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QUADRATIC PROGRAMMING VERSUS SECOND ORDER CONE PROGRAMMING IN PORTFOLIO OPTIMIZATION

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Abstract

Despite the extensive literature in finding new models to replace the Markowitz model or trying to increase the accuracy of its input estimations, there is less studies about the impact on the results of using different optimization algorithms. This paper aims to add some research to this field by comparing the performance of two optimization algorithms in drawing the Markowitz Efficient Frontier and in real world investment strategies. Second order cone programming is a faster algorithm, appears to be more efficient, but is impossible to assert which algorithm is better. Quadratic Programming often shows superior performance in real investment strategies.

Key words: Portfolio Optimization, Second Order Cone Programming, Quadratic Programming

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1. Introduction

Portfolio Optimization is about allocation. Which investments should one choose to maximize return and bear the minimum risk? The Modern Portfolio Theory by Markowitz (1954) tries to solve this problem by creating the Mean – Variance framework. In order to solve the Markowitz optimization problem, the most used algorithm is the Quadratic Programming. The hypothetical problem of using Quadratic Programming, is the existence of previous work signalling a superior investment return by using a different optimization algorithm, the Second Order Cone Programming.

This paper objective is to test two hypothesis by increasing the dataset and number of strategies in relation to the previous work aiming to draw robust results. The hypothesis tested are:

H1: Does the Mean-Variance Efficient Frontier depend on the optimization algorithm used?

H2: If H1 is true, does it translates into a superior risk adjusted return?

The thesis is divided in four chapters. In chapter 1 there is a literature review, where the main portfolio optimization concepts are contextualized and evince the latest's works in the area. In chapter 2 the optimization problems and methodology used to test the two hypothesis are explained. In chapter 3, the aggregated results are presented and the possible conclusions are drawn and discussed. Finally in chapter 4, the conclusion, it is made a wrap-up of the paper, the main results are reinforced while discussing some of the limitations and suggested improvements.

2. Literature review

Portfolio Optimization is about how the investor's initial wealth should be allocated among the available financial assets in order to find the optimal combination of weights that maximizes/minimizes the investor's requirements while respecting their constraints (Markowitz, 1952; Sharpe, 2000).

Markowitz (1952) developed the Mean – Variance approach or Markowitz Model to tackle the portfolio optimization problem. The Mean – Variance Theory (MVT) is the foundation of the Modern Portfolio Theory (MPT) and is still used nowadays. In the Mean – Variance framework it is assumed that the investor only cares about return and risk, so the portfolio optimization problem is resumed to maximize return given a certain level of risk or the opposite, minimize the risk given a certain level of return. In order to do so, the model needs inputs, such as, assets returns and the assets variance-covariance matrix. The latter one, is one of the differences comparing with prior portfolio optimization theories. The variance-covariance matrix allows the model to take into account the diversification gains when constructing a portfolio. Diversification has a key role in selecting the optimal combination of investable assets. By spreading the investment among assets it is possible to reduce the portfolio's overall risk, since it is reducing the negative impact on portfolio's return of stock specific risks.

The Mean – Variance theory assumes that the inputs, assets returns and variance-covariance matrix are known, but that is not true. Two of the most accepted and used models to estimate the expected return of an asset are the Capital Asset Pricing Model (CAPM) by Sharpe (1964) and Lintner (1965), and Arbitrage Pricing Theory (APT) by Ross (1976).

The CAPM, is a method to estimate the expected rate of return of an asset given the risk free rate, the sensitivity of a security to market movements (systematic risk) and the market return. Since it only takes into account the systematic risk of the security, the expected return is the return required if that asset was to be added to an already diversified strategy.

The APT, states that the expected return of an asset can be determined by its sensitivity to some economic or financial factors. It can be viewed as a less restrictive model than CAPM, since it allows more "risk" factors to take part in determining the asset return. For example, Fama and French (1993) added two factors to the traditional CAPM model. They discovered a superior performance of small capitalization stocks versus high capitalization stock and high book to market ratio stocks versus low book to market ratio stocks, measured by the Small minus Big (SMB) factor and high minus low (HML) factor respectively. By adding these two factors to the model, the estimations of the expected returns were more accurate and the model was more robust.

When looking to prior portfolio optimization studies, one can find a lot of research about new models to increase the accuracy of input estimations or different approaches to the Mean – Variance framework, e.g. Intertemporal CAPM by Merton (1973), Four Factor Model by Carhart (1997) and Black – Litterman by Black and Litterman (1991), but few on how the actual portfolio optimal combination is computed. The traditional Markowitz optimization model is solved by a Quadratic Programming (QP) optimization algorithm. It is used for problems where the objective function is quadratic and the constraints are linear. The problem with this optimization tool, is the increase in computing time with the increase of decision variables. A possible solution to this problem is to use Second Order Cone Programming (SOCP). By doing a mathematical transformation, the Markowitz problem can be suited to solve through SOCP.

Davidsson (2011), tested the performance of two investment strategies using both algorithms. By selecting a dataset of one hundred SP500 stocks returns from 2005 to 2010, for one certain level of expected return and a portfolio rebalancing strategy of twenty days without short selling, Davidsson spotted a difference of 2.8 percent return premium by using SOCP instead of QP as the portfolio optimization algorithm. Although Davidsson results are interesting, he only uses one level of portfolio expected return and one rebalancing scheme, so it is impossible to say that SOCP will always be better than QP when solving the Markowitz optimization problem.

The aim of this paper is to explore Davidsson results using a bigger data set, various levels of return, rebalancing intervals, weight constraints and with or without short selling in order to better understand the results and draw more robust conclusions.

3. Methodology

3.1. Theoretical Models

3.1.1. Quadratic Programming

The traditional Markowitz optimization model is solved by a Quadratic Programming (QP) optimization algorithm. The goal is to minimize the risk given a certain level of expected return. The model to be solved by QP can be formulated as:

$$Min\ \sigma_p^2 = Min\ \vec{w}^T \Sigma\, \vec{w}$$

$$\text{Subject to } \vec{w}^T \vec{\bar{r}} = r_p\ ;$$

$$\vec{1}^T \vec{w} = 1\ ;$$

Where $\vec{w}$ is the column vector of securities' weights, T is the transpose notation, Σ is the variance – covariance matrix, $\vec{\bar{r}}$ is the column vector of the securities' estimated returns, r_p is the aimed portfolio return and $\vec{1}$ is a column vector of ones. Due to the quadratic objective function (e.g. $w_j^2 \sigma_j^2$) and the linear constraints, QP is the recommended approach to solve the stated optimization problem.

3.1.2. Second Order Cone Programming

To solve the traditional Markowitz with a SOCP algorithm, the optimization problem needs to be rearranged. A new variable needs to be introduced, e.g. a, the objective function becomes to minimize a, and the old quadratic objective function becomes a constraint transformed into a second order cone. This is possible since we can decompose the variance – covariance matrix as a product of two matrices, using the Cholesky decomposition matrix[1]. Then the constraint can be changed to:

$$\vec{w}^T \Sigma \, \vec{w} < a^2 \Leftrightarrow \vec{w}^T R R^T \, \vec{w} < a^2 \Leftrightarrow (R\vec{w})^T R\vec{w} < a^2 \Leftrightarrow \sqrt{(R\vec{w})^T R\vec{w}} < \sqrt{a^2} \Leftrightarrow$$

$$^{2}norm(R\vec{w}) < a$$

Then, the model to be solved by SOCP can be formulated as:

$$Min \; a$$

Subject to $norm(R\vec{w}) < a$;

$$\vec{w}^T \vec{\vec{r}} = r_p \; ;$$

$$\vec{1}^T \vec{w} = 1 \; ;$$

The advantage of using a conic constraint is that the conic constraint is convex. "*For a convex problem any locally optimal point is a globally optimal hence the optimization becomes fast*" (Davidsson, 2011).

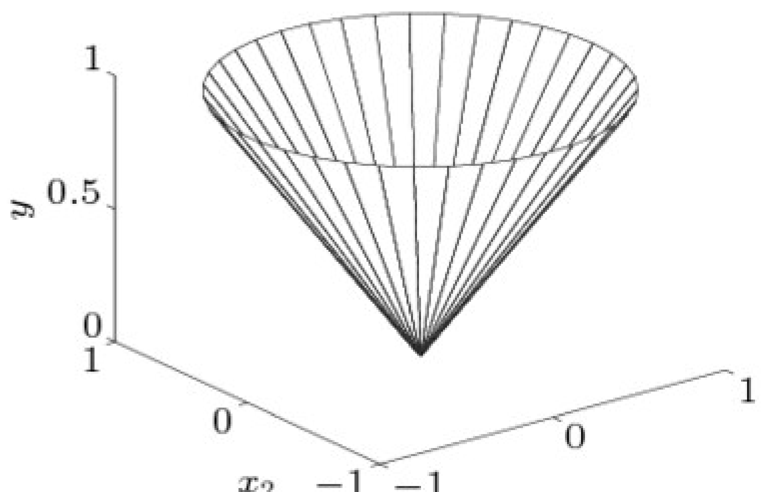

Figure 1- Cone constraint norm(x) = y

[1] The factorization of a positive-definite matrix into one lower triangular matrix and its transpose;
[2] $norm(x) = \sqrt{x^T x}$;

3.2. Data

The assets chosen for the comparison of performance between the two algorithms, QP and SOCP, was all the securities daily returns in the SP500 from 1997 to 2015. The option for the total composition of this index was to increase the robustness of the results. The SP500 is composed of very liquid large cap securities, probably the ones more scrutinized with an "*average of 23 analysts covering each stock*" (O'Shaughnessy, 2013). The high liquidity and information analysis reduce information noise and mispricing turning this market into one of the most efficient in the world.

The time span, 1997 to 2015, was defined to allow five years of historical data to estimate the model inputs and to simulate a 13 years investment strategy, beginning in 2002 until 2015.

3.3. Model Inputs

The portfolio optimization problem, independently of the algorithm used needs to have inputs in order to find the best solution. In the Markowitz Model the inputs are the securities returns and the variance – covariance matrix.

To estimate the expected returns one could use the historical returns, CAPM or other more complex model. In this paper the method used was the CAPM. Each security return premium was regressed against the market premium return to find the beta coefficient, sensitivity of the security in relation to the market. The proxies used were, the historical SP500 index returns for the market return and the 3 month US dollar LIBOR interest rate[3] for the risk free rate.

$$r_i - R_f = B_i(r_m - R_f)$$

After finding the beta for each security, the expected return for each security was estimated using the same equation.

The advantage of using CAPM to estimate the expected returns versus the assuming the historical returns as the future returns is that the past does not necessarily means the future and

[3] Average interest rate London based banks are willing to lend American dollars to others for a 3 month maturity;

it does not allow the expected returns to be negative. In theory the expected returns should not be negative or the investor would prefer to invest in the risk free rate. Another advantages are that the CAPM expected return is the required rate of a security to be added in a diversified portfolio, which is the output of a Markowitz model.

To estimate the variance-covariance matrix the historical returns were used. The reason behind it was the inferior volatility of the estimations through time and to simplify the computations for the sake of the work.

3.4. Research Design

This paper's aim is to test two hypothesis:

H1: Does the Mean-Variance Efficient Frontier depend on the optimization algorithm used?

H2: If H1 is true, does it translates into a superior risk adjusted return?

To test H1, the methodology used consisted in running the Markowitz optimization problem using 5 years of historical data, from 1997 until 2002, to estimate the model inputs, as stated above, and running both algorithms, QP and SOCP, for different levels of return. The output was the Markowitz efficient frontier. If the efficient frontiers were different that would confirm hypothesis one and mean that the optimal combination of weights to achieve the expected return would have to be different too.

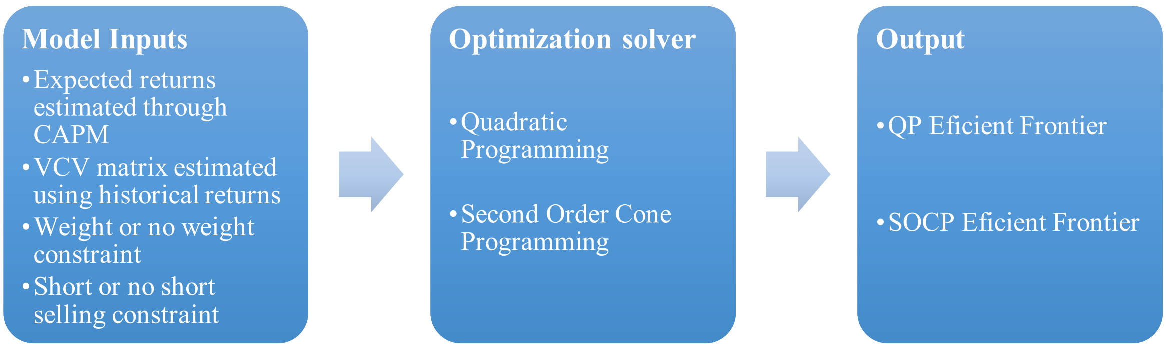

Figure 2 - Scheme of the research design to test first hypothesis

To test H2, which algorithm would provide a higher return when testing an investment strategy using different algorithms with real data, the methodology used was to assume an investor wanted to achieve a certain level of return and could only invest in stocks that belonged to the SP500 with five years of historical data[4] and the portfolio would be rebalanced every *x* days. Five different portfolio rebalancing schemes, daily, weekly, monthly, biannually and annually were tested for four levels of expected portfolio returns [5], approximately, 9%, 13%, 17% and 23%, with and without short selling constraint and security allocation weight constraint. The security allocation constraints tested were free weights, limited to 1 and limited to 0.5. The weight constraints were introduced to test a type of more conservative strategies and to reduce the extremal weight allocations to some securities that the Markowitz optimization approach can lead. The strategy took place from January 1st of 2002 until ends of 2015.

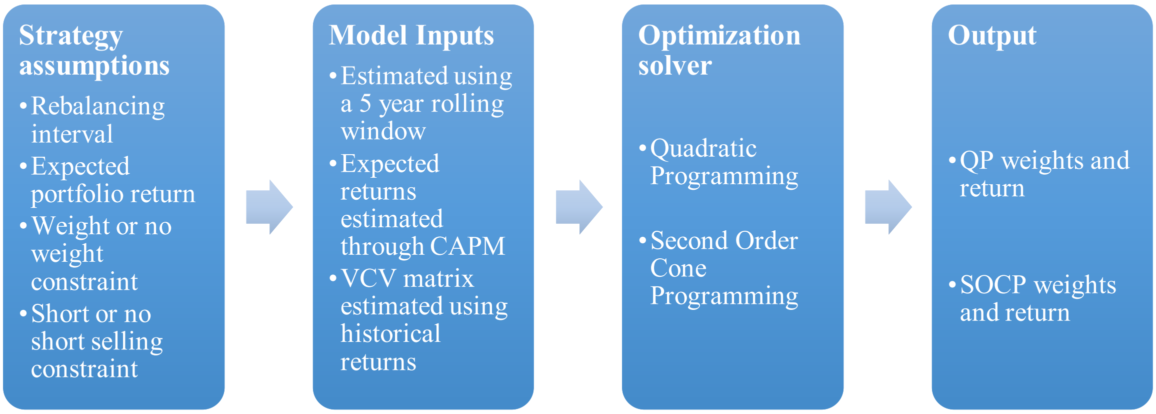

Figure 3 - Scheme of the research design to test second hypothesis

4. Results and Discussion

The original and first hypothesis of this paper was if one used different optimization algorithms to find the optimal allocation that could minimize the risk for various levels of expected portfolio would lead to different efficient frontier results.

H1: Does the Mean-Variance Efficient Frontier depend on the optimization algorithm used?

[4] The five year historical data filter was to introduced to increase the VCV matrix robustness;

[5] The expected portfolio returns were defined by computing the 10th, 25th, 50th and 75th percentile of the average annual stock returns. If the 10th percentile is equal to 9% that would mean that 90% of the stocks have an average annual returns superior to 9%;

By following the research design stated above to test H1, the results obtained were one efficient frontier for every algorithm and combination of constraints used. The difference in the efficient frontier was so small that was impossible to see with the naked eye. Figure 2, 3 and 4 show the difference between the QP and SOCP standard deviation for different levels of return. From the figures it is possible to conclude that the SOCP algorithm can almost always find a better combination of asset allocation than the QP algorithm. For the same level of return, the SOCP provides a solution bearing less risk. With the short selling constraint the difference in efficiency ranges from 0% to almost 0.20%.

Another interesting result is that by reducing the maximum allocation weights there is an increase in the maximum difference of efficiency. For example, with free weights, the maximum difference of performance was approximately 0.11%, with weights limited to 1, the maximum difference of performance was approximately 0.15% and with weights limited to 0.5, the maximum difference of performance was approximately 0.18%.

For the efficient frontiers with short selling the SOCP continues to be the most efficient but with much smaller differences and without the pattern seen with no short selling, Appendix A.

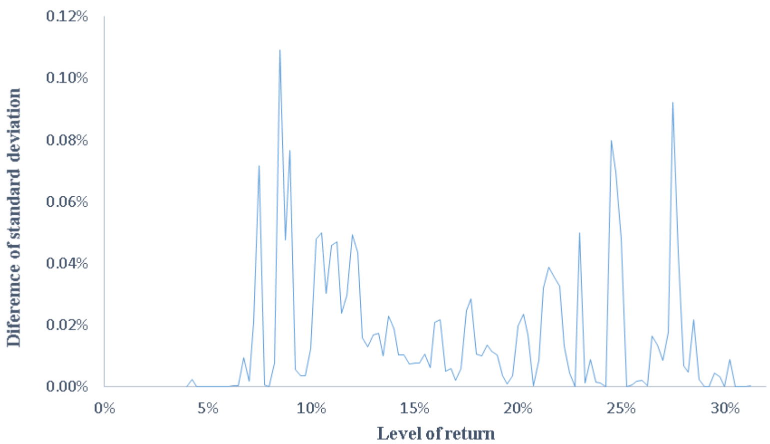

Figure 4 - Difference between QP and SOCP Markowitz Efficient frontiers for free weights and no short selling

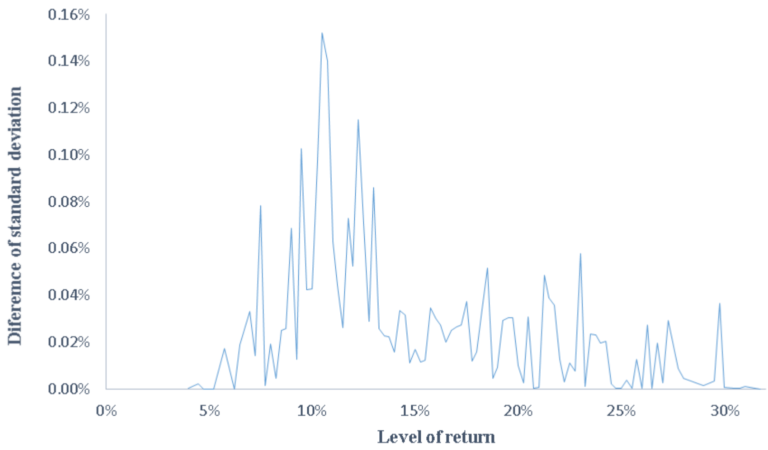

Figure 5 - Difference between QP and SOCP Markowitz Efficient frontiers for weights limited to 1 and no short selling

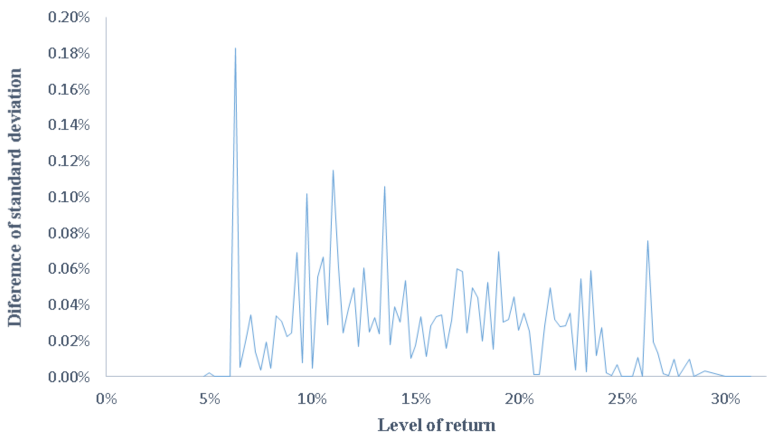

Figure 6 - Difference between QP and SOCP Markowitz Efficient frontiers for weights limited to 0.5 and no short selling

The superior performance of SOCP in relation to the QP when drawing the efficient frontier confirms H1 and leads us to formulate a second hypothesis. If the SOCP can theoretically reach the same return as QP but with less risk, how this difference translates in risk adjusted performance in a real world investment strategy?

H2: If H1 is true, does it translates into a superior risk adjusted return?

Applying the methodology stated in the research design for the second hypothesis it was to obtain optimal weights, returns and returns standard deviation to every strategy tested. A lot accumulated performances were negative leading to negative annual mean returns, so in order to analyse the differences in the risk adjusted return of the algorithms, the traditional Sharpe Ratio (S.R.) is not the best measure. When the annual mean return is negative the S.R. will be negative. The problem with negative Sharpe Ratios is that they can be misleading. For example, assuming a risk free rate of 3%, one investment, A, with -10% annual return and 10% standard deviation and another investment, B, with -10% annual return and 16.25% standard deviation. When applying the traditional Sharpe Ratio, the investment A will have a S.R. of -1.3 while investment B has a S.R. of -0.8 leading to a conclusion that investment B is better. A possible solution to this problem is to use the Modified Sharpe Ratio (M.S.R.) proposed by Israelson (2009). The Modified Sharpe Ratio is not different from the original Sharpe Ratio when the excess returns are positive, but when they are negative it "corrects" the value in order to be possible to draw right conclusions.

$$\mathrm{M.S.R.} = \frac{Excess\ return}{Standard\ deviation^{\frac{Excess\ return}{|Excess\ return|}}}$$

The drawdown of using the M.S.R. is that the output range is wider and if an investment has a Modified Sharpe Ratio two times the M.S.R. of other, it is not possible to affirm that one has two times better risk adjusted returns.

The results shown in Table 1, 2 and 3 are the difference between the Modified Sharpe Ratio of the Q.P and SOCP algorithm for the different strategies. For example, 0.90% in Table 1 means, that for a 9% expected portfolio return, a daily rebalancing scheme without short selling and no weight constraint, the QP Modified Sharpe Ratio was 0.9% superior to the SOCP Modified Sharpe Ratio. Negative result values mean SOCP had better risk adjusted returns. When analysing the results, H2 does not hold. The results are too much dispersed, but it is possible to see that for more than 50% of the strategies, the QP showed to reward better risk adjusted returns. Although QP has more often better performances it is impossible to conclude that a specific algorithm is better in solving the Markowitz optimization model.

Table 1 - Difference between QP and SOCP Modified Sharpe Ratio with free weights and no short selling

Level of return	Rebalancing interval				
	Daily	Weekly	Monthly	Biannually	Annually
9%	0.90%	0.34%	0.52%	0.03%	1.30%
13%	- 0.01%	0.21%	-0.01%	-0.01%	0.02%
17%	-0.16%	0.03%	0.01%	0.01%	-0.03%
23%	0.15%	0.05%	0.00%	0.01%	-0.02%

Table 2 - Difference between QP and SOCP Modified Sharpe Ratio with weights limited to 1 and no short selling

Level of return	Rebalancing interval				
	Daily	Weekly	Monthly	Biannually	Annually
9%	0.62%	7.43%	25.44%	0.01%	0.25%
13%	-0.06%	-8.44%	0.04%	0.00%	-0.01%
17%	0.24%	-0.52%	0.81%	0.00%	-0.06%
23%	-0.07%	0.66%	-0.78%	0.01%	-0.03%

Table 3 - Difference between QP and SOCP Modified Sharpe Ratio with weights limited to 0.5 and no short selling

Level of return	Rebalancing interval				
	Daily	Weekly	Monthly	Biannually	Annually
9%	1.32%	-0.54%	-1.99%	0.01%	-0.49%
13%	-0.36%	10.42%	-0.04%	0.01%	-0.01%
17%	-0.08%	-0.62%	0.01%	0.11%	0.02%
23%	0.18%	0.46%	-0.47%	0.03%	-0.06%

A similar pattern to H1, the increase in the maximum difference efficiency algorithms with the reduction of maximum weight allocation, is present in H2. With free weights and no short selling the SOCP was better 6 times out of 20, Table 1.With free weights and no short selling the SOCP was better 8 times out of 20, Table 2. With free weights and no short selling the SOCP was better 9 times out of 20, Table 3. The results suggest again the existence of a relation between the SOCP performance and the number/restrictiveness of constraints.

Other finding came from the number of feasible solutions from QP versus SOCP. When running the optimization problem with the model inputs and for a certain level of expected return, the algorithm can reach two types of solutions. A feasible, where an optimal solution is found while respecting the constraints or infeasible, the target return is impossible to achieve with the active constraints. It was expected and confirmed that with the increase of the number/restrictiveness of the constraints the number of infeasible solutions would be higher. By comparing the number of feasible solution from QP and SOCP, the QP often outperforms the SOCP, Appendix C. This result is in a sense opposed to the previous one. If the SOCP was better than QP when the number/restrictiveness of the constraints increase, then it would be expected that SOCP would find feasible solutions more often for more constrained problems which is not true.

The results of this paper, do not check with Davidsson's paper results, difference of 2.8 percent return premium by using SOCP instead of QP, Appendix B. The fact that Davidsson only tried one strategy with a small dataset could have influenced the results. Other missing point in the paper is the comparison of the risk adjusted returns of the two algorithms.

The simulations results for H2 with short selling will not be shown or discussed in this section. The reason behind is because the limitations of the Markowitz framework. For example, the output suffers a lot from estimation errors. Since the model inputs are estimations for the future, an estimation error is almost always present. When optimizing the portfolio allocation based in the future estimations, estimation errors can lead to extreme portfolio positions which in turn can cause great or catastrophic returns. Another known problem is the instability of the optimized solution. A small change in some of the future estimations can result in major changes in the portfolio allocation. One way to limit this problem is to constraint the possible asset allocation, as it was made by not allowing to allocate more than 1 or 0.5 of the wealth to each asset. Even constraining the weights, allowing short selling, results in a lot of allocations for the maximum and minimum weight allowed, causing abnormal accumulated performances and high volatility leading to unrealistic results and investment strategies.

5. Conclusion

The work developed in this paper had as main objective the comparison between the performances of Quadratic Programming versus Second Order Cone Programming within the Mean –Variance optimization problem proposed by Markowitz (1952). Previous work, had detected a 2.8% superior investment performance by using the Second Order Cone Programming over Quadratic Programming in a single investment strategy, (Davidsson, 2011). To test the robustness of the result, new variables were introduced, such as weight constraints, various levels of portfolio return and different rebalancing intervals. The first hypothesis tested was if the Markowitz Efficient Frontier was different depending on the algorithm used. The results clearly showed a difference in optimization efficiency between algorithms, where Second Order Cone Programming was the "winner". It reached the same level of expected return with less risk. Based on this result, a second hypothesis was tested. If the risk adjusted return of an investment strategy within Markowitz framework would be different depending of the algorithm used. The results were highly dispersed making it hard to draw affirmative conclusions. In more than 50% of the strategies tested Quadratic Programming had a superior risk adjusted return but the percentage decreases when the problem restrictions increase, suggesting that Second Order Cone Programming could lead to better returns if the problem is more restrict. Opposed to this suggested relation, the second result showed the Quadratic

Programming obtaining more feasible solutions than Second Order Cone Optimization, even when the restrictions increased.

Given the results of this paper and the previous work it would be advantageous to continue the study of how an investment performance can differ with the choice of the optimization algorithm. Since it is impossible to select a winner from this research and Quadratic Programming still accounts for the majority of the better performances, there is no incentive to propose a change from the traditional algorithm used in the Mean – Variance optimization problem, Quadratic Programming, to Second Order Cone Programming. The only exception is when the data set is very large, since Second Order Cone Programming is a much faster algorithm than Quadratic Programming.

Some of the limitations of this paper were limited use of constraints, if the weights were more restricted possibly the conclusions could be more robust and there is no study on the changes of the rebalancing weight allocation. The volatility of the weight allocation can tip about the effect of transaction costs in the investment return.

To continue the research on this subject, I suggest to test again both algorithms with even more constraining problems to confirm or reject the suggested relation drawn from the results and to include transaction costs. The introduction of transaction costs to the model can alter the results completely, seeing that if one algorithm's output is more volatile it will significantly increase the costs of the one investment strategy over the other.

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Appendix A – Difference of QP and SOCP standard deviation for the various levels of returns with short selling

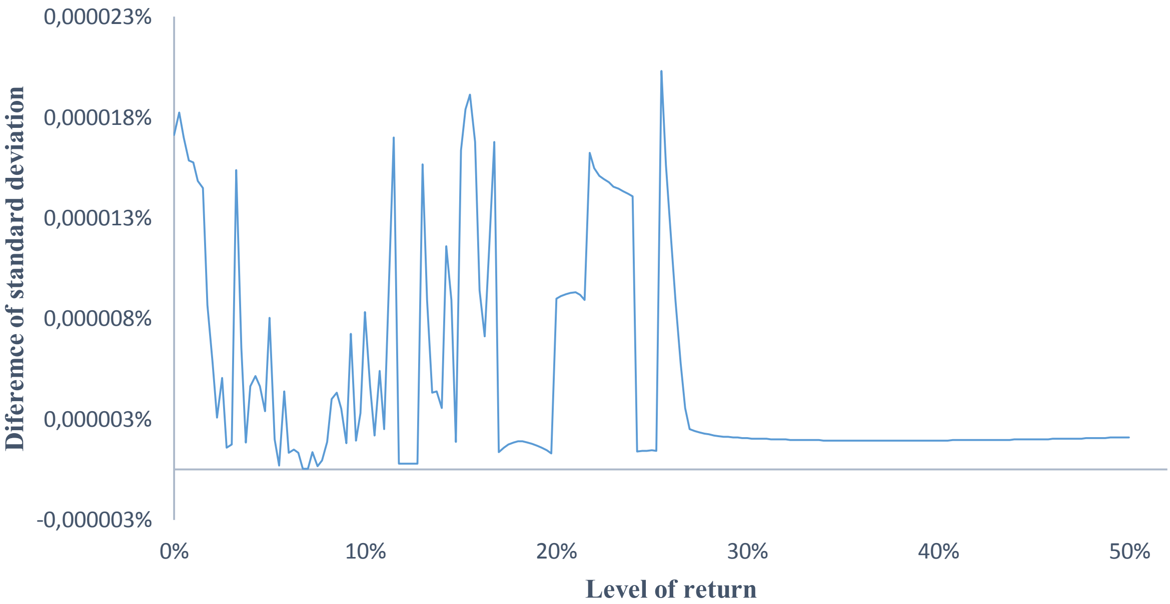

Figure 1 - Difference between QP and SOCP Markowitz Efficient frontiers with free weights and short selling

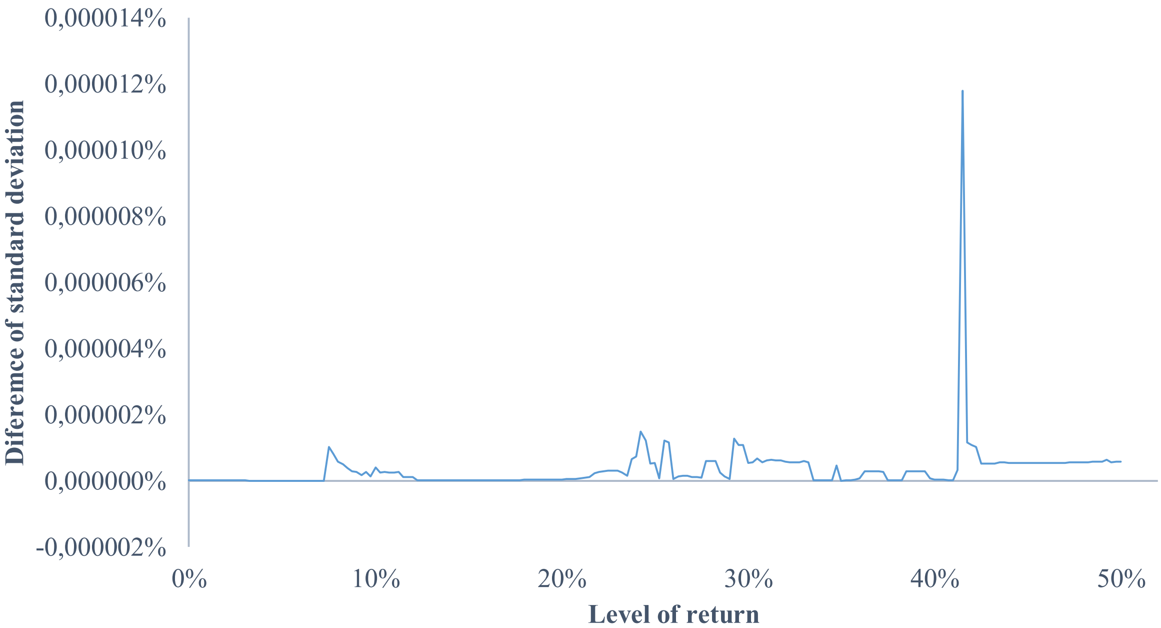

Figure 2 - Difference between QP and SOCP Markowitz Efficient frontiers for weights limited to 1 with short selling

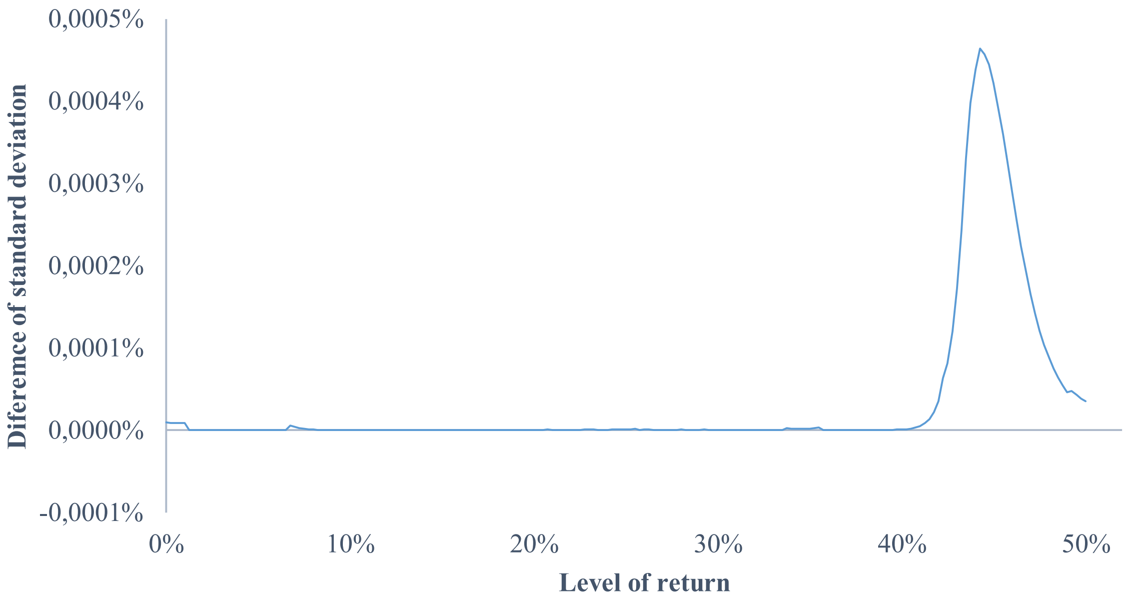

Figure 3 - Difference between QP and SOCP Markowitz Efficient frontiers for weights limited to 0.5 with short selling

Appendix B – Difference between QP and SOCP Accumulated performance with no short selling

Table 2 - Difference between QP and SOCP accumulated performance with free weights and no short selling

Level of return	Rebalancing interval				
	Daily	Weekly	Monthly	Biannually	Annually
9%	1.56%	0.54%	1.23%	1.71%	1.39%
13%	-0.16%	0.29%	-0.13%	-0.09%	0.64%
17%	-1.69%	0.48%	0.25%	0.18%	0.33%
23%	0.77%	0.33%	0.02%	0.06%	0.03%

Table 2 - Difference between QP and SOCP accumulated performance with weights limited to 1 and no short selling

Level of return	Rebalancing interval				
	Daily	Weekly	Monthly	Biannually	Annually
9%	1.48%	23.62%	44.63%	0.54%	0.25%
13%	-1.40%	-11.67%	0.36%	-0.05%	0.62%
17%	1.43%	-9.81%	13.86%	-0.03%	-0.01%
23%	-1.29%	3.63%	-7.43%	0.09%	0.07%

Table 3 - Difference between QP and SOCP accumulated performance with weights limited to 0.5 and no short selling

Level of return	Rebalancing interval				
	Daily	Weekly	Monthly	Biannually	Annually
9%	2.46%	-0.78%	-2.68%	0.77%	-0.55%
13%	-6.54%	12.93%	-0.77%	0.16%	0.94%
17%	-1.35%	-10.00%	0.21%	1.30%	0.47%
23%	0.23%	2.15%	-4.81%	0.40%	0.52%

Appendix C – Difference between QP and SOCP number of feasible solutions with no short selling

Table 3 - Difference between QP and SOCP number of feasible solutions with free weights and no short selling

Level of return	Rebalancing interval				
	Daily	Weekly	Monthly	Biannually	Annually
9%	0.09%	0.00%	0.00%	0.00%	0.00%
13%	0.00%	0.00%	0.00%	0.00%	0.00%
17%	0.03%	0.14%	0.00%	0.00%	0.00%
23%	0.03%	0.00%	0.00%	0.00%	0.00%

Table 2 - Difference between QP and number of feasible solutions with weights limited to 1 and no short selling

Level of return	Rebalancing interval				
	Daily	Weekly	Monthly	Biannually	Annually
9%	1.58%	2.58%	2.42%	0.00%	0.00%
13%	0.52%	0.86%	0.61%	0.00%	0.00%
17%	1.15%	0.86%	1.21%	0.00%	0.00%
23%	1.35%	1.29%	0.61%	0.00%	0.00%

Table 3 - Difference between QP and SOCP number of feasible solutions with weights limited to 0.5 and no short selling

Level of return	Rebalancing interval				
	Daily	Weekly	Monthly	Biannually	Annually
9%	0.29%	0.43%	0.61%	0.00%	0.00%
13%	0.20%	0.29%	0.00%	0.00%	0.00%
17%	0.37%	0.86%	0.00%	0.00%	0.00%
23%	0.57%	0.86%	0.61%	0.00%	0.00%