DEPARTMENT OF
MATHEMATICS

NOVA SCHOOL OF
SCIENCE & TECHNOLOGY

MARIANA MAGALHÃES CÂNDIDO
BSc in Matemática Aplicada à Economia e à Gestão

ASSET PRICING IN THE GENERAL EQUILIBRIUM MODEL WITH MEAN-VARIANCE PREFERENCES WHEN INVESTORS HAVE RESTRICTIONS ON TRANSACTIONS

MASTER IN MATEMÁTICA E APLICAÇÕES COM ESPECIALIZAÇÃO EM
MATEMÁTICA FINANCEIRA

NOVA University Lisbon
September, 2023

DEPARTMENT OF
MATHEMATICS

ASSET PRICING IN THE GENERAL EQUILIBRIUM MODEL WITH MEAN-VARIANCE PREFERENCES WHEN INVESTORS HAVE RESTRICTIONS ON TRANSACTIONS

MARIANA MAGALHÃES CÂNDIDO

BSc in Matemática Aplicada à Economia e à Gestão

Adviser: Marta Cristina Vieira Faias Mateus
Associate Professor, NOVA University Lisbon

MASTER IN MATEMÁTICA E APLICAÇÕES COM ESPECIALIZAÇÃO EM MATEMÁTICA FINANCEIRA

NOVA University Lisbon
September, 2023

Asset Pricing in the General Equilibrium Model with Mean-Variance Preferences when Investors Have Restrictions on Transactions

This document was created with the (pdf/Xe/Lua)LaTeX processor and the NOVAthesis template (v7.1.5) (Lourenço, 2021).

To the people who have always supported me.

Acknowledgements

Since day one, writing this dissertation was not an easy task. It was a long journey, with many ups and downs. After crossing the finish line, I realised that the ups were what gave me the strength and motivation to keep going. I had to combine the dissertation with a full-time job, getting home after a long day at work and having a long night of studying ahead of me.

Today I clearly realise that I only had enough strength to see this through because I had very important and special people who gave me all the strength I needed to carry on.

My first acknowledgement goes to my thesis adviser, Professor Marta Faias, who was always very willing to help me and teach me everything I needed to do to finish my dissertation. Without her, it wouldn't have been possible. My sincerest thanks for everything you did and for all the hours you spent with me. You were the best adviser I could have had.

To FCT for being my home for two years, where I was very happy. Thank you to all the teachers I met on this journey, you all contributed to my development.

To my parents. Minha querida mãe, meu querido pai. Sem eles nada disto teria sido possível. Deram-me a oportunidade de estudar no ensino superior e suportaram as despesas necessárias para mudar de cidade, ir viver para Lisboa, de forma a poder realizar o mestrado que sempre quis. Deram-me força quando não tinha, incentivaram-me a escrever a dissertação quando essa não era a minha vontade, e mostraram-me que sempre que nos esforçamos conseguimos atingir os objetivos. Desistir nunca esteve no dicionário deles, e por isso também nunca esteve no meu. Obrigada, mãe e pai.

I would also like to thank my brother, Henrique. Que sempre me apoiou e ajudou, ainda que não diretamente, na realização desta dissertação. Obrigada por toda a paciência e atenção que me deste.

To my boyfriend, who always supported me, who helped me find motivation when I couldn't. His support, patience and dedication to me were crucial. Without him, completing this dissertation would have been much more difficult.

To my family, grandparents, uncles and cousins, thank you for your support.

I would also like to thank my friends for being so real and for always showing me that if you have the will, you have the way.

Abstract

This dissertation shows how the pricing of financial products varies when investors have transaction restrictions and in the context of the CAPM, i.e., when investors' portfolios are subject to restrictions. The study is restricted to the case in which investors have mean-variance preferences, since in this case it is possible to deduce closed formulas for pricing. For example, transaction restrictions prevent agents from investing in certain assets or, for instance, only being able to invest in a certain asset if they also invest in other specific ones. By building a model in which there are portfolios of bonds and equities, it is possible to calculate the price of these financial products using the CAPM, and by adapting the agents' budget restrictions we can calculate the same prices when the agents have transaction restrictions and the risk premiums associated with these prices are also calculated.

Keywords: Trading Restrictions, CAPM, Pricing, Risk Premium

Resumo

Nesta dissertação mostra-se de que forma varia o apreçamento de produtos financeiros quando os investidores tem restrições à transação e no contexto do CAPM, ou seja quando as carteiras dos investidores estão sujeitas a restrições. O estudo é restrito ao caso em que os investidores têm preferências do tipo média-variância, uma vez que neste caso é possível deduzir fórmulas fechadas para o apreçamento. As restrições à transação impedem, por exemplo, os agentes de investir em alguns ativos ou de, por exemplo, só poder investir num certo ativo se investir também noutros ativos específicos. Construindo um modelo em que existem portefolios referentes a bonds e equities é possível calcular através do CAPM o preço destes produtos financeiros, e adaptando a restrição orçamental dos agentes podemos calcular os mesmos preços quando os agentes têm restrições às transações. Associado a estes preços calculam-se ainda os prémios de risco associados a cada um destes.

Palavras-chave: Restrições às Transações, CAPM, Apreçamento, Prémio de Risco

Contents

Symbols

α aggregate preference parameters *(p. 27)*

B^i budget set *(p. 5)*

δ initial endowment of equities *(p. 28)*

$\mathcal{E}$ economy *(p. 5)*

γ security *(p. 18)*

h increment in consumption stream *(p. 18)*

I number of consumers *(p. 4)*
i consumer *(p. 4)*

J number of assets *(p. 5)*
j equity *(p. 4)*

K maximum loans *(p. 37)*

λ Lagrange multiplier *(p. 13)*

m income stream *(p. 16)*

ω initial endowment of the commodity *(p. 4)*

p prices *(p. 4)*
π vector of state prices *(p. 9)*

q	security prices *(p. 5)*
R	return per unit *(p. 37)*
r	risk free rate *(p. 19)*
ρ	probability of occurrence of each state of nature *(p. 15)*
$\mathcal{R}$	excess return *(p. 37)*
S	set of all states of nature *(p. 4)*
s	state of nature *(p. 4)*
σ	function of equities *(p. 28)*
t	period *(p. 4)*
τ	income transfer vector between the states *(p. 8)*
u^i	utility function of consumer *(p. 4)*
V	matrix that represents payment of J securities *(p. 4)*
W	combination between the two dates matrix of security payoffs *(p. 5)*
x	consumption plan of agent i *(p. 4)*
Y	matrix of payoff of each equity *(p. 33)*
y	payoff of each equity *(p. 10)*
z	portfolio of consumer i *(p. 5)*

1

Introduction

Real world economic activity is an ongoing process that cannot be predicted. A realistic equilibrium model must be able to describe the transactions happening in the markets. The theory of general equilibrium allows us to explain the relationship between demand and supply and how these determine price. In the equilibrium economy, we consider a vector of prices, initial allocations for each agent, and equal supply and demand for each good in a competitive market.

A very important result in the study of an economy is the existence of equilibrium, because it gives us a framework that allows us to understand how markets work and how the decisions of agents influence prices, quantities of goods, services and the equilibrium of the economy. The theory of the existence of equilibrium is often studied and demonstrated in various contexts. There are various theorems that help proving the existence of equilibrium, for example whether the market is complete or incomplete, and whether or not the market is competitive. However, it is important to note that the existence of equilibrium is not always guaranteed, as there are specific conditions that must be respected. In addition to these factors, prove the existence of equilibrium does not imply that this equilibrium is unique. There are cases in which there can be several equilibrium in an economy, with different prices and different quantities of goods. (You can see more about the existence of equilibrium at Geanakoplos and Polemarchakis, 1986, Geanakoplos, 1990 and Araújo and Villalba, 2016). In the case of this dissertation, proving the existence of equilibrium is not the focus. Nevertheless, it is important to note that equilibrium also exists when there are restrictions on transactions. More on this topic can be found at Polemarchakis and Siconolfi, 1997, Aouani and Cornet, 2009 and Cornet and Gopalan, 2010.

The aim of this dissertation is to study the pricing of financial assets when consumers have trading restrictions. In order to, more accurately, capture the dynamics of contemporary financial markets, certain authors have embarked on the examination of trading constraints within the context of general equilibrium. The objective is to create a framework that can effectively simulate the individualized and limited access that market participants experience. There are two primary approaches for modeling trading constraints: one involves treating these constraints as externally imposed, while the other posits that constraints can naturally arise within the system, potentially influenced by pricing dynamics or the acquisition of commodities. (You can see more about restrictions to transactions in Cea-Echenique and Torres-Martínez, 2016 and Faias and Torres-Martínez, 2017). Restrictions on consumers can take different forms and restrict consumers in different ways. These restrictions are divided into exogenous and endogenous. In the specific case we are studying, the restrictions are exogenous (You can read more about exogenous constraints in Balasko et al., 1990), since they do not depend on the direct decisions of the agents but are related to the market itself, in other words, they are external factors to consumers. It should be noted that both situations are very important in the economic context as they influence choices and financial results. Therefore, it is also important to understand how endogenous restrictions are influenced, for which you can find more information on this type of restriction at Sabarwal, 2003, Carosi et al., 2009, Seghir and Torres-Marínez, 2011, Hoelle et al., 2012, Gori et al., 2014 and Cea-Echenique and Torres-Martínez, 2018.

To reach the value of the price associated with income streams when there are restrictions on the transaction, we must first explain in detail what a standard economy. To do this, we first need to describe an economy, using some of the basic concepts normally associated with economics: time and uncertainty, commodities, consumers and their initial endowments. We also need to describe how prices and payoffs are presented. After this, we are now able to understand how we can arrive at the formula for the equilibrium valuation of an income stream that belongs to the space generated by the assets in an economy. Subsequently, we can conclude that the ideal security is given by the market portfolio, in other words, it is given by a linear combination between a risk-free security and the aggregate endowment of an economy. Once all these concepts have been presented, we can then go on to present the model that serves as the basis for pricing financial products and how it differs from the classic CAPM. Finally, we can move on to the maths required to price financial products where investors have restrictions on transactions and the returns associated with each of these prices.

This dissertation is organised as follows. Chapter 2 describes an economy with financial markets, the first sub-chapter describes the economy, the second sub-chapter explains the absence of arbitrage and state prices and the third sub-chapter helps to interpret state prices. Chapter 3 describes the valuation of income streams, the first sub-chapter briefly explains some concepts relating to probabilities, the second sub-chapter presents individual security pricing and the third sub-chapter specifically explains the ideal security of each agent i. Chapter 4 explains how the ideal security works in the equilibrium context of the Capital Asset Pricing Model. Chapter 5 presents the study model and some of the main results. Chapter 6 talks about the pricing of financial products where consumers have restrictions on transactions. The first sub-chapter introduces the pricing part and the second chapter calculates the return associated with the prices. Chapter 7 presents the conclusions of the work and possible future work related to this topic.

2

A standard economy with financial markets

The main objective of this dissertation is to arrive at the price of a financial product when there are restrictions on transactions. To get to this point, we must first explain in detail what a standard economy is like. To do this, we first describe an economy, using some of the basic concepts normally associated with economics: time and uncertainty, commodities, consumers and their initial endowments. We also describe how prices and payoffs are presented. In order to better understand basic economics and how we can arrive at the associated equilibrium prices, some examples are presented throughout this chapter.

In an economy the possibility of getting something without giving something in return should not exist. That's why it's important to study arbitrage and its absence. We must realise that for an economy to fully function, there must be no opportunities for arbitrage. To support this idea, we present the fundamental theorem of asset pricing, which is a very important theorem in economics. As before, we are now presented with explicit examples that allow us to understand whether there is arbitrage in an economy.

In order to proceed, we must first understand what state prices are and how their interpretation can influence what we think about an economy and how the consumer behaves, i.e. what they are willing to give to achieve their objectives. From what is presented in this chapter we can see that state prices are the marginal rate of substitution between consumption at date 0 and consumption in the state of nature $s = 1, \ldots, S$ at date 1.

(This chapter is based on Magill and Quinzii, 1996 where more detailed information on the valuation of income streams can also be found).

2.1 The Economy and Equilibrium

In order to understand the concepts of a standard economy, we now explain how it is described and what notations we use for each concept involved in an economy. The entire introduction to this topic will be useful in understanding the following chapters, so a detailed description is given in this sub-chapter.

We consider a pure exchange economy with two dates, $t = 0, 1$ and uncertainty at date 1 that is described by S ($s = 1, \ldots, S$) states of nature. There is one commodity at date 0 and at each state of nature of date 1. The consumption set is $\mathbb{R}_+^{S+1}$ and we denote by $x = (x_0, x_1, \ldots, x_S) \in \mathbb{R}_+^{S+1}$ a consumption bundle.

There are I ($i = 1, \ldots, I$) consumers in the economy. Each consumer i is characterized by a preference relation on $\mathbb{R}_+^{S+1}$ represented by the utility function u^i, which is continuously differentiable, and an endowment vector $\omega^i \in \mathbb{R}_+^{S+1}$ with $\omega^i = (\omega_0^i, \omega_1^i, \ldots, \omega_S^i)$ where ω_0^i denotes the initial endowment in $t = 0$ and ω_s^i denotes the initial endowment in $t = 1$ for the state $s = 1, \ldots, S$.

So, we have defined the most basic concepts of an economy: time, uncertainty, commodities, the agents present in the economy and the initial endowments associated with them. We can then go on to show how a financial structure can be described.

Financial Structure

A clearly defined financial structure is essential if the economy is to be stable. For this to happen, the economy must be defined as a whole, and all its details must be detailed, including, above all, all the details of the financial contracts that are permitted in the economy.

A contingency contract can be used to model an insurance contract, if its contingencies are specific. These dangers are typically physical risks that are beyond a person's control. Assuming that this sort of financial contract is the only one that can be signed and that it covers every possible economic contingency, the next step is to expand the definition of financial contracts to include this kind of arrangement.

Being in possession of a financial contract j, ($j = 1, \ldots, J$) that guarantees the delivery of $V_{s,j}$ units of a single good or income at state s, ($s = 1, \ldots, S$). Its cost q_j, which was paid at date 0, is expressed in the currency in use at that time. In this case, since we have only one good we assume that $p = 1$. We now go into more detail about the previously clearly stated conditions that apply to all buyers and sellers. Now concentrating on financial contracts, including bonds, corporate actions, futures contracts, options, and insurance, which can be traded through financial intermediaries like banks, savings and loan institutions, and insurance firms or sold on major financial markets.

In the analyses that follow standard assumptions, the markets in which financial contracts are traded are competitive, so that agents think they can buy and sell all the contracts they want without having any change in prices. Each contract can be bought or sold in split quantities and for these types of purchases and sales there are no transaction costs. We can consider that the penalties for not fulfilling a contract are very high and having this in account we can conclude that agents should have no interest in reneging on contracts.

The exogenously given date 1 payment of J securities can be represented by matrix $S \times J$:

$$V = \begin{bmatrix} V_{1,1} & \cdots & V_{1,J} \\ \vdots & \ddots & \vdots \\ V_{S,1} & \cdots & V_{S,J} \end{bmatrix}.$$

In an exchange economy that consists of I agents with the following characteristics: $(u^i, \omega) = (u^1, \ldots, u^I, \omega^1, \ldots, \omega^I)$, trading J bonds with payments at repayment date 1, given by matrix V, is denoted by:

$$\mathcal{E} = (V, (u^i, \omega^i),\ i = 1, \ldots, I).$$

Note that with S contingent contracts the matrix V is of dimension $S \times S$.

Let $q = (q_1, \ldots, q_J)$ denote the vector of security prices. A portfolio of assets is a vector $z = (z_1, \ldots, z_J) \in R^J$ where z_j, $j = 1, \ldots, J$ indicates the units of security j that an agent buys ($z_j > 0$) or sells ($z_j < 0$). The budget set for an agent i is given by:

$$B^i(q, \omega^i, V) = \left\{ (x, z) \in \mathbb{R}_+^{S+1} \times \mathbb{R}^J : \begin{array}{l} x_0 = \omega_0 - q \cdot z \\ x_s = \omega_s^i + V_s \cdot z, s = 1, \ldots, S \end{array} \right\} \tag{2.1}$$

Assuming W denotes the combination between date 0 and date 1 of the matrix of security payoffs:

$$W = W(q, V) = \begin{bmatrix} -q \\ V \end{bmatrix} = \begin{bmatrix} -q_1 & \cdots & -q_J \\ V_{1,1} & \cdots & V_{1,J} \\ \vdots & \ddots & \vdots \\ V_{S,1} & \cdots & V_{S,J} \end{bmatrix}$$

Then the budget set can be written based on the matrix W:

$$B^i\left(\omega^i, W\right) = \left\{ (x, z) \in \mathbb{R}_+^{S+1} \times \mathbb{R}^J : x - \omega^i = Wz \right\} \tag{2.2}$$

$B^i(q, \omega^i, V)$ denotes the budget set in the financial market for agent i. Each agent chooses the consumption bundle x and the portfolio z, so that agent i chooses the pair (x, z). Thus, when agent i chooses a portfolio z, it is said that z finances x.

Given a price vector for securities, $q = (q_1, \ldots, q_J)$, each individual chooses a consumption bundle x and a portfolio z in his budget set, $(x, z) \in B^i(q, \omega^i, V)$, that maximize the utility function $u^i(x)$, that is, each individual solves the problem:

$$\max_{(x,z) \in \mathbb{R}_+^{S+1} \times \mathbb{R}^J} u^i(x), \text{ subject to } (x, z) \in B^i(q, \omega^i, V).$$

If (x^i, z^i) is the solution of the problem of individual i then we can say that z^i is a portfolio that funds x^i and we can write as follows:

$$\left(x^i, z^i\right) \in \arg\max\left\{u^i\left(x\right) : \left(x, z\right) \in B^i\left(q, \omega^i, V\right)\right\}$$

Definition 2.1.1 (Financial Market Equilibrium). *A financial market equilibrium for the economy $\mathcal{E}$ is a price vector, an allocation of commodities and a portfolio allocation, $(\bar{q}, (\bar{x}^i, \bar{z}^i)_{i=1}^I) \in \mathbb{R}_+^J \times \mathbb{R}_+^{(S+1)I} \times \mathbb{R}^{JI}$, such that:*

1. $\left(\bar{x}^i; \bar{z}^i\right) \in arg\max\left\{u^i\left(x\right) \mid \left(x, z\right) \in B^i\left(\bar{q}, \omega^i, V\right)\right\}, i = 1, \ldots, I;$

2. $\sum_{i=1}^{I} \bar{z}^i = 0.$

Note that the equation $\sum_{i=1}^{I} \bar{z}^i = 0$ implies that the allocation of commodities is feasible, that is,

$$\sum_{i=1}^{I}\left(\bar{x}^i - \omega^i\right) = 0 \tag{2.3}$$

In order to apply the concepts and definitions presented in practice, here is an example that allows us to realise that there is in fact equilibrium in the economy and how we can arrive at the equilibrium prices of goods and the equilibrium consumption of each agent, present in the equilibrium of the economy.

Example 2.1.1. *Consider an economy with two dates, $t = 0$ and $t = 1$, two states of nature at $t = 1$, one good and two consumers. Given that the consumers are characterized by the following utility functions continuously differentiable:*

$$u_1\left(x_0, x_1, x_2\right) = \ln\left(x_0\right) + \frac{1}{2}\left(\frac{1}{2}\ln(x_1) + \frac{1}{2}\ln(x_2)\right)$$
$$u_2\left(x_0, x_1, x_2\right) = \ln\left(x_0\right) + \frac{1}{3}\left(\frac{1}{2}\ln(x_1) + \frac{1}{2}\ln(x_2)\right)$$

and with the following initial endowments: $\omega^1 = \left(\frac{19}{8}, 1, 3\right)$ and $\omega^2 = \left(\frac{21}{8}, 5, 3\right)$.

In this case the financial structure is described by the matrix $V = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and the assets are paid in units of good.

To reach the equilibrium of this economy let us first focus on the maximization problem of the agent 1:

$max\ ln(x_0) + \frac{1}{2}\left(\frac{1}{2}\ln(x_1)) + \frac{1}{2}\ln(x_2)\right)$

$$s.a \begin{cases} x_0 = \frac{19}{8} - qz \\ x_1 = 1 + z \\ x_2 = 3 + 0 \times z \end{cases} \Leftrightarrow \begin{cases} x_0 = \frac{19}{8} - qz \\ x_1 = 1 + z \\ x_2 = 3 \end{cases}$$

From this we can conclude that $z^1 = \frac{19}{40q} - \frac{4}{5}$.

We now turn our focus on the maximization problem of agent 2*:*
$max\ ln(x_0) + \frac{1}{6}\ln(x_1) + \frac{1}{6}\ln(x_2)$

$$s.a \begin{cases} x_0 = \frac{21}{8} - qz \\ x_1 = 5 + z \\ x_2 = 3 + 0 \times z \end{cases} \Leftrightarrow \begin{cases} x_0 = \frac{21}{8} - qz \\ x_1 = 5 + z \\ x_1 = 3 \end{cases}$$

From this we can conclude that $z^2 = \frac{21}{56q} - \frac{30}{7}$
Then:
$z^1 + z^2 = 0 \Leftrightarrow q = 0,17$

Thus we have the following solutions to the problem:
$(\bar{p}, \bar{q}) = (1, 1, 1, 0.17)$
$(\bar{x}_0^1, \bar{x}_1^1, \bar{x}_2^1, z^1) = (2.035, 3, 3, 2)$
$(\bar{x}_0^2, \bar{x}_1^2, \bar{x}_2^2, z^2) = (2.965, 3, 3, -2)$

The calculations needed to solve this exemple can be found in the Appendix.

2.2 Absence of Arbitrage and State Prices

Thinking about a real economic case, it's quite easy to realise that there are no free goods, i.e. you can't get anything if you don't actually give something in return. In other words, arbitrage exists when investors take advantage, for example, of price differences between similar assets to make a profit without any risk. The absence of arbitrage is when these profit opportunities do not exist. There are various consequences for the economy if there is arbitrage, including market inefficiency, which makes asset prices unfair.

In other words, in an environment where investment opportunities have constant returns to scale, we can say that there are no arbitrage opportunities if there is no investment strategy that causes the agent i to have a positive return in at least one of the states and non-negative returns in all the other states. Since there are constant returns to scale on investments in competitive financial markets, an investment strategy with an arbitrage opportunity can create a very high return in some state without the corresponding sacrifice in any state. Since financial markets are characterised by the matrix $W(q, V)$, this leads us to the following definition:

Definition 2.2.1. *Given (q,V) then there are no arbitrage opportunities in financial markets if there is no portfolio generating a semi-positive income stream, i.e. there is no* $z \in \mathbb{R}^J$ *such that* $\omega_z > 0$.

Definition 2.2.2. *q is a vector of arbitrage-free security prices, relative to V, if given* (q, V) *there are no arbitrage opportunities in the financial market.*

As seen earlier the financial model can be interpreted in two ways: as a single-good version of the general model or as an income model. The interpretation of the first option can be useful to explore the formal structure of the financial model and its relationship. But when the intent is to understand what the economic role of financial markets is, it is better to use the second interpretation, seeing the financial model as an income model.

In financial markets agents are allowed to redistribute their income across S+1 states. Thus, agent i by purchasing the portfolio of securities:

$$z^i = (z_1^i, \ldots, z_J^i) \in \mathbb{R}^J$$

which leads us to an income transfer vector between the states:

$$\tau^i \in \mathbb{R}^{S+1}, \tau^i = Wz^i$$

The set of all income transfers that are possible can be obtained in this way and the subspace generated by the columns J and matrix W that is denoted by $\langle W \rangle$, thus the generated subspace, which summarizes the opportunities offered by financial markets, can be given by:

$$\langle W \rangle = \left\{ \tau \in \mathbb{R}^{S+1} \mid \tau = Wz, \quad z \in \mathbb{R}^J \right\}$$

The following example serves to explain how the space generated by a matrix is obtained, since the concept of space generated by a matrix is essential to understanding the next chapters.

Example 2.2.1. *Considering the matrix A:*

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 3 & 1 \end{bmatrix}$$

In this matrix A we have two assets and three states of nature, the space generated by it is as follows:

$$\langle A \rangle = \{\alpha(1,2,3) + \beta(1,0,1), \alpha, \beta \in \mathbb{R}\}$$

Thus, the absence of arbitrage in financial markets can be expressed as a geometric property of the market subspace, having only one point in common with the non-negative octant, zero. Thus the absence of arbitrage can be written as follows:

$$\langle W \rangle \cap \mathbb{R}_+^{S+1} = \{0\}$$

The requirement that the subspace generated by the market has no point in common with the semi-positive octant is easily interpreted as the implication that there exists a strictly positive vector of state prices that is at right angles to the market.

Considering the simplest case, where there is one state of nature and only one asset, then the matrix W representing this asset is as follows:

$$W = \begin{bmatrix} -q_1 \\ 1 \end{bmatrix}$$

and $\langle W \rangle$ is the line through the origin and intersects with the point $(-q_1, 1)$. In order for the no-arbitrage condition to be satisfied then price q_1 must be positive, that is, if there is a unit of payment at date t=1, then there must be some cost to agent i at date t=0. If the no-arbitrage condition is satisfied, we are then faced with a vector $\bar{\pi} = (\bar{\pi}_0, \bar{\pi}_1) \in \mathbb{R}^2_{++}$, a vector that makes a right angle with the vector W (in this case with a single column), and so we can write that:

$$\bar{\pi}W = 0 \tag{2.4}$$

If we normalize p and write it as $p = (1, p_1)$, then we can take from the equation above that $q_1 = p_1$. From the equation above we can also derive that q_1 is the present value of p_1 at date 0 of one unit of income at date 1.

If the agents' preferences are strongly monotone, then the equation $\langle W \rangle \cap \mathbb{R}^{S+1}_{+} = \{0\}$ must guarantee that if each agent i wants to have a solution then its maximum problem must be as follows:

$$\max \left\{ u^i \left(x^i \right) \mid x^i \in \mathrm{B} \left(\bar{q}, \omega^i, V \right) \right\}.$$

Theorem 2.2.1 (Fundamental theorem of asset pricing)**.** *There are no arbitrage opportunities in the financial market if and only if there exists a positive state price vector $\bar{\pi} \in \mathbb{R}^{S+1}_{++}$ such that $\bar{\pi}W(\bar{q}, V) = 0$.*

This is the fundamental theorem concerning the prices of financial securities, through this theorem we see that the absence of arbitrage opportunities is equivalent to the existence of a state price vector $\bar{\pi} \in \mathbb{R}^{S+1}_{++}$. For each security j, the state price vector $\bar{\pi}$ defines the investment profit, which consists of buying one unit of security j, for price q_1 at date $t = 0$ and making yield $V_{s,j}$ at date $t = 1$. Considering the matrix W already defined above, then the equality

$$\bar{\pi}W_j = 0 \tag{2.5}$$

implies that there is zero profit on one unit of investment in each security $j = 1, \ldots, J$. Thus, the theorem states that non-arbitrage is equivalent to the existence of a positive vector of state prices over which the investment in each security has no profit.

In order to better understand the concept of state prices and how we can see whether or not there is arbitrage in an order, here's an example.

Example 2.2.2. *Considering the return matrix* $V = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$ *and the price vector* $q = \begin{bmatrix} 1 \\ 2 \\ 0.5 \end{bmatrix}$*. We can show that there is no arbitrage by using* $\bar{\pi}W = 0$*, defined earlier, from which we then proceed to have:*

$$\begin{bmatrix} 1 & \bar{\pi}_1 & \bar{\pi}_2 & \bar{\pi}_3 \end{bmatrix} \times \begin{bmatrix} -2 & -1 & -0.5 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} -2 + 2\bar{\pi}_1 + \bar{\pi}_2 = 0 \\ -1 + \bar{\pi}_1 + \bar{\pi}_2 + \bar{\pi}_3 = 0 \\ -0.5 + \bar{\pi}_1 - \bar{\pi}_3 = 0 \end{cases} \Leftrightarrow \begin{cases} \bar{\pi}_2 = 2 - 2\bar{\pi}_1 \\ -1 + \bar{\pi}_1 + 2 - 2\bar{\pi}_1 - \bar{\pi}_1 + 0.5 = 0 \\ \bar{\pi}_3 = \bar{\pi}_1 - 0.5 \end{cases} \Leftrightarrow \begin{cases} \bar{\pi}_2 = 0.5 \\ \bar{\pi}_1 = 0.75 \\ \bar{\pi}_3 = 0.25 \end{cases}$$

This market does not allow arbitrage since the solution of the system is strictly positive, $(\bar{\pi}_1, \bar{\pi}_2, \bar{\pi}_3) = (0.75, 0.5, 0.25)$*, therefore this is the state price vector for this financial structure.*

The null profit condition can be written as:

$$\pi_0 \bar{q} = \pi_1 V$$

or with $\bar{\pi} = (\bar{\pi}_0, \bar{\pi}_1)$ normalized so that $\bar{\pi}_0 = 1$, as

$$\bar{q} = \bar{\pi}_1 V \Longleftrightarrow \bar{q}_j = \bar{\pi}_1 V_{1,j} + \ldots + \bar{\pi}_s V_{s,j}, \quad j = 1, \ldots, J$$

Thus $\bar{\pi}_s$ represents the present value, at date 0, of a unit of income in state s, for every $s = 1, \ldots, S$. From the previous equation we can derive that the security price q_j is the present value of its future income stream $V_{s,j}$. We can also use the previous formula to obtain a price formula for any type of flows, which can be obtained by creating a portfolio of securities J. The choice of this portfolio gives at date 1 the following returns:

$$y = \sum_{j=1}^{J} V_j z_j \tag{2.6}$$

where $V_j = \left(V_{1,j}, \ldots, V_{s,j}\right)$ is the j^{th} column of payoffs of matrix V, already defined earlier. The set of all portfolios $z \in \mathbb{R}^J$ generate a subspace $\mathbb{R}^s$ consisting of the columns V which can be represented by:

$$\begin{aligned} \langle V \rangle &= \left\{ y \in \mathbb{R}^s \mid y = \sum_{j=1}^{J} V_j z_j, \quad z \in \mathbb{R}^J \right\} \\ &= \left\{ y \in \mathbb{R}^s \mid y = Vz, \quad z \in \mathbb{R}^J \right\} \end{aligned}$$

Let's now look at the price formula for an income stream and how it can be obtained in an economy.

Definition 2.2.3. *Assuming $q = (q_1, \ldots, q_J)$ a vector of prices for J bonds, then the cost $c_q(y)$ of a yield y is defined by:*

$$c_q(y) = q \cdot z, such that y = Vz \tag{2.7}$$

There are two ways to calculate the price of an income stream, as seen above. To prove this, follow the examples below.

Example 2.2.3. *Considering the matrix* $V = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, *the price vector* $q = (1.2, 1.8, 1.2)$ *and a new income stream* $\begin{bmatrix} 2 & 10 & 4 \end{bmatrix}^{\perp}$ *we start by determining the price of this income stream through the replication portfolio (Definition 2.2.3.):*

$$\begin{bmatrix} 3 & 4 & 2 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 10 \\ 4 \end{bmatrix}$$

$$\begin{cases} 3z_1 + 4z_2 + 2z_3 = 2 \\ 2z_1 + z_3 = 10 \\ z_3 = 4 \end{cases} \Leftrightarrow \begin{cases} z_1 = -6 \\ z_2 = 3 \\ z_3 = 4 \end{cases}$$

Then the replica portfolio is given by: $(z_1, z_2, z_3) = (-6, 3, 4)$ *and the price is:* $1,2 \times (-6) + 1,8 \times 3 + 1,2 \times 4 = 3$.

Turning now to calculating the price of the income stream through state prices we start by calculating the state prices:

$$\begin{cases} 1,2 = 3\pi_1 \\ 1,8 = 4\pi_1 + 2\pi_2 \\ 1,2 = 2\pi_1 + \pi_2 + \pi_3 \end{cases} \Leftrightarrow \begin{cases} \pi_1 = 0,4 \\ \pi_2 = 0,1 \\ \pi_3 = 0,3 \end{cases}$$

So the income stream prices are given by: $0,4 \times 2 + 0,1 \times 10 + 0,3 \times 4 = 3$

Thus concluding that the price of an income stream can be obtained in two different ways and that both give the same result.

Example 2.2.4. *Considering the matrix* $V = \begin{bmatrix} 0 & 1 \\ 2 & 2 \\ 0 & 3 \end{bmatrix}$, *the price vector* $q = (1, 2)$ *and two new income streams* $z_1 = \begin{bmatrix} 1 & 4 & 3 \end{bmatrix}^{\perp}$ *and* $\begin{bmatrix} 2 & 5 & 3 \end{bmatrix}^{\perp}$ *we start by determining the price of income*

m_1 through the replication portfolio (Definition 6.4):

$$\begin{bmatrix} 0 & 1 \\ 2 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} \Leftrightarrow \begin{cases} z_2 = 1 \\ 2z_1 + 2z_2 = 4 \\ 3z_2 = 3 \end{cases} \Leftrightarrow \begin{cases} z_2 = 1 \\ z_1 = 1 \end{cases}$$

Then the replica portfolio exist and is given by: $(z_1, z_2) = (1, 1)$, then the price is: $1\times1+1\times2 = 3$.
Let us now determine the price of the income y_2 through the replica portfolio:

$$\begin{bmatrix} 0 & 1 \\ 2 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix} \Leftrightarrow \begin{cases} z_2 = 1 \\ 2z_1 + 2z_2 = 5 \\ 3z_2 = 3 \end{cases}$$

This system is not possible, which means that there is no replica portfolio to buy today that will allow us to get the income stream return z_2 "tomorrow". This is because the market is incomplete, that is, there are more states of nature than there are assets.

Turning now to calculating the price of the z_1 and z_2 income stream through state prices we start by calculating the state prices:

$$\begin{cases} 1 = 2\pi_2 \\ 2 = \pi_1 + 2\pi_2 + 3\pi_3 \end{cases} \Leftrightarrow \begin{cases} \pi_2 = \frac{1}{2} \\ \pi_1 = 1 - 3\pi_3 \end{cases}$$

Then:

$q_1 = (1 - 3\pi_3) \times 1 + \frac{1}{2} \times 4 + 3 \times \pi_3 = 1 - 3\pi_3 + 2 + 3\pi_3 = 3$
$q_2 = (1 - 3\pi_3) \times 2 + \frac{1}{2} \times 5 + 3 \times \pi_3 = 2 - 6\pi_3 + \frac{5}{2} + 3\pi_3 = \frac{9}{2} - 3\pi_3$

Income stream z_1 is redundant as it has a replica portfolio and in turn is single priced. The z_2 income stream is non-redundant since it has no replica portfolio.

In the case of z_2, there are two ways to arrive at prices for this income stream, either through state prices, as shown, or through sub-replication and super-replication portfolios. Since we know that $1 - 3\pi_3 > 0$ then $\pi_3 < \frac{1}{3} \Leftrightarrow 0 < \pi_3 < \frac{1}{3} \Leftrightarrow -1 < -3\pi_3 < 0 \Leftrightarrow \frac{7}{2} < q_2 < \frac{9}{2}$. So the non-arbitrage price for z_2 is undetermined, and is given by any value in the interval $\left]\frac{7}{2}, \frac{9}{2}\right[$.

This example illustrates that when the market is complete the price of the income stream is unique, but when the market is incomplete the price of the income stream is unique if the income stream is generated by a portfolio of the assets, i.e., it has a replica portfolio, otherwise, the price is not unique. It should be noted that in these two examples, a different notation was used for the income stream, each one being considered as z.

2.3 Interpretation and Computation of the State Prices

State prices are a very important concept to consider and understand about an economy. As we'll see throughout this sub-chapter, state prices are nothing more than the price an investor would be willing to pay or receive to switch between different future states of well-being or wealth. State prices also reflect consumers' preferences regarding risk and uncertainty. In order to arrive at this interpretation, we must first define the first-order conditions to be considered in order to solve the maximization problem for each agent i.

The maximization problem for an agent i is given by:

$$\max_{(x^i,z^i)\in\mathbb{R}_+^{S+1}\times\mathbb{R}^J}\left\{u^i\left(x^i\right)\mid \begin{array}{l} x_0^i-\omega_0^i=-\bar{q}z^i \\ x_s^i-\omega_s^i=V_s z^i, \quad s=1,\ldots,S \end{array}\right\} \tag{2.8}$$

Considering that the utility function u^i is continuously differentiable, we can then move on to the next step.

Denoting the Lagrange multipliers by $\lambda^i=\left(\lambda_0^i,\lambda_1^i,\ldots,\lambda_s^i\right)$ (only in this chapter are the Lagrange multipliers represented by λ) we can form the Lagrangian function as follows:

$$L^i\left(x^i,z^i,\lambda^i\right)=u^i\left(x^i\right)-\lambda_0^i\left(x_0^i-\omega_0^i+qz^i\right)-\sum_{s=1}^{S}\lambda_s^i\left(x_s^i-\omega_s^i-V_s z^i\right) \tag{2.9}$$

The necessary and sufficient first-order conditions for $\left(\bar{x}^i,\bar{z}^i\right)$ to be the solution of the utility maximization problem of agent i, is that there exists $\bar{\lambda}\in\mathbb{R}_{++}^{S+1}$ such that:

$$\nabla L^i\left(\bar{x}^i,\bar{z}^i,\bar{\lambda}^i\right)=0 \tag{2.10}$$

which is equivalent to saying that

$$\nabla u^i\left(\bar{x}^i\right)=\bar{\lambda}^i \tag{2.11}$$

$$-\bar{\lambda}_0^i\bar{q}+\sum_{s=1}^{S}\bar{\lambda}_s^i V_s=0 \tag{2.12}$$

$$\bar{x}_0^i-\omega_0^i=-\bar{q}\bar{z}^i, \quad \bar{x}_s^i-\omega_s^i=V_s\bar{z}^i, \quad s=1,\ldots,S \tag{2.13}$$

Assuming that the accounts of each state are kept in units of goods, we can conclude from the equation $\nabla u^i\left(\bar{x}^i\right)=\bar{\lambda}^i$ that the vector of marginal utilities of income is equal to the vector of marginal utilities of the good. If we divide this same equation by the marginal utility of the good at date 0 we get:

$$\bar{\pi}^i=\pi^i\left(\bar{x}^i\right)\text{ with }\bar{\pi}^i=\left(\frac{1}{\bar{\lambda}_0^i}\right)\bar{\lambda}^i \tag{2.14}$$

which tells us that the normalized vector of multipliers is equal to the vector of the current value, and so we can now write the equation as follows:

$$\bar{q}_j=\sum_{s=1}^{S}\bar{\pi}_s^i V_{s,j}, \quad j=1,\ldots,J \tag{2.15}$$

Analyzing the three equivalences that we can draw from the maximization problem of agent i, we know that these accounts allow us to interpret the state prices. Developing the second equation we now have the following:

$$q_j = \frac{\lambda_1}{\lambda_0} V_{1,j} + \cdots + \frac{\lambda_s}{\lambda_0} V_{s,j} + \cdots + \frac{\lambda_S}{\lambda_0} V_{S,j}, \quad j = 1, \cdots, J \tag{2.16}$$

From the first equation, we have that:

$$\lambda_s = \frac{\partial u^i(x^i)}{\partial x_s}, \quad s = 0, 1, \ldots, S \tag{2.17}$$

Then:

$$q_j = \frac{\frac{\partial u^i(x^i)}{\partial x_1}}{\frac{\partial u^i(x^i)}{\partial x_0}} V_{1,j} + \cdots + \frac{\frac{\partial u^i(x^i)}{\partial x_s}}{\frac{\partial u^i(x^i)}{\partial x_0}} V_{s,j} + \cdots + \frac{\frac{\partial u^i(x^i)}{\partial x_S}}{\frac{\partial u^i(x^i)}{\partial x_0}} V_{S,j}, \quad j = 1, \cdots, J \tag{2.18}$$

If we take into account non-arbitrage pricing, referred to earlier in this chapter, we know that:

$$q_j = \pi_1 V_{1,j} + \cdots + \pi_s V_{s,j} + \cdots + \pi_S V_{S,j}, \quad j = 1, \cdots, J \tag{2.19}$$

where (π_s) represents the vector of state prices, so one can conclude that:

$$\pi_s = \frac{\frac{\partial u^i(x^i)}{\partial x_s}}{\frac{\partial u^i(x^i)}{\partial x_0}}, \quad s = 1, \ldots, S. \tag{2.20}$$

Which is the marginal rate of substitution between consumption at date 0 and consumption in the state of nature $s = 1, \ldots, S$ at date 1.

As we said before, state prices are the prices that an investor is willing to pay or receive to exchange wealth between the present and different states of nature in the future. As we have seen, by computing state prices we conclude that what an investor is willing to pay or receive in this case is simply the marginal rate of substitution between consumption at date 0 and the state of nature $s = 1, \ldots, S$ at date 1. This marginal rate of substitution is represented by the amount of one good that the individual is willing to sacrifice in order to obtain an additional unit of the good in the future in a given state of nature.

3

Valuation of Income Streams

Our goal is to price assets under the CAPM context. One of the main assumptions of the CAPM model is that preferences only depend on the expected value and variance of the returns of income streams and then, we need to fix a probability measure to describe our uncertainty structure.

In this Chapter, we introduce the probability measure over the set of states of nature. Then we will derive a formula for the equilibrium valuation of an income stream that belongs to the subspace generated by the assets in the economy.

To obtain the formula we first compute the individual valuation of an income stream belonging to $\langle V \rangle$. Then we compute the ideal security of an agent i and finally, we obtain the equilibrium valuation of an income stream in $\langle V \rangle$ as a function of the ideal security, which is the same for every agent in equilibrium, and the risk-free rate. More precisely, we obtain that the equilibrium value of an income stream in $\langle V \rangle$ is the expected value of the income stream times the value of the risk-less asset plus the covariance between the ideal security and the income stream.

The next chapter will be closely related to this one, where we'll show that under the CAPM assumptions, the ideal security is the market portfolio, that is, a linear combination between the risk-less security and the aggregate endowment of the economy.

(This chapter is based on Magill and Quinzii, 1996 where more detailed information on the valuation of income streams can also be found).

3.1 Introduction to Probability Concepts

In order to give a brief introduction about probability concepts that are fundamental to understanding the following sub-chapters, we begin by considering a market with $s = (1, \ldots, S)$ states of nature, $y = (y_1, \ldots, y_s, \ldots, y_S)$ income streams (the notation y associated with income streams is used only in this sub-chapter) and $\rho = (\rho_1, \ldots, \rho_s, \ldots, \rho_S)$, which represents the probability of occurrence of each state of nature.

We then know that the expected value of Y is given by:

$$E[Y] = \rho_1 y_1 + \cdots + \rho_s y_s + \cdots + \rho_S y_S = \sum_{s=1}^{S} \rho_s y_s \tag{3.1}$$

and the variance of Y is given by:

$$Var[Y] = E[Y-E[Y]]^2 = E[Y^2]-E[Y]^2 = \rho_1(y_1-E[Y])^2+\cdots+\rho_s(y_s-E[Y])^2+\cdots+\rho_S(y_S-E[Y])^2 \tag{3.2}$$

Thus, the covariance between Y and a random variable Z is given by:

$$cov(Y,Z) = E[[Y - E[Y]][Z - E[Z]]] = E[YZ] - E[Y]E[Z] \tag{3.3}$$

The expected value given by the product of Y by Z is:

$$E[YZ] = \rho_1 y_1 z_1 + \cdots + \rho_s y_s z_s + \cdots + \rho_S y_S z_S \tag{3.4}$$

Let us consider two inner products,

$$Y \cdot Z = y_1 z_1 + \cdots + y_s z_s + \cdots + y_S z_S = [\![Y, Z]\!] \tag{3.5}$$

And:

$$[\![Y, Z]\!]_\rho = \rho_1 y_1 z_1 + \cdots + \rho_s y_s z_s + \cdots + \rho_S y_S z_S = E[YZ] = E[Y]E[Z] + cov(Y,Z) \tag{3.6}$$

Having presented the basic concepts of probability, understanding these terms is fundamental to being able to move on to the following sub-chapters and understand everything. The main idea, wich is fundamental to understand this introduction of concepts is in relation to the ρ, which represents the probability of occurrence in each state of nature, the inner product associated with two assets when considering this new probability.

3.2 Individual Security Pricing

Individual security pricing is an essential component in the study of an economy. It allows investors to assess the risk and return associated with a specific asset, which is very important when making investment decisions. The CAPM model is the most widely used in these cases and relates the expected return of an asset to its associated risk and risk-free return. This model is fundamental for pricing assets in efficient markets.

The value that an individual i assigns to an income stream $m_\mathbf{1} = (m_1, \ldots, m_S) \in \mathbb{R}^S$ is the amount m_0 that he can give up of the good at date 0 maintaining the same level of utility. Let us denote the consumption at date 1 by, $x_\mathbf{1} = (x_1, \ldots, x_S)$. Therefore we are imposing that $u^i(x_0, x_\mathbf{1}) = u^i(x_0 - m_0, x_\mathbf{1} + m_\mathbf{1})$. When the utility function u^i is continuously differentiable, m_0 may be obtained by solving the linear equation,

$$du^i_x(-m_0, m_\mathbf{1}) = 0. \tag{3.7}$$

$$du_x^i(-m_0, m_{\mathbf{1}}) = -\frac{\partial u^i(x^i)}{\partial x_0}(x)m_0 + [\![\Delta_{\mathbf{1}} u^i(x), m_{\mathbf{1}}]\!] = 0 \tag{3.8}$$

where

$$\Delta_{\mathbf{1}} u^i(x) = \left(\frac{\partial u^i(x^i)}{\partial x_1}(x), \dots, \frac{\partial u^i(x^i)}{\partial x_S}(x)\right). \tag{3.9}$$

Thus, $m_0 = c^i(m_{\mathbf{1}}; x) = [\![\pi_{\mathbf{1}}^i(x), m_{\mathbf{1}}]\!]$ where $\pi_{\mathbf{1}}^i(x) = \left(1/\frac{\partial u^i(x^i)}{\partial x_0}(x)\right)\Delta_{\mathbf{1}} u^i(x)$.

If the income stream belongs to the subspace $\langle V\rangle$, that is, if the income stream has a replica portfolio, then,

$$du_x^i(-m_0, m_{\mathbf{1}}) = -\frac{\partial u^i(x^i)}{\partial x_0}(x)m_0 + [\![\Delta_V u^i(x), m_{\mathbf{1}}]\!] = 0 \tag{3.10}$$

where $\Delta_V u^i(x)$ is the orthogonal projection of the date 1 gradient $\Delta_{\mathbf{1}} u^i(x)$ onto $\langle V\rangle$.

We can now write that $m_0 = c^i(m_{\mathbf{1}}; x) = [\![\pi_V^i(x), m_{\mathbf{1}}]\!]$ where

$$\pi_V^i(x) = \left(1/\frac{\partial u^i(x^i)}{\partial x_0}(x)\right)\Delta_V u^i(x), \quad \forall m_{\mathbf{1}} \in \langle V\rangle. \tag{3.11}$$

In order to make appear means, variances and covariances in the cost of an income stream, $c^i(m_{\mathbf{1}}; x)$, we are going to use the probability inner product $[\![,]\!]_\rho$ instead of the standard inner product $[\![,]\!]$ on $\mathbb{R}^{S+1}$.

Let

$$\Delta_{\mathbf{1}}^\rho u^i(x) = \left(\frac{1}{\rho}\frac{\partial u^i(x^i)}{\partial x_1}(x), \dots, \frac{1}{\rho}\frac{\partial u^i(x^i)}{\partial x_S}(x)\right), \tag{3.12}$$

$\Delta_V^\rho u^i(x)$ be the ρ-ortogonal projection of $\Delta_{\mathbf{1}}^\rho u^i(x)$ onto $\langle V\rangle$ and $\pi_V^{\rho,i} = \left(1/\frac{\partial u^i(x^i)}{\partial x_0}(x)\right)\Delta_V^\rho u^i(x)$. Then,

$$m_0 = c^i(m_{\mathbf{1}}; x) = [\![\pi_V^{\rho,i}, m_{\mathbf{1}}]\!]_\rho = E(\pi_V^{\rho,i})E(m_{\mathbf{1}}) + cov(\pi_V^{\rho,i}, m_{\mathbf{1}}), \quad \forall m_{\mathbf{1}} \in \langle V\rangle. \tag{3.13}$$

$\pi_V^{\rho,i}$ is the ρ-orthogonal projection of $\pi_{\mathbf{1}}^{\rho,i}$ onto $\langle V\rangle$, $\pi_V^{\rho,i}$ is unique and lies in $\langle V\rangle$, then can be interpreted as the pricing security of individual i at x.

This sub-chapter provides us with knowledge about individual security pricing, i.e. after defining a function u^i by imposition and differentiating it, we arrive at a value of m_0 and from which, consequently, we can arrive at a state price $\pi_V^i(x)$ and from which we can derive, by making an orthogonal ρ projection, a value of $\pi_V^{\rho,i}(x)$. Since m_0 is the cost of an income stream we can then conclude from what it is given, which will be important knowledge in understanding the next sub-chapter.

3.3 Ideal Security of Agent i

Starting in this sub-chapter by calculating the optimal security of each agent in $\langle V\rangle$, and associating this with the subject studied previously, we will now obtain the equilibrium valuation of an income stream in $\langle V\rangle$, as a function of the optimal security, which, as we can conclude later, is the same for all agents in equilibrium, and of the risk-free rate. Specifically, we will obtain that the equilibrium value of an income stream in $\langle V \rangle$ is the expected value of the income stream times the value of the risk-free asset plus the covariance between the optimal security and the income stream.

This security has a very important meaning for agent i. We start by considering that the agent is allowed to change its consumption stream, h:

$$x^i \to x^i + h$$

subject to the condition that the increment $h = (h_0, h_1)$ satisfies:

$$h_1 \in \langle V \rangle \quad |h_0|^2 + ||h_1||_\rho^2 = k^2$$

Since k is a small value we can then assume that $u^i(x^i + h) \simeq u^i(x^i) + du^i_{x^i}(h)$ and so the objective of agent i is to maximize the following differential:

$$du^i_{x^i}(h)) = [\![\nabla u^i(x^i), h]\!] = ||\nabla u^i(x^i)||\, ||h||\, \cos(\widehat{\nabla u^i(x^i), h})$$

Since we know that $||h|| = k$ and $||\nabla u^i(x^i)||$ does not depend on h, then the maximization of the differential is such that h^* is collinear with $\nabla u^i(x^i)$ and hence it follows that h^* is given by:

$$h^* = (h_0^*, h_1^*) = \alpha(\nabla u^i(x^i)) = \alpha(\lambda_0^i(x^i), \nabla_V^\rho u^i(x^i)) \tag{3.14}$$

Definition 3.3.1. *The security $\gamma^i(x^i) = \frac{h_1^*}{h_0^*}$, where (h_0^*, h_1^*) is the income stream that maximizes the change in utility $du^i_{x^i}(h)$ subject to $h_1 \in \langle V \rangle$ and $|h_0|^2 + ||h_1||_\rho^2 = k^2$, for a non-zero value of k, is called the ideal security of agent i at $x^i \in \mathbb{R}^{S+1}_{++}$.*

The following proposition relates the pricing security of an agent, defined above, to the ideal security concept of each agent i.

Proposition 3.3.1 (Equality of Pricing and Ideal Security)**.** *For any $x^i \in \mathbb{R}^{S+1}_{++}$, the pricing security and the ideal security of agent i coincide:*

$$\gamma^i(x^i) = \pi_V^{\rho,i}(x^i) \tag{3.15}$$

Since $\gamma^i = \frac{h_1^*}{h_0^*} = \frac{\nabla_V^\rho u^i(x^i)}{\frac{\partial u^i(x^i)}{\partial x_0}(x^i)} = \pi_V^{\rho,i}(x^i)$.

Equilibrium Valuation

Let $((x, z), q)$ be the equilibrium of the financial economy. In the equilibrium the basic bonds J, whose redemption date 1 are the columns J, $[V_1, \ldots, V_J]$ of matrix V, are traded in the market at prices $q = (q_1, \ldots, q_J)$. Any income stream in the generated space $\langle V \rangle$ can be obtained through a portfolio of these securities J. The market value of an income stream $y \in \langle V \rangle$ (here we denote an income stream as y), denoted by $c_q(y)$ is the cost of the portfolio z that generates income stream y:

$$c_q(y) = q \cdot z \tag{3.16}$$

The function $c_q : \langle V \rangle \longrightarrow \mathbb{R}$ is called the market equilibrium validation function since agents trade J securities in the common price vector, and thus each agent's personal validation for each income stream in the market subspace will coincide with the market equilibrium validation of the income stream. Thus, from the first order equilibrium conditions (sub-chapter 2.3) we have that for each agent $i = 1, \ldots, I$:

$$c_q(y) = q \cdot z = \sum_{s=1}^{s} \bar{\pi}_s^i V_{s,j} \cdot z = [\![\pi_1^i(x^i), y]\!]_\rho = c^i(y, x^i) \tag{3.17}$$

The equilibrium value of an income stream $\mathbb{1} = (1, \ldots, 1)$ without risk that defines the interest rate equilibrium r:

$$c_q(\mathbb{1}) = \frac{1}{1+r} = \gamma \cdot \mathbb{1} \tag{3.18}$$

Theorem 3.3.1. *Let $\mathcal{E} = (V, (u^i, \omega^i),\ i = 1, \ldots, I)$ be the economy, if $((x, z), q)$ is the financial market equilibrium, then the optimal security of all agents in their consumption equilibrium coincides:*

$$\gamma^1(x^1) = \cdots = \gamma^I(x^I) = \gamma \tag{3.19}$$

and the common vector γ is called the ideal security at the equilibrium. The market value at the equilibrium of any income stream y in the subspace $\langle V \rangle$ is given by:

$$c_q(y) = \frac{E[y]}{1+r} + cov(\gamma, y) \tag{3.20}$$

where r is the equilibrium interest rate and considering that $\gamma \cdot y = E[y]E[\gamma] + cov(\gamma, y)$.

Proof:

Knowing that

$$c_q(y) = c^i(y, x^i) = [\![\pi_V^{\rho,i}, y]\!]_\rho \tag{3.21}$$

and that through the Proposition 3.3.1.:

$$\pi_V^{\rho,i} = \gamma^i(x^i) \tag{3.22}$$

then we have that:

$$[\![\pi_V^{\rho,i}, y]\!]_\rho = [\![\gamma^i(x^i), y]\!]_\rho \tag{3.23}$$

which implies that:

$$c_q(y) = [\![\gamma^i(x^i), y]\!]_\rho \tag{3.24}$$

from which it emerges that:

$$c_q(y) = E[\gamma^i(x^i)]E[y] + cov(\gamma, y) \tag{3.25}$$

we have that:

$$E[\gamma] = [\![\gamma, \mathbb{1}]\!]_\rho = c^i(\mathbb{1}, x^i) = \frac{1}{1+r} \tag{3.26}$$

and so:

$$c_q(y) = \frac{E[y]}{1+r} + cov(\gamma, y) \tag{3.27}$$

as we wanted to prove.

The result of Theorem 3.3.1 is very important because it tells us that the optimal security of all agents, in equilibrium, coincides. Since this is the same for everyone in equilibrium, it can then be called optimal security in equilibrium. At equilibrium the pricing security is the same for every agent then the ideal security should be the same. This theorem has a result that will be useful later on, so it's important to understand it.

4

Capital Asset Pricing Model

In this chapter, contrary to what was presented in the previous chapter, we show that under the CAPM assumptions, also presented in this chapter, the ideal security is the market portfolio, i.e. it is given by a linear combination between the risk-free asset and the aggregate endowment of the economy. In addition, in this chapter we will also find what the market price of an income stream is.

In the following chapters, the aim is to calculate the return (or risk premium) of financial products, so it is important to present the formula for the CAPM model.

(This chapter is based on Magill and Quinzii, 1996 where more detailed information on the valuation of income streams can also be found).

The Capital Asset Pricing Model is like a model of an equity economy with risk-tolerant utility functions, as studied earlier. The similarity between these two models translates into investors' attitudes toward risk, and this is modeled by the assumption of mean-variance preferences. Investors with medium variance preferences do not care about the exact profile of a random consumption stream at date 1. The income stream is evaluated only by the properties of the averages summarized in their mean and variance. This assumption seems acceptable if we consider a typical consumer, for whom the case of no consumption in some states of nature would be a matter of concern. This insinuates that the model must be interpreted as one in which agents are institutional investors, and their earnings, as opposed to their consumption, are qualifiable.

The price formula obtained in the previous Chapter implies that the value of the covariance of a security is a linear decreasing function with respect to the covariance of the income stream with aggregate. Thus, we now have a more precise result since in the previous Chapter the value of the covariance could only be determined through the income streams, which are positive or negative depending on aggregate output.

When it comes to finance it follows the practice of allowing utility functions to be defined over the entire $\mathbb{R}^{S+1}$, the utility of negative values is reflected in the disutility assigned to losses because in the CAPM investors rate their gains.

4.1 Ideal Security in Equilibrium

As already mentioned, this point shows that the ideal security is the same as the market portfolio. It is important to make a clear distinction between what is covered in this chapter and what was covered in the previous chapter. To do this, let's remember that in the previous chapter the aim was to explain that ideal security was the term given to the optimal security of all agents, and that it was the same for everyone. In this case, we're going to detail the optimal security, which can be given by the combination of the risk-free asset and the aggregate endowment of the economy.

To do this, we'll start by defining a mean-variance utility function, which will help us understand the proof of the theorem later on.

Definition 4.1.1. *$u : \mathbb{R}^{S+1} \to \mathbb{R}$ is called a mean-variance utility function if for any $x = (x_0, x_1), x' = (x_0', x_1') \in \mathbb{R} \times \mathbb{R}^S$ satisfies the following inequality:*

$$(x_0, E[x_1], -var(x_1)) > (x_0', E[x_1'], -var(x_1')) \tag{4.1}$$

then $u(x) > u(x')$. Which means that x is preferred to x' if any of the following conditions are satisfied:

$$(x_0, E[x_1]) = (x_0', E[x_1']) \quad and \quad var(x_1) < var(x_1')$$

$$(x_0, E[x_1]) > (x_0', E[x_1']) \quad and \quad var(x_1) = var(x_1')$$

Defining which assumptions need to be considered when studying the CAPM model is very important if we are then to be able to study whether there is equilibrium in the economy. Although the focus of this dissertation is not on proving equilibrium, in order to come to plausible and acceptable results, the economy must be in equilibrium.

Definition 4.1.2. *An economy $\mathcal{E} = (V, (u^i, \omega^i))$ satisfies the assumptions of the Capital Asset Pricing Model (CAPM) if:*

(i) There are probabilities in the states of nature

(ii) There is a risk-less security: $\mathbb{1} \in \langle V \rangle$

(iii) $\omega_1^i \in \langle V \rangle, i = 1, \dots, I$

(iv) $u^i : \mathbb{R}^{S+1} \to \mathbb{R}$ is mean-variance, $i = 1, \dots, I$.

Having defined the assumptions of the model, we are now able to present and understand the following theorem, which is quite important, since it tells us how the ideal security in equilibrium is given and what the market price in equilibrium is for an income stream that is in the space $\langle V \rangle$. The proof of this theorem is also presented, which allows you to better understand the whole theorem.

Theorem 4.1.1 (CAPM Equilibrium). *If (x,z,q) is a financial market equilibrium of an economy $\mathcal{E} = (u, \omega, V)$ satisfying the CAPM assumptions with $var(\omega_1) > 0$, then:*

(i) there exist strictly positive constants c' and c, such that the ideal security, $\bar{y}$ is given by:

$$\bar{y} = c'\mathbb{1} - c\omega_{\mathbf{1}} \tag{4.2}$$

(ii) the equilibrium market value of any income stream y in the marketed subspace $\langle V \rangle$ is given by:

$$c_{\bar{q}}(y) = \frac{E[Y]}{1+\bar{r}} + c\, cov(\omega_{\mathbf{1}}, y) \tag{4.3}$$

Proof:

Since $\omega_{\mathbf{1}} \in \langle V \rangle$, the budget constraints of an agent i can be written as follows:

$$\begin{cases} x_0^i - \omega_0^i = -\bar{q}z^i \\ x_{\mathbf{1}}^i - \omega_1^i = Vz^i, z^i \in \mathbb{R}^J \end{cases} \Longleftrightarrow \begin{cases} x_0^i - \omega_0^i = c_{\bar{q}}\left(\omega_1^i\right) - c_{\bar{q}}\left(x_{\mathbf{1}}^i\right) \\ x_{\mathbf{1}}^i \in \langle V \rangle \end{cases} \tag{4.4}$$

Let $\bar{y}$ be the security in $\langle V \rangle$ such that:

$$c_{\bar{q}}(y) = [\![\bar{y}, y]\!]_\rho = \frac{E[y]}{1+\bar{r}} + cov(\bar{y}, y), \quad \forall y \in \langle V \rangle \tag{4.5}$$

since $E[\bar{y}] = c_{\bar{q}}(\mathbb{1}) = \frac{1}{1+\bar{r}}$. For each agent $i = 1, \ldots, I$, let us take $\hat{x}_{\mathbf{1}}^i$ as the ρ-projection of $\bar{x}_{\mathbf{1}}^i$ in the subspace $\langle \mathbb{1}, \bar{y}_1^i \rangle$.

Considering that $\hat{x}_{\mathbf{1}}^i \in \langle \mathbb{1}, \bar{y} \rangle$ and $\bar{x}_{\mathbf{1}}^i - \hat{x}_{\mathbf{1}}^i \in \langle \mathbb{1}, \bar{y} \rangle^\perp$ we can then write the following orthogonal relations:

$$\left[\!\!\left[\bar{y}, \bar{x}_{\mathbf{1}}^i - \hat{x}_{\mathbf{1}}^i\right]\!\!\right]_\rho = 0 \Longleftrightarrow c_{\bar{q}}\left(\bar{x}_{\mathbf{1}}^i\right) = c_{\bar{q}}\left(\hat{x}_{\mathbf{1}}^i\right) \tag{4.6}$$

$$\left[\!\!\left[\mathbb{1}, \bar{x}_{\mathbf{1}}^i - \hat{x}_{\mathbf{1}}^i\right]\!\!\right]_\rho = 0 \Longleftrightarrow E\left(\bar{x}_{\mathbf{1}}^i\right) = E\left(\hat{x}_{\mathbf{1}}^i\right) \tag{4.7}$$

$[\![\hat{x}_{\mathbf{1}}^i, \bar{x}_{\mathbf{1}}^i - \hat{x}_{\mathbf{1}}^i]\!]_\rho = 0$ and $[\![\mathbb{1}, \bar{x}_{\mathbf{1}}^i - \hat{x}_{\mathbf{1}}^i]\!]_\rho = 0 \Longleftrightarrow E\left[\bar{x}_{\mathbf{1}}^i\right] = E\left[\hat{x}_{\mathbf{1}}^i\right] \Rightarrow \mathrm{cov}\left(\hat{x}_{\mathbf{1}}^i, \bar{x}_{\mathbf{1}}^i - \hat{x}_{\mathbf{1}}^i\right) = 0$, which implies that:

$$var(\bar{x}_{\mathbf{1}}^i) = var(\hat{x}_{\mathbf{1}}^i) + var(\bar{x}_{\mathbf{1}}^i - \hat{x}_{\mathbf{1}}^i) \tag{4.8}$$

so that $var(\bar{x}_{\mathbf{1}}^i) > var(\hat{x}_{\mathbf{1}}^i)$ if $var(\bar{x}_{\mathbf{1}}^i - \hat{x}_{\mathbf{1}}^i) > 0$. Since:

$$\left\|\bar{x}_{\mathbf{1}}^i - \hat{x}_{\mathbf{1}}^i\right\|_\rho = \left(\left(E\left[\bar{x}_{\mathbf{1}}^i - \hat{x}_{\mathbf{1}}^i\right]\right)^2 + \mathrm{var}\left(\bar{x}_{\mathbf{1}}^i - \hat{x}_{\mathbf{1}}^i\right)\right)^{\frac{1}{2}} \tag{4.9}$$

Given that $E[\bar{x}_{\mathbf{1}}^i] = E[\hat{x}_{\mathbf{1}}^i]$, it implies that $\bar{x}_{\mathbf{1}}^i - \hat{x}_{\mathbf{1}}^i = 0$ if and only if $var(\bar{x}_{\mathbf{1}}^i - \hat{x}_{\mathbf{1}}^i) = 0$. Since both $\bar{x}_{\mathbf{1}}^i$ and $\omega_{\mathbf{1}}^i$ are elements of $\langle V \rangle$, and the costs of $\bar{x}_{\mathbf{1}}^i$ are equal to the costs of $\hat{x}_{\mathbf{1}}^i$, it follows that $\hat{x}^i = (\bar{x}_0^i, \hat{x}_{\mathbf{1}}^i)$ satisfies the budget constraints of all agents.

As the utility function u_i is mean-variance, if $\bar{x}_{\mathbf{1}}^i \neq \hat{x}_{\mathbf{1}}^i$, then $u(\bar{x}^i) < u(\hat{x}^i)$ since $E[\bar{x}_{\mathbf{1}}^i] = E[\hat{x}_{\mathbf{1}}^i]$ and $var(\bar{x}_{\mathbf{1}}^i) > var(\hat{x}_{\mathbf{1}}^i)$, which contradicts the optimally of $\bar{x}_{\mathbf{1}}^i$. Therefore, we conclude that $\bar{x}_{\mathbf{1}}^i = \hat{x}_{\mathbf{1}}^i$, and both are elements of $\langle \mathbb{1}, \bar{y} \rangle$ for $i = 1, \ldots, I$, which implies that $\sum_{i=1}^I \bar{x}_{\mathbf{1}}^i = \omega_{\mathbf{1}} \in \langle 1, \bar{y} \rangle$.

Since $var(\omega_1) > 0$, ω_1 is not proportional to 1. Consequently, $\langle \mathbb{1}, \bar{y} \rangle$ has a dimension of 2, and thus, $\langle \mathbb{1}, \bar{y} \rangle = \langle \mathbb{1}, \omega_1 \rangle$.

Thus, saying that $\bar{y} \in \langle \mathbb{1}, \omega_1 \rangle$ implies the existence of constants c' and c such that $\bar{y} = c'\mathbb{1} - c\omega_1$. To establish the positivity of these constants, we begin by examining the cost formula previously explored, now adjusted to incorporate the newly derived insights:

$$\begin{aligned} c_{\bar{q}} \left(\sum_{i=1}^{I} \bar{x}_1^i \right) &= \frac{E\left(\sum_{i=1}^{I} \bar{x}_1^i \right)}{1+\bar{r}} + \operatorname{cov}\left(\bar{y}, \sum_{i=1}^{I} \bar{x}_1^i \right) \\ &= \frac{E\left(\sum_{i=1}^{I} \bar{x}_1^i \right)}{1+\bar{r}} - c \operatorname{var}(\omega_1) \end{aligned} \tag{4.10}$$

Since $\sum_{i=1}^{I} \bar{x}_1^i = \omega_1$ and $\operatorname{cov}(\bar{y}, \omega_1) = -c \operatorname{var}(\omega_1)$. Since $var(\omega_1) > 0$, if $c > 0$, then must be some agent for whom $var(\bar{x}_1^i) > 0$ and $c_{\bar{q}} \geqslant \frac{E[x_1^i]}{1+\bar{r}}$. If this were true, then the consumption stream defined at the beginning of this proof $\bar{x}^i = (\bar{x}_0^i, E[\bar{x}_1^i]\mathbb{1})$, since it is less costly than x^i belonging to $\langle \bar{V} \rangle$, contradicting the optimally of x^i. Thus $c > 0$. Mean-variances preference implies that if an agent's maximization problem has a solution then $c_{\bar{q}}(\mathbb{1}) > 0$ must be satisfied, since $c_{\bar{q}} = E[\bar{y}] = c' - cE[\omega_1] > 0$, and $E[\omega_1] \geqslant 0$ which makes $c' > 0$ and so (i) is satisfied.

Proposition 4.1.1 (CAPM Formula for Equilibrium Risk Premium)**.** *Under the assumptions of Theorem 4.1.1., the equilibrium risk premium on a security y in the marketed subspace $\langle V \rangle$ is given by*

$$E\left(R_y\right) - \bar{R} = \frac{\operatorname{cov}\left(R_m, R_y\right)}{\operatorname{var}\left(R_m\right)} \left(E\left(R_m\right) - \bar{R}\right) \tag{4.11}$$

(See the proof in Magill and Quinzii, 1996)

Using the price formula presented in Theorem 4.1.1, it is possible to arrive at the return formula presented in Proposition 4.1.1. As we will see later, the formula given by Theorem 4.1.1. is very important and it was considered necessary to introduce it in this Chapter as it is the return or risk premium associated with a security y in the CAPM model.

4.2 Summary of The Ideal Security in CAPM

There are several important points to retain from the previous sub-chapter. Firstly, the CAPM model, which is like the basic economics model presented earlier. The two are similar when they refer to investors' attitudes towards risk, which is represented by the assumption that preferences have mean-variance characteristics. Secondly, we must obviously refer to the ideal security and the fact that it is equal to the market portfolio, i.e. it is given by the linear combination of the risk-free asset and the aggregate endowment of an economy.

We can then see that the optimal security presented in the Theorem 4.1.1. takes us a step further than what was presented in the previous chapter. Since in the previous chapter we were able to equate individual security pricing with the optimal security of each agent i and then conclude through Theorem 3.3.1. that optimal security is the same for all agents in equilibrium, this leads us to conclude that it is the same as optimal security in equilibrium. On the other hand, in this chapter, through Theorem 4.1.1., the optimal security is equalled to the market portfolio, i.e. it can be represented by a linear combination, as mentioned above. The ideal security and the price associated with an income stream y in the space generated by $\langle V \rangle$ are important in later chapters, so it's important to make the whole subject clear.

Associated with a price we can always get a return. In this case, associated with the price of the income stream y presented in Theorem 4.1.1, we can obtain a return and through a few calculations we can easily arrive at the formula presented in Proposition 4.1.1. (You can see more in Magill and Quinzii, 1996).

A relevant interpretation to make, related to the result of Proposition 4.1.1., is that the risk premium of a given individual security is the increase in expected return that goes beyond the risk-free return obtained by being exposed to the risk of investing in this security instead of investing risk-free in another safer security. The capital premium associated with the economy-wide portfolio q_m is the reward that is given in addition to the return achieved by risk-free assets. The result of Proposition 4.1.1 is well known for evaluating risk premiums in an equilibrium CAPM model.

5

The Finance Model with Trading Constraints

In this chapter we are going to explain in detail the model with constraints to be studied, which we will use in the next chapter to price financial products, adding trading restrictions. This model has similarities when compared to the CAPM model.

In order to present the model, we begin by presenting the basic concepts, such as time and uncertainty, since it is important to describe how time and uncertainty occur at any given moment. In this case, we present a simpler model where there are only two periods of uncertainty and, like the CAPM, there are finite states of nature in the second period. Commodities and equities are all goods that are traded on the market. Consumers are an essential part of the economy, and they are defined by their preferences.

In the case of this model, there are restrictions on transactions, which is the main difference between this model and the CAPM. Trading restrictions basically mean that there is limited access to all consumers in the market since consumers are restricted in their access to certain assets.

To understand all these concepts, the following sub-chapters show us in detail the model we need to study in order to be able to do the pricing in the next chapter. (This model is based on the model presented in Eichberger et al., 2014).

5.1 Time and Uncertainty

In this model there are two dates $t \in 0, 1$. At $t = 0$ there is no uncertainty since there is only the state of nature $s = 0$. At $t = 1$ there is uncertainty since there are $s = \{1, \ldots, s, \ldots, S\}$ states of nature.

5.2 Commodities

There is only one commodity being traded on each of the dates in this model. We assume that there can be negative consumption imposing bankruptcy penalties.

5.3 Equities

In order to achieve the optimal consumption of consumers by adapting to their preferences agents can trade equities. At $t = 0$ there are $j = 1, \dots, J$, equities to be traded that at date $t = 1$ pay dividends, $y_j = (Y_{1,j}, \dots, Y_{S,j}) \in \mathbb{R}^S_+$. The matrix $S \times J$ of all payoffs y of the equities is denoted by Y. The price of these equities at date $t = 0$ is denoted by $q = (q_1, \dots, q_J) \in \mathbb{R}^J_+$. All payments for equities are made in units of the commodity concerned by the transaction. Equities cannot be sold in the short term. Let $(z_b, z_e) = (z_b, z_1, \dots, z_j, \dots, z_J)$ denote a portfolio of assets, being z_b the units of the bond and z_e the equities portfolio.

5.4 Consumers

There are I agents or consumers. Each agent is characterized by the consumption set $\mathbb{R}^{1+S}_+$, a utility function $u^i : \mathbb{R}^{1+S}_+ \to \mathbb{R}$ that is continuous, continuously differentiable, strictly increasing and concave and by an initial endowment of the commodity in each state of nature. We denote by ω_0 the endowment at date $t = 0$ and by $\omega_{\mathbf{1}} = (\omega^i_1, \dots, \omega^i_S)$ endowment vector at date $t = 1$. The initial endowment related to equities is denoted by $\delta^i_j \in \mathbb{R}$.

We denote by $x^i = (x^i_0, x^i_{\mathbf{1}}) \in \mathbb{R}^{1+S}_+$ the consumption bundle of each agent i, $i = 1, \dots, I$.

To make the model more useful from an economic point of view, we add sufficient structure to the model to be able to study equilibrium prices and their changes. Thus, instead of general preferences, we assume that preferences are defined by a model with a linear quadratic utility function continuously differentiable:

$$u^i\left(x^i\right) = \alpha^i_0 x^i_0 - \frac{1}{2}\sum_{s=1}^{S} \rho_s \left(\alpha^i_1 - x^i_s\right)^2, \quad i = 1, \dots, I \tag{5.1}$$

(See more about this utility function in Eichberger et al., 2014).

Defining as $\alpha_0 = \sum_{i=1}^I \alpha^i_0, \alpha_1 = \sum_{i=1}^I \alpha^i_1$ the aggregate preference parameters and by $\omega_{\mathbf{1}} = \sum_{i=1}^I \omega^i_{\mathbf{1}}, \delta = \sum_{i=1}^I \delta^i$ and $K = |I|\kappa$ the aggregate endowments of goods, stocks and maximum loans.

5.5 Trading Constraints

In this model, trading restrictions will be introduced, agents are no longer allowed to trade all equities or bonds. Trading restrictions are a widely accepted practice in the realm of securities trading. Particularly in the case of derivative trading, investors are mandated to maintain margin accounts, which essentially serve as a performance guarantee. These transactions restrictions are determined by regulatory authorities, clearinghouses, and intermediaries. The specifics of these margin requirements may be encapsulated by the term (CON), and the precise format can vary depending on the margin requirement under consideration.

The trading restrictions we will study are the following (inspired by restrictions on transactions presented in Elsinger and Summer, 2001 and Gori et al., 2013):

(i) $\mathbf{z_b} \geqslant -\alpha\sigma(\mathbf{q})$: How much we can borrow on the bond is limited, i.e. in this case, we assume that each agent has only partial access, in a personalizes manner to asset markets. We suppose that each agent cannot sell, that is, cannot borrow, more than a fixes quantity, depending on asset prices. More precisely the maximum amount we can sell at prices q of the bond, is given by a function that depends on the prices of equities multiplied by a constant α.

(ii) $\mathbf{q_b z_b} \geqslant -\nu\mathbf{q_e} \cdot (\mathbf{z_e} + \delta^{\mathbf{i}}_{\mathbf{e}})$: How much we can borrow on the bond is limited, i.e. in this case, there is a requirement to maintain an income margin, the maximum amount we can borrow on the bond depends on the investment on equities and their price multiplied by a constant ν.

5.6 Assumptions

Let $u = (u^1, \ldots, u^I)$, $\omega = (\omega^1, \ldots, \omega^I)$, and $\delta = (\delta^1, \ldots, \delta^I)$ be the individual utility, the initial endowments of the good and of equities, respectively, and denote by $V = [\mathbb{1}\, Y]$ the matrix that gives us the payoffs of bond and equities. Let $Z^i = \mathbb{R} \times (\mathbb{R}^J_+ - \{\delta^i\})$, $i = 1, \ldots, I$, the set where the portfolios are chosen. We denote by $\mathcal{E} = (u, \omega, \delta, V)$ the economy. In this economy we have the following assumptions:

$(A.1)$ (consumption sets) X^i is a closed, convex and nonempty subset of $\mathbb{R}^{S+1}$.

$(A.2)$ (preferences) $u^i : X^i \to \mathbb{R}$ is continuous, continuously differentiable, strictly increasing, and strictly concave.

$(A.3)$ (endowments) $\omega^i = \left(\omega^i_0, \omega^i_1\right) \in X^i \cap \mathbb{R}^{S+1}_{++}$ and $\delta^i \in \mathbb{R}^J_{++}$.

$(A.4)$ (asset returns) the matrix $V = [\mathbb{1}\, Y]$ has full column rank.

5.7 Budget Set

Since there is only one commodity in the first period and in each state of nature of second period, and assets payoffs are given in units of the commodity, we set $(p_0, p_1, \ldots, p_s, \ldots, p_S) = (1, 1, \ldots, 1, \ldots, 1)$. We can then define the budget set as follows:

$$B^i(q) = \left\{ \left(x^i_0, x^i_{\mathbf{1}}, z^i_b, z^i_e\right) \in \mathbb{R}^{1+S} \times Z^i \;\middle|\; \begin{array}{l} x^i_0 - \omega^i_0 \leq -q_b z^i_b - q_e \cdot z^i_e \\ x^i_{\mathbf{1}} - \omega^i_{\mathbf{1}} - Y\delta^i \leq \mathbb{1} z^i_b + Y z^i_e \end{array} \right\} \tag{5.2}$$

Definition 5.7.1 (Equilibrium). *The equilibrium of an economy $\mathcal{E} = (u, \omega, \delta, V)$ is a collection of consumption plans, portfolio choices, equity and bond prices $(\bar{x}, \bar{z}_b, \bar{z}_e, \bar{q}) \in \mathbb{R}^{(1+S)I} \times Z^{JI} \times \mathbb{R}_+^{J+1}$ such that for all agents $i = 1, \ldots, I$:*

(i) $\bar{x}^i \in \arg\max \left\{u^i\left(x^i\right) \mid (x^i, Z^I) \in \mathbb{B}^i(\bar{q})\right\}$, (optimal behavior)

(ii) $\sum_{i=1}^{I} \bar{z}_b^i = 0$ and $\sum_{i=1}^{I} \bar{z}_e^i = 0$, (asset market clearing)

(iii) $\sum_{i=1}^{I} \bar{x}^i = \sum_{i=1}^{I} \omega^i + Y\delta^i$, (commodity market clearing)

Theorem 5.7.1 (Existence of equilibrium). *Under the assumptions of $A.1 - A.4$ there exists an equilibrium for the economy $\mathcal{E} = (u, \omega, \delta, V)$.*

In the specific case of this economy presented without restrictions, we can see the proof of this theorem in Geanakoplos and Polemarchakis, 1986 and Geanakoplos, 1990. On the other hand, we can see evidence of equilibrium when there are limits on access to credit, i.e. trading restrictions in Carosi et al., 2009, Cea-Echenique and Torres-Martínez, 2016, Cea-Echenique and Torres-Martínez, 2018, Cornet and Gopalan, 2010 and Faias and Torres-Martínez, 2017.

We are now at the point where we have explained in detail the model to be studied in the next chapter. As we have already seen, this model is very similar to the classic CAPM model. This model differs from the CAPM and the standard economics, presented earlier because in that it contains trading restrictions, i.e., consumers have limited access to the financial products on the market.

6

Pricing of financial products where investors have restrictions on transactions

In this chapter, the main objective is to explain at the prices associated with the model described in the previous Chapter, considering account the two trading restrictions that have been presented and understanding how prices may vary when these restrictions are added to the model.

In previous chapters we concluded what the price of a financial product associated with the classic CAPM should be. In this case, where we add trading restrictions, the aim is to understand how the equilibrium price of an income stream varies when we add restrictions to consumer transactions in the model. Therefore, we present two propositions and their respective interpretations, which lead us to conclude that the equilibrium price varies when restrictions are present.

When we know the price of a financial product, we can then find out the return associated with this financial product, considering its price. As with the price, the return of a financial product also varies when restrictions are added to transactions. The return associated with a financial product when there are restrictions is very similar to the return of the same product when calculated taking into account the classic CAPM, although in the case where there are restrictions there is one more term to be considered in the return equation.

To interpret the variation of these prices we can use the envelope theorem (This can be seen in more detail in Simon and Blume, 1985), which helps us to interpret how trading restrictions influence each consumer's utility function. This interpretation is done through the Lagrange multipliers used to calculate the price associated with the income stream.

In the next sub-chapter, we will review the envelope theorem, which allows us to interpret the Lagrange multipliers associated to the trading restrictions. These Lagrange multipliers will appear in the cost and return of an income stream when the restrictions are biding.

In order to explain in depth what has been said above, let's move on to the following sub-chapters, where we can see the prices and yields associated with an income stream. (All the calculations and conclusions presented in sub-chapters 6.2 and 6.3 were based on the article Almeida et al., 2018).

6.1 Envelope Theorem

Consider the following optimization problem with two variables and a restriction, where f and h are continuously differentiable functions defined on $\mathbb{R}^2$:

$$\begin{cases} \max_{(x,y)} f(x,y) \\ \text{subject to} \quad h(x,y) = a \end{cases} \tag{6.1}$$

Considering a as a parameter that changes depending on the problem at hand. For any fixed value a, denoted by $(x^*(a), y^*(a))$ the solution of problem (6.1) and let $\mu(a)$ be the solution to the Lagrange multiplier corresponding to this solution. Thus $\mu(a)$ measures the rate of change of the optimal value of f with respect to the parameter a, i.e. the effect of a unit increase in a on $f\,(x^*(a), y^*(a))$.

Theorem 6.1.1 (Envelope Theorem). *Let f and h be two continuous differentiable functions of two variables. For each fixed value a, let $(x^*(a), y^*(a))$ be the solution of problem (6.1) and with the corresponding Lagrange multiplier $\mu^*(a)$. Suppose that x^*, y^* and μ are functions of a. Then:*

$$\mu(a) = \frac{df}{da}(x^*(a), y^*(a))$$

The proof of Theorem 6.1.1 is in the Appendix.

The Lagrange multiplier associated with the restriction represents the change in the optimal value of the function resulting from a small increase in the parameter a. Or, in other words, the Lagrange multiplier represents the rate of change of $f(x^*(a), y^*(a))$ with respect to a. The Lagrange multiplier can also be interpreted as the shadow price related to parameter a, that is, it indicates how much the consumer is willing to pay for an increase in the parameter a.

In general, the restrictions can be more complex than the one presented in the problem (6.1), as it is the case of the trading restrictions that we have stated in sub-chapter 5.5. The envelope theorem also holds to problems where the restrictions are more general functions of a parameter, $h(x; a)$.

Theorem 6.1.2. *Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be continuously differentiable functions. Let $x^*(a)$ denote the solution of the problem of maximizing $f(x)$ on the constraint set $h(x; a) = 0$, for any fixed choice of the parameter a. Suppose that $x^*(a)$ and the Lagrange multiplier $\mu(a)$ are a continuously differentiable functions of a. Then,*

$$\frac{df}{da}(x^*(a); a) = \frac{\partial L}{\partial a}(x^*(a), \mu(a); a)$$

where L is the Lagrangian function for this problem.

See the Proof in Simon and Blume, 1985

We are going to use this last theorem in the sub-chapters 6.2 and 6.3 to better understand the formulas that will be deduced for the cost or return of an income stream.

6.2 Equilibrium Pricing of an Income Stream

It is necessary, for this chapter, to make a connection between what is being done here and the previous chapters that in some way serve as the basis for the realization of this chapter. We can start by noticing that the following proposition refers to, as previously defined, ideal security, but now this γ stands for the equilibrium marginal evaluation of income streams that can be generated with the financial structure given for this economy.

The following proposition gives us a formula for calculating the price of an income stream m generated by a viable portfolio z^m for a given agent i with the respective restrictions on transactions.

Proposition 6.2.1. *If $(\bar{x}, \bar{z}, \bar{q})$ is an equilibrium of the economy $\mathcal{E}$ under the restrictions , $\bar{z}_b^i \geq -\alpha\sigma(\bar{q})$, $i = 1, \ldots, I$, then:*

- *$\bar{\gamma} = (\bar{\gamma}_0, \bar{\gamma}_1)$ satisfies the following condition: $\bar{\gamma} = (\alpha_0, \mathbb{1}\alpha_1 - \tilde{\omega}_1)$*
- *If $c(m)$ is the equilibrium price of an income stream m generated by a linear combination of the existing $J + 1$ assets, with $z^m = (z_b^m, z_e^m)$ being the respective portfolio, then $c(m)$ satisfies the following condition:*

$$c(m) = E\left[\frac{\bar{\gamma}_1}{\alpha_0}\right] E[m] - \frac{1}{\alpha_0} cov(m, \tilde{\omega}_{\mathbf{1}}) + \frac{\overline{\mu} z_b^m}{\alpha_0} \tag{6.2}$$

where $\tilde{\omega}_{\mathbf{1}} = \sum_{i=1}^{I} \omega_{\mathbf{1}}^i + Y\delta^i$, $\overline{\mu} = \sum_{i=1}^{I} \mu^i$ and μ^i, $i = 1, \ldots, I$ are the equilibrium Lagrange multipliers associated to the restriction.

Proof:

We have the following constraints:

$$\begin{cases} x_0^i - \omega_0^i \leq -q_b z_b - q_e \cdot z_e \\ x_{\mathbf{1}}^i - \omega_{\mathbf{1}}^i - Y\delta \leq \mathbb{1} z_b^i + Y z_e \\ z_b \geq -\alpha\sigma(q) \end{cases}$$

Consider the following notation, $x^i = (x_0^i, x_1^i, \ldots, x_s^i, \ldots, x_S^i)$ and $z^i = (z_b^i, z_e^i) = (z_b^i, z_1^i, \ldots, z_j^i, \ldots, z_J^i)$. The vector $\pi^i = (\pi_0^i, \pi_1^i, \ldots, \pi_s^i, \ldots, \pi_S^i)$ denotes the Lagrange multipliers associated to the budget constraints and μ^i the Lagrange multiplier associated to the trading restriction.

We saw in sub-chapter 2.3 how we should apply the Lagrangian function and how it should be constructed. So, the Lagrangian function associated to the consumer problem is given by, $L^i(x^i, z^i, \pi^i, \mu^i) = u^i(x^i) - \pi_0^i(x_0^i - \omega_0^i + q_b z_b^i + q_e \cdot z_e^i) - \sum_{s=1}^{S} \pi_s^i (x_s^i - \omega_s^i - \sum_{j=1}^{J} Y_{s,j}\delta_j^i - z_b^i - \sum_{j=1}^{J} Y_{s,j}^i z_j^i) + \mu(z_b + \alpha\sigma(q))$.

Since $\bar{x}^i$ and $\bar{z}^i$ are the optimal choice of consumer i, the following conditions have to be satisfied:

$$\frac{\partial L^i(\bar{x}^i, \bar{z}^i, \pi^i, \mu^i)}{\partial x_0} = 0 \Leftrightarrow \frac{\partial u^i(\bar{x}^i)}{\partial x_0} = \pi_0^i \tag{6.3}$$

$$\frac{\partial L^i(\bar{x}^i, \bar{z}^i, \pi^i, \mu^i)}{\partial x_s} = 0 \Leftrightarrow \frac{\partial u^i(\bar{x}^i)}{\partial x_s} = \pi_s^i,\ s = 1, \ldots, S \tag{6.4}$$

$$\frac{\partial L^i(\bar{x}^i, \bar{z}^i, \pi^i, \mu^i)}{\partial z_b} = 0 \Leftrightarrow -\pi_0^i \bar{q}_b + \sum_{s=1}^{S} \pi_s^i + \mu^i = 0 \tag{6.5}$$

$$\frac{\partial L^i(\bar{x}^i, \bar{z}^i, \pi^i, \mu^i)}{\partial z_j} = 0 \Leftrightarrow -\pi_0^i \bar{q}_j + \sum_{s=1}^{S} \pi_s^i Y_{s,j} = 0, \quad j = 1, \ldots, J \tag{6.6}$$

$$\frac{\partial L^i(\bar{x}^i, \bar{z}^i, \pi^i, \mu^i)}{\partial \pi_0} = 0 \Leftrightarrow \bar{x}_0^i - \omega_0^i + \bar{q}_b \bar{z}_b^i + \bar{q}_e \cdot \bar{z}_e^i = 0 \tag{6.7}$$

$$\frac{\partial L^i(\bar{x}^i, \bar{z}^i, \pi^i, \mu^i)}{\partial \pi_s} = 0 \Leftrightarrow \bar{x}_s^i - \omega_s^i - \sum_{j=1}^{J} Y_{s,j} \delta_j^i - \bar{z}_b^i - \sum_{j=1}^{J} Y_{s,j} \bar{z}_j^i = 0,\ s = 1, \ldots, S \tag{6.8}$$

$$\frac{\partial L^i(\bar{x}^i, \bar{z}^i, \pi^i, \mu^i)}{\partial \mu} = 0 \Leftrightarrow \bar{z}_b^i + \alpha \sigma(\bar{q}) = 0 \tag{6.9}$$

Using (6.3) and (6.4) to replace into the equations (6.5) and (6.6) we then have:

$$\frac{\partial L^i(\bar{x}^i, \bar{z}^i, \pi^i, \mu^i)}{\partial z_b} = 0 \Leftrightarrow -\frac{\partial u^i(\bar{x}^i)}{\partial x_0} \bar{q}_b + \sum_{s=1}^{S} \frac{\partial u^i(\bar{x}^i)}{\partial x_s} = -\mu^i \tag{6.10}$$

$$\frac{\partial L^i(\bar{x}^i, \bar{z}^i, \pi^i, \mu^i)}{\partial z_j} = 0 \Leftrightarrow -\frac{\partial u^i(\bar{x}^i)}{\partial x_0} \bar{q}_j + \sum_{s=1}^{S} \frac{\partial u^i(\bar{x}^i)}{\partial x_s} Y_{s,j} = 0,\ j = 1, \ldots, J \tag{6.11}$$

Let

$$Y = \begin{bmatrix} Y_{1,1} & \cdots & Y_{1,j} & \cdots & Y_{1,J} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Y_{s,1} & \cdots & Y_{s,j} & \cdots & Y_{s,J} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Y_{S,1} & \cdots & Y_{S,j} & \cdots & Y_{S,J} \end{bmatrix}$$

Taking T_b and T_e as:

$$T_b = \begin{bmatrix} -\bar{q}_b \\ \mathbb{1} \end{bmatrix} \text{ and } T_e = \begin{bmatrix} -\bar{q}_e \\ Y \end{bmatrix}$$

Then from (6.10) and (6.11), respectively, we can conclude that:

$$\langle \nabla u^i(\bar{x}^i), T_b \rangle = -\mu^i$$

$$\langle \nabla u^i(\bar{x}^i), T_e \rangle = (0, \dots, 0)$$

These are the inner product between the gradient of the utility function and the T_b vector and the inner product between the gradient of the utility function and the T_e vectors, respectively.

Let $\tau = Tz = \begin{bmatrix} -\bar{q}_b & -\bar{q}_e \\ \mathbb{1} & Y \end{bmatrix} \begin{bmatrix} z_b^m \\ z_e^m \end{bmatrix}$

Making $\langle \nabla u^i(\bar{x}^i), \tau \rangle$ we then have:

$$\langle \nabla u^i(\bar{x}^i), \tau \rangle = \begin{bmatrix} \frac{\partial u^i(\bar{x}^i)}{\partial x_0} \frac{\partial u^i(\bar{x}^i)}{\partial x_1} \cdots \frac{\partial u^i(\bar{x}^i)}{\partial x_s} \end{bmatrix} \begin{bmatrix} -\bar{q}_b & -\bar{q}_e \\ \mathbb{1} & Y \end{bmatrix} \begin{bmatrix} z_b^m \\ z_e^m \end{bmatrix} =$$

$$= \begin{bmatrix} -\frac{\partial u^i(\bar{x}^i)}{\partial x_0}\bar{q}_b + \frac{\partial u^i(\bar{x}^i)}{\partial x_1} + \cdots + \frac{\partial u^i(\bar{x}^i)}{\partial x_s} & -\frac{\partial u^i(\bar{x}^i)}{\partial x_0}\bar{q}_e + \frac{\partial u^i(\bar{x}^i)}{\partial x_1}Y_1 + \cdots + \frac{\partial u^i(\bar{x}^i)}{\partial x_S}Y_S \end{bmatrix} \begin{bmatrix} z_b^m \\ z_e^m \end{bmatrix} =$$

$$= \left(-\frac{\partial u^i(\bar{x}^i)}{\partial x_0}\bar{q}_b + \frac{\partial u^i(\bar{x}^i)}{\partial x_1} + \cdots + \frac{\partial u^i(\bar{x}^i)}{\partial x_s}\right) z_b^m + \left(-\frac{\partial u^i(\bar{x}^i)}{\partial x_0}\bar{q}_1 + \frac{\partial u^i(\bar{x}^i)}{\partial x_1}Y_{1,1} + \cdots + \frac{\partial u^i(\bar{x}^i)}{\partial x_S}Y_{S,1}\right) z_1^m + \cdots +$$
$$\left(-\frac{\partial u^i(\bar{x}^i)}{\partial x_0}\bar{q}_j + \frac{\partial u^i(\bar{x}^i)}{\partial x_1}Y_{1,j} + \cdots + \frac{\partial u^i(\bar{x}^i)}{\partial x_S}Y_{S,j}\right) z_j^m + \cdots + \left(-\frac{\partial u^i(\bar{x}^i)}{\partial x_0}\bar{q}_J + \frac{\partial u^i(\bar{x}^i)}{\partial x_1}Y_{1,J} + \cdots + \frac{\partial u^i(\bar{x}^i)}{\partial x_S}Y_{S,J}\right) z_J^m =$$
$$-\mu^i z_b^m$$

Given that $\nabla u^i(\bar{x}^i) = \left(\frac{\partial u^i(\bar{x}^i)}{\partial x_0}, \left(\frac{1}{\rho_s}\frac{\partial u^i(\bar{x}^i)}{\partial x_s}\right)_{s=1}^S\right)$ and that $u^i\left(\bar{x}^i\right) = \alpha_0^i \bar{x}_0^i - \frac{1}{2}\sum_{s=1}^S \rho_s \left(\alpha_1^i - \bar{x}_s^i\right)^2, \quad i = 1, \dots, I$, then we have that:

$$\frac{\partial u^i(\bar{x}^i)}{\partial x_0} = \alpha_0^i \text{ and } \frac{\partial u^i(\bar{x}^i)}{\partial x_s} = \rho_s(\alpha_1^i - \bar{x}_s^i),\ s = 1, \dots, S.$$

That is,

$$\nabla u^i(\bar{x}^i) = (\alpha_0^i, \alpha_1^i - \bar{x}_1^i, \cdots, \alpha_1^i - \bar{x}_S^i).$$

If we add in all the agents we get:

$$\left(\sum_{i=1}^I \alpha_0^i, \sum_{i=1}^I \alpha_1^i - \bar{x}_1^i, \dots, \sum_{i=1}^I \alpha_1^i - \bar{x}_S^i\right)$$
$$= \left(\alpha_0, \alpha_1 - \sum_{i=1}^I \bar{x}_1^i, \dots, \alpha_1 - \sum_{i=1}^I \bar{x}_S^i\right) = (\alpha_0, \alpha_1\mathbb{1} - (\omega_{\mathbf{1}} + Y\delta)).$$

Finally, we have:

$$\langle \sum_{i=1}^I \nabla u^i(\bar{x}^i), \tau \rangle = -\sum_{i=1}^I \mu^i z_b^m \Leftrightarrow \langle (\alpha_0, \alpha_1\mathbb{1} - \omega_{\mathbf{1}} - Y\delta), (-c(m), m) \rangle = \sum_{i=1}^I \mu^i z_b^m \Leftrightarrow$$
$$-c(m)\alpha_0 + \langle (\alpha_1\mathbb{1} - \omega_{\mathbf{1}} - Y\delta), m \rangle = -\sum_{i=1}^I \mu^i z_b^m \Leftrightarrow$$
$$\Leftrightarrow -c(m)\alpha_0 = -\sum_{i=1}^I \mu^i z_b^m - \langle (\alpha_1\mathbb{1} - \omega_{\mathbf{1}} - Y\delta), m \rangle \Leftrightarrow$$
$$\Leftrightarrow c(m) = \langle \frac{\bar{\gamma}_1}{\alpha_0}, m \rangle + \frac{\overline{\mu} z_m}{\alpha_0} = E\left[\frac{\bar{\gamma}_1}{\alpha_0}\right] E[m] - \frac{1}{\alpha_0} cov(m, \tilde{\omega}_1) + \frac{\overline{\mu} z_b^m}{\alpha_0}.$$

The equation 6.2. defines the equilibrium price of an income stream generated by the $1+J$ assets in the economy $\mathcal{E}$. In the case of this pricing formula, if we have $\mu^i = 0, i = 1, \ldots, I$, i.e. the restriction is not biding for any consumer, and/or $z_b^m = 0$, i.e. the income stream m is generated by a portfolio in which the bond is not traded, then the pricing formula would coincide with the classic CAPM formula, already presented above.

If an income stream m is generated by a portfolio where the bond is traded and for at least one consumer the restriction is binding, that is, $\mu^i > 0$ for some i, then the term that differentiates equation 6.2 of the classic CAPM formula, $\frac{\overline{\mu} z_b^m}{\alpha_0}$ is different from zero. If the bond is being used to make a credit, i.e. $z_b^m < 0$, then the cost of the income stream is lower than in the classic CAPM. If the bond is being used for investment, i.e. $z_b^m > 0$, then the income stream cost is higher than in the seminal case. Therefore, the disutility of at least one consumer for being constrained (he cannot trade the portfolio that would be optimal if it were not subject to restrictions) has an impact on the price of any income stream generated by the financial market, if this income stream involves the trading of the bond.

Let us apply the envelope theorem with the aim of interpreting the Lagrange multiplier. The Lagrangian function for the consumer problem with the restriction (i), is given by, $L^i(x^i, z^i, \pi^i, \mu^i) = u^i(x^i) - \pi_0^i(x_0^i - \omega_0^i + q_b z_b^i + q_e \cdot z_e^i) - \sum_{s=1}^{S} \pi_s(x_s^i - \omega_s^i - \sum_{j=1}^{J} Y_{s,j}\delta_j^i - z_b^i - \sum_{j=1}^{J} Y_{s,j} z_j^i) + \mu^i(z_b^i + \alpha\sigma(q))$. If the restriction is biding, from the envelope theorem we obtain,

$$\frac{\partial u^i}{d\alpha}(x^i; \alpha) = \frac{\partial L^i}{d\alpha}(x^i, z^i, \pi^i, \mu^i; \alpha) = \mu^i \sigma(q).$$

Then, we can interpret the Lagrange multiplier $\mu^i, i = 1, .., I$, as the ratio between the shadow price for consumer i, associated to the parameter α and the function σ evaluated at the assets equilibrium prices. Thus, the additional term in the pricing formula depends on the shadow price of the parameter α.

Proposition 6.2.2. *If $(\bar{x}, \bar{z}, \bar{q})$ is an equilibrium of the economy $\mathcal{E}$ under the restrictions , $\bar{q}_b \bar{z}_b \geqslant -\nu \bar{q}_e \cdot (\bar{z}_e + \delta_e^i)$, $i = 1, \ldots, I$, then:*

- *$\bar{\gamma} = (\bar{\gamma}_0, \bar{\gamma}_1)$ satisfies the following condition: $\bar{\gamma} = (\alpha_0, \mathbb{1}\alpha_1 - \tilde{\omega}_1)$*
- *If $c(m)$ is the equilibrium price of an income stream m generated by a linear combination of the existing $J + 1$ assets, with $z^m = (z_b^m, z_e^m)$ being the respective portfolio, then $c(m)$ satisfies the following condition:*

$$c(m) = E\left[\frac{\bar{\gamma}_1}{\alpha_0}\right] E[m] - \frac{1}{\alpha_0} cov(\tilde{\omega}_{\mathbf{1}}, m) + \frac{\bar{\mu}(\bar{q}_b z_b^m + \nu \bar{q}_e \cdot z_e^m)}{\alpha_0}. \tag{6.12}$$

where $\tilde{\omega}_{\mathbf{1}} = \sum_{i=1}^{I} \omega_{\mathbf{1}}^i + Y\delta^i$, $\overline{\mu} = \sum_{i=1}^{I} \mu^i$ and μ^i, $i = 1, \ldots, I$ are the equilibrium Lagrange multipliers associated to the restriction.

The Proof of the Proposition 6.2.2. can be found in the Appendix.

The first part of this pricing formula is the same as that we have obtained in Proposition 6.2.1 and is equivalent to the formula of the classical CAPM model. However, the cost of an income stream with the trading restriction (ii) has also an additional term. This term involves not only the Lagrange multipliers associated with the restriction and the replica portfolio of the income stream, as it is in the case of restriction (i), but also the prices of the underlying assets of the economy. Actually, the term $\bar{q}_b z_b^m + \nu \bar{q}_e \cdot z_e^m$ seems like the value of the income stream but with the prices of the equities being modified through the parameter of the restriction. If for at least one consumer the restriction is biding, that is, $\mu^i > 0$ for some i, and the income stream is such that the portfolio replica satisfies $z_b^m > 0$ and $z_e^m > 0$ then the income stream cost is higher than in the classical CAPM. The opposite holds when the portfolio replica satisfies $\bar{q}_b z_b^m + \nu \bar{q}_e \cdot z_e^m < 0$. In particular, if the bond and the stocks are being traded to make credit, the cost of the income stream is smaller when in the case without restrictions. Regarding the parameter ν, we can conclude that if ν increases and the restriction still biding for some consumer, then the additional term is higher. This is related with the great demand for credit.

This leads us to conclude that the price rises when agents have limited access to assets in these two specific cases.

Applying the envelope theorem to the Lagrangian function for the consumer problem with the restriction (ii), $L^i(x^i, z^i, \pi^i, \mu^i) = u^i(x^i) - \pi_0^i(x_0^i - \omega_0^i + q_b z_b^i + q_e \cdot z_e^i) - \sum_{s=1}^{S} \pi_s^i (x_s^i - \omega_s^i - \sum_{j=1}^{J} Y_{s,j}^i \delta_j^i - z_b^i - \sum_{j=1}^{J} Y_{s,j}^i z_j^i) + \mu^i (q_b z_b^i + \nu q_e \cdot (z_e^i + \delta_e^i))$, we obtain,

$$\frac{du}{d\nu}(x^i; \nu) = \frac{\partial L}{\partial \nu}(x^i, z^i, \pi^i, \mu^i; \nu) = \mu^i q_e \cdot (z_e^i + \delta_e^i).$$

The shadow price for an increase in the parameter ν is given by the product between the Lagrange multiplier associated to the restriction and the value invested in the equities. The higher the value invested in the equities the higher the shadow price associated with the parameter ν.

Notice that the Lagrange multiplier μ^i is the ratio between the shadow price associated to the parameter ν and the value of the portfolio of equities, that is, it represents how much is the consumer willing to pay, per unit of the value invested in equities, to weaken the short sales restriction of the bond. The value that investors are willing to pay to weaken the short sell restriction on the bond is reflected in the price of an income stream. The higher the value, the higher the cost of an income stream.

6.3 Equilibrium Risk Premium of an Income Stream

In this sub-chapter the goal is to know the return (or risk premium) associated with the prices we attained in the previous sub-chapter. To understand more about the return or risk premium, this topic was already covered in Chapter 4.

We define the return per unit of income invested associated with portfolio z^m as $R_m = \frac{m}{c(m)}$, if the price of an income stream m is positive ($c(m) > 0$) and we define the return of the market portfolio $z^{\tilde{\omega}_1}$ as $R_{\tilde{\omega}_1} = \tilde{\omega}_1 / c_{\bar{q}}(\tilde{\omega}_1)$ when $c_{\bar{q}}(\tilde{\omega}_1) > 0$. The excess return of portfolio z^m is the random variable $\mathcal{R} : S \longrightarrow \mathbb{R}$ defined by $\mathcal{R}_m = R - \bar{R}$, where $\bar{R} = 1 + \bar{r}$ and $\bar{r}$ is the risk-free rate. The expected value of $\mathcal{R}_m$, defined by $E(\mathcal{R}_m) = E(R_m) - \bar{R}$, is called the risk-premium of portfolio z^m. Similarly, we define the risk-premium of the market portfolio as $E(\mathcal{R}_{\tilde{\omega}_1}) = E(R_{\tilde{\omega}_1}) - \bar{R}$.

Proposition 6.3.1. *If $(\bar{x}, \bar{z}, \bar{q})$ is an equilibrium of the economy $\mathcal{E}$ under the restrictions $\bar{z}_b^i \geq -\alpha\sigma(\bar{q})$, $i = 1, \ldots, I$, then the equilibrium risk premium of a portfolio z^m associated to the income stream m generated by the asset returns matrix $[\,\mathbb{1} \;\; Y\,]$ is:*

$$E[\mathcal{R}_m] = \beta_{m,\tilde{\omega}_1} E[\mathcal{R}_{\tilde{\omega}_1}] + f(\mu) g(\beta_{m,\tilde{\omega}_1})$$

where:

$$\beta_{m,\tilde{\omega}_1} = \frac{cov(R_m, R_{\tilde{\omega}_1})}{var(R_{\tilde{\omega}_1})}$$

is the beta coefficient associated with the income stream m.

$$f(\mu) = \frac{\bar{\mu} z_b^m \bar{R}}{\alpha_0 - \bar{\mu} z_b^m}$$

and $g(\beta_{m,\tilde{\omega}_1}) = \left(1 - \frac{1}{c(m)}\right) - \left(1 - \frac{1}{c(\tilde{\omega}_1)}\right) \beta_{m,\tilde{\omega}_1}$ is an increasing function of $\beta_{m,\tilde{\omega}_1}$.

Proof:

Consider the price of an income stream m settled in Proposition 6.2.1.,
$c(m) = E\left[\frac{\bar{\gamma}_1}{\alpha_0}\right] E[m] - \frac{1}{\alpha_0} cov(m, \tilde{\omega}_1) + \frac{\bar{\mu} z_b^m}{\alpha_0} = E\left[\frac{\bar{\gamma}_1}{\alpha_0}\right] E[m] - \frac{1}{\alpha_0} cov(m, \tilde{\omega}_1) + K$
From where $K = \frac{\bar{\mu} z_b^m}{\alpha_0}$.
Considering $\bar{R}$ the interest rate for a risk-free asset we obtain:

$$1 = E\left[\frac{\bar{\gamma}_1}{\alpha_0}\right](1 + \bar{r}) + K \Leftrightarrow E\left[\frac{\bar{\gamma}_1}{\alpha_0}\right] = \frac{1 - K}{1 + \bar{r}}$$

Substituting this expression into the initial price formula we then have:
$c(m) = \frac{1-K}{1+\bar{r}} E[m] - \frac{1}{\alpha_0} cov(m, \tilde{\omega}_1) + K \Leftrightarrow \frac{c(m) + \frac{1}{\alpha_0} cov(m, \tilde{\omega}_1) - K}{\frac{1-K}{1+\bar{r}}} = E[m] \Leftrightarrow$
$\Leftrightarrow E[m] = \frac{1+\bar{r}}{1-K}\left[c(m) + \frac{1}{\alpha_0} cov(m, \tilde{\omega}_1) - K\right] \Leftrightarrow$
$\Leftrightarrow \frac{E[m]}{c(m)} = \frac{1+\bar{r}}{1-K}\left[1 + \frac{1}{\alpha_0 c(m)} cov(m, \tilde{\omega}_1) - \frac{K}{c(m)}\right] \Leftrightarrow$
$\Leftrightarrow \frac{E[m]}{c(m)} - (1 + \bar{r}) = \frac{1+\bar{r}}{1-K}\left[1 + \frac{1}{\alpha_0 c(m)} cov(m, \tilde{\omega}_1) - \frac{K}{c(m)}\right] - \frac{1+\bar{r}}{1-K}(1 - K) \Leftrightarrow$
$\Leftrightarrow \frac{E[m]}{c(m)} - (1 + \bar{r}) = \frac{1+\bar{r}}{1-K}\left[1 + \frac{1}{\alpha_0 c(m)} cov(m, \tilde{\omega}_1) - \frac{K}{c(m)} - (1 - K)\right] \Leftrightarrow$
$\Leftrightarrow \frac{E[m]}{c(m)} - (1 + \bar{r}) = \frac{1+\bar{r}}{1-K}\left[K + \frac{1}{\alpha_0 c(m)} cov(m, \tilde{\omega}_1) - \frac{K}{c(m)}\right] \Leftrightarrow$
$\Leftrightarrow \frac{E[m]}{c(m)} - (1 + \bar{r}) = \frac{1}{1-K} \frac{1+\bar{r}}{\alpha_0 c(m)} cov(m, \tilde{\omega}_1) + (1 + \bar{r}) \frac{K}{1-K}\left(1 - \frac{1}{c(m)}\right)$
Rewriting this formula for the market portfolio then gives us:

$$\frac{E[\tilde{\omega}_1]}{c(\tilde{\omega}_1)} - (1+\bar{r}) = \frac{1}{1-K}\frac{1+\bar{r}}{\alpha_0 c(\tilde{\omega}_1)} var(\tilde{\omega}_1) + (1+\bar{r})\frac{K}{1-K}\left(1 - \frac{1}{c(\tilde{\omega}_1)}\right) \Leftrightarrow$$
$$\Leftrightarrow \frac{1}{1-K}\frac{1+\bar{r}}{\alpha_0}\frac{var(\tilde{\omega}_1)}{c(\tilde{\omega}_1)} = \frac{E[\tilde{\omega}_1]}{c(\tilde{\omega}_1)} - (1+\bar{r}) - (1+\bar{r})\frac{K}{1-K}\left(1 - \frac{1}{c(\tilde{\omega}_1)}\right) \Leftrightarrow$$
$$\Leftrightarrow \frac{1}{1-K}\frac{1+\bar{r}}{\alpha_0} = \frac{c(\tilde{\omega}_1)}{var(\tilde{\omega}_1)}\left[\frac{E[\tilde{\omega}_1]}{c(\tilde{\omega}_1)} - (1+\bar{r}) - (1+\bar{r})\frac{K}{1-K}\left(1 - \frac{1}{c(\tilde{\omega}_1)}\right)\right]$$

And so we have:

$$\frac{E[m]}{c(m)} - (1+\bar{r}) = \frac{1}{1-K}\frac{1+\bar{r}}{\alpha_0}\frac{cov(m,\tilde{\omega}_1)}{c(m)} + (1+\bar{r})\frac{K}{1-K}\left(1 - \frac{1}{c(m)}\right) \Leftrightarrow$$
$$\Leftrightarrow \frac{E[m]}{c(m)} - (1+\bar{r}) = \frac{cov(m,\tilde{\omega}_1)}{c(m)}\frac{c(\tilde{\omega}_1)}{var(\tilde{\omega}_1)}\left[\frac{E[\tilde{\omega}_1]}{c(\tilde{\omega}_1)} - (1+\bar{r}) - (1+\bar{r})\frac{K}{1-K}\left(1 - \frac{1}{c(\tilde{\omega}_1)}\right)\right] + (1+\bar{r})\frac{K}{1-K}\left(1 - \frac{1}{c(m)}\right) \Leftrightarrow$$

$$\Leftrightarrow \frac{E[m]}{c(m)} - (1+\bar{r}) = \frac{cov(m,\tilde{\omega}_1)}{c(m)}\frac{c(\tilde{\omega}_1)}{var(\tilde{\omega}_1)}\left[\frac{E[\tilde{\omega}_1]}{c(\tilde{\omega}_1)} - (1+\bar{r})\right] +$$
$$+ (1+\bar{r})\frac{K}{1-K}\left[\left(1 - \frac{1}{c(m)}\right) - \left(1 - \frac{1}{c(\tilde{\omega}_1)}\right)\frac{cov(m,\tilde{\omega}_1)}{c(m)}\frac{c(\tilde{\omega}_1)}{var(\tilde{\omega}_1)}\right]$$

We therefore have that:

$$E[\mathcal{R}_m] = \frac{cov(R_m, R_{\tilde{\omega}_1})}{var(R_{\tilde{\omega}_1})}E[\mathcal{R}_{\tilde{\omega}_1}] + \frac{K\bar{R}}{1-K}\left[\left(1 - \frac{1}{c(m)}\right) - \left(1 - \frac{1}{c(\tilde{\omega}_1)}\right)\frac{cov(R_m, R_{\tilde{\omega}_1})}{var(R_{\tilde{\omega}_1})}\right]$$

From where we have to:

$$E[\mathcal{R}_m] = \frac{cov(R_m, R_{\tilde{\omega}_1})}{var(R_{\tilde{\omega}_1})}E[\mathcal{R}_{\tilde{\omega}_1}] + \frac{\bar{\mu} z_b^m \bar{R}}{\alpha_0 - \bar{\mu} z_b^m}\left[\left(1 - \frac{1}{c(m)}\right) - \left(1 - \frac{1}{c(\tilde{\omega}_1)}\right)\frac{cov(R_m, R_{\tilde{\omega}_1})}{var(R_{\tilde{\omega}_1})}\right] \Leftrightarrow$$

$$E[\mathcal{R}_m] = \beta_{m,\tilde{\omega}_1}E[\mathcal{R}_{\tilde{\omega}_1}] + f(\mu)g(\beta_{m,\tilde{\omega}_1}) \tag{6.13}$$

where $\beta_{m,\tilde{\omega}_1} = \frac{cov(R_m, R_{\tilde{\omega}_1})}{var(R_{\tilde{\omega}_1})}$, $f(\mu) = \frac{\bar{\mu} z_b^m \bar{R}}{\alpha_0 - \bar{\mu} z_b^m}$ and $g(\beta_{m,\tilde{\omega}_1}) = \left(1 - \frac{1}{c(m)}\right) - \left(1 - \frac{1}{c(\tilde{\omega}_1)}\right)\beta_{m,\tilde{\omega}_1}$.

Taking equation (6.13) into account, we can see that the term $\beta_{m,\tilde{\omega}_1}E[\mathcal{R}_{\tilde{\omega}_1}]$ is equal to the term present in the classic CAPM. This risk premium term is the inner product between two terms, the parameter $\beta_{m,\tilde{\omega}_1}$ and the expected risk premium $E[\mathcal{R}_{\tilde{\omega}_1}]$ that investors demand from the market portfolio. The $\beta_{m,\tilde{\omega}_1}$ can be called the quantity of risk, which measures the volatility, the systematic risk of a z^m portfolio when compared to the market as a whole. Thus, we can interpret $\beta_{m,\tilde{\omega}_1}$ as the sensitivity of the expected risk premium of an income stream m to the market income stream $\tilde{\omega}_1$. When this term is less than 1 then it refers to an income stream m with a lower volatility than the market or to income streams whose price movements are less related to the market. When the term is greater than 1 then generally the z^m portfolio its volatility is more correlated with the market portfolio.

Considering this second term in equation (6.13) we can conclude that it is new and not present in the classic CAPM formula. In this case, when the restrictions are not active for all agents, i.e. $\mu^i = 0$, then the additional term is zero and we recover the risk premium of the classic CAPM, just as it does when the income stream m is being generated by a portfolio where the bond is not being traded, i.e. $z_b^m = 0$. Conversely, when income stream m is being generated by a portfolio where the bond is being traded and the restriction is active for at least one consumer, then the restriction has an impact on the risk premium.

Corollary 6.3.1. *When trading constraints are binding the influence of the systematic risk on the risk premium is greater.*

This corollary is the result of what can be concluded from Proposition 6.3.1, since when there are trading restrictions the risk premium associated with an income stream has an additional term that also depends on $\beta_{m,\tilde{\omega}_1}$.

Proposition 6.3.2. *If $(\bar{x}, \bar{z}, \bar{q})$ is an equilibrium of the economy $\mathcal{E}$ under the restrictions $\bar{q}_b \bar{z}_b \geqslant -\beta \bar{q}_e \cdot (\bar{z}_e + \delta_e^i)$, $i = 1, \ldots, I$, then the equilibrium risk premium of a portfolio z^m associated to the income stream m generated by the asset returns matrix $[\,\mathbb{1} \;\; Y\,]$ is:*

$$E[\mathcal{R}_m] = \beta_{m,\tilde{\omega}_1} E[\mathcal{R}_{\tilde{\omega}_1}] + f(\mu) g(\beta_{m,\tilde{\omega}_1})$$

where:

$$\beta_{m,\tilde{\omega}_1} = \frac{cov(R_m, R_{\tilde{\omega}_1})}{var(R_{\tilde{\omega}_1})}$$

is the beta coefficient associated with portfolio z^m,

$$f(\mu) = \frac{\bar{\mu}\bar{R}(q_b z_b^m \bar{R} + \nu q_e \cdot z_e^m)}{\alpha_0 - \bar{\mu}(q_b z_b^m + \nu q_e \cdot z_e^m)}$$

and $g(\beta_{m,\tilde{\omega}_1}) = \left(1 - \frac{1}{c(m)}\right) - \left(1 - \frac{1}{c(\tilde{\omega}_1)}\right) \beta_{m,\tilde{\omega}_1}$ is an increasing function of $\beta_{m,\tilde{\omega}_1}$.

As in the previous case, when the restriction is active for at least one of the consumers, then the restriction also has an impact on the risk premium.

In the same way that it was possible to relate the two propositions presented for pricing, we can now relate the previous propositions, where we obtained the risk premium in equilibrium. In these two propositions, we obtained a risk premium result that is similar to the classic CAPM formula, although in this case we also have a new term that differentiates the two formulas. So, generalizing from the two propositions presented, we can conclude that the risk premium associated with models where there are restrictions on transactions has a part that is identical to the classic CAPM, and a new term that comes from the restrictions and includes the Lagrange multipliers associated with the restrictions. Thus, we see that there is clearly a difference between the result of these propositions and the risk premium result that we showed in Chapter 4.

7

Concluding Remarks

The purpose of this dissertation was to show how the pricing of financial products varies when investors have trading restrictions and in the context of the CAPM. When there are trading restrictions, agents have limited access to market assets. In order to derive closed price formulas, it was necessary to consider that investors have mean-variance preferences. Therefore, we gradually explained all the concepts needed to understand the main objective of this dissertation. In addition to the prices, it was also possible to calculate the return associated with each price, which allowed us to extend our conclusions.

Presenting a standard economy in such detail was very important in order to understand the chapters that follow, and to better understand this topic, we also presented some examples. We also described the ideal security of an agent i and its result of this in equilibrium, in the CAPM model. Based on all the concepts that had already been explained, it was easy to introduce the financial model where we introduced restrictions and which we used to study pricing and the risk premium.

For this dissertation we considered a model based on the one presented by Eichberger et al., 2014, with the introduction of restrictions on securities transactions in this instance. With limited access to securities on the market, we tried to understand how this would affect their price.

Throughout this dissertation we have been able to realise that the prices associated with income streams do indeed vary when there are restrictions on transactions and the same applies when it comes to the return associated with them. As we can see, the formulae obtained are similar to the results of the classic CAPM model, which allows us to compare the new prices associated with income streams when there are restrictions with the prices when there are none.

In a future work, it would be very interesting to study other restrictions. It would be interesting, for example, to study a model in which two restrictions were included, or even to combine the two restrictions studied in this dissertation in a single model and see how the price would change. It would also be interesting to see how the price changes, not only in comparison to when there are no restrictions, but also in comparison to when there are fewer restrictions.

We could also consider a case in which there are endogenous and exogenous restrictions in the same model, and to understand, based on models that consider exogenous restrictions (in this dissertation) and models that consider endogenous restrictions (as in Sabarwal, 2003, Carosi et al., 2009, Seghir and Torres-Marínez, 2011, Hoelle et al., 2012, Gori et al., 2014 and Cea-Echenique and Torres-Martínez, 2018), how the price of financial products varies.

Bibliography

Almeida, A., Faias, M., & Luque, J. (2018). Mortgage default risk amplifies the effect of systemic risk on risk premium. https://works.bepress.com/luque/35/

Aouani, Z., & Cornet, B. (2009). Existence of financial equilibria with restricted participation. *Journal of Mathematical Economics*, *45*, 772–786.

Araújo, A. P., & Villalba, J. M. (2016). Bankruptcy equilibrium : Secured and unsecured assets, 1–23.

Balasko, Y., Cass, D., & Siconolfi, P. (1990). The structure of financial equilibrium with exogenous yields : The case of restricted participation. *Journal of Mathematical Economics*, *19*, 195–216.

Carosi, L., Gori, M., & Villanacci, A. (2009). Endogenous restricted participation in general financial equilibrium. *Journal of Mathematical Economics*, *45*, 787–806.

Cea-Echenique, S., & Torres-Martínez, J. P. (2016). Credit segmentation in general equilibrium. *Journal of Mathematical Economics*, *62*, 19–27.

Cea-Echenique, S., & Torres-Martínez, J. P. (2018). General equilibrium with endogenous trading constraints. *PLoS ONE*, *13*(9), 1–10.

Cornet, B., & Gopalan, R. (2010). Arbitrage and equilibrium with portfolio constraints. *Economic Theory*, *45*(1/2), 227–252.

Eichberger, J., Rheinberger, K., & Summer, M. (2014). Credit risk in general equilibrium. *Economic Theory*, *57*(2), 407–435.

Elsinger, H., & Summer, M. (2001). Arbitrage and optimal portfolio choice with financial constraints. *Oesterreichische Nationalbank, Wien.*

Faias, M., & Torres-Martínez, J. P. (2017). Credit market segmentation, essentiality of commodities, and supermodularity. *Journal of Mathematical Economics*, *70*, 115–122.

Geanakoplos, J. D. (1990). An introduction to general equilibrium with incomplete asset markets. *Journal of Mathematical Economics*, *19*(1-2), 1–38.

Geanakoplos, J. D., & Polemarchakis, H. M. (1986). Existence, regularity, and constrained suboptimality of competitive allocations when the asset market is incomplete. In W. Heller, R. Starr, & D. Starret (Eds.), *Essays in honor of kenneth arrow* (pp. 65–96, Vol. 3). Cambridge University Press.

Gori, M., Pireddu, M., & Villanacci, A. (2013). Regularity and pareto improving on financial equilibria with price-dependent borrowing restrictions. *Research in Economics*, *67*, 100–110.

Gori, M., Pireddu, M., & Villanacci, A. (2014). Existence of financial equilibria with endogenous short selling restrictions and real assets. *Decisions in Economics and Finance*, *37*(2), 349–371.

Hoelle, M., Pireddu, M., & Villanacci, A. (2012, June). *Incomplete financial markets with real assets and endogenous credit limits* (Purdue University Economics Working Papers No. 1271). Purdue University, Department of Economics.

Lourenço, J. M. (2021). *The NOVAthesis LaTeX Template User's Manual*. NOVA University Lisbon. https://github.com/joaomlourenco/novathesis/raw/main/template.pdf

Magill, M., & Quinzii, M. (1996). *Theory of incomplete markets*. MIT Press.

Polemarchakis, H. M., & Siconolfi, P. (1997). Generic existence of competitive equilibria with restricted participation. *Journal of Mathematical Economics*, *28*(3), 289–311.

Sabarwal, T. (2003). Competitive equilibria with incomplete markets and endogenous bankruptcy. *Contributions to Theoretical Economics*, *3*(1).

Seghir, A., & Torres-Marínez, J. P. (2011). On equilibrium existence with endogenous restricted financial participation. *Journal of Mathematical Economics*, *47*(1), 37–42.

Simon, C. P., & Blume, L. (1985). *Mathematics for economists*. W. W. Norton Company, Inc.

A

Appendix

This appendix contains calculations of Exercises or Proofs of Propositions presented throughout this dissertation.

Exercise 2.1.1.

To reach the equilibrium of this economy let us first focus on the maximization problem of the agent 1:

$max\ ln(x_0) + \frac{1}{2}(\frac{1}{2}\ln(x_1)) + \frac{1}{2}\ln(x_2))$

$$\text{s.a}\begin{cases} x_0 = \frac{19}{8} - qz \\ x_1 = 1 + z \\ x_2 = 3 + 0 \times z \end{cases} \Leftrightarrow \begin{cases} x_0 = \frac{19}{8} - qz \\ x_1 = 1 + z \\ x_2 = 3 \end{cases}$$

Then

$max\ ln(\frac{19}{8} - qz) + \frac{1}{2}(\frac{1}{2}\ln(1+z) + \frac{1}{2}\ln(3))$

$\dfrac{\delta}{\delta z} = 0 \Leftrightarrow \dfrac{-q}{\frac{19}{8} - qz} + \dfrac{1}{4} \times \dfrac{1}{1+z} = 0 \Leftrightarrow \dfrac{-q}{\frac{19}{8} - qz} + \dfrac{1}{4+4z} = 0 \Leftrightarrow -4q - 4qz + \dfrac{19}{8} - qz = 0 \Leftrightarrow -5qz = 4q - \dfrac{19}{8} \Leftrightarrow z^1 = \dfrac{19}{40q} - \dfrac{4}{5}$

We now turn our focus on the maximization problem of agent 2:

$max\ ln(x_0) + \frac{1}{6}\ln(x_1) + \frac{1}{6}\ln(x_2)$

$$\text{s.a}\begin{cases} x_0 = \frac{21}{8} - qz \\ x_1 = 5 + z \\ x_2 = 3 + 0 \times z \end{cases} \Leftrightarrow \begin{cases} x_0 = \frac{21}{8} - qz \\ x_1 = 5 + z \\ x_1 = 3 \end{cases}$$

Then

$max\ ln(\frac{21}{8} - qz) + \frac{1}{6}\ln(5+z) + \frac{1}{6}\ln(3)$

$\dfrac{\partial}{\partial z} = 0 \Leftrightarrow \dfrac{-q}{\frac{21}{8} - qz} + \dfrac{1}{6} \times \dfrac{1}{5+z} = 0 \Leftrightarrow -30q - 6qz + \dfrac{21}{8} - qz = 0 \Leftrightarrow -7qz = 30q - \dfrac{21}{8} \Leftrightarrow z^2 = \dfrac{21}{56q} - \dfrac{30}{7}$

$z^1 + z^2 = 0 \Leftrightarrow \dfrac{19}{40q} - \dfrac{4}{5} + \dfrac{21}{56q} - \dfrac{30}{7} = 0 \Leftrightarrow \dfrac{1064q + 840q}{2240q^2} = \dfrac{178}{35} \Leftrightarrow \dfrac{1064 + 840}{\frac{178}{35}} = 2240q \Leftrightarrow$
$q = 0,17$

Thus we have the following solutions to the problem:
$(\bar{p}, \bar{q}) = (1, 1, 1, 0.17)$
$(\bar{x}_0^1, \bar{x}_1^1, \bar{x}_2^1, z^1) = (2.035, 3, 3, 2)$
$(\bar{x}_0^2, \bar{x}_1^2, \bar{x}_2^2, z^2) = (2.965, 3, 3, -2)$

Proof of Theorem 6.1.1.

The Lagrangean function associated with the optimization problem (6.1) is as follows:

$$L(x, y, \mu, a) = f(x, y) - \mu(h(x, y) - a) \tag{A.1}$$

with a taken as parameter. Thus, the solution $(x^*(a), y^*(a))$ satisfies the following conditions:

$$\frac{\partial L(x, y, \mu, a)}{\partial x}(x^*(a), y^*(a), \mu^*(a); a) = 0 \Leftrightarrow \frac{\partial f}{\partial x}(x^*(a), y^*(a), \mu^*(a)) - \mu^*(a)\frac{\partial h}{\partial x}(x^*(a), y^*(a), \mu^*(a)) = 0$$

$$\frac{\partial L(x, y, \mu, a)}{\partial y}(x^*(a), y^*(a), \mu^*(a); a) = 0 \Leftrightarrow \frac{\partial f}{\partial y}(x^*(a), y^*(a), \mu^*(a)) - \mu^*(a)\frac{\partial h}{\partial y}(x^*(a), y^*(a), \mu^*(a)) = 0$$

for all a. Furthermore, since $h\left(x^*(a), y^*(a)\right) = a$ for all a,

$$\frac{\partial h}{\partial x}(x^*, y^*)\frac{dx^*}{da}(a) + \frac{\partial h}{\partial y}(x^*, y^*)\frac{dy^*}{da}(a) = 1$$

for every a. Therefore, using the Chain Rule and equations (4) and (5),

$$\begin{aligned}
\frac{d}{da} f\left(x^*(a), y^*(a)\right) &= \frac{\partial f}{\partial x}(x^*(a), y^*(a))\frac{dx^*}{da}(a) + \frac{\partial f}{\partial y}(x^*(a), y^*(a))\frac{dy^*}{da}(a) \\
&= \mu^*\frac{\partial h}{\partial x}(x^*(a), y^*(a))\frac{dx^*}{da}(a) + \mu^*\frac{\partial h}{\partial y}(x^*(a), y^*(a))\frac{dy^*}{da}(a) \\
&= \mu^*\left[\frac{\partial h}{\partial x}(x^*(a), y^*(a))\frac{dx^*}{da}(a) + \frac{\partial h}{\partial y}(x^*(a), y^*(a))\frac{dy^*}{da}(a)\right] \\
&= \mu^* \cdot 1
\end{aligned}$$

Proof of Proposition 6.2.2.

We have the following constraints:

$$\begin{cases}
x_0^i - \omega_0^i \leq -q_b z_b - q_e \cdot z_{e,j} \\
x_{\mathbf{1}}^i - \omega_{\mathbf{1}}^i - \Upsilon\delta \leq \mathbb{1} z_b + \Upsilon z_e \\
q_b z_b \geqslant -\nu q_e \cdot (z_e + \delta_e^i)
\end{cases}$$

Consider the following notation, $x^i = (x_0^i, x_1^i, \ldots, x_s^i, \ldots, x_S^i)$ and $z^i = (z_b^i, z_e^i) = (z_b^i, z_1^i, \ldots, z_j^i, \ldots, z_J^i)$. The vector $\pi^i = (\pi_0^i, \pi_1^i, \ldots, \pi_s^i, \ldots, \pi_S^i)$ denotes the Lagrange multipliers associated to the budget constraints and μ^i the Lagrange multiplier associated to the transaction restriction.

The Lagrangian function associated with the consumer problem is given by, $L^i(x^i, z^i, \pi^i, \mu^i) = u^i(x^i) - \pi_0^i(x_0^i - \omega_0^i + q_b z_b^i + q_e \cdot z_e^i) - \sum_{s=1}^S \pi_s^i(x_s^i - \omega_s^i - \sum_{j=1}^J Y_{s,j}\delta_j^i - z_b^i - \sum_{j=1}^J Y_{s,j} z_j^i) + \mu^i(q_b z_b^i + \nu q_e \cdot (z_e^i + \delta_e^i))$

Since $\bar{x}^i$ and $\bar{z}^i$ are the optimal choice of consumer i, the following conditions have to be satisfied:

$$\frac{\partial L^i(\bar{x}^i, \bar{z}^i, \pi^i, \mu^i)}{\partial x_0} = 0 \Leftrightarrow \frac{\partial u^i(\bar{x}^i)}{\partial x_0} = \pi_0 \tag{A.2}$$

$$\frac{\partial L^i(\bar{x}^i, \bar{z}^i, \pi^i, \mu^i)}{\partial x_s} = 0 \Leftrightarrow \frac{\partial u^i(\bar{x}^i)}{\partial x_s} = \pi_s, s = 1, \dots, S \tag{A.3}$$

$$\frac{\partial L^i(\bar{x}^i, \bar{z}^i, \pi^i, \mu^i)}{\partial z_b} = 0 \Leftrightarrow -\pi_0^i \bar{q}_b + \sum_{s=1}^{s} \pi_s^i + \mu^i \bar{q}_b = 0 \tag{A.4}$$

$$\frac{\partial L^i(\bar{x}^i, \bar{z}^i, \pi^i, \mu^i)}{\partial z_j} = 0 \Leftrightarrow -\pi_0^i \bar{q}_j + \sum_{s=1}^{s} \pi_s^i Y_{s,j} + \mu^i \nu \bar{q}_j = 0, \quad j = 1, \dots, J \tag{A.5}$$

$$\frac{\partial L^i(\bar{x}^i, \bar{z}^i, \pi^i, \mu^i)}{\partial \pi_0} = 0 \Leftrightarrow \bar{x}_0^i - \omega_0^i + \bar{q}_b \bar{z}_b^i + \bar{q}_e \cdot \bar{z}_e^i = 0, \tag{A.6}$$

$$\frac{\partial L^i(\bar{x}^i, \bar{z}^i, \pi^i, \mu^i)}{\partial \pi_s} = 0 \Leftrightarrow \bar{x}_s^i - \omega_s^i - \sum_{j=1}^{J} Y_{s,j}\delta_j^i - \bar{z}_b^i - \sum_{j=1}^{J} Y_{s,j}\bar{z}_j^i = 0, s = 1, \dots, S \tag{A.7}$$

$$\frac{\partial L^i(\bar{x}^i, \bar{z}^i, \pi^i, \mu^i)}{\partial \mu} = 0 \Leftrightarrow \bar{q}_b \bar{z}_b^i + \nu \bar{q}_e \cdot (\bar{z}_e^i + \delta_e^i) = 0 \tag{A.8}$$

Using (A.1) and (A.2) to substitute into the equations of (A.3) and (A.4) we then have:

$$\frac{\partial L^i}{\partial z_b} = 0 \Leftrightarrow -\frac{\partial u^i(\bar{x}^i)}{\partial x_0}\bar{q}_b + \frac{\partial u^i(\bar{x}^i)}{\partial x_s} = -\mu^i \bar{q}_b \tag{A.9}$$

$$\frac{\partial L^i}{\partial z_j} = 0 \Leftrightarrow -\frac{\partial u^i(\bar{x}^i)}{\partial x_0}\bar{q}_e + \sum_{s=1}^{S} \frac{\partial u^i(\bar{x}^i)}{\partial x_s} Y_{s,j} = -\mu^i \nu \bar{q}_j \tag{A.10}$$

Let $Y = \begin{bmatrix} Y_{1,1} & \cdots & Y_{1,j} & \cdots & Y_{1,J} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Y_{s,1} & \cdots & Y_{s,j} & \cdots & Y_{s,J} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Y_{S,1} & \cdots & Y_{S,j} & \cdots & Y_{S,J} \end{bmatrix}$

Taking T_b and T_e as:

$T_b = \begin{bmatrix} -\bar{q}_b \\ \mathbb{1} \end{bmatrix}$ e $T_e = \begin{bmatrix} -\bar{q}_e \\ Y \end{bmatrix}$

Then from (A.8) and (A.9), respectively, we can conclude that:

$$\langle \nabla u^i(\bar{x}^i), T_b \rangle = -\mu^i \bar{q}_b$$

$$\langle \nabla u^i(\bar{x}^i), T_e \rangle = -\mu \nu \bar{q}_e$$

Let $\tau = Tz = \begin{bmatrix} -\bar{q}_b & -\bar{q}_e \\ \mathbb{1} & Y \end{bmatrix} \begin{bmatrix} z_b^m \\ z_e^m \end{bmatrix}$

Making $\langle \nabla u^i(\bar{x}^i), \tau \rangle$ we then have:

$\langle \nabla u^i(\bar{x}^i), \tau \rangle = \left[\frac{\partial u^i(\bar{x}^i)}{\partial x_0} \frac{\partial u^i(\bar{x}^i)}{\partial x_1} \cdots \frac{\partial u^i(\bar{x}^i)}{\partial x_s} \right] \begin{bmatrix} -\bar{q}_b & -\bar{q}_e \\ \mathbb{1} & Y \end{bmatrix} \begin{bmatrix} z_b^m \\ z_e^m \end{bmatrix} =$
$= \begin{bmatrix} -\frac{\partial u^i(\bar{x}^i)}{\partial x_0}\bar{q}_b + \frac{\partial u^i(\bar{x}^i)}{\partial x_1} + \cdots + \frac{\partial u^i(\bar{x}^i)}{\partial x_s} & -\frac{\partial u^i(\bar{x}^i)}{\partial x_0}\bar{q}_e + \frac{\partial u^i(\bar{x}^i)}{\partial x_1}Y_1 + \cdots + \frac{\partial u^i(\bar{x}^i)}{\partial x_s}Y_s \end{bmatrix} \begin{bmatrix} z_b^m \\ z_e^m \end{bmatrix} =$
$= \left(-\frac{\partial u^i(\bar{x}^i)}{\partial x_0}\bar{q}_b + \frac{\partial u^i(\bar{x}^i)}{\partial x_1} + \cdots + \frac{\partial u^i(\bar{x}^i)}{\partial x_s} \right) z_b^m + \left(-\frac{\partial u^i(\bar{x}^i)}{\partial x_0}\bar{q}_1 + \frac{\partial u^i(\bar{x}^i)}{\partial x_1}Y_{1,1} + \cdots + \frac{\partial u^i(\bar{x}^i)}{\partial x_s}Y_{s,1} \right) z_1^m + \cdots +$
$\left(-\frac{\partial u^i(\bar{x}^i)}{\partial x_0}q_j + \frac{\partial u^i(\bar{x}^i)}{\partial x_1}Y_{1,j} + \cdots + \frac{\partial u^i(\bar{x}^i)}{\partial x_s}Y_{s,j} \right) z_j^m + \cdots + \left(-\frac{\partial u^i(\bar{x}^i)}{\partial x_0}q_J^m + \frac{\partial u^i(\bar{x}^i)}{\partial x_1}Y_{1,J} + \cdots + \frac{\partial u^i(\bar{x}^i)}{\partial x_s}Y_{s,J} \right) z_J^m =$
$-\mu^i \bar{q}_b z_b^m - \mu^i \nu \bar{q}_e \cdot z_e^m$

Given that $\nabla u^i(\bar{x}^i) = \left(\frac{\partial u^i(\bar{x}^i)}{\partial x_0}, \left(\frac{1}{\rho_s} \frac{\partial u^i(\bar{x}^i)}{\partial x_s} \right)_{s=1}^S \right)$ and that $u^i\left(\bar{x}^i\right) = \alpha_0^i \bar{x}_0^i - \frac{1}{2}\sum_{s=1}^S \rho_s \left(\alpha_1^i - \bar{x}_s^i\right)^2, i = 1, \ldots, I$, then we have that:

$\frac{\partial u^i(\bar{x}_0^i)}{\partial x_0} = \alpha_0^i$ and $\frac{\partial u^i(\bar{x}^i)}{\partial x_s} = \rho_s(\alpha_1^i - \bar{x}_s^i),\ s = 1, \ldots, S$. That is,

$$\nabla u^i(\bar{x}^i) = (\alpha_0^i, \alpha_1^i - \bar{x}_1^i, \ldots, \alpha_1^i - \bar{x}_S^i).$$

If we add in all the agents we get:
$\left(\sum_{i=1}^I \alpha_0^i, \sum_{i=1}^I \alpha_1^i - \bar{x}_1^i, \ldots, \sum_{i=1}^I \alpha_1^i - \bar{x}_S^i \right)$
$= \left(\alpha_0, \alpha_1 - \sum_{i=1}^I \bar{x}_1^i, \ldots, \alpha_1 - \sum_{i=1}^I \bar{x}_S^i \right) = (\alpha_0, \alpha_1\mathbb{1} - (\omega_{\mathbf{1}} + Y\delta)).$

Finally we have:
$\langle \sum_{i=1}^I \nabla u^i(\bar{x}^i), \tau \rangle = -\sum_{i=1}^I \bar{\mu}\bar{q}_b z_b^m - \sum_{i=1}^I \bar{\mu}\nu\bar{q}_e \cdot z_e^m \Leftrightarrow \langle (\alpha_0, \alpha_1\mathbb{1} - \omega^i - Y\delta^i), (-c(m), m) \rangle =$
$-\bar{\mu}\bar{q}_b z_b^m - \nu\bar{\mu}\bar{q}_e \cdot z_e^m \Leftrightarrow -c(m)\alpha_0 + \langle (\alpha_1\mathbb{1} - \omega^i - Y\delta^i), m \rangle = -\bar{\mu}\bar{q}_b z_b^m - \bar{\mu}\nu\bar{q}_e \cdot z_e^m \Leftrightarrow$
$\Leftrightarrow -c(m)\alpha_0 = -\bar{\mu}\bar{q}_b z_b^m - \bar{\mu}\nu\bar{q}_e \cdot z_e^m - \langle (\alpha_1\mathbb{1} - \omega^i - Y\delta^i), m \rangle \Leftrightarrow$
$\Leftrightarrow c(m) = \langle \frac{\gamma_1}{\alpha_0}, m \rangle + \frac{\bar{\mu}\bar{q}_b z_b^m}{\alpha_0} + \frac{\bar{\mu}\nu\bar{q}_e \cdot z_e^m}{\alpha_0} = E\left[\frac{\gamma_1}{\alpha_0}\right] E[m] - \frac{1}{\alpha_0} cov(m, \tilde{\omega_{\mathbf{1}}}) + \frac{\bar{\mu}(\bar{q}_b z_b^m + \nu\bar{q}_e \cdot z_e^m)}{\alpha_0}$

Proof of Proposition 6.3.2.

Consider the price of an income stream m settled in Proposition 6.1.2:
$c(m) = E\left[\frac{\gamma_1}{\alpha_0}\right] E[m] - \frac{1}{\alpha_0} cov(m, \tilde{\omega_{\mathbf{1}}}) + \frac{\bar{\mu}(q_b z_b^m + \nu q_e \cdot z_e^m)}{\alpha_0} \Leftrightarrow c(m) = E\left[\frac{\gamma_1}{\alpha_0}\right] E[m] - \frac{1}{\alpha_0} cov(m, \tilde{\omega_{\mathbf{1}}}) + K$

From where $K = \frac{\bar{\mu}(q_b z_b^m + \nu q_e \cdot z_e^m)}{\alpha_0}$.

Considering $\bar{R}$ the interest rate for a risk-free asset we obtain:

$$1 = E[\frac{\bar{\gamma_1}}{\alpha_0}](1+\bar{R}) + K \Leftrightarrow E[\frac{\bar{\gamma_1}}{\alpha_0}] = \frac{1-K}{1+\bar{R}}$$

Substituting this expression into the initial price formula we then have:

$c(m) = \frac{1-K}{1+\bar{R}}E[m] - \frac{1}{\alpha_0}cov(\tilde{w}_1, m) + K \Leftrightarrow \frac{c(m)+\frac{1}{\alpha_0}cov(\tilde{w}_1,m)+K}{\frac{1-K}{1+\bar{R}}} = E[m] \Leftrightarrow$

$\Leftrightarrow E[m] = \frac{1+\bar{R}}{1-K}[c(m) + \frac{1}{\alpha_0}cov(\tilde{w}_1, m) + K] \Leftrightarrow$

$\Leftrightarrow \frac{E[m]}{c(m)} = \frac{1+\bar{R}}{1-K}[1 + \frac{1}{\alpha_0 c(m)}cov(\tilde{w}_1, m) + \frac{K}{c(m)}] \Leftrightarrow$

$\Leftrightarrow \frac{E[m]}{c(m)} - (1+\bar{R}) = \frac{1+\bar{R}}{1-K}[1 + \frac{1}{\alpha_0 c(m)}cov(\tilde{w}_1, m) + \frac{K}{c(m)}] - \frac{1+\bar{R}}{1-K}(1-K) \Leftrightarrow$

$\Leftrightarrow \frac{E[m]}{c(m)} - (1+\bar{R}) = \frac{1+\bar{R}}{1-K}[1 + \frac{1}{\alpha_0 c(m)}cov(\tilde{w}_1, m) + \frac{K}{c(m)} - (1+K)] \Leftrightarrow$

$\Leftrightarrow \frac{E[m]}{c(m)} - (1+\bar{R}) = \frac{1+\bar{R}}{1-K}[K + \frac{1}{\alpha_0 c(m)}cov(\tilde{w}_1, m) + \frac{K}{c(m)}] \Leftrightarrow$

$\Leftrightarrow \frac{E[m]}{c(m)} - (1+\bar{R}) = \frac{1}{1-K}\frac{1+\bar{R}}{\alpha_0 c(m)}cov(\tilde{w}_1, m) + (1+\bar{R})\frac{K}{1-K}(1 - \frac{1}{c(m)})$

Rewriting this formula for the market portfolio then gives us:

$\frac{E[\tilde{w}_1]}{c(\tilde{w}_1)} - (1+\bar{R}) = \frac{1}{1-K}\frac{1+\bar{R}}{\alpha_0 c(\tilde{w}_1)}var(\tilde{w}_1) + (1+\bar{R})\frac{K}{1-K}(1 - \frac{1}{c(\tilde{w}_1)}) \Leftrightarrow$

$\Leftrightarrow \frac{1}{1-K}\frac{1+\bar{R}}{\alpha_0}\frac{var(\tilde{w}_1)}{c(\tilde{w}_1)} = \frac{E[\tilde{w}_1]}{c(\tilde{w}_1)} - (1+\bar{R}) - (1+\bar{R})\frac{K}{1-K}(1 - \frac{1}{c(\tilde{w}_1)}) \Leftrightarrow$

$\Leftrightarrow \frac{1}{1-K}\frac{1+\bar{R}}{\alpha_0} = \frac{c(\tilde{w}_1)}{var(\tilde{w}_1)}[\frac{E[\tilde{w}_1]}{c(\tilde{w}_1)} - (1+\bar{R}) - (1+\bar{R})\frac{K}{1-K}(1 - \frac{1}{c(\tilde{w}_1)})]$

And so we have:

$\frac{E[m]}{c(m)} - (1+\bar{R}) = \frac{1}{1-K}\frac{1+\bar{R}}{\alpha_0}\frac{cov(\tilde{w}_1,m)}{c(m)} + (1+\bar{R})\frac{K}{1-K}(1 - \frac{1}{c(m)}) \Leftrightarrow$

$\Leftrightarrow \frac{E[m]}{c(m)} - (1+\bar{R}) = \frac{cov(\tilde{w}_1,m)}{c(m)}\frac{c(\tilde{w}_1)}{var(\tilde{w}_1)}[\frac{E[\tilde{w}_1]}{c(\tilde{w}_1)} - (1+\bar{R}) - (1+\bar{R})\frac{K}{1-K}(1 - \frac{1}{c(\tilde{w}_1)})] + (1+\bar{R})\frac{K}{1-K}(1 - \frac{1}{c(m)}) \Leftrightarrow$

$\Leftrightarrow \frac{E[m]}{c(m)} - (1+\bar{R}) = \frac{cov(\tilde{w}_1,m)}{c(m)}\frac{c(\tilde{w}_1)}{var(\tilde{w}_1)}[\frac{E[\tilde{w}_1]}{c(\tilde{w}_1)} - (1+\bar{R})] + (1+\bar{R})\frac{K}{1-K}[(1 - \frac{1}{c(m)}) - (1 - \frac{1}{c(\tilde{w}_1)})\frac{cov(\tilde{w}_1,m)}{c(m)}\frac{c(\tilde{w}_1)}{var(\tilde{w}_1)}]$

We therefore have that:

$E[\mathcal{R}_m] = \frac{cov(R_{\tilde{w}_1},R_m)}{var(R_{\tilde{w}_1})}E[R_{\tilde{w}_1}] + \frac{K\bar{R}}{1-K}[(1 - \frac{1}{c(m)}) - (1 - \frac{1}{c(\tilde{w}_1)})\frac{cov(R_{\tilde{w}_1},R_m)}{var(R_{\tilde{w}_1})}]$

From where we have to:

$E[\mathcal{R}_m] = \frac{cov(R_{\tilde{w}_1},R_m)}{var(R_{\tilde{w}_1})}E[R_{\tilde{w}_1}] + \frac{\mu q_b z_b \bar{R} + \mu q_e \nu z_e \bar{R}}{\alpha_0 - \mu q_b z_b + \mu \nu q_e \cdot z_e}[(1 - \frac{1}{c(m)}) - (1 - \frac{1}{c(\tilde{w}_1)})\frac{cov(R_{\tilde{w}_1},R_m)}{var(R_{\tilde{w}_1})}] \Leftrightarrow E[\mathcal{R}_m] = \beta_{\tilde{w}_1 m}E[R_{\tilde{w}_1}] + f(\mu)g(\beta_{\tilde{w}_1 m})$

Where $\beta_{\tilde{w}_1 m} = \frac{cov(R_{\tilde{w}_1},R_m)}{var(R_{\tilde{w}_1})}$, $f(\mu) = \frac{\bar{\mu}\bar{R}(q_b z_b \bar{R} + \nu q_e \cdot z_e)}{\alpha_0 - \bar{\mu}(q_b z_b + \nu q_e \cdot z_e)}$ and $g(\beta_{\tilde{w}_1 m}) = [(1 - \frac{1}{c(m)}) - (1 - \frac{1}{c(\tilde{w}_1)})\frac{cov(R_{\tilde{w}_1},R_m)}{var(R_{\tilde{w}_1})}]$.