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A NECESSARY CONDITION FOR THE BOUNDEDNESS OF THE MAXIMAL OPERATOR ON $L^{p(\cdot)}$ OVER REVERSE DOUBLING SPACES OF HOMOGENEOUS TYPE

O. KARLOVYCH* and A. SHALUKHINA†

Centro de Matemática e Aplicações, Departamento de Matemática, Faculdade de Ciências e Tecnologia, Universidade Nova de Lisboa, Quinta da Torre, 2829–516 Caparica, Portugal
e-mails: oyk@fct.unl.pt, a.shaukhina@campus.fct.unl.pt

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Abstract. Let (X, d, μ) be a space of homogeneous type and $p(\cdot)\colon X \to [1, \infty]$ be a variable exponent. We show that if the measure μ is Borel-semiregular and reverse doubling, then the condition $\operatorname{ess\,inf}_{x\in X} p(x) > 1$ is necessary for the boundedness of the Hardy–Littlewood maximal operator M on the variable Lebesgue space $L^{p(\cdot)}(X, d, \mu)$.

1. Introduction and the main result

Let (X, d, μ) be a space of homogeneous type (see Section 2). For $x \in X$ and $r > 0$, consider the ball $B(x, r) := \{y \in X : d(x, y) < r\}$ centered at x of radius r. By definition, the Borel measure μ has the doubling property, that is, for every ball $B(x, r)$ with respect to the quasi-metric d, the relation

$$\mu(B(x, r)) \le A\mu(B(x, r/2))$$

holds true with an absolute constant $A > 1$. We will assume that $0 < \mu(B) < \infty$ for every ball B. If, additionally, the reverse inequality

$$\mu(B(x, r/2)) \le \delta\mu(B(x, r))$$

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* Corresponding author.

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is valid for some absolute constant $0 < \delta < 1$, it is said that the measure μ is also reverse doubling (see, e.g., [5, Section 2.6]).

Given a complex-valued function $f \in L^1_{\rm loc}(X, d, \mu)$, we define its Hardy–Littlewood maximal function Mf by

$$Mf(x) := \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f(y)|\, d\mu(y), \quad x \in X, \tag{1.1}$$

where the supremum is taken over all balls $B \subset X$ containing $x \in X$. The Hardy–Littlewood maximal operator M is a sublinear operator acting by the rule $f \mapsto Mf$.

Let $L^0(X, d, \mu)$ denote the set of all complex-valued measurable functions on X and let $\mathcal{P}(X)$ denote the set of all measurable functions $p(\cdot)\colon X \to [1, \infty]$. The functions in $\mathcal{P}(X)$ are called variable exponents. Let $X_\infty := \{x \in X : p(x) = \infty\}$. For a function $f \in L^0(X, d, \mu)$ and $p(\cdot) \in \mathcal{P}(X)$, consider the functional, which is called modular, given by

$$m_{p(\cdot)}(f) := \int_{X \setminus X_\infty} |f(x)|^{p(x)}\, d\mu(x) + \operatorname*{ess\,sup}_{x \in X_\infty} |f(x)|.$$

By definition, the variable Lebesgue space $L^{p(\cdot)}(X, d, \mu)$ consists of all functions $f \in L^0(X, d, \mu)$ such that $\varrho_{p(\cdot)}(f/\lambda) < \infty$ for some $\lambda > 0$ depending on f. It is a Banach space with respect to the Luxemburg–Nakano norm given by

$$\|f\|_{p(\cdot)} := \inf\bigl\{\lambda > 0 : \varrho_{p(\cdot)}(f/\lambda) \le 1\bigr\}.$$

If $p(\cdot) \in \mathcal{P}(X)$ is constant, then $L^{p(\cdot)}(X, d, \mu)$ is nothing but the standard Lebesgue space $L^p(X, d, \mu)$. Variable Lebesgue spaces are often called Nakano spaces. We refer to Maligranda's paper [11] for the role of Hidegoro Nakano in the study of variable Lebesgue spaces and to the monographs [3,7] for the basic properties of these spaces.

Cruz-Uribe, Fiorenza and Neugebauer proved in [2, Theorem 1.7] that if $\Omega \subset \mathbb{R}^d$ is an open set and $p(\cdot)\colon \Omega \to [1, \infty)$ is upper semi-continuous, then the boundedness of M on $L^{p(\cdot)}(\Omega)$ implies that $\inf_{x\in\Omega} p(x) > 1$ (it is supposed that Ω is equipped with the Lebesgue measure and the usual Euclidean distance). The upper semi-continuity assumption was removed by Deining et al. [6, Theorem 6.3]. Another proof of this fact was given by Izuki, Nakai and Sawano in [8, Proposition 3.3] and [9, Proposition 21.2]. Proofs of the fact that the boundedness of M on $L^{p(\cdot)}(\mathbb{R}^d)$ implies that

$$p_-(\mathbb{R}^d) := \operatorname*{ess\,inf}_{x \in \mathbb{R}^d} p(x) > 1$$

are given in [7, Theorem 4.7.1] and [3, Theorem 3.19]. Note also that very recently Roberts [13, Theorem 3.3] (added in proof: see also [4, Theorem 1.1]) extended the above result to the setting of the fractional maximal operator

$$M_\alpha f(x) := \sup_{Q \ni x} |Q|^{\alpha/d-1} \int_Q |f(y)|\,dy,$$

where $0 \le \alpha < d$ and the supremum is taken over all cubes $Q \subset \mathbb{R}^d$ containing x, and proved that if M_α is bounded from $L^{p(\cdot)}(\mathbb{R}^d)$ to $L^{q(\cdot)}(\mathbb{R}^d)$ with $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$ and $q(\cdot)$ defined by $1/p(\cdot) - 1/q(\cdot) = \alpha/d$, then $p_-(\mathbb{R}^d) > 1$.

In August of 2019, during the ISAAC Congress held in Aveiro, Portugal, Stefan Samko asked the first author, under which conditions on a quasi-metric space (X, d) and a measure μ on X, the boundedness of the Hardy–Littlewood maximal operator M on $L^{p(\cdot)}(X, d, \mu)$ implies that

$$p_-(X) := \operatorname*{ess\,inf}_{x \in X} p(x) > 1.$$

Surprisingly enough, we were not able to find any result on necessary conditions for the boundedness of M on variable Lebesgue spaces beyond the Euclidean setting. Moreover, we are not aware of a proof that M is unbounded on $L^1(X, d, \mu)$ in the setting of spaces of homogeneous type.

The aim of this paper is to address this open problem. Our main result is the following.

THEOREM 1. *Suppose (X, d, μ) is a space of homogeneous type which has the property that the measure μ is Borel-semiregular and reverse doubling. Given an exponent function $p(\cdot) \in \mathcal{P}(X)$, if the Hardy–Littlewood maximal operator M is bounded on the variable Lebesgue space $L^{p(\cdot)}(X, d, \mu)$, then $p_-(X) > 1$.*

Although the assumption that the measure μ is reverse doubling is essential in our proof, we believe that Theorem 1 should be true without it.

The paper is organized as follows. In Section 2, we provide necessary background on spaces of homogeneous type and the Lebesgue differentiation theorem in this setting. Section 3 contains the proof of Theorem 1. We conclude this paper observing in Section 4 that if the variable exponent $p(\cdot)$ is upper semi-continuous, then one can avoid the use of the Lebesgue differentiation theorem. Hence the hypothesis of Borel-semiregularity of μ can be dropped under this assumption on $p(\cdot)$.

2. Preliminaries on spaces of homogeneous type

Following [1, Section 2.1], given a nonempty set X, call a function $\varrho\colon X \times X \to [0, \infty)$ a quasi-distance (or a quasi-metric) provided there exist constants $C_0, C_1 \in (0, \infty)$ such that for all $x, y, z \in X$ the following axioms hold:

(a) $\varrho(x, y) = 0$ if and only if $x = y$;

(b) $\varrho(y, x) \le C_0 \varrho(x, y)$;

(c) $\varrho(x, y) \le C_1 \max\{\varrho(x, z), \varrho(z, y)\}$.

If X has cardinality at least 2, then necessarily $C_0, C_1 \ge 1$. A pair (X, ϱ) is called a quasi-metric space. Given $r > 0$ and $x \in X$, let

$$B_\varrho(x, r) := \{y \in X : \varrho(x, y) < r\}$$

be the quasi-metric ball related to ϱ of radius r and with center x. If (X, ϱ) is a quasi-metric space, then $\mathcal{T}_\varrho$, the topology on X induced by the quasi-metric ϱ is canonically defined by declaring $G \subset X$ to be open if and only if for every $x \in G$, there exists $r > 0$ such that $B_\varrho(x, r) \subset G$. The quasi-metric balls themselves need not be open (unless ϱ is a genuine metric) even if $C_0 = 1$ (see, e.g., an example in [12, p. 4310]). According to a refined version of the theorem by Macías and Segovia (see [10, Theorem 2]) available in [1, Theorem 2.1], given a quasi-metric ϱ, there exists a constant $c \in (0, \infty)$ and a quasi-metric d on X such that for all $x, y \in X$, one has

$$c^{-1}\varrho(x, y) \le d(x, y) \le c\varrho(x, y), \quad d(x, y) = d(y, x),$$

and all balls $B_d(x, r)$ with respect to d are open in the topology $\mathcal{T}_\varrho = \mathcal{T}_d$.

From now on, we will assume that X is equipped with this equivalent quasi-metric d with the property that all quasi-metric balls $B_d(x, r)$ are open in the topology $\mathcal{T}_d$. For simplicity, we will write $B(x, r) := B_d(x, r)$.

Let $\mathfrak{M}$ be a σ-algebra of subsets of X and $\mu\colon \mathfrak{M} \to [0, \infty]$ be a measure. Following [1, Definition 2.9], a measure μ on the topological space $(X, \mathcal{T}_d)$ is said to be Borel if $\mathfrak{M}$ contains all Borel subsets of X. A Borel measure μ on X is said to be doubling if there exists a constant $A \in (1, \infty)$ such that

$$0 < \mu(B(x, r)) \le A\mu(B(x, r/2)) < \infty$$

for all $x \in X$ and $r > 0$. In this case the triple (X, d, μ) is called the space of homogeneous type.

One says that a measurable function f on X belongs to $L^1_{\mathrm{loc}}(X, d, \mu)$ if

$$\int_{B(x,r)} |f(y)|\, d\mu(y) < \infty$$

for every $x \in X$ and $r > 0$. If $f \in L^1_{\mathrm{loc}}(X, d, \mu)$, then the Hardy–Littlewood maximal function Mf defined by (1.1) is measurable on X because the quasi-metric balls $B \subset X$ (with respect to the quasi-metric d) are open and so Mf is lower semi-continuous. Further, if $p(\cdot) \in \mathcal{P}(X)$ and $f \in L^{p(\cdot)}(X, d, \mu)$, then $f \in L^1_{\mathrm{loc}}(X, d, \mu)$. This can be proved as in the Euclidean setting (see, e.g., [3, Proposition 2.41]).

Following [1, Definition 3.9], a Borel measure μ on $(X, \mathcal{T}_d)$ is said to be Borel-semiregular if for any measurable set E of finite measure there exists a Borel set B such that $\mu(E\Delta B) = 0$, where $E\Delta B := (E \setminus B) \cup (B \setminus E)$.

We will need the following sharp version of the Lebesgue differentiation theorem (see [1, Theorem 3.14]).

THEOREM 2. *Let (X, d, μ) be a space of homogeneous type. Then the measure μ is Borel-semiregular on $(X, \mathcal{T}_d)$ if and only if for every $f \in L^1_{\mathrm{loc}}(X, d, \mu)$,*

$$\lim_{r\to 0^+} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f(y)\, d\mu(y) = f(x)$$

for μ-almost every $x \in X$.

3. Proof of the main result

Assume that $p_-(X) = 1$. Following the general idea of the proof from [3, Theorem 3.19], to show that the maximal operator is not bounded, we will construct a sequence of functions $\{f_k\}$ such that for all k, $f_k \in L^{p(\cdot)}(X, d, \mu)$ but the norms $\|Mf_k\|_{p(\cdot)}$ can not be uniformly bounded by $\|f_k\|_{p(\cdot)}$.

Since $p_-(X) = 1$, for each $k \in \mathbb{N}$ the set

$$E_k = \{x \in X : p(x) < 1 + 1/k\}$$

has positive measure. Given that μ is assumed to be Borel-semiregular, applying Theorem 2 to the function χ_{E_k}, we can choose a point $x_k \in E_k$ such that

$$\lim_{r\to 0^+} \frac{\mu(E_k \cap B(x_k, r))}{\mu(B(x_k, r))} = 1,$$

that is, a density point of E_k. This choice implies, in particular, that for each k, there exists a radius R_k, $0 < R_k < 1$, such that if $0 < r \le R_k$, then

$$\frac{\mu(E_k \cap B(x_k, r))}{\mu(B(x_k, r))} > \frac{1+\delta}{2}, \tag{3.1}$$

where $\delta \in (0, 1)$ is the reverse doubling constant.

Let $B_k^0 := B(x_k, R_k)$ be a ball, sufficiently densely—in the sense of (3.1)—filled with the points of E_k. For $i \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$, consider the balls

$$B_k^i := B(x_k, R_k/2^i)$$

and split B_k^0 into the disjoint union of dyadic annular regions $B_k^i \setminus B_k^{i+1}$. Using the doubling property of μ with the constant $A > 1$ and the reverse doubling property with the constant δ, for each $i \in \mathbb{N}_0$ we estimate beforehand

$$\mu(B_k^i \setminus B_k^{i+1}) \geq (1-\delta)\mu(B_k^i) \geq \frac{1-\delta}{A^i}\mu(B_k^0). \tag{3.2}$$

Finally, define the sequence of functions

$$f_k(x) = \Bigg(\sum_{i=0}^{\infty} \frac{\chi_{B_k^i \setminus B_k^{i+1}}(x)}{A^{i/k}\mu(B_k^i \setminus B_k^{i+1})}\Bigg)\chi_{E_k}(x) \tag{3.3}$$

on X. Note that outside $E_k \cap B_k^0$, the function f_k is identically zero.

To show that $f_k \in L^{p(\cdot)}(X, d, \mu)$, we use the simple observation that

$$f_k(x)^{p(x)} \leq \max\{1, f_k(x)\}^{p(x)} \leq \max\{1, f_k(x)^{1+1/k}\} \leq 1 + f_k(x)^{1+1/k}$$

for all $x \in E_k$, and this, together with (3.2), gives us

$$\begin{aligned} m_{p(\cdot)}(f_k) &= \int_{E_k \cap B_k^0} f_k(x)^{p(x)}\,d\mu(x) \leq \mu(B_k^0) + \int_{B_k^0} f_k(x)^{1+1/k}\,d\mu(x) \\ &= \mu(B_k^0) + \sum_{i=0}^{\infty} \frac{\mu((B_k^i \setminus B_k^{i+1}) \cap E_k)}{[A^{i/k}\mu(B_k^i \setminus B_k^{i+1})]^{1+1/k}} \leq \mu(B_k^0) + \sum_{i=0}^{\infty} \frac{[\mu(B_k^i \setminus B_k^{i+1})]^{-1/k}}{(A^{1/k+1/k^2})^i} \\ &\leq \mu(B_k^0) + [(1-\delta)\mu(B_k^0)]^{-1/k} \sum_{i=0}^{\infty} \frac{A^{i/k}}{(A^{1/k+1/k^2})^i} \\ &= \mu(B_k^0) + [(1-\delta)\mu(B_k^0)]^{-1/k} \sum_{i=0}^{\infty} \left(A^{-1/k^2}\right)^i, \end{aligned}$$

where the last expression is finite since $A^{-1/k^2} < 1$ for each $k \in \mathbb{N}$.

To estimate the norm of Mf_k, first fix $x \in E_k \cap B_k^0$. Clearly, there exists $i \in \mathbb{N}_0$ such that $x \in B_k^i \setminus B_k^{i+1}$ and hence

$$f_k(x) = \frac{1}{A^{i/k}\mu(B_k^i \setminus B_k^{i+1})}. \tag{3.4}$$

Note that no less than a certain "portion" of each annulus $B_k^j \setminus B_k^{j+1}$ is filled with the points of E_k: more precisely, since the radius of each dyadic ball B_k^j, $j \in \mathbb{N}_0$, does not exceed R_k, it follows from (3.1) and the reverse doubling that

$$\mu((B_k^j \setminus B_k^{j+1}) \cap E_k) \geq \mu(B_k^j \cap E_k) - \mu(B_k^{j+1})$$

$$> \frac{1+\delta}{2}\mu(B_k^j) - \delta\mu(B_k^j) \geq \frac{1-\delta}{2}\mu(B_k^j \setminus B_k^{j+1}).$$

Then

$$Mf_k(x) \geq \frac{1}{\mu(B_k^i)}\int_{B_k^i} f_k(y)\,d\mu(y) = \frac{1}{\mu(B_k^i)}\sum_{j=i}^{\infty}\frac{\mu((B_k^j \setminus B_k^{j+1})\cap E_k)}{A^{j/k}\mu(B_k^j \setminus B_k^{j+1})}$$

$$\geq \frac{1}{\mu(B_k^i)}\cdot\frac{1-\delta}{2}\sum_{j=i}^{\infty}(A^{-1/k})^j = \frac{1}{\mu(B_k^i)}\cdot\frac{1-\delta}{2}\cdot\frac{A^{-i/k}}{1-A^{-1/k}},$$

which implies, along with (3.2) and (3.4), that for $x \in E_k \cap B_k^0$,

$$Mf_k(x) \geq f_k(x)\cdot\frac{A^{i/k}\mu(B_k^i \setminus B_k^{i+1})}{\mu(B_k^i)}\cdot\frac{1-\delta}{2}\cdot\frac{A^{-i/k}}{1-A^{-1/k}}$$

$$\geq \frac{(1-\delta)^2}{2(1-A^{-1/k})}f_k(x).$$

Trivially, this inequality also holds if $x \notin E_k \cap B_k^0$. Hence, we have shown that

$$\|Mf_k\|_{p(\cdot)} \geq \frac{(1-\delta)^2}{2(1-A^{-1/k})}\|f_k\|_{p(\cdot)},$$

but since $A^{-1/k} \to 1$ as $k \to \infty$, we can not get the uniform boundedness of the norms $\|Mf_k\|_{p(\cdot)}$, and this completes the proof.

4. Final remark

If we additionally assume that $p(\cdot) \in \mathcal{P}(X)$ is upper semi-continuous, then the hypothesis of Borel-semiregularity of μ can be dropped because we can avoid using the Lebesgue differentiation theorem in this case. More precisely, we have the following.

THEOREM 3. *Suppose (X, d, μ) is a space of homogeneous type which has the property that the measure μ is reverse doubling. Given an upper semi-continuous exponent function $p(\cdot) \in \mathcal{P}(X)$, if the Hardy–Littlewood maximal operator M is bounded on the variable Lebesgue space $L^{p(\cdot)}(X, d, \mu)$, then $p_-(X) > 1$.*

PROOF. Assume that $p_-(X) = 1$. Since X is open and $p(\cdot)$ is upper semi-continuous, for every $k \in \mathbb{N}$, there exist $x_k \in X$ and $R_k > 0$ such that if $x \in B_k^0 := B(x_k, R_k)$, then $p(x) < 1 + 1/k$. Now define f_k replacing χ_{E_k} by $\chi_{B_k^0}$ in (3.3). After this the proof goes as that of Theorem 1 with minor changes. $\square$

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