member having made no transitions. Otherwise, the next state j ($j \neq i$) is randomly chosen, with probability μ_x^{ij}/μ_x^i. The trajectory then ends when the absorbing state (Dead) is reached or when the aggregated sojourn times reach the end of the observation period.

We represent a trajectory by a sequence starting with the initial state on the left-hand side and then the time spent in that state, followed by the next state and the time spent in that state, and so on. Below, we present some examples of the obtained trajectories as well as the corresponding plots in Figure 4.8:

- Trajectory 1: {1, 2.19, 2, 3.34, 3, 1.14, 4, 1.71}
  In this case, the individual begins the trajectory in the Healthy state, remaining in that state for $t = 2.19$ months. Then, transitions to the Mildly Disabled state, remaining in that state for $t = 3.34$ months. Then, transitions to the Severely Disabled state, remaining in that state for $t = 1.14$ months. Finally, the individual transitions to the Dead state, remaining in that state, obviously, until the end of the observation period, for $t = 1.71$ months. We note that this individual has an exposure to risk equal to 8.38 months.

- Trajectory 2: {2, 1.98, 1, 1.23, 2, 3.97, 3, 1.41}
  In this case, the individual begins the trajectory in the Mildly Disabled state, remaining in that state for $t = 1.98$ months. Then, makes a recovery to the Healthy state, remaining in that state for $t = 1.23$ months. Then, transitions back to the Mildly Disabled state, remaining in that state for $t = 3.97$ months. Finally, the individual transitions to the Severely Disabled state, remaining in that state for $t = 1.41$ months, and where the trajectory ends. We note that this individual has an exposure to risk equal to 8.59 months.



(a) Plot of Trajectory 1 (b) Plot of Trajectory 2

Figure 4.8: Examples of simulated trajectories.

Having simulated the trajectories for the 100,000 individuals in our population, with observation periods between 6 months and 12 months, we have determined the number of transitions of each type and the times spent in each state observed for each individual.

Therefore, we now have the required data for the estimation and graduation of the transition intensities.

4.2 Results and Discussion

Finally, applying the GLM framework presented in Section 3.3.2 and considering as explanatory variables the risk factors presented in Section 4.1.1, we shall obtain graduated estimates of the transition intensities for the Disability multiple state model in Section 2.2.2. Furthermore, our goal is to determine a function for the transition intensities for the different risk profiles in our population.

4.2.1 GLM for Transition Intensities

Firstly, following the graduation process in Section 3.3.2, we shall organise the available data in a grid of units u, in which the categorical variables are the following risk factors and the corresponding levels:

- **Age**, with 12 levels (age groups):
  $[0, 45)$, $[45, 50)$, $[50, 55)$, $[55, 60)$, $[60, 65)$, $[65, 70)$, $[70, 75)$, $[75, 80)$, $[80, 85)$, $[85, 90)$, $[90, 95)$, $[95, 101)$;

- **Sex**, with 2 levels:
  Male and Female;

- **Region**, with 6 levels:
  0-1,999, 2,000-4,999, 5,000-9,999, 10,000-19,999, 20,000-99,999, and 100,000 or more;

- **BMI**, with 5 levels:
  Underweight, Normal weight, Overweight I, Overweight II, and Obese;

- **Smoking habits**, with 2 levels:
  Smoker and Non-smoker;

- **Exercise habits**, with 4 levels:
  Less than 2 hours, 2 to less than 3 hours, 3 to less than 5 hours, and 5 or more hours.

Note that we have considered a larger age group for the initial ages, $[0, 45)$, instead of smaller 5-year age groups as for the remaining ages. This was decided after having performed multiple comparisons of means between the smaller age groups in the fitted models and verifying that the means are not significantly different from each other. Moreover, we observed that the smaller age groups are not statistically significant in none of the fitted models, but become significant by being joined in the larger age group.

Then, a brief analysis demonstrates that the most frequent levels in the grid of units u for each explanatory variable are the ones in Table 4.5. This set of levels constitutes the intercept of the fitted models and we shall call it the *base level*.

Table 4.5: Base level of the fitted models.

Age Group	Sex	Region	BMI	Smoking	Exercise
$[0, 45)$	Male	0-1,999	Normal weight	Non-smoker	Less than 2 hours

4.2.1.1 Transition Intensities from Healthy to Mildly Disabled

Let us start by graduating the transition intensities from the Healthy state to the Mildly Disabled state, μ_u^{12}. For each cell u of the grid, the required data consists of the observed number of transitions from the Healthy state to the Mildly Disabled state, i_u, accruing from corresponding exposure e_u (total waiting time in the Healthy state).

Then, to proceed with the graduation process, by assuming that i_u are independent overdispersed Poisson variables, we fit a quasi-Poisson regression to the data, using a log-link function and with $\log(e_u)$ offsetting the model. The covariate structure of the model should be the one that best adjusts the observed data and thus we perform tests to the following covariate structures:

- **Model 1:** Age + Age:Sex + Age:Region + Age:BMI + Age:Smoking + Age:Exercise
- **Model 2:** Age + Age:Sex + Region + BMI + Smoking + Exercise
- **Model 3:** Age + Sex + Region + BMI + Smoking + Exercise
- **Model 4:** Age + Sex + BMI + Smoking + Exercise

where ':' symbolises an interaction between the two involved variables.

The obtained results are presented in Table 4.6. The residual deviance is calculated by (3.34), the null deviance is the deviance of the intercept-only model and the next column states the degrees of freedom of the models. The last column represents the Akaike Information Criterion (AIC) for the corresponding Poisson regression models (since the AIC is not calculated for quasi-Poisson models).

Table 4.6: Summary results of the tested GLMs.

Model	Residual Deviance	Null Deviance	Degrees of Freedom	AIC
Model 1	3647.236	45308.74	3371	10758.22
Model 2	3797.122	45308.74	3514	10622.11
Model 3	3821.414	45308.74	3525	10624.40
Model 4	3826.060	45308.74	3530	10519.04

In summary, the process to choose the most appropriate covariate structure for the regression model was the following (bearing in mind that all the tested models are quasi-Poisson regression models with a log-link function and the exposure as the offset):

1. We started with a covariate structure in which the Age is in interaction with all the remaining variables. Since the majority of the parameters of this model were

not significant, we then tested a covariate structure in which the Age is only in interaction with the Sex variable. Even though the deviance of the second model is higher, a likelihood-ratio test between the two models as well as the value of the AIC, confirms that the added complexity of the first model does not significantly improve the goodness of fit;

2. Then, since not all parameters of the second model were statistically significant, we tested a covariate structure with no interactions between the variable Age and the remaining variables. Again, although the deviance of the third model is higher, the likelihood-ratio test confirms that the difference between the two models is not significant and that the simpler model is sufficient to explain the data;

3. Next, since the effect of the variable Region was not statistically significant in the third model, we tested the removal of that variable for the final model. The likelihood-ratio test between the last two models, as well as the lower AIC value, confirms that we may remove this variable from the model;

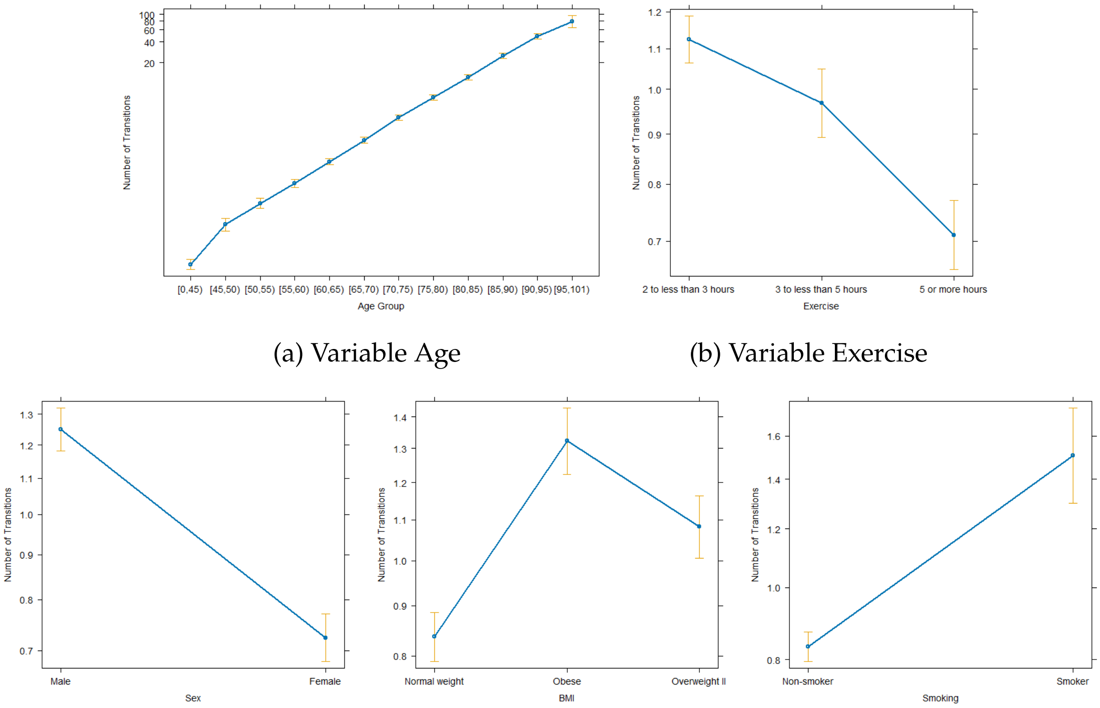
4. Finally, since some levels of the remaining explanatory variables are not statistically significant, we shall join the statistically similar levels in each variable. For this, we perform multiple comparisons of means, using Tukey's test, between pairs of the levels in each covariate. Then, we join the levels for which the means are not significantly different, until all variables and levels are statistically significant in the fitted model. In this case, we joined the levels "Underweight", "Normal weight" and "Overweight I" of the variable BMI, and the levels "Less than 2 hours" and "2 to less than 3 hours" for the variable Exercise habits.

In Table 4.7, we present the summary of the final GLM for the transition intensities μ_u^{12}, which includes the estimates of the coefficients of the model (obtained by maximum likelihood estimation), along with their standard error and statistical significance. Furthermore, we obtain the dispersion parameter of the quasi-Poisson variable ($\phi = 2.082165$) and the AIC of the corresponding Poisson model (10627.89), as well as the residual deviance (3840.9) and the degrees of freedom (3533) of the final model.

We note that all levels of the explanatory variables are statistically significant and we represent the effects of each covariate on the response variable in the final model in Figure 4.9. As expected, we observe that the number of transitions from the Healthy state (State 1) to the Mildly Disabled state (State 2) highly increases with the Age of the individuals. There is also an increase in the response variable between Smokers and Non-smokers as well as between Females and Males. Moreover, a decrease is observed with the increase in time spent performing physical activity as well as for the individuals with "Normal weight" when compared with the remaining BMI classes.

Table 4.7: Summary of the final GLM for the transition intensities μ_u^{12}.

Variable	Estimate	Std. Error	t-value	Pr(> |t|)
(Intercept)	-6.87333	0.08792	-78.181	$< 2 \times 10^{-16}$
Age [45,50)	1.32306	0.13766	9.611	$< 2 \times 10^{-16}$
Age [50,55)	2.02961	0.11867	17.104	$< 2 \times 10^{-16}$
Age [55,60)	2.69289	0.10464	25.735	$< 2 \times 10^{-16}$
Age [60,65)	3.41294	0.09685	35.240	$< 2 \times 10^{-16}$
Age [65,70)	4.13175	0.09738	42.427	$< 2 \times 10^{-16}$
Age [70,75)	4.87628	0.09334	52.243	$< 2 \times 10^{-16}$
Age [75,80)	5.55835	0.09352	59.432	$< 2 \times 10^{-16}$
Age [80,85)	6.22471	0.09291	66.999	$< 2 \times 10^{-16}$
Age [85,90)	6.93904	0.09352	74.201	$< 2 \times 10^{-16}$
Age [90,95)	7.58535	0.09616	78.886	$< 2 \times 10^{-16}$
Age [95,101)	8.07791	0.13251	60.959	$< 2 \times 10^{-16}$
Sex Female	-0.54602	0.03026	-18.041	$< 2 \times 10^{-16}$
BMI Obese	0.45861	0.03906	11.742	$< 2 \times 10^{-16}$
BMI Overweight II	0.25769	0.03602	7.155	1.01×10^{-12}
Smoking Smoker	0.59303	0.07884	7.522	6.81×10^{-14}
Exercise 3 to less than 5 hours	-0.14942	0.03980	-3.755	0.000176
Exercise 5 or more hours	-0.45964	0.03991	-11.518	$< 2 \times 10^{-16}$



(a) Variable Age

(b) Variable Exercise

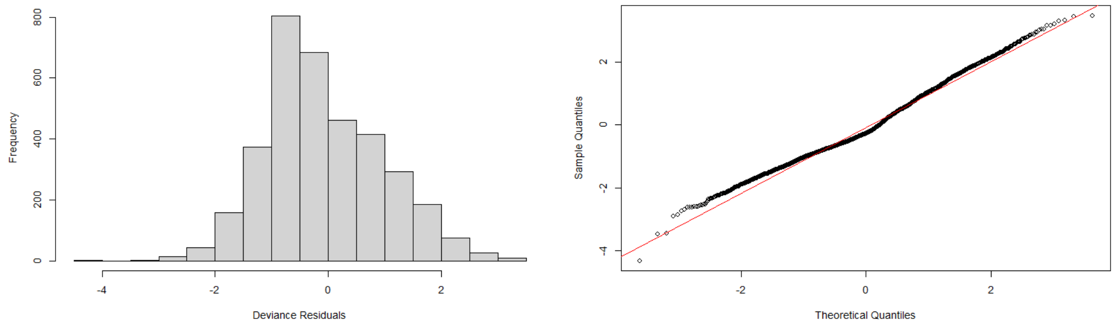
(c) Variable Sex

(d) Variable BMI

(e) Variable Smoking

Figure 4.9: Effects of the explanatory variables on the response variable in the final GLM for the transition intensities μ_u^{12}.

To test the goodness of fit of the final model, we perform some residual analysis and present the histogram and Q-Q plot of the deviance residuals in Figure 4.10.



(a) Histogram of deviance residuals (b) Q-Q plot of deviance residuals

Figure 4.10: Deviance residual plots of the final GLM for the transition intensities μ_u^{12}.

In the histogram, we observe that the deviance residuals are centred around 0, with a large concentration between −1 and 0, and the remaining values focused between −2 and 2, which indicates a reasonable fit to the observed data. The fact that the predicted values of the model are always positive, even though an individual might not make any transitions (the observed value of the response variable is 0), explains the concentration of the deviance residuals between −1 and 0. Additionally, a deeper analysis shows that the higher residuals are more frequent in the values corresponding to the age groups of the more advanced ages, where the values of the response variable are expected to have a higher variability (due to the observed overdispersion).

Then, we observe the Q-Q plot to check whether the deviance residuals of the final model are approximately Normal distributed. The slight deviation of some residuals from the straight line, combined with the higher frequency of negative residuals observed in the histogram, might indicate that the deviance residuals do not follow a Normal distribution. However, we note that for GLMs with Poisson response variables, the deviance residuals do not necessarily have to be Normally distributed for the model to present a satisfactory fit, see [19, p. 77].

Finally, when performing the Chi-Square test for goodness of fit, we reject the null hypothesis, for a significance level of $\alpha = 0.05$, of the values of the model representing the actual observed values. This might be explained by the fact that the model does not adjust well to the advanced ages due to the observed variability in the response variable, as mentioned above.

4.2.1.2 Remaining Transition Intensities

We then follow the analogous process for the remaining transition intensities, considering the corresponding definitions for the response variables i_u and the exposures e_u. As before, the fitted model for each intensity is a quasi-Poisson regression, with log-link function and $\log(e_u)$ as an offset, in which all covariates and the respective levels are statistically significant.

The obtained results of the final GLMs for all transition intensities are presented in Table 4.8 and the summary of the final models for each transition intensity, including the significant explanatory variables and levels, can be found in Appendix A.

Table 4.8: Summary results of the final GLMs for the transition intensities.

Intensity	Residual Deviance	Null Deviance	Degrees of Freedom	AIC
μ_u^{12}	3840.9062	45 308.738	3533	10627.890
μ_u^{13}	3119.0462	10 783.795	3537	6473.392
μ_u^{14}	3747.0100	16 849.595	3537	9725.720
μ_u^{21}	2609.5969	23 481.200	3341	7114.899
μ_u^{23}	3059.6136	15 236.161	3345	7171.716
μ_u^{24}	3196.1659	11 475.233	3342	7795.169
μ_u^{32}	996.8889	1702.032	2814	1467.997
μ_u^{34}	3329.4120	39 033.051	2798	10096.894

The various dimension of the values of the residual deviance as well as the different number of degrees of freedom are explained by the fact that the models consider different explanatory variables, with different levels being joined in each model.

Regarding the covariate structures and explanatory variables, we observe that the variables Age and Sex are statistically significant in all the models, as well as the variables BMI and Smoking (except for the transition intensity between Severely Disabled and Mildly Disabled). We also note that the variable Region is only significant for the transitions to the Dead state and that the variable Exercise is not significant for transitions from the Mildly Disabled state (except to the Healthy state).

Finally, we are able to generate predicted values of the response variables, based on the fitted GLMs, for all the 100,000 individuals in our population. Following (3.30), by dividing the predicted number of transitions by the total exposure to risk, we obtain the value of the transition intensities for each individual, as predicted by the fitted models.

In summary, starting with the trajectories of a given population and following the GLM framework, we are able to obtain values of the transition intensities for each individual based on their demographic characteristics. Using the predicted values from the GLMs, we shall determine a function for the transition intensities for the different risk profiles in our population.

4.2.2 Function for Transition Intensities Depending on Risk Factors

The final goal of this work is to present a form for the transition intensities, as a function of the age of the individuals, in which the parameters depend on the risk profile of the selected group of individuals. For this, we shall again assume that we have transition intensities of the Gompertz-Makeham type, i.e. of the form

$$\mu_x^{ij} = \gamma_{ij} + 10^{\alpha_{ij}x+\beta_{ij}},\ \alpha_{ij}, \beta_{ij}, \gamma_{ij} \in \mathbb{R}, \tag{4.5}$$

and we shall fit the corresponding non-linear least squares (NLS) regression model to the predicted values from the GLMs. Using as starting values the initial parameters in Table 4.2, the parameters α_{ij}, β_{ij}, γ_{ij} in the function are estimated by minimising the residual sum of squares.

Let us again start with the transition intensities from the Healthy state to the Mildly Disabled state, μ_x^{12}. First, we shall determine a form for μ_x^{12}, as a function of age, for the whole population.

Fitting a NLS regression model to the predicted values (from the GLM) of the transition intensities, with function of the form (4.5) and starting parameters $\alpha_{12} = 0.06$, $\beta_{12} = -5.46$, $\gamma_{12} = 0.0004$, we obtain the function

$$\mu^{12}_{x\,\textbf{Population}} = -0.001986 + 10^{0.056674x-4.996130} \tag{4.6}$$

resulting from the model represented in Table 4.9. The last two columns of Table 4.9 include the lower and upper bound, respectively, of a confidence interval, with a confidence level of 95%, for the estimated parameters.

Table 4.9: Summary of the NLS model for the function for the transition intensity μ_x^{12} for the whole population.

Parameter	**Estimate**	**Std. Error**	***t*-value**	**Pr(> |*t*|)**	**Confidence Interval**	
					Lower Bound	Upper Bound
α_{12}	0.056674	0.000135	420.548	$< 2 \times 10^{-16}$	0.056410	0.056938
β_{12}	-4.996130	0.012301	-406.170	$< 2 \times 10^{-16}$	-5.020239	-4.972021
γ_{12}	-0.001986	0.000487	-4.076	$< 4.59 \times 10^{-5}$	-0.002941	-0.001031

We note that value of the parameter γ_{12} is negative, probably caused by the model adjusting the exponential term to the higher values in the advanced ages, resulting in a negative constant term. However, this would result in negative values of μ_x^{12} for members under around 40 years old, which is not possible. To account for this fact, we could fit different NLS regression models, with functions of the Gompertz-Makeham type, for the members under and over 40 years old. This is supported by the fact that the individuals between 0 and 45 years old were grouped for the construction of the GLM and thus have similar predicted values.

We shall now compare the obtained parameters with the ones of the transition intensities used to simulate the trajectories in Section 4.1.2. As a reminder, the latter were generated by adjusting the starting parameters in Table 4.2 with the factors in Table 4.4 depending on the characteristics of each individual. For this, we compute the mean estimate of each parameter for the whole population, as well as a confidence interval, with a confidence level of 95%, and present the results in Table 4.10.

Table 4.10: Mean estimates and 95% confidence interval for the parameters of the simulated transition intensities μ_x^{12}.

Parameter	Estimate	Confidence Interval Lower Bound	Confidence Interval Upper Bound
$\overline{\alpha_{12}}$	0.062116	0.062130	0.062144
$\overline{\beta_{12}}$	−5.458460	−5.458428	−5.458395
$\overline{\gamma_{12}}$	4.930498×10^{-4}	4.936969×10^{-4}	4.943440×10^{-4}

We observe that the confidence intervals for the estimated parameters of the NLS model do not intersect with the confidence intervals for the mean of the parameters used in the simulation. The differences are likely caused by the high variability in the transition intensities values at the advanced ages. As previously mentioned, this leads to the GLM not adjusting well to the advanced ages, as well as to the negative parameter γ_{12} in the estimated function. Nevertheless, the plots of both functions are quite similar, as seen in Figure 4.11.

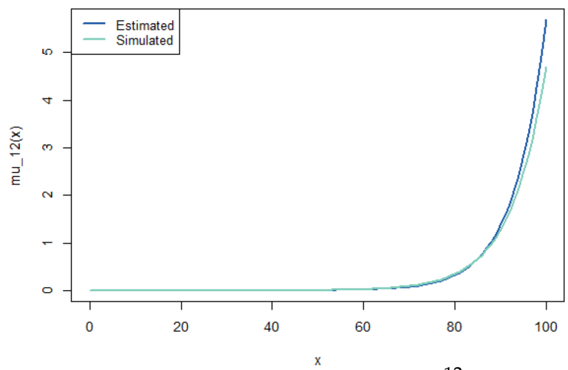


Figure 4.11: Function for the transition intensity μ_x^{12} for the whole population

Following the analogous approach for the remaining transition intensities, we obtain a function of the form (4.5) for each transition intensity. In Figure 4.12, we compare the transition intensity from the Healthy state to the Mildly Disabled state, μ_x^{12}, to the transition intensity from the Healthy state to the Severely Disabled state, μ_x^{13}. We note that, for the individuals over 75 years old, the value of μ_x^{12} increases at a higher rate than the value of μ_x^{13}, which is consistent with the observed in [6].

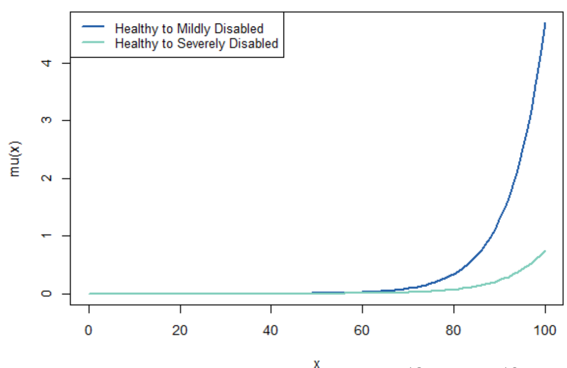


Figure 4.12: Functions for the transition intensities μ_x^{12} and μ_x^{13} for the whole population

Furthermore, for a given risk factor, if we apply the same method to the group of individuals corresponding to each level, we obtain a form for the transition intensities, as a function of age, specific for that level.

For example, let us determine the functions for the transition intensity μ_x^{12} specific for the Male and the Female individuals. We start by splitting the individuals by Sex and fitting a NLS regression model to the predicted transition intensities values (from the GLM) of each group, with function of the form (4.5) and starting parameters $\alpha_{12} = 0.06$, $\beta_{12} = -5.46$, $\gamma_{12} = 0.0004$. We obtain the functions

$$\mu^{12}_{x\,\textbf{Male}} = -0.003155 + 10^{0.056806x-4.881030} \tag{4.7}$$

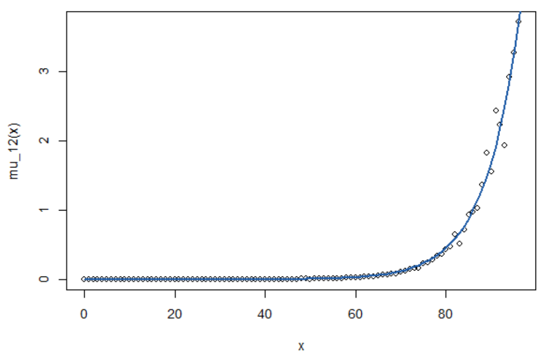
$$\mu^{12}_{x\,\textbf{Female}} = -0.001589 + 10^{0.056815x-5.118482} \tag{4.8}$$

resulting from the models represented in Table 4.11.

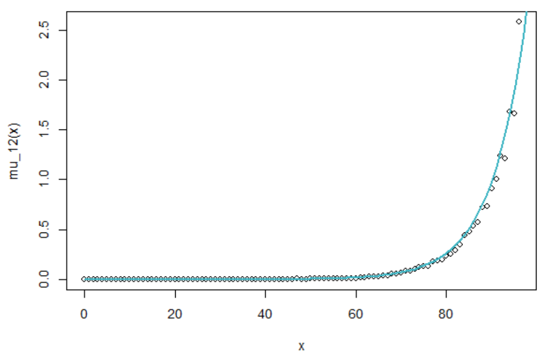
Table 4.11: Summary of the NLS models for the function for the transition intensities μ_x^{12} differentiated by Sex.

Sex	Parameter	Estimate	Std. Error	t-value	Pr(> |t|)
	α_{12}	0.056806	0.000148	384.534	$< 2 \times 10^{-16}$
Male	β_{12}	-4.881030	0.013471	-362.328	$< 2 \times 10^{-16}$
	γ_{12}	-0.003155	0.000661	-4.772	1.83×10^{-6}
	α_{12}	0.056815	0.000137	413.710	$< 2 \times 10^{-16}$
Female	β_{12}	-5.118482	0.012547	-407.933	$< 2 \times 10^{-16}$
	γ_{12}	-0.001589	0.000409	-3.884	1.03×10^{-4}

Additionally, in Figure 4.13 we observe the functions (4.7) and (4.8), respectively, as well as the raw estimates of the transition intensities for each age. As a reminder, the crude estimates can be calculated, in this case, by dividing the number of transitions between the Healthy state and the Mildly Disabled state by the waiting time in the Healthy state, as seen in (3.9). We note that the obtained functions demonstrate a satisfactory fit to the raw estimates for each year of age.



(a) Function for Male individuals



(b) Function for Female individuals

Figure 4.13: Functions for the transition intensities μ_x^{12} and corresponding raw estimates differentiated by Sex.

Proceeding in the same manner for each of the significant risk factors in the final modelling of the transition intensities, we obtain transition intensities functions specific for all the available characteristics in our population.

The obtained functions for the transition intensity μ_x^{12}, for the levels of each risk factor, are listed below and represented in in Figure 4.14. We note that the variable Region is not considered since it is not an explanatory variable in the GLM for this transition intensity, and thus all levels of that variable would have a similar function.

- **Sex**

$$\mu^{12}_{x\,\textbf{Male}} = -0.003155 + 10^{0.056806x - 4.881030}$$
$$\mu^{12}_{x\,\textbf{Female}} = -0.001589 + 10^{0.056815x - 5.118482}$$

  We observe in Figure 4.14 that, as expected, the function for the transition intensity μ_x^{12} is higher for the Male individuals than for the Female individuals, especially in the advanced ages;

- **BMI**

$$\mu^{12}_{x\,\textbf{Obese}} = -0.005716 + 10^{0.055494x - 4.758877}$$
$$\mu^{12}_{x\,\textbf{Overweight}} = -0.001821 + 10^{0.057277x - 5.001222}$$
$$\mu^{12}_{x\,\textbf{Normal weight}} = -0.001811 + 10^{0.056449x - 5.045657}$$

  We note that the “Normal weight" level also includes the individuals in the levels “Underweight" and “Overweight I", since these levels were joined in the construction of the final GLM for this transition intensity, and thus would have a similar function. As expected, the function for the transition intensity μ_x^{12} is higher for the individuals in the “Obese" level, followed by individuals in the “Overweight II" level, and the individuals in the “Normal weight" range present lower values;

- **Smoking habits**

$$\mu^{12}_{x\,\textbf{Smoker}} = -0.002659 + 10^{0.054392x - 4.503004}$$
$$\mu^{12}_{x\,\textbf{Non-smoker}} = -0.002292 + 10^{0.056535x - 4.994203}$$

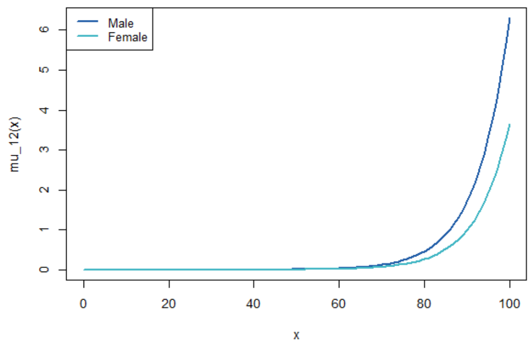
  As expected, the function for the transition intensity μ_x^{12} is higher for the Smoker individuals when comparing to the Non-smoker individuals, with the difference being more significant in the advanced ages;

- **Exercise habits**

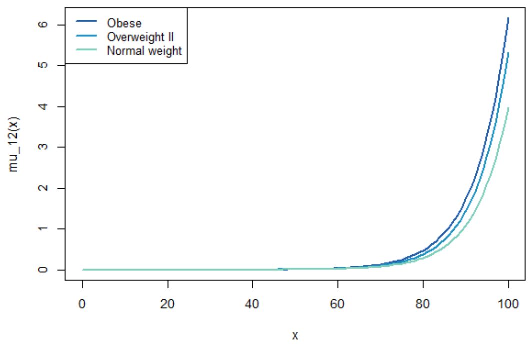
$$\mu^{12}_{x\,\textbf{Less than 3 hours}} = -0.002389 + 10^{0.056392x - 4.943397}$$
$$\mu^{12}_{x\,\textbf{3 to less than 5 hours}} = -0.001479 + 10^{0.057796x - 5.085398}$$
$$\mu^{12}_{x\,\textbf{5 or more hours}} = -0.001668 + 10^{0.055266x - 5.021899}$$

We note that the “Less than 3 hours" level consists of the individuals in the levels “Less than 2 hours" and “2 to less than 3 hours", since these levels were joined in the construction of the final GLM for this transition intensity, and thus would have a similar function.

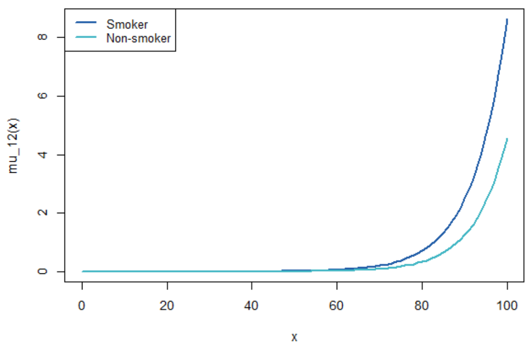
The values of the function for the transition intensity μ_x^{12} decrease with the increase in frequency of physical activity, as one would expect, with the individuals in the level “5 or more hours" having lower values.



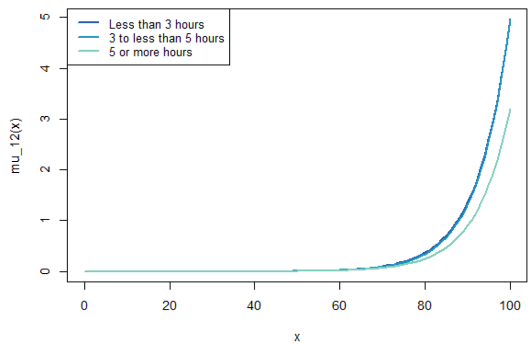
(a) Variable Sex



(b) Variable BMI



(c) Variable Smoking



(d) Variable Exercise

Figure 4.14: Functions for the transition intensities μ_x^{12} differentiated by each risk factor.

Finally, by applying the same method to a group of individuals with a desired set of characteristics (in more than one risk factor), we can also obtain a form for the transition intensities, as a function of age, specific for that risk profile.

For example, let us consider the two risk profiles presented in Table 4.12: a “Low risk" profile, with the levels of each explanatory variable that we have seen result in a lower value of the transition intensities, and a “High risk" profile, with the characteristics that lead to higher values.

Table 4.12: Example risk profiles corresponding to low and high risk.

Risk Profile	Sex	BMI	Smoking	Exercise
Low risk	Female	Normal weight	Non-smoker	5 or more hours
High risk	Male	Obese	Smoker	Less than 3 hours

Following the same process as before, we fit a NLS regression model to the predicted transition intensities (from the GLM) of each group (where a group is formed by the individuals from the population that present the characteristics of each risk profile), and obtain the following functions for the transition intensity μ_x^{12} for the two selected risk profiles:

$$\mu^{12}_{x\,\textbf{Low risk}} = -0.001125 + 10^{0.056449x - 5.324668} \tag{4.9}$$

$$\mu^{12}_{x\,\textbf{High risk}} = -0.008275 + 10^{0.055920x - 4.364620} \tag{4.10}$$

In Figure 4.15, we have represented functions (4.9) and (4.10) and observe that the function for the transition intensity μ_x^{12} is higher for the high risk profile, especially in the advanced ages, as expected.

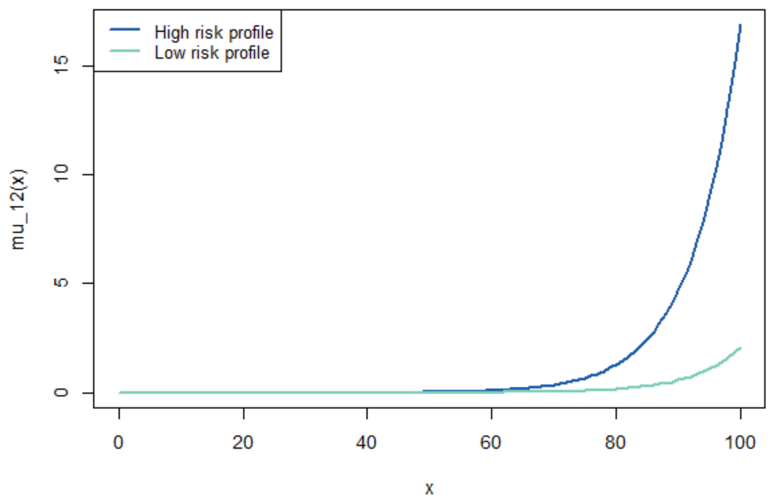


Figure 4.15: Functions for the transition intensities μ_x^{12} differentiated by risk profile.

In summary, by considering different risk factors and the respective levels, and grouping the individuals according to the desired risk profile, we are be able to obtain each transition intensity as functions of the age of the individuals, depending on their characteristics. This process allows us to differentiate individuals based on their risk profile when determining their transition intensities and, consequently, the transition probabilities that are used in the calculation of premiums and reserves in the scope of Disability or LTC Insurance.

5

Conclusions and Future Work

Disability Insurance and, in particular, Long-Term Care Insurance have recently become research areas of interest due to the accelerated population ageing, and consequent increase of the elderly and dependent population, observed in the developed countries. In Portugal, although the same social and economic problems are present, there are no private insurance companies offering LTC Insurance products yet, mostly due to the complexities involved with modelling the underlying risks of this type of insurance, as well as to the lack of the data that allows for an adequate risk measurement.

In this work, we developed a robust study of the existing actuarial models used to describe Disability or LTC Insurance benefits, i.e. Multiple State Models. We studied the implementation of these models using continuous-time Markov chains, while focusing on the estimation and graduation of the transition intensities (which we regarded as the models' parameters, following the TI-approach). Additionally, we presented an original practical application of the studied methods, using our own Disability multiple state model and a simulated dataset based on the Portuguese population, since the required data was not available. The use of a GLM framework for the graduation of the transition intensities allowed to incorporate the demographic characteristics of the population into the modelling structure, and thus differentiate individuals by their risk profile in the calculations required for insurance purposes.

In Chapter 2, we presented the definitions and the underlying properties of continuous-time Markov chains, which served as the basis for the implementation of the studied actuarial models. In particular, the Kolmogorov Forward Equations establish an important relation between the transition probabilities and the transitions intensities, since they provide an approach to calculate the transition probabilities given the intensities. Furthermore, we studied the formulation of Multiple State Models through the use of continuous-time Markov chains, including the required assumptions. Additionally, we presented our own Disability multiple state model in Section 2.2.2, which considers two disability states to account for different degrees of disability, as well as the corresponding transition matrices and Kolmogorov Forward Equations.

In Chapter 3, we studied the Transition Intensities Approach, introduced in [30], which regards the transition intensities as the parameters of Multiple State Models, thus preserving their stochastic nature. Considering the assumption, proposed by the TI-approach, that the transition intensities are constant between exact ages, we developed a maximum likelihood method for the estimation of the transition intensities of our Disability multiple state model. We concluded that the raw estimates, for each year of age, of the transition intensity between two states can be obtained by dividing the number of transitions observed by the waiting time in the first state.

Although the transition intensity function is expected to have a smooth evolution across all ages, the proposed assumption and estimation method builds a piece-wise constant function between ages, and thus has some undesirable fluctuations. As a result, graduation techniques must be applied in order to obtain more realistic and regular values as well as the expected degree of smoothness in the transition intensities estimates. Therefore, in Chapter 3, we also presented a parametric approach based on Generalised Linear Models, first applied to Multiple State Models in [28], to graduate the obtained raw estimates of the transition intensities. The fitted GLMs consist of overdispersed Poisson response variables and predictors of the Gompertz-Makeham type, and use the logarithm as the link function as well as the exposure to risk as an offset of the model.

Furthermore, due to the versatility of the GLM framework, we were able to generalise the approach presented in [28] to our Disability multiple state model and extend the explanatory variables of the GLM to other demographic factors besides solely the individuals' age. This approach allowed for the development of a practical application in which the heterogeneity of a population and the effect of different observable risk factors were included in the determination of functions for the transition intensities. Finally, in Chapter 4, we develop a practical application of the methods regarding the estimation and graduation of the transition intensities for the Disability model introduced in Section 2.2.2.

Since the required data to apply the studied graduation techniques was not available, we resorted to data simulation to produce a dataset of a population of 100,000 individuals, that mimics the demographic characteristics of the Portuguese population. The risk factors (or explanatory variables) considered for the modelling of the transition intensities were: Age, Sex, Region, Body Mass Index, Smoking habits, and Exercise habits, each categorised in appropriate levels. The characteristics for each individual were sampled according to the latest available statistical data on the Portuguese population (namely, health statistics and census).

Furthermore, we simulated transition intensities for each individual in our population, to be used for the simulation of their trajectory through the states of the Disability multiple state model. For this, we assumed that the transition intensities were of the Gompertz-Makeham type and that the initial parameters were the ones of the calibrated transitions intensities obtained in [6]. Then, we applied multiplicative adjustment factors to the initial parameters based on the demographic characteristics of each individual, thus obtaining simulated transition intensity values for each risk profile. Using the transition intensity matrix of our Disability model, with the generated transition intensities, as well as simulated initial states and observation periods, we simulated trajectories for the individuals in our population. Therefore, we generated a dataset with 100,000 individuals containing their demographic characteristics, the number of transitions of each type and the waiting time in each state of the Disability model, which is the necessary data for the studied estimation and graduation methods.

Finally, following the studied GLM framework, for each of the available transition intensities, we fitted a quasi-Poisson regression model to the corresponding number of transitions observed, in which the explanatory variables included the considered risk factors (and not only the age of the individuals), using a log-link function and the corresponding exposure to offset the model. Then, the covariate structure of the GLM was adjusted until all the explanatory variables and the respective levels were statistically significant. Although we obtained a satisfactory fit to the observed values, we noted that the models have some difficulty in adjusting to the more advanced ages due to the variability in the response variable. Regarding the obtained covariate structures and explanatory variables, we noted that the variables Age and Sex are statistically significant in all the models, as well as the variables BMI and Smoking (except for the transition intensity between Severely Disabled and Mildly Disabled, which is the transition intensity with fewer explanatory variables). We also observed that the variable Region is only significant for the transitions to the Dead state and that the variable Exercise is not significant for transitions from the Mildly Disabled state (except to the Healthy state).

Based on the fitted GLMs, we generated predicted values of the transition intensities for the 100,000 individuals in our population depending on their demographic characteristics. The final goal was to present a form for the transition intensities, as a function of the age of the individuals, for the different risk profiles in our population. For this, we assumed again that the function of the transition intensities we wanted to obtain was of the Gompertz-Makeham type. Then, we fitted a non-linear least squares regression model to the predicted values (from the GLMs) of the group of individuals with the desired risk profile. That way, the parameters of the NLS models depend on the characteristics of the selected group of individuals. Regarding the goodness of fit of the NLS models, we observed again that the models have some difficulty in adjusting to the advanced ages, but nevertheless the obtained functions demonstrate a satisfactory fit to the crude estimates for each year of age.

For a given risk factor, we applied this method to the group of individuals corresponding to each level, and obtained a form for the transition intensities, as a function of age, specific for that level. Proceeding in the same manner for all the significant risk factors in the final modelling of the transition intensities, we obtained transition intensities functions specific for all the available characteristics in our population. Regarding the transition intensity from the Healthy to the Mildly Disabled state, we observed that, as expected, the obtained function is higher for the Male individuals than for the Female individuals, as well as for the Smoker individuals when compared with the Non-Smoker individuals, especially in the advanced ages. Moreover, the function is higher for the individuals in the “Obese" and “Overweight II” levels when compared to the remaining BMI classes, and that the values of the transition intensity decrease with the increase in frequency of physical activity, as one would expect.

At last, by applying the same method to a group of individuals with a desired set of characteristics (in more than one risk factor), we can also obtain a form for the transition intensities, as a function of age, specific for that risk profile. We considered two risk profiles, with distinct degrees of risk, and we fitted a NLS regression model to the predicted transition intensities (from the GLMs) of each group, thus obtaining a function for the transition intensities, particular for each risk profile. As expected, the obtained transition intensity function is higher for the high risk profile, especially in the advanced ages. We thus concluded that this process allows us to differentiate individuals based on their risk profile when determining a function for the transition intensities.

After the work developed in this thesis, a natural continuation would be, given the obtained transition intensity functions, to derive the corresponding transition probabilities, using the Kolmogorov Forward Equations deduced in (2.32). Then, we would have all the tools required for the computation of the actuarial values used for modelling insurance products, namely in the calculations of premiums and reserves.

Since the proposed process differentiates individuals based on their risk profile when determining their transition intensities (and consequently their premiums), we would be able to build a reliable pricing structure for an insurance product based on the studied Disability multiple state model.

Furthermore, an interesting topic of investigation would obviously be the application of the studied models and proposed methods to real data for the Portuguese population. We would be able to test the accuracy of the data simulation, not only of the population characteristics but also of the trajectories, as well as search for other relevant risk factors to be included in the analysis. Moreover, we could assess how the proposed models adjust to the real data and calibrate the covariate structure of the GLMs if required. A final goal would be the design of a Disability or LTC Insurance product for the Portuguese insurance industry, with an accurate modelling of the risks and pricing structure designed accordingly.

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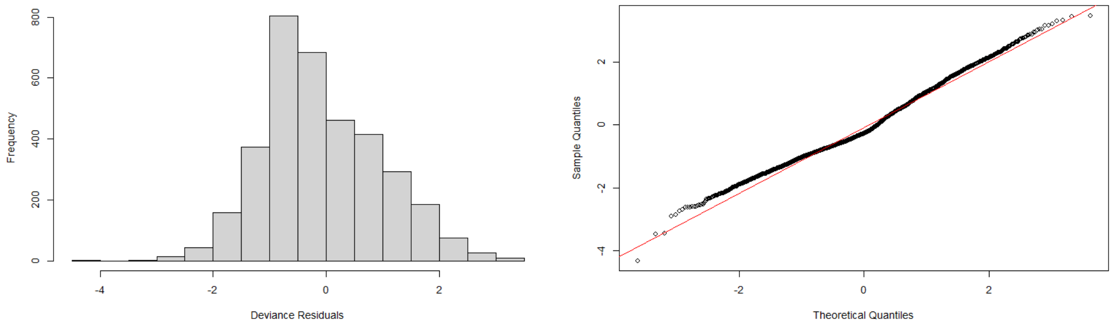
A

Complete Results

A.1 GLM for Transition Intensities from Healthy to Mildly Disabled

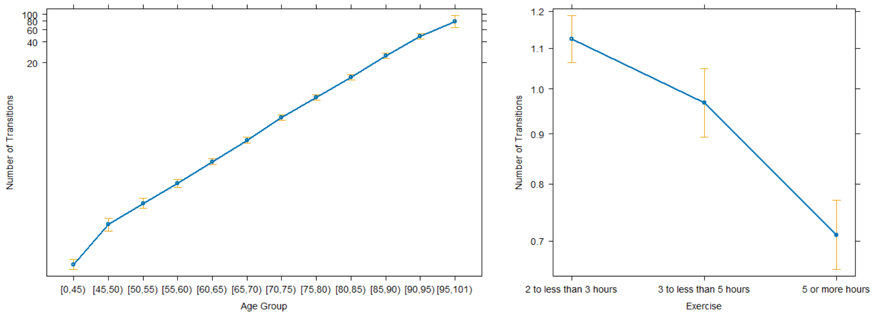
Table A.1: Summary of the final GLM for the transition intensities μ_u^{12}.

Variable	Estimate	Std. Error	t-value	Pr(> $\|t\|$)
(Intercept)	-6.87333	0.08792	-78.181	$< 2 \times 10^{-16}$
Age [45,50)	1.32306	0.13766	9.611	$< 2 \times 10^{-16}$
Age [50,55)	2.02961	0.11867	17.104	$< 2 \times 10^{-16}$
Age [55,60)	2.69289	0.10464	25.735	$< 2 \times 10^{-16}$
Age [60,65)	3.41294	0.09685	35.240	$< 2 \times 10^{-16}$
Age [65,70)	4.13175	0.09738	42.427	$< 2 \times 10^{-16}$
Age [70,75)	4.87628	0.09334	52.243	$< 2 \times 10^{-16}$
Age [75,80)	5.55835	0.09352	59.432	$< 2 \times 10^{-16}$
Age [80,85)	6.22471	0.09291	66.999	$< 2 \times 10^{-16}$
Age [85,90)	6.93904	0.09352	74.201	$< 2 \times 10^{-16}$
Age [90,95)	7.58535	0.09616	78.886	$< 2 \times 10^{-16}$
Age [95,101)	8.07791	0.13251	60.959	$< 2 \times 10^{-16}$
Sex Female	-0.54602	0.03026	-18.041	$< 2 \times 10^{-16}$
BMI Obese	0.45861	0.03906	11.742	$< 2 \times 10^{-16}$
BMI Overweight II	0.25769	0.03602	7.155	1.01×10^{-12}
Smoking Smoker	0.59303	0.07884	7.522	6.81×10^{-14}
Exercise 3 to less than 5 hours	-0.14942	0.03980	-3.755	0.000176
Exercise 5 or more hours	-0.45964	0.03991	-11.518	$< 2 \times 10^{-16}$

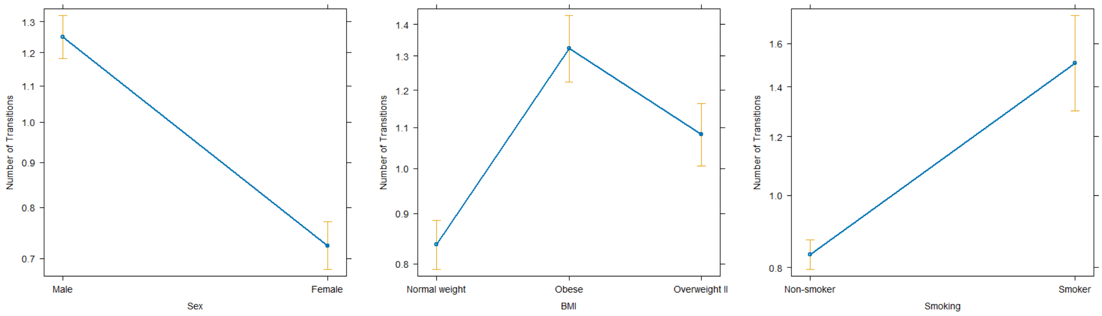


(a) Histogram of deviance residuals (b) Q-Q plot of deviance residuals

Figure A.1: Deviance residual plots of the final GLM for the transition intensities μ_u^{12}.



(a) Variable Age (b) Variable Exercise



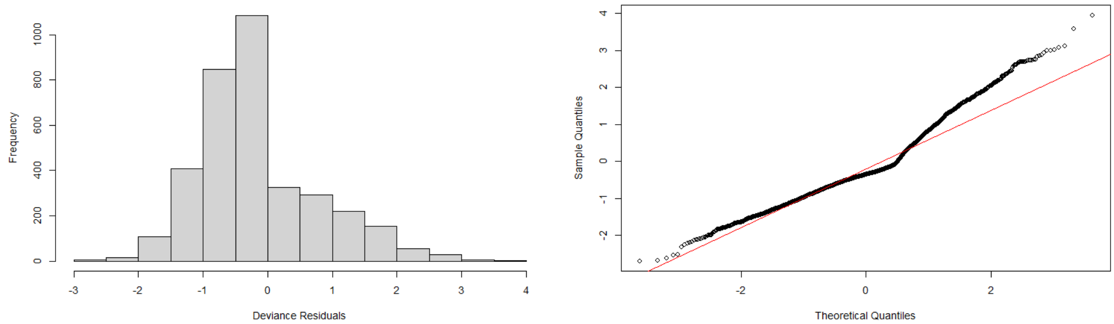
(c) Variable Sex (d) Variable BMI (e) Variable Smoking

Figure A.2: Effects of the explanatory variables on the response variable in the final GLM for the transition intensities μ_u^{12}.

A.2 GLM for Transition Intensities from Healthy to Severely Disabled

Table A.2: Summary of the final GLM for the transition intensities μ_u^{13}.

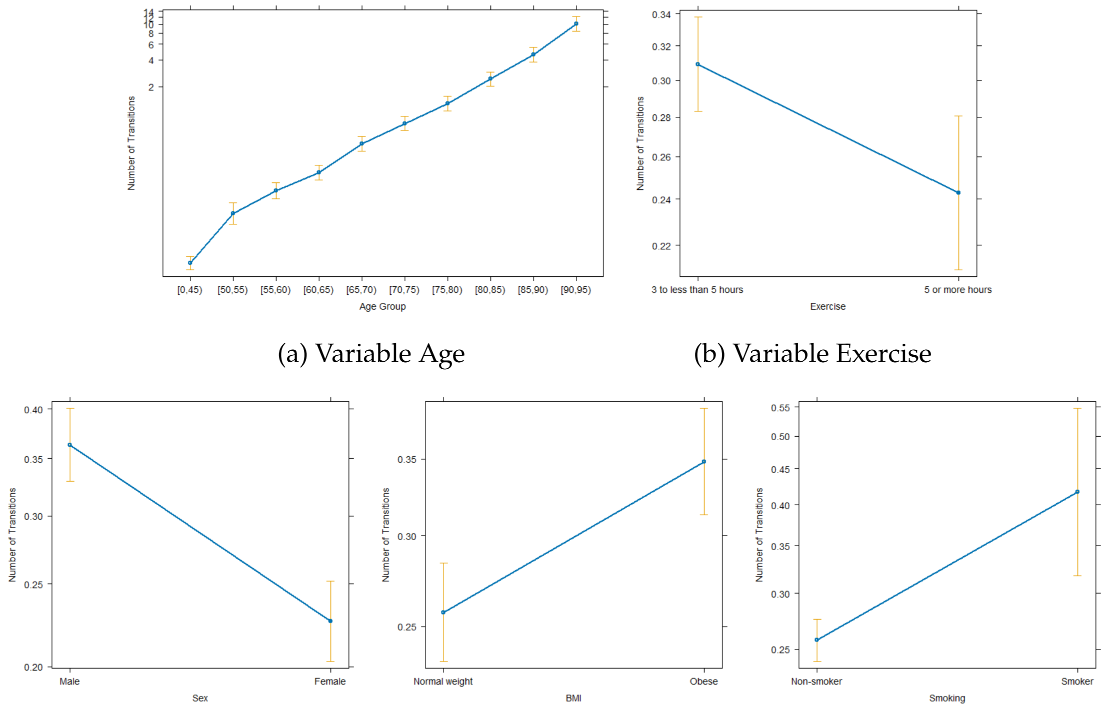
Variable	Estimate	Std. Error	t-value	Pr(> $\|t\|$)
(Intercept)	-7.08794	0.09988	-70.964	$< 2 \times 10^{-16}$
Age [50,55)	1.27903	0.16176	7.907	3.50×10^{-15}
Age [55,60)	1.86906	0.13706	13.637	$< 2 \times 10^{-16}$
Age [60,65)	2.32632	0.12746	18.251	$< 2 \times 10^{-16}$
Age [65,70)	3.08132	0.12810	24.055	$< 2 \times 10^{-16}$
Age [70,75)	3.60239	0.12253	29.401	$< 2 \times 10^{-16}$
Age [75,80)	4.11340	0.12795	32.148	$< 2 \times 10^{-16}$
Age [80,85)	4.74307	0.12575	37.717	$< 2 \times 10^{-16}$
Age [85,90)	5.37162	0.13044	41.182	$< 2 \times 10^{-16}$
Age [90,95)	6.16580	0.13132	46.953	$< 2 \times 10^{-16}$
Sex Female	-0.47488	0.06107	-7.776	9.71×10^{-15}
BMI Obese	0.30254	0.06017	5.028	5.20×10^{-7}
Smoking Smoker	0.48355	0.14524	3.329	0.000879
Exercise 5 or more hours	-0.24204	0.07490	-3.232	0.001242



(a) Histogram of deviance residuals

(b) Q-Q plot of deviance residuals

Figure A.3: Deviance residual plots of the final GLM for the transition intensities μ_u^{13}.



(a) Variable Age (b) Variable Exercise

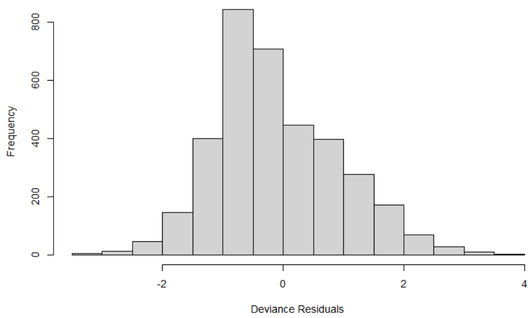
(c) Variable Sex (d) Variable BMI (e) Variable Smoking

Figure A.4: Effects of the explanatory variables on the response variable in the final GLM for the transition intensities μ_u^{13}.

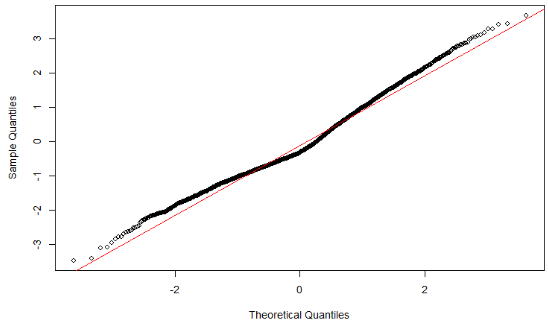
A.3 GLM for Transition Intensities from Healthy to Dead

Table A.3: Summary of the final GLM for the transition intensities μ_u^{14}.

Variable	**Estimate**	**Std. Error**	**t-value**	**Pr(> $\|t\|$)**
(Intercept)	-5.99394	0.08737	-68.607	$< 2 \times 10^{-16}$
Age [45,50)	1.55028	0.09408	16.478	$< 2 \times 10^{-16}$
Age [55,60)	2.18784	0.09858	22.193	$< 2 \times 10^{-16}$
Age [60,65)	2.56409	0.09448	27.140	$< 2 \times 10^{-16}$
Age [65,70)	2.98246	0.10259	29.072	$< 2 \times 10^{-16}$
Age [70,75)	3.49946	0.09804	35.694	$< 2 \times 10^{-16}$
Age [75,80)	4.05923	0.10055	40.371	$< 2 \times 10^{-16}$
Age [80,85)	4.52411	0.09419	48.032	$< 2 \times 10^{-16}$
Age [90,95)	5.31089	0.12838	41.368	$< 2 \times 10^{-16}$
Sex Female	-0.32332	0.04632	-6.981	3.49×10^{-12}
Region 5000-9999	-0.25148	0.04515	-5.569	2.75×10^{-8}
BMI Obese	0.21937	0.04583	4.787	1.76×10^{-6}
Smoking Smoker	0.26558	0.10853	2.447	0.01445
Exercise 5 or more hours	-0.12527	0.04597	-2.725	0.00646

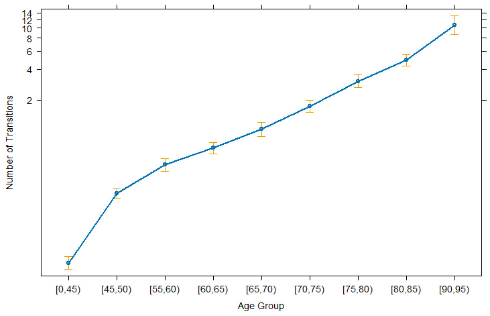


(a) Histogram of deviance residuals

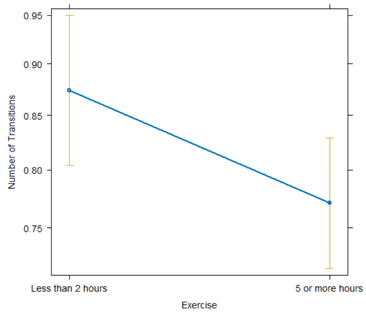


(b) Q-Q plot of deviance residuals

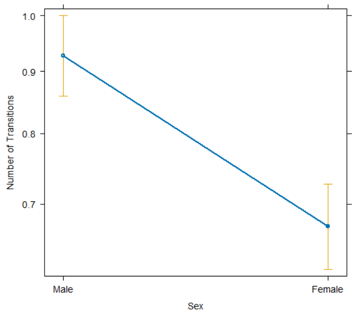
Figure A.5: Deviance residual plots of the final GLM for the transition intensities μ_u^{14}.



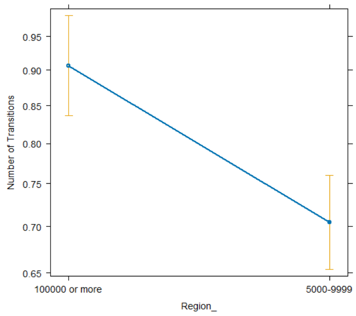
(a) Variable Age



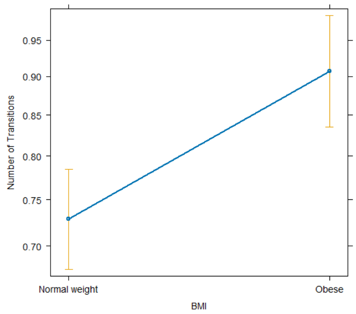
(b) Variable Exercise



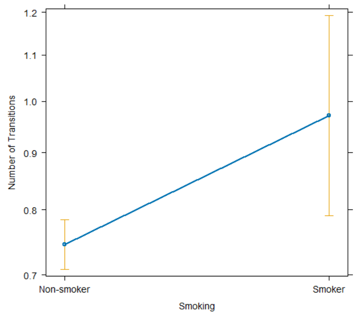
(c) Variable Sex



(d) Variable Region



(e) Variable BMI



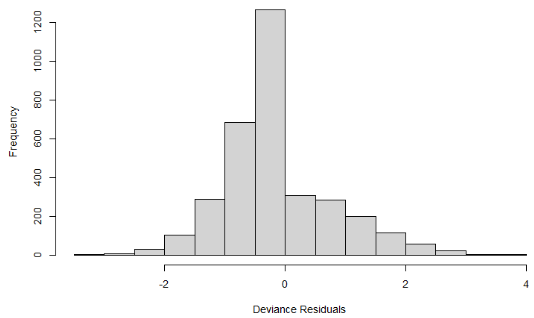
(f) Variable Smoking

Figure A.6: Effects of the explanatory variables on the response variable in the final GLM for the transition intensities μ_u^{14}.

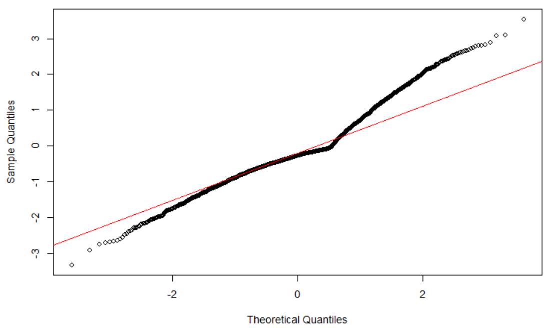
A.4 GLM for Transition Intensities from Mildly Disabled to Healthy

Table A.4: Summary of the final GLM for the transition intensities μ_u^{21}.

Variable	Estimate	Std. Error	t-value	Pr(> $\|t\|$)
(Intercept)	-7.73021	0.16115	-47.968	$< 2 \times 10^{-16}$
Age [45,50)	1.13871	0.27395	4.157	3.31×10^{-5}
Age [50,55)	2.21250	0.18317	12.079	$< 2 \times 10^{-16}$
Age [60,65)	3.23600	0.17701	18.281	$< 2 \times 10^{-16}$
Age [65,70)	3.85452	0.16508	23.350	$< 2 \times 10^{-16}$
Age [70,75)	4.49781	0.16270	27.645	$< 2 \times 10^{-16}$
Age [75,80)	5.20484	0.16213	32.104	$< 2 \times 10^{-16}$
Age [80,85)	5.86117	0.16156	36.278	$< 2 \times 10^{-16}$
Age [85,90)	6.49617	0.16170	40.175	$< 2 \times 10^{-16}$
Age [90,95)	7.20018	0.16280	44.228	$< 2 \times 10^{-16}$
Age [95,101)	7.66899	0.18093	42.386	$< 2 \times 10^{-16}$
Sex Female	0.58868	0.03239	18.175	$< 2 \times 10^{-16}$
BMI Obese	-0.50894	0.04385	-11.607	$< 2 \times 10^{-16}$
BMI Overweight II	-0.27563	0.03595	-7.668	2.28×10^{-14}
Smoking Smoker	-0.53043	0.14605	-3.632	0.000286
Exercise 3 to less than 5 hours	0.14312	0.03939	3.634	0.000284
Exercise 5 or more hours	0.45877	0.03628	12.646	$< 2 \times 10^{-16}$

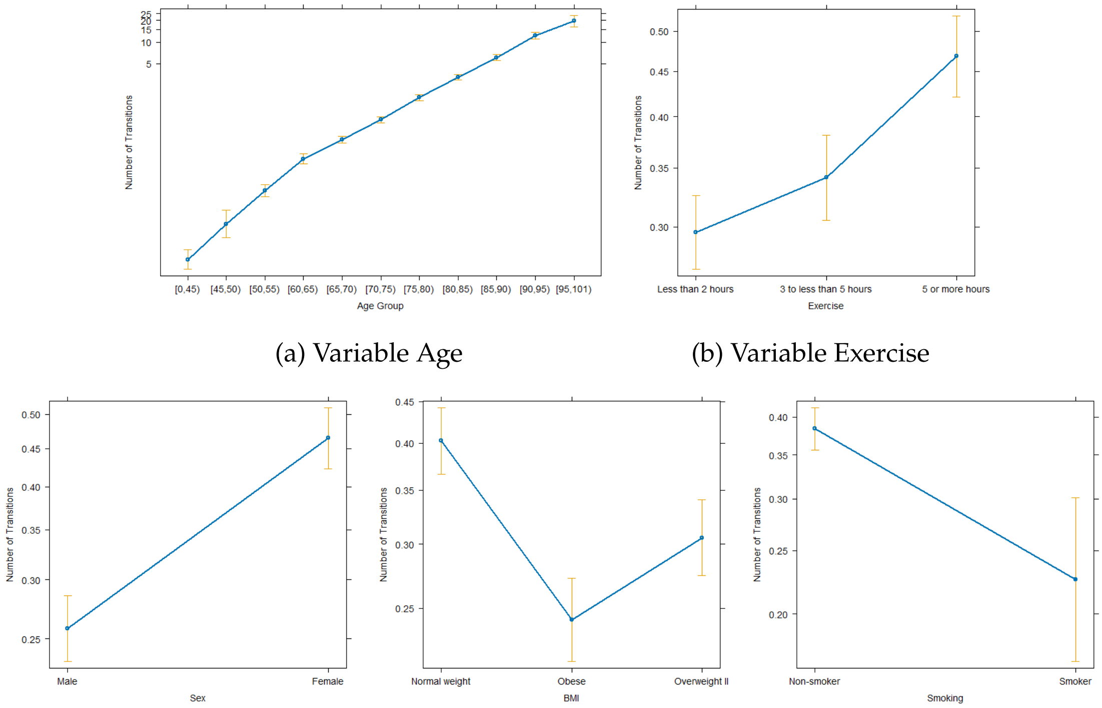


(a) Histogram of deviance residuals



(b) Q-Q plot of deviance residuals

Figure A.7: Deviance residual plots of the final GLM for the transition intensities μ_u^{21}.



(a) Variable Age (b) Variable Exercise

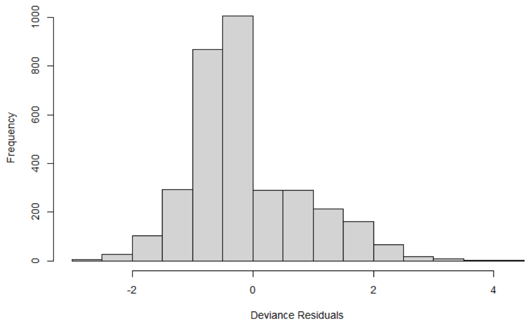
(c) Variable Sex (d) Variable BMI (e) Variable Smoking

Figure A.8: Effects of the explanatory variables on the response variable in the final GLM for the transition intensities μ_u^{21}.

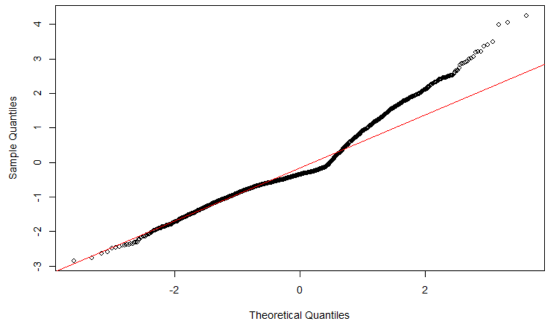
A.5 GLM for Transition Intensities from Mildly Disabled to Severely Disabled

Table A.5: Summary of the final GLM for the transition intensities μ_u^{23}.

Variable	**Estimate**	**Std. Error**	***t*-value**	**Pr(> |t|)**
(Intercept)	-7.24478	0.29299	-24.727	$< 2 \times 10^{-16}$
Age [45,50)	1.80614	0.33427	5.403	7.00×10^{-8}
Age [60,65)	2.81912	0.33970	8.299	$< 2 \times 10^{-16}$
Age [65,70)	3.46311	0.30729	11.270	$< 2 \times 10^{-16}$
Age [70,75)	4.15368	0.30061	13.817	$< 2 \times 10^{-16}$
Age [75,80)	4.81944	0.29983	16.074	$< 2 \times 10^{-16}$
Age [80,85)	5.37452	0.29965	17.936	$< 2 \times 10^{-16}$
Age [85,90)	6.08471	0.30014	20.273	$< 2 \times 10^{-16}$
Age [90,95)	6.78148	0.30233	22.430	$< 2 \times 10^{-16}$
Sex Female	-0.50526	0.06648	-7.600	3.83×10^{-14}
BMI Obese	0.44667	0.08236	5.423	6.27×10^{-8}
BMI Overweight II	0.21041	0.08076	2.605	0.00922
Smoking Smoker	0.55937	0.17292	3.235	0.00123

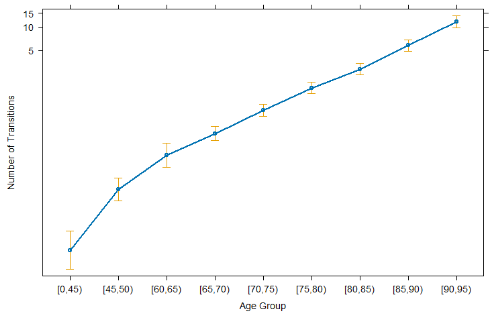


(a) Histogram of deviance residuals

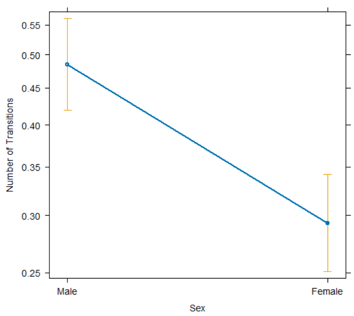


(b) Q-Q plot of deviance residuals

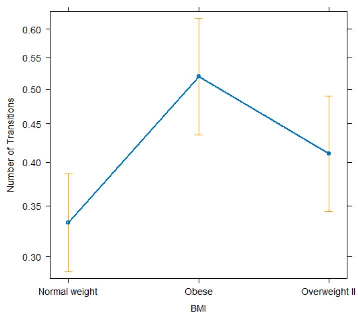
Figure A.9: Deviance residual plots of the final GLM for the transition intensities μ_u^{23}.



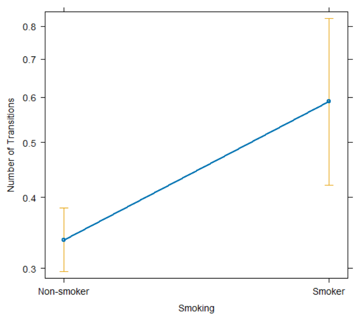
(a) Variable Age



(b) Variable Sex



(c) Variable BMI



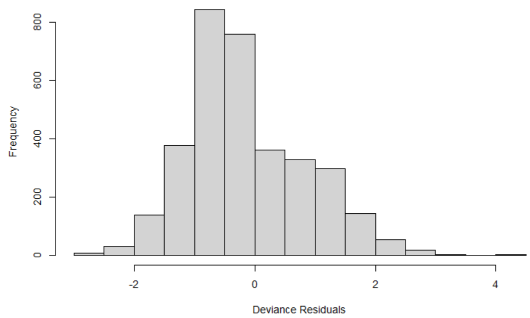
(d) Variable Smoking

Figure A.10: Effects of the explanatory variables on the response variable in the final GLM for the transition intensities μ_u^{23}.

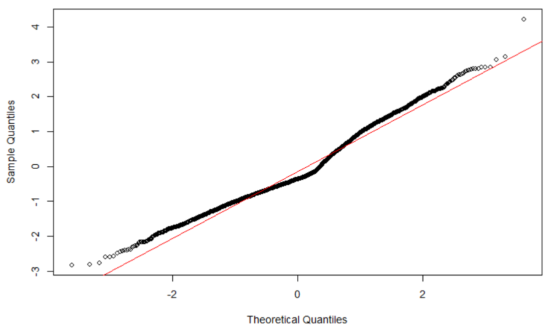
A.6 GLM for Transition Intensities from Mildly Disabled to Dead

Table A.6: Summary of the final GLM for the transition intensities μ_u^{24}.

Variable	Estimate	Std. Error	t-value	Pr(> $|t|$)
(Intercept)	-6.19478	0.11863	-52.218	$< 2 \times 10^{-16}$
Age [45,50)	1.31045	0.17531	7.475	9.80×10^{-14}
Age [50,55)	2.09526	0.12774	16.403	$< 2 \times 10^{-16}$
Age [60,65)	2.71701	0.12912	21.042	$< 2 \times 10^{-16}$
Age [65,70)	3.02606	0.11872	25.489	$< 2 \times 10^{-16}$
Age [70,75)	3.44995	0.11694	29.502	$< 2 \times 10^{-16}$
Age [75,80)	3.94461	0.11768	33.520	$< 2 \times 10^{-16}$
Age [80,85)	4.35849	0.11876	36.701	$< 2 \times 10^{-16}$
Age [85,90)	4.84102	0.12175	39.763	$< 2 \times 10^{-16}$
Age [90,95)	5.29609	0.12802	41.369	$< 2 \times 10^{-16}$
SexFemale	-0.33586	0.03698	-9.081	$< 2 \times 10^{-16}$
Region 2000-4999	-0.31495	0.05243	-6.006	2.10×10^{-9}
Region 20000-99999	-0.12715	0.05385	-2.361	0.018271
BMI Obese	0.31333	0.04628	6.770	1.52×10^{-11}
BMI Overweight II	0.15863	0.04482	3.539	0.000407
SmokingSmoker	0.31360	0.10143	3.092	0.002006

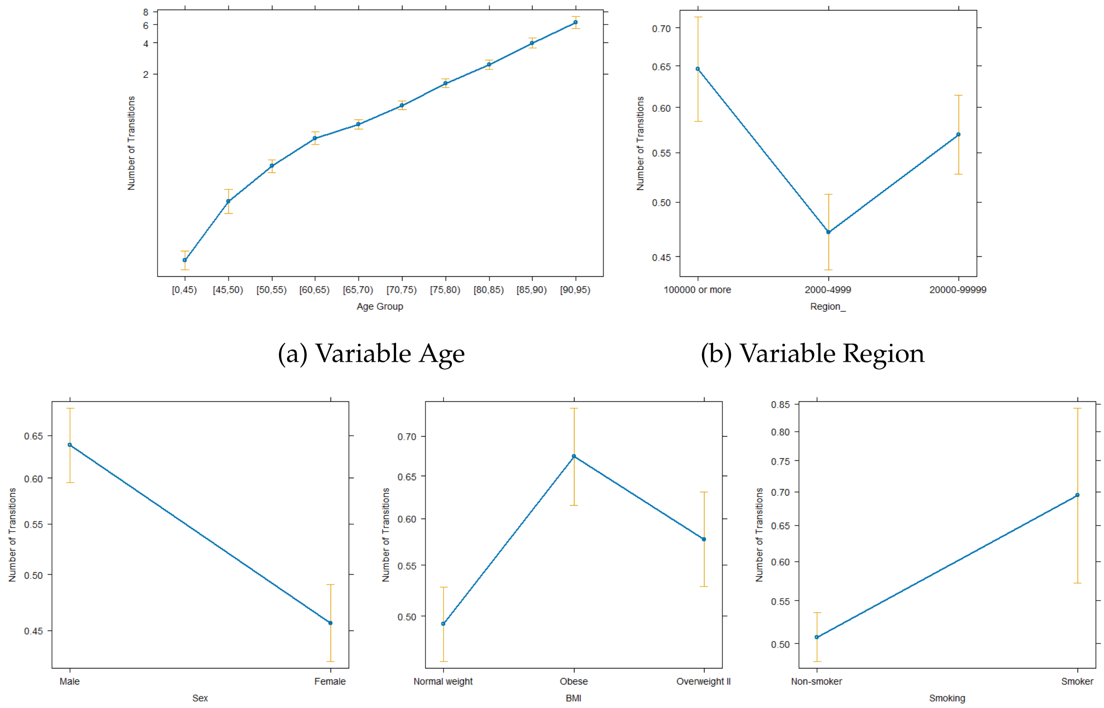


(a) Histogram of deviance residuals



(b) Q-Q plot of deviance residuals

Figure A.11: Deviance residual plots of the final GLM for the transition intensities μ_u^{24}.



(a) Variable Age

(b) Variable Region

(c) Variable Sex

(d) Variable BMI

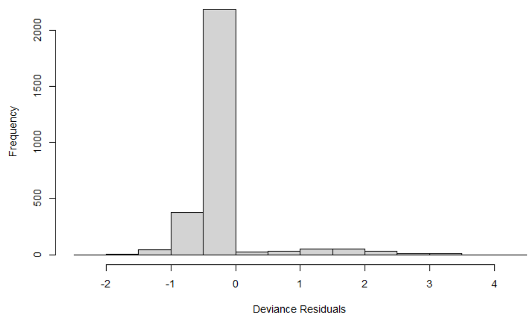
(e) Variable Smoking

Figure A.12: Effects of the explanatory variables on the response variable in the final GLM for the transition intensities μ_u^{24}.

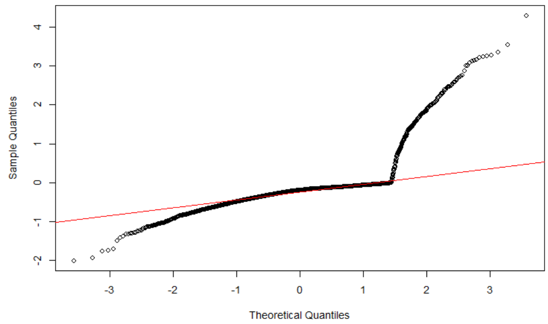
A.7 GLM for Transition Intensities from Severely Disabled to Mildly Disabled

Table A.7: Summary of the final GLM for the transition intensities μ_u^{32}.

Variable	Estimate	Std. Error	t-value	Pr(> $\|t\|$)
(Intercept)	-7.2253	0.3356	-21.530	$< 2 \times 10^{-16}$
Age [60,65)	2.4750	0.3757	6.588	5.29×10^{-11}
Age [70,75)	3.5056	0.3546	9.887	$< 2 \times 10^{-16}$
Age [80,85)	4.6662	0.3598	12.970	$< 2 \times 10^{-16}$
Sex Female	0.4038	0.2151	1.877	0.0606

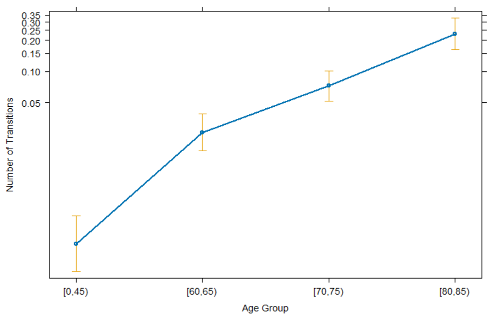


(a) Histogram of deviance residuals

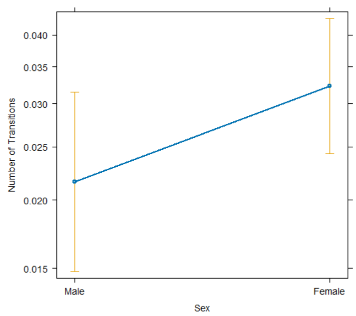


(b) Q-Q plot of deviance residuals

Figure A.13: Deviance residual plots of the final GLM for the transition intensities μ_u^{32}.



(a) Variable Age



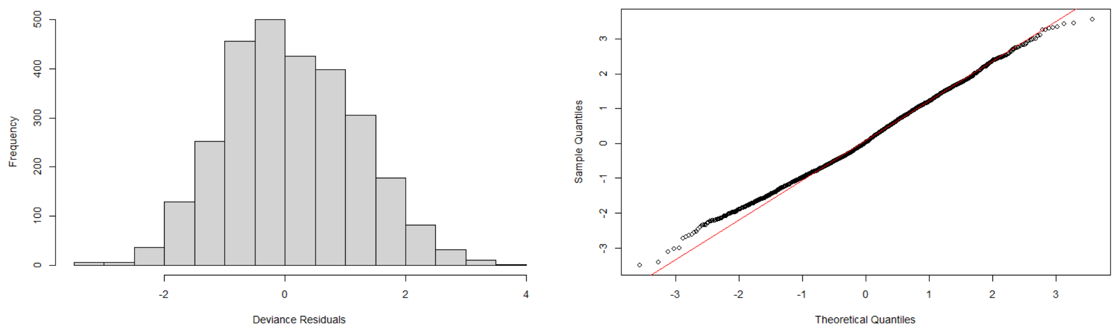
(b) Variable Sex

Figure A.14: Effects of the explanatory variables on the response variable in the final GLM for the transition intensities μ_u^{32}.

A.8 GLM for Transition Intensities from Severely Disabled to Dead

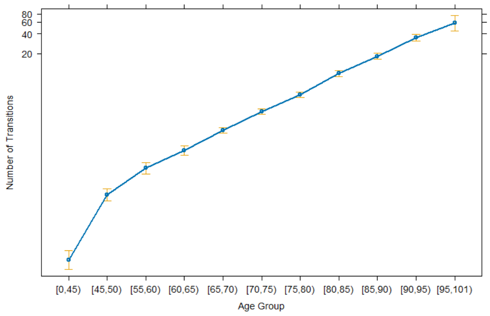
Table A.8: Summary of the final GLM for the transition intensities μ_u^{34}.

Variable	Estimate	Std. Error	t-value	Pr(> |t|)
(Intercept)	-5.25677	0.17435	-30.151	$< 2 \times 10^{-16}$
Age [45,50)	2.26929	0.19786	11.469	$< 2 \times 10^{-16}$
Age [55,60)	3.20371	0.19569	16.371	$< 2 \times 10^{-16}$
Age [60,65)	3.82183	0.18618	20.528	$< 2 \times 10^{-16}$
Age [65,70)	4.51543	0.17460	25.862	$< 2 \times 10^{-16}$
Age [70,75)	5.17514	0.17356	29.817	$< 2 \times 10^{-16}$
Age [75,80)	5.75366	0.17468	32.939	$< 2 \times 10^{-16}$
Age [80,85)	6.49562	0.17481	37.157	$< 2 \times 10^{-16}$
Age [85,90)	7.09436	0.17585	40.344	$< 2 \times 10^{-16}$
Age [90,95)	7.74841	0.17882	43.330	$< 2 \times 10^{-16}$
Age [95,101)	8.25315	0.21605	38.201	$< 2 \times 10^{-16}$
Sex Female	-0.50080	0.03781	-13.243	$< 2 \times 10^{-16}$
Region 100000 or more	0.50977	0.05695	8.951	$< 2 \times 10^{-16}$
Region 20000-99999	0.31372	0.04599	6.821	1.11×10^{-11}
Region 5000-9999	0.13896	0.05092	2.729	0.00640
BMI Obese	0.49146	0.05257	9.348	$< 2 \times 10^{-16}$
BMI Overweight II	0.33459	0.05107	6.552	6.73×10^{-11}
BMI Underweight	0.12981	0.04933	2.631	0.00855
SmokingSmoker	0.50395	0.10429	4.832	1.42×10^{-06}
Exercise 3 to less than 5 hours	-0.14996	0.05109	-2.935	0.00336
Exercise 5 or more hours	-0.38900	0.05105	-7.620	3.45×10^{-14}

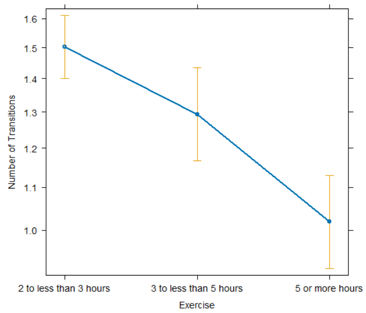


(a) Histogram of deviance residuals (b) Q-Q plot of deviance residuals

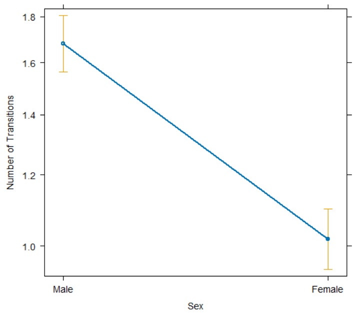
Figure A.15: Deviance residual plots of the final GLM for the transition intensities μ_u^{34}.



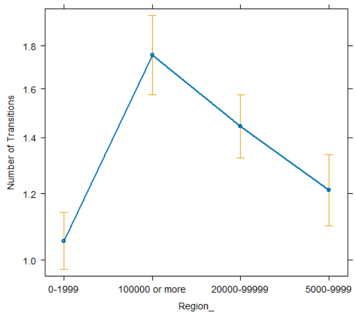
(a) Variable Age



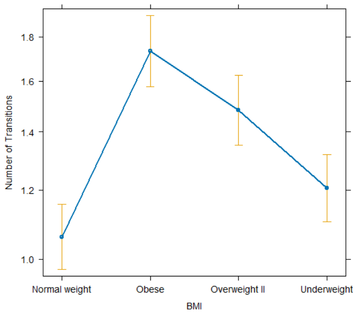
(b) Variable Exercise



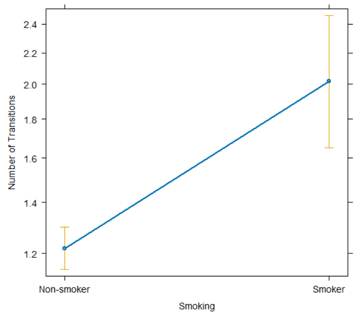
(c) Variable Sex



(d) Variable Region



(e) Variable BMI



(f) Variable Smoking

Figure A.16: Effects of the explanatory variables on the response variable in the final GLM for the transition intensities μ_u^{34}.