DEPARTMENT OF
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ON VOEVODSKY AND FARRELL-JONES CONJECTURES

JOSÉ FRANCISCO DE VASCONCELOS TEODÓSIO NUNES DOS REIS

Master in Pure Mathematics

DOCTORATE IN MATHEMATICS

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To my parents.

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"A university training is the great ordinary means to a great but ordinary end." (John Henry Newman)

Abstract

This thesis consists of two parts. In the first part we improve the Voevodsky nilpotence conjecture, a problem on algebraic geometry. In the second part we improve the Farrell-Jones isomorphism conjecture, a problem on algebraic topology.

The Voevodsky nilpotence conjecture claims that the nilpotence and the numerical equivalence relations on the algebraic cycles of a smooth proper scheme agree. Bernardara-Marcolli-Tabuada generalized this problem to the setting of differential graded categories and showed that this conjecture holds for certain quadric fibrations, intersections of quadrics, linear sections of Grassmannians and of determinantal varietie and Moishezon manifolds. Later on, Ornaghi-Pertusi showed that the conjecture holds for cubic fourfolds and for Gushel-Mukai fourfolds. Working on this setting of differential graded categories, we refine these results by showing that the algebraic and the numerical equivalence relations also agree for those cases. Moreover, we work with quotients between equivalence relations. This allows us not only to obtain a refinement of the aforementioned results but also to compute, for example, the quotient between the rational and the algebraic equivalence relations of certain five-folds. These results lead to preprint [74].

The Farrell-Jones isomorphism conjecture is a local-to-global statement that claims that the algebraic K-theory of a group ring RG, where R stands for a commutative ring, is completely determined by the algebraic K-theory of the group rings RV with V a virtually cyclic subgroup of G. This important conjecture simplifies the computation of the algebraic K-theory of a group ring. Following the approach of Bunke-Kasprowski-Winges, we work on a generalized version of the Farrell-Jones isomorphism conjecture that is stated in the setting of ∞-categories. Using tools from the theory of noncommutative motives we were able to show that the Farrell-Jones isomorphism conjecture holds, under certain conditions, for finitary localising invariants. This result lead to the publication [75].

Keywords: Algebraic cycles, Voevodsky nilpotence conjecture, Homological projective duality, noncommutative motives, Farrell-Jones isomorphism conjecture, quotients of equivalence relations, K-theory, ∞-categories, finitary localising invariants, lax monoidal.

Resumo

Esta tese consiste em duas partes. Na primeira parte melhoramos resultados na conjectura nilpotente de Voevodsky, um problema de geometria algébrica. Na segunda parte melhoramos um resultado na conjectura de Farrell-Jones, um problema de topologia algébrica.

A conjectura nilpotente de Voevodsky afirma que as relações de equivalência nilpotente e numérica nos ciclos algébricos de um esquema próprio e suave identificam-se. Bernardara-Marcolli-Tabuada generalizaram este problema para o contexto de categorias diferenciais graduadas e mostraram que a conjectura é válida para certas fibrações quádricas, intersecções de quádricas, secções lineares de variedades Grassmanianas e determinantais, e variedades de Moishezon. Mais tarde, Ornaghi-Pertusi provaram que a conjectura é válida para variedades cúbicas e de Gushel-Mukai de dimensão quarto. Trabalhando no contexto de categorias diferenciais graduadas, melhorámos estes resultados ao provar que as relações de equivalência algébrica e númerica identificam-se naqueles casos. Mais ainda, consideramos quocientes entre as relações de equivalências. Isto permite-nos não só que obtenhamos um refinamento dos resultados mencionados mas também que calculemos, por exemplo, o quociente entre as relações de equivalência racional e algébrica de certas variedades de dimensão cinco. Estes resultados levaram à pre-publicação [74].

A conjectura de Farrell-Jones afirma que a K-teoria algébrica da álgebra de grupo RG, R anel comutativo, determina-se pela K-teoria algébrica de RV, com V subgrupo virtualmente cíclico de G. Esta importante conjectura simplifica o cálculo da K-theory algébrica da álgebra de grupo. Seguindo o raciocínio de Bunke-Kasprowski-Winges, consideramos a conjectura de Farrell-Jones no contexto de categorias ∞. Usando resultados da teoria de motivos não comutativos provamos, em certas condições, a conjectura de Farrell-Jones para invariantes finitários localizantes. Este resultado foi publicado em [75].

Palavras-chave: Ciclos algébricos, conjectura nilpotente de Voevodsky, dualidade homológica projectiva, motivos não comutativos, conjectura de Farrell-Jones, quocientes de relações de equivalência, K-teoria, categorias ∞, invariantes finitamente localizantes, lax monoidal.

Contents

1 Introduction

In this thesis, making use of the theory of noncommutative motives we obtain results in two different areas: algebraic geometry and algebraic topology. More concretely, we are going to obtain results related to the Voevodsky nilpotence conjecture and to the Farrell-Jones isomorphism conjecture.

1.1 Voevodsky nilpotence conjecture

In [87], Voevodsky introduced the nilpotence equivalence relation on the algebraic cycles $\mathcal{Z}^*(X)_{\mathbb{Q}}$ of smooth projective k-schemes X and conjectured that it agrees with the numerical equivalence relation.

It is important to point out that there are some well-known equivalence relations on algebraic cycles: rational, algebraic, nilpotence and numerical equivalence relations. Moreover, for α and β algebraic cycles of a smooth projective k-scheme one has the implications

$$\alpha \sim_{\mathrm{rat}} \beta \Rightarrow \alpha \sim_{\mathrm{alg}} \beta \Rightarrow \alpha \sim_{\mathrm{nil}} \beta \Rightarrow \alpha \sim_{\mathrm{num}} \beta,$$

where $\sim_{\mathrm{rat}}$, $\sim_{\mathrm{alg}}$, $\sim_{\mathrm{nil}}$ and $\sim_{\mathrm{num}}$ stand for, respectively, the rational, algebraic, nilpotence and numerical equivalence relations.

After noticing this comparison among these equivalence relations, an immediate question is whether those implications hold in the reverse order. Voevodsky conjectured that one would also have $\alpha \sim_{\mathrm{nil}} \beta \Leftarrow \alpha \sim_{\mathrm{num}} \beta$. Thanks to Kahn-Sebastian [39], Matsusaka [70], Voevodsky [87] and Voisin [88], it is known that the Voevodsky nilpotence conjecture holds for curves, surfaces, and abelian 3-folds (when k is of characteristic zero).

More recently, in [17], Bernadara, Marcolli and Tabuada extended this conjecture to the setting of smooth proper differential graded categories. This generalization allowed them to prove the Voevodsky nilpotence conjecture in some new cases: quadric fibrations, intersections of quadrics, linear sections of Grassmannians, linear sections of determinantal varieties, and Moishezon manifolds. Building on this noncommutative approach, Ornaghi and Pertusi [73] were then able to prove the Voevodsky nilpotence conjecture for cubic fourfolds and Gushel-Mukai fourfolds.

Bearing in mind these results, and the way they were obtained, it is reasonable to ask if one could go even further. For instance, does the converse implication $\alpha \sim_{\text{alg}} \beta \Leftarrow \alpha \sim_{\text{num}} \beta$ also holds? Building on the approach given in [17] we were able to show that this implication actually holds for all the cases mentioned above and some new other cases: certain Küchle fourfolds, families of Sextic del Pezzo surfaces and families of Fano fourfolds of $K3$ type, see §5.5 below.

Note that thanks to the work of Griffiths [34] we have that if X is a general quintic threefold, then the algebraic and numerical equivalence relations on the algebraic cycles of X do not agree. One can find more examples where the algebraic and numerical equivalence relations do not agree on later works of Clemens [24] and Voisin [89].

Nonetheless, even though the algebraic and numerical equivalence relations do not agree in general, it is important to stress that we have obtained these results by determining the quotient group of $\mathcal{Z}^*_{\sim_1}(X)_{\mathbb{Q}}$ over $\mathcal{Z}^*_{\sim_2}(X)_{\mathbb{Q}}$, where $\mathcal{Z}^*_{\sim}(X)_{\mathbb{Q}}$ stands for the algebraic cycles α such that $\alpha \sim 0$ for the equivalence relation $\sim$. As a consequence, some of our results can be used even if this quotient is not trivial (i.e., if the equivalence relations $\sim_1$ and $\sim_2$ do not agree). Indeed, we have computed this quotient for certain fivefolds while considering the rational and algebraic equivalence relations. In addition, we have shown that this quotient satisfies some nice properties: it is invariant under Morita equivalences (Proposition 5.4.6) and it factors (in a certain sense) along semi-orthogonal decompositions (Proposition 5.4.5).

In chapter 2 we introduce some basic definitions and results on differential graded categories. Note that we will translate the question on algebraic cycles on a smooth scheme to a problem on smooth proper differential graded categories. Hence, this chapter is important to have a (almost) self-contained thesis. The algebraic K-theory on differential graded categories is an important functor for this thesis and happens to be an additive invariant. In chapter 3 we define what is meant by an additive invariant and we also show the existence of a universal additive invariant. Moreover, we introduced the semi-orthogonal decomposition in the sense of Bondal and Orlov. Ultimately, this will allow us to factor the problem of showing that two equivalence relations agree into smaller problems. In chapter 4 we introduce some notions on classical motives and on noncommutative motives. This provides us with more tools to relate the classical equivalence relation on algebraic cycles with the equivalence relations defined in this noncommutative setting. Finally, in chapter 5 we apply all the theory introduced before to show that the algebraic and numerical equivalence relations agree for many cases. Part I of this thesis is based on the preprint [74].

1.2 Farrell-Jones isomorphism conjecture

The computation of the algebraic K-theory of a group ring RG, for R a commutative ring and G a group (that can be infinite), is a very difficult problem. The Farrell-Jones isomorphism conjecture is a local-to-global statement that simplifies this problem. Roughly speaking, it claims that the algebraic K-theory of a group ring RG is completely determined by the algebraic K-theory of the group rings RV, with V a virtually cyclic subgroup of G, see Definition 7.1.1. This important conjecture was first formulated by Farrell and Jones in the 90's (see [30]) and is nowadays known to be true in several cases: hyperbolic groups, finite dimensional CAT(0)-groups, virtually solvable groups, mapping class groups, etc. For details on the applications of the Farrell-Jones isomorphism conjecture, we invite the reader to consult the surveys [60, 61, 64].

Since 1993 this conjecture is having plenty generalizations: Davis and Lück [25] proposed a general setting that allowed to replace the algebraic K-theory functor by other functors (and the family of subgroups by other families), Bartels and Reich [12] introduced the Farrell-Jones isomorphism conjecture with coefficients in R-linear categories and, finally, Bunke, Kasprowski and Winges [22] generalized this conjecture to admit coefficients in ∞-categories.

In Part II we are going to improve a result of Bunke, Kasprowski and Winges that states that this general version of the Farrell-Jones isomorphism conjecture holds, under certain condition, for lax monoidal finitary localising invariants. By making use of the theory of noncommutative motives we are going to show that one can drop the lax monoidal assumption.

Chapters 6 and 7 correspond to the second part of this thesis. In chapter 6 one provides the necessary background on ∞-categories. In chapter 7 we introduce the Farrell-Jones isomorphism conjecture and several generalizations that it has been subjected to in the last almost 30 years. At last, we show that the Farrell-Jones isomorphism conjecture with coefficients in ∞-category holds, under certain conditions, for finitary localising invariants. This second part of the thesis is based on the publication [75].

Part I
Voevodsky nilpotence conjecture

2

Differential graded categories

In this chapter we introduce all concepts on differential graded categories that we are going to require throughout this thesis. Note that in the last 20 years differential graded categories were instrumental to prove that some conjectures due to Grothendieck, Voevodsky, Beilinson, Weil, Tate, Parshin, Kimura, Schur, etc... hold for certain cases; consult [78] for details. This has been done, in part, thanks to generalizations of these conjectures to the setting of differential graded categories. This thesis is inspired by this approach. Unless otherwise mentioned, most definitions are taken from Keller's survey [42], our main reference for this chapter.

2.1 Definitions

Throughout this thesis k denotes a field and $\otimes$ stands for the tensor product over k.

Definition 2.1.1. *A k-category* A *is a category such that the class of morphisms* $\mathrm{A}(X,Y)$ *is a k-vector space for all* $X,Y \in \mathrm{obj}(\mathrm{A})$ *objects of* A *and it has k-linear associative maps*

$$\begin{aligned} \mathrm{A}(Y,Z)\otimes \mathrm{A}(X,Y) &\longrightarrow \mathrm{A}(X,Z), \quad X,Y,Z \in \mathrm{obj}(\mathrm{A}) \\ (f,g) &\longmapsto fg, \end{aligned}$$

admitting a unit in $\mathrm{A}(X,X)$.

From now on we shall write $X \in A$ to mean an object in a category A.

Definition 2.1.2. *A k-vector space V together with a decomposition $V = \bigoplus_{n\in\mathbb{Z}} V^n$ indexed by the integers is called a* graded k-vector space. *A* morphism of degree m between graded k-vector spaces

$$f : V = \bigoplus_{n\in\mathbb{Z}} V^n \longrightarrow W = \bigoplus_{n\in\mathbb{Z}} W^n$$

is a k-linear morphism such that $f(V^n) \subseteq W^{n+m}$ for all $n \in \mathbb{Z}$. We denote the category of graded k-vector spaces *as $\mathcal{G}(k)$.*

For V a graded k-vector space, the *shifted vector space* $V[1]$ is determined by $V[1]^n := V^{n+1}$, for all $n \in \mathbb{Z}$.

Definition 2.1.3. *Given graded k-vector spaces V and W, the* tensor product $V \otimes W$ *is the graded k-vector space defined by:*

$$(V \otimes W)^n = \bigoplus_{p+q=n} V^p \otimes W^q.$$

For $f : V \to V'$ and $g : W \to W'$ morphisms of graded k-vector spaces, the tensor product $f \otimes g$ *is defined as*

$$(f \otimes g)(v \otimes w) = (-1)^{pq} f(v) \otimes g(w),$$

where p is the degree of g and $v \in V^q$.

Definition 2.1.4. *A graded k-vector space A is called a* graded k-algebra *if it is equipped with an associative multiplication morphism $A \otimes A \to A$ of degree zero that admits a unit in A^0.*

We can now define a differential graded (dg) vector space over k.

Definition 2.1.5. *A* dg k-vector space *is a graded k-vector space V which is endowed with a* differential map $d_V : V \to V$ *of degree* 1 *such that $d_V \circ d_V = 0$. Equivalently, a* dg k-vector space *is a complex of k-vector spaces.*

Given a dg k-vector space V we define the *shifted dg k-vector space* $V[1]$ to be the shifted graded vector space whose differential is $d_{V[1]} := -d_V$. We also define the *tensor product of dg k-vector spaces* $V \otimes W$ to be the graded k-vector space $V \otimes W$ endowed with the differential $d_V \otimes \mathrm{id}_W + \mathrm{id}_V \otimes d_W$.

At this point, we can define a dg category over k.

Definition 2.1.6. *A* dg k-category $\mathcal{A}$ *is a k-category whose class of morphisms are dg k-vector spaces and the compositions are morphisms of dg k-vector spaces:*

$$\mathcal{A}(Y, Z) \otimes \mathcal{A}(X, Y) \longrightarrow \mathcal{A}(X, Z), \quad X, Y, Z \in \mathcal{A}.$$

A dg k-category is said to be small if $\mathrm{obj}(\mathcal{A})$ *is a set.*

Note that most of the time we will omit k from the notation. We now give three examples of dg categories.

Example 2.1.7. (i) Take a k-algebra A. It is a graded k-algebra concentraded at degree zero. Hence, it determines a dg category $\mathcal{A}$. Indeed, let $\mathcal{A}$ be the dg category whose unique object is X and the class of morphisms is $\mathcal{A}(X, X) := A$.

(ii) Take a k-algebra B and consider $C(B)$ the category of complexes of right B-modules

$$\cdots \to M^{n-1} \to M^n \to M^{n+1} \to \cdots .$$

Given complexes L and M and an integer n, we define $\mathcal{H}om(L,M)^n$ to be the k-vector space whose elements are graded morphisms $f : L \to M$ of degree n. That is, $f = (f^p)$ where $f^p : L^p \to M^{p+n}$ is a morphism of B-modules for all $p \in \mathbb{Z}$. Hence, define $\mathcal{H}om(L,M) = \oplus_{n\in\mathbb{Z}}\mathcal{H}om(L,M)^n$ and, for $f \in \mathcal{H}om(L,M)^n$, let the differential d be as follows:

$$d(f) := d_M \circ f - (-1)^n f \circ d_L,$$

where d_L and d_M are, respectively, the differentials of complexes L and M. With this, we can define the dg category $C_{\text{dg}}(B)$: its objects are complexes of right B-modules and the class of morphisms of two complexes L and M is defined as $C_{\text{dg}}(B)(L,M) = \mathcal{H}om(L,M)$. The composition is induced by the composition of graded morphisms of graded modules.

(iii) Given a dg category $\mathcal{A}$, we define the opposite dg category $\mathcal{A}^{\text{op}}$. The objects of $\mathcal{A}^{\text{op}}$ are the same as the objects of $\mathcal{A}$ and, for $X, Y \in \mathcal{A}^{\text{op}}$,

$$\mathcal{A}^{\text{op}}(X,Y) = \mathcal{A}(Y,X).$$

For $f \in \mathcal{A}^{\text{op}}(Y,Z)^p$ and $g \in \mathcal{A}^{\text{op}}(X,Y)^q$ we define composition as

$$(-1)^{pq} gf \in \mathcal{A}^{\text{op}}(X,Z)^{p+q}.$$

From time to time, we will make use of the following definitions.

Definition 2.1.8. *For a dg category $\mathcal{A}$, we define the category $Z^0(\mathcal{A})$ whose objects are the same as those of $\mathcal{A}$ and, for $X, Y \in Z^0(\mathcal{A})$, the class of morphisms is defined as:*

$$\big(Z^0(\mathcal{A})\big)(X,Y) = Z^0\big(\mathcal{A}(X,Y)\big),$$

where $Z^0\big(\mathcal{A}(X,Y)\big)$ stands for the kernel of $d : \mathcal{A}(X,Y)^0 \to \mathcal{A}(X,Y)^1$.

Definition 2.1.9. *For a dg category $\mathcal{A}$, we define the category $H^0(\mathcal{A})$ whose objects are the same as those of $\mathcal{A}$ and, for $X, Y \in H^0(\mathcal{A})$, the class of morphisms is defined as the zeroth homology of the complex $\mathcal{A}(X,Y)$:*

$$\big(H^0(\mathcal{A})\big)(X,Y) = H^0\big(\mathcal{A}(X,Y)\big).$$

Definition 2.1.10. *For a dg category $\mathcal{A}$, the* homology category *$H^*(\mathcal{A})$ is the category with the same objects of $\mathcal{A}$ and where, for $X, Y \in H^*(\mathcal{A})$, the class of morphisms is the cohomology of the complex $\mathcal{A}(X,Y)$:*

$$\big(H^*(\mathcal{A})\big)(X,Y) = H^*\big(\mathcal{A}(X,Y)\big).$$

Definition 2.1.11. *Given dg categories $\mathcal{A}$ and $\mathcal{B}$, we define a* dg functor *$F : \mathcal{A} \to \mathcal{B}$ to be a map $F : \text{obj}(\mathcal{A}) \to \text{obj}(\mathcal{B})$ and morphisms of dg k-vector spaces:*

$$F(X,Y) : \mathcal{A}(X,Y) \to \mathcal{B}\big(F(X),F(Y)\big), \quad X, Y \in \mathcal{A},$$

compatible with differentials, composition and units.

Remark 2.1.12. Note that, given dg categories $\mathcal{A}$ and $\mathcal{B}$, a dg functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ induces a functor on the categories defined in Definitions 2.1.8-2.1.10.

Definition 2.1.13. *The* category of small dg categories over a ring k *has the small dg k-categories as objects and dg functors as morphisms. We shall denote this category by* $\mathrm{dgcat}(k)$*.*

From now on, we assume all dg categories to be small, except if otherwise stated.

Definition 2.1.14. *Take $\mathcal{A}, \mathcal{B} \in \mathrm{dgcat}(k)$. A dg functor $F : \mathcal{A} \to \mathcal{B}$ is called a* quasi-equivalence *if:*

(i) $F(X,Y) : \mathcal{A}(X,Y) \to \mathcal{B}\big(F(X), F(Y)\big)$ is a quasi-isomorphism of complexes for all objects $X, Y \in \mathcal{A}$.

(ii) The induced functor $H^0(F) : H^0(\mathcal{A}) \to H^0(\mathcal{B})$ is an equivalence of categories.

Definition 2.1.15. *For $\mathcal{A}, \mathcal{B} \in \mathrm{dgcat}(k)$, the* tensor product $\mathcal{A} \otimes \mathcal{B}$ of dg categories *has* $\mathrm{obj}(\mathcal{A}) \times \mathrm{obj}(\mathcal{B})$ *as the class of objects and, for $(X,Y), (X',Y') \in \mathcal{A} \otimes \mathcal{B}$, the class of morphisms is*

$$(\mathcal{A} \otimes \mathcal{B})\big((X,Y),(X',Y')\big) = \mathcal{A}(X,X') \otimes \mathcal{B}(Y,Y'),$$

with the induced compositions and units.

Given dg categories $\mathcal{A}$ and $\mathcal{B}$ and dg functors $F, G : \mathcal{A} \to \mathcal{B}$, we define $\mathcal{H}om(F,G)^n$ to be the k-vector space whose elements are natural transformations of degree n from F to G. That is, for each $X, Y \in \mathcal{A}$ and $f \in \mathcal{A}(X,Y)$ there exists a family of morphisms

$$\phi_X \in \mathcal{B}\big((F(X), G(X)\big)^n$$

such that $\big(G(f)\big) \circ \phi_X = \phi_Y \circ \big(F(f)\big)$. So, we define the *complex of graded morphisms* to be $\mathcal{H}om(F,G) = \oplus_{n \in \mathbb{Z}} \mathcal{H}om(F,G)^n$ and the differential is induced by that of $\mathcal{B}\big((F(X), G(X)\big)$, for all $X \in \mathcal{A}$. We define the morphisms $F \to G$ to be those $\phi \in \mathcal{H}om(F,G)^0$ such that $d\phi = 0$, i.e., $\phi \in Z^0\big(\mathcal{H}om(F,G)\big)$. Thus, we have defined the following dg category.

Definition 2.1.16. *Take $\mathcal{A}, \mathcal{B} \in \mathrm{dgcat}(k)$. We define $\mathcal{H}\mathrm{om}(\mathcal{A}, \mathcal{B})$ as the dg category whose objects are dg functors $F : \mathcal{A} \to \mathcal{B}$ and for dg functors F and G define the class of morphisms to be $\mathcal{H}\mathrm{om}(F,G)$.*

Two important definitions follow from Definition 2.1.16. These will be introduced in chapter 2.2 below where we define dg modules over a dg category.

Remark 2.1.17. We write here a list of some properties of $\mathrm{dgcat}(k)$:

(i) The initial object is the empty dg category $\emptyset$.

(ii) The final object is the dg category with one object whose endomorphisms form the zero dg k-algebra.

(iii) This category is both complete (i.e. admits all small limits) and cocomplete (i.e. admits all small colimits).

(iv) The tensor product of dg categories gives rise to a symmetric monoidal structure on dgcat(k), with $\otimes$-unit k. Consult [29, §2.1 and §8.1] for the notion of symmetric monoidal category.

(v) This category is a symmetric tensor category that admits an internal Hom-functor. Indeed, for $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathrm{dgcat}(k)$ one has

$$\mathrm{Hom}(\mathcal{A} \otimes \mathcal{B}, \mathcal{C}) = \mathrm{Hom}\big(\mathcal{A}, \mathcal{H}om(\mathcal{B}, \mathcal{C})\big).$$

2.2 Differential graded modules

We start by defining left and right dg modules.

Definition 2.2.1. *Given a dg k-category $\mathcal{A}$ we define a* left dg $\mathcal{A}$-module *L to be a dg functor*

$$L : \mathcal{A} \longrightarrow \mathcal{C}_{\mathrm{dg}}(k).$$

Equivalently, a left dg $\mathcal{A}$-module *L is given by complexes of k-vector spaces $L(X)$ and morphisms of complexes*

$$\mathcal{A}(X, Y) \otimes L(X) \longrightarrow L(Y), \textit{for } X \textit{ and } Y \in \mathcal{A}.$$

Similarly, a right dg $\mathcal{A}$-module *M is a dg functor*

$$M : \mathcal{A}^{\mathrm{op}} \longrightarrow \mathcal{C}_{\mathrm{dg}}(k).$$

Equivalently, a right dg $\mathcal{A}$-module *M is given by complexes of k-vector spaces $M(X)$ and a morphism of complexes*

$$M(Y) \otimes \mathcal{A}(X, Y) \longrightarrow M(X), \textit{for } X \textit{ and } Y \in \mathcal{A}.$$

Whenever we refer to a dg $\mathcal{A}$-module we assume that it is a right dg $\mathcal{A}$-module.

Example 2.2.2. For each object X of a dg category $\mathcal{A}$ one has *Yoneda right dg $\mathcal{A}$-module*:

$$\begin{aligned} X^{\wedge} = \mathcal{A}(-, X) : \mathcal{A}^{\mathrm{op}} &\longrightarrow \mathcal{C}_{\mathrm{dg}}(k) \\ Y &\longmapsto \mathcal{A}(Y, X). \end{aligned}$$

Definition 2.2.3. *Given $\mathcal{A} \in \mathrm{dgcat}(k)$ and a dg $\mathcal{A}$-module M, the* homology of M *is the induced functor:*

$$\begin{aligned} H^*(\mathcal{A}) &\longrightarrow \mathcal{G}(k) \\ X &\longmapsto H^*\big(M(X)\big), \end{aligned}$$

with values in the category of graded k-vector spaces.

As said before, a particular instance of Definition 2.1.16 is Definition 2.2.4 below.

Definition 2.2.4. *For $\mathcal{A}$ a dg category we define the following dg category*

$$C_{\mathrm{dg}}(\mathcal{A}) = \mathcal{H}\mathrm{om}\big(\mathcal{A}^{\mathrm{op}}, C_{\mathrm{dg}}(k)\big),$$

whose objects are right dg $\mathcal{A}$-modules.

In turn, this allows us to define the category of (right) dg $\mathcal{A}$-modules.

Definition 2.2.5. *Let $\mathcal{A} \in$ dgcat(k). The* category of dg $\mathcal{A}$-modules $C(\mathcal{A})$ *is defined to be the category $Z^0\big(C_{\mathrm{dg}}(\mathcal{A})\big)$. More explicitly, the objects of $C(\mathcal{A})$ are dg $\mathcal{A}$-modules and for F and G right dg $\mathcal{A}$-modules a morphism $\phi : F \to G$ consists of a family of morphisms $\phi_X \in C_{\mathrm{dg}}\big(F(X), G(X)\big)^0$ such that $\big(G(f)\big) \circ \phi_X = \phi_Y \circ \big(F(f)\big)$ and $d\phi = 0$, where this differential is induced by the differential in C_{dg}. In other words, ϕ_X is a map of complexes for all X.*

Definition 2.2.6. *For L, M in $C(\mathcal{A})$, where $\mathcal{A}$ is a dg category, we say that a morphism $f : L \to M$ is a quasi-isomorphism of dg $\mathcal{A}$-modules if it induces an isomorphism in homology. That is, for all $X \in \mathcal{A}$, the map of complexes $f_X : L(X) \to M(X)$ induces an isomorphism in the homology of those complexes. In this case, L and M are said to be* quasi-isomorphic.

Definition 2.2.7. *Take $\mathcal{A} \in$ dgcat(k), and let M be a dg $\mathcal{A}$-module. M is called a* quasi-representable dg $\mathcal{A}$-module *if there exists a $X \in \mathcal{A}$ such that M and $X^\wedge$ are quasi-isomorphic.*

2.3 Tensor product

In this section we follow [27] in order to define the tensor product of dg modules.

Definition 2.3.1. *Take $\mathcal{A}$ a dg category and let F be a left dg $\mathcal{A}$-module and G a right dg $\mathcal{A}$-module. We define a* dg pairing $G \times F \to C$, *with $C \in C_{\mathrm{dg}}(k)$, to be a morphism from the dg bifunctor (a dg functor is obtained by fixing one co-ordinate) $(X, Y) \mapsto \mathcal{A}(X, Y)$ to the dg bifunctor $(X, Y) \mapsto C_{\mathrm{dg}}(k)\big(G(Y) \otimes F(X), C\big)$, which stands for morphism of complexes.*

Definition 2.3.2. *Take $\mathcal{A} \in$ dgcat *and let $F : \mathcal{A} \to C_{\mathrm{dg}}(k)$ and $G : \mathcal{A}^{\mathrm{op}} \to C_{\mathrm{dg}}(k)$ be, respectively, left, and right, dg $\mathcal{A}$-modules. The universal dg pairing $G \times F \to C_0$, where $C_0 \in C_{\mathrm{dg}}(k)$, is called the* tensor product *of G and F and it is denoted by $G \otimes_{\mathcal{A}} F$. Explicitly,*

$$G \otimes_{\mathcal{A}} F = \Big(\bigoplus_{X \in \mathcal{A}} G(X) \otimes F(X)\Big)/\sim,$$

where $\sim$ stands for the following equivalence relation: for every morphism $f \in \mathcal{A}(X, Y)$, $u \in G(Y)$ and $v \in F(X)$ one has $f^*(u) \otimes v \sim u \otimes f_*(v)$.

Remark 2.3.3. Note that $\Big(\bigoplus_{X \in \mathcal{A}} G(X) \otimes F(X)\Big)/\sim$ is indeed the universal dg pairing.

Example 2.3.4. We work out an example from [27, §14.4]. Let $\mathcal{A}$ be a dg category and for $Y \in \mathcal{A}$ one has the Yoneda right dg $\mathcal{A}$-module $Y^\wedge = \mathcal{A}(-, Y)$, see Example 2.2.2. Let F be a left dg $\mathcal{A}$-module. We want to show that $Y^\wedge \otimes_{\mathcal{A}} F \simeq F(Y)$.

We start by noticing that, as F is a left dg $\mathcal{A}$-module, for all $X \in \mathcal{A}$ there is a natural morphism:

$$\begin{aligned} Y^\wedge(X) \otimes F(X) = A(X, Y) \otimes F(X) &\longrightarrow F(Y) \\ f \otimes v &\longmapsto f_*(v). \end{aligned} \tag{2.1}$$

Hence, this induces a morphism

$$\Big(\bigoplus_{X \in \mathcal{A}} Y^\wedge(X) \otimes F(X) \Big) \longrightarrow F(Y) \tag{2.2}$$

We show that it respects the equivalence relation $\sim$.

Take $f \in \mathcal{A}(X_1, X_2)$, $u \in Y^\wedge(X_2) = \mathcal{A}(X_2, Y)$ and $v \in F(X_1)$, we want to show that the image of $f^*(u) \otimes v$ and $u \otimes f_*(v)$ are the same, through the appropriate functor (2.1). Note that $f^*(u)$ belongs to $Y^\wedge(X_1)$ and $f_*(v)$ to $F(X_2)$.

We have:

$$f^*(u) \otimes v \overset{(2.1)}{\longmapsto} \big(f^*(u)\big)_*(v) = (u \circ f)_*(v) = u_*\big(f_*(v)\big) \overset{(2.1)}{\longleftarrow} u \otimes f_*(v).$$

Consequently, (2.2) respects the equivalence relation $\sim$.

It is clear that (2.2) is surjective.

Now, take $f \in Y^\wedge(X_1) = \mathcal{A}(X_1, Y)$, $g \in Y^\wedge(X_2) = \mathcal{A}(X_2, Y)$, $v \in F(X_1)$ and $u \in F(X_2)$. We now show that if $f \otimes v$ and $g \otimes u$ have the same image under the corresponding map (2.1), i.e., $f_*(v) = g_*(u)$, then $f \otimes v \sim g \otimes u$. It is enough do consider $\mathrm{id}_{\mathcal{A}(Y,Y)} \in \mathcal{A}(Y, Y) = Y^\wedge(Y)$, for then we have:

$$f \otimes v = f^*(\mathrm{id}_{\mathcal{A}(Y,Y)}) \otimes v \sim \mathrm{id}_{\mathcal{A}(Y,Y)} \otimes f_*(v) = \mathrm{id}_{\mathcal{A}(Y,Y)} \otimes g_*(u) = g^*(\mathrm{id}_{\mathcal{A}(Y,Y)}) \otimes u = g \otimes u.$$

This is enough to conclude that $\big(\bigoplus_{X \in \mathcal{A}} Y^\wedge(X) \otimes F(X)\big)/\!\sim = Y^\wedge \otimes_{\mathcal{A}} F \simeq F(Y)$.

2.4 Derived category

In this section we define three important notions: derived category of a dg category, Morita equivalence of dg categories and dg quotient of dg categories (which leads to the definition of derived dg category). For sake of completeness, we start by considering two different constructions.

Example 2.4.1. Let $\mathcal{A}$ and $\mathcal{B}$ be two dg categories and take $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ a dg functor. It is clear that there is an induced functor in the category of dg modules:

$$\begin{aligned} \mathcal{F}^* : C(\mathcal{B}) &\longrightarrow C(\mathcal{A}) \\ M &\longmapsto M \circ F. \end{aligned} \tag{2.3}$$

Example 2.4.2. Similarly to Example 2.4.1, let $\mathcal{A}$ and $\mathcal{B}$ be two dg categories and take $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ a dg functor. We want to define an induced functor in the category of dg modules:

$$\begin{aligned} \mathcal{F}_! : C(\mathcal{A}) &\longrightarrow C(\mathcal{B}) \\ M &\longmapsto \mathcal{F}_!(M). \end{aligned} \tag{2.4}$$

Where $\mathcal{F}_!(M)$ is the following right dg $\mathcal{B}$-module:

$$\begin{aligned} \mathcal{F}_!(M) : \mathcal{B}^{\mathrm{op}} &\longrightarrow C_{\mathrm{dg}}(k) \\ X &\longmapsto M \otimes_{\mathcal{A}} \mathcal{B}\big(X, \mathcal{F}(-)\big). \end{aligned} \tag{2.5}$$

Note that M is a right dg $\mathcal{A}$-module and $\mathcal{B}\big(X, F(-)\big)$ is a left dg $\mathcal{A}$-module for any $X \in \mathcal{B}$. So, Definition 2.3.2 guarantees that $\mathcal{F}_!(M)$ is indeed a right dg $\mathcal{B}$-module

Remark 2.4.3. For a dg functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ we have that $\mathcal{F}_!$ is the left adjoint of F^*:

$$\begin{tikzcd} C(\mathcal{B}) \arrow[d, shift left, "\mathcal{F}^*"] \\ C(\mathcal{A}). \arrow[u, shift left, "\mathcal{F}_!"] \end{tikzcd}$$

Definition 2.4.4. *For $\mathcal{A}$ a dg category, the* derived category $\mathcal{D}(\mathcal{A})$ *is the localization of the category of dg $\mathcal{A}$-modules $C(\mathcal{A})$ with respect to quasi-isomorphism.*

Remark 2.4.5. Thanks to Examples 2.4.1 and 2.4.2, Remark 2.4.3 and Definition 2.4.4, a dg functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ induces adjoint functors:

$$\begin{tikzcd} \mathcal{D}(\mathcal{B}) \arrow[d, shift left, "\mathcal{F}^*"] \\ \mathcal{D}(\mathcal{A}). \arrow[u, shift left, "\mathbf{L}\mathcal{F}_!"] \end{tikzcd}$$

Definition 2.4.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be dg categories and $F : \mathcal{A} \to \mathcal{B}$ a dg functor. F is called a* Morita equivalence *if the induced functor $\boldsymbol{L}\mathcal{F}_! : \mathcal{D}(\mathcal{A}) \xrightarrow{\sim} \mathcal{D}(\mathcal{B})$ is an equivalence on the derived categories. The dg categories $\mathcal{A}$ and $\mathcal{B}$ are called* Morita equivalent *if it exists a Morita equivalence between them.*

Remark 2.4.7. Some important functors on dg categories are Morita invariant. For example, the algebraic K-theory, the Hochschild cohomology and the Hochschild homology are invariant by Morita equivalences, consult [84, §5.2]. In Proposition 5.4.6 below we show that "algebraic K_0 functor quotient by certain equivalence relations" is Morita invariant.

In order to introduce the notion of derived dg category we recall the definition of dg quotients of dg categories, consult [27] for more details.

Definition 2.4.8 (Drinfeld's quotient). *Let $\mathcal{A}$ be a dg category and take $\mathcal{B}$ a full dg category of $\mathcal{A}$. We define the dg quotient $\mathcal{A}/\mathcal{B}$ to be the dg category whose object are the objects of $\mathcal{A}$ and for every object $U \in \mathcal{B}$ we add a morphism $\epsilon_U \in \mathcal{A}(U,U)$ of degree -1 such that $d(\epsilon_U) = \mathrm{id}_U$.*

Following [42, §4.4], we define:

Definition 2.4.9. *Let $\mathcal{E}$ be an abelian (or exact) category. The* derived dg category $\mathcal{D}_{\mathrm{dg}}(\mathcal{E})$ *of $\mathcal{E}$ is defined as the dg quotient $C_{\mathrm{dg}}(\mathcal{E})/Ac_{\mathrm{dg}}(\mathcal{E})$ of the dg category of complexes over $\mathcal{E}$ by its full dg subcategory of acyclic complexes.*

Example 2.4.10. Let X be a k-scheme and denote by $\mathrm{Mod}(X)$ be the Grothendieck category of sheaves of (right) $\mathcal{O}_X$-modules, $\mathrm{Qcoh}(X)$ the full subcategory of quasi-coherent $\mathcal{O}_X$-modules, $\mathcal{D}(X) := \mathcal{D}\big(\mathrm{Mod}(X)\big)$ the derived category of X, and $\mathcal{D}_{\mathrm{Qcoh}}(X) \to \mathcal{D}(X)$ the full triangulated subcategory of those complexes of $\mathcal{O}_X$-modules with quasi-coherent cohomology. When X is separated, we have $\mathcal{D}_{\mathrm{Qcoh}}(X) \simeq \mathcal{D}\big(\mathrm{Qcoh}(X)\big)$. With this in mind, we can now define the notion of a perfect complex of X.

A complex of $\mathcal{O}_X$-modules $C \in \mathcal{D}(X)$ is called *perfect* if there exists a covering $X = \cup_i V_i$ of X by affine open subschemes $V_i \subseteq X$ such that for every i the restriction $C_{|V_i}$ of C to V_i is quasi-isomorphic to a bounded complex of finitely generated projective $\mathcal{O}_{X|V_i}$-modules. Let us denote by $\mathrm{perf}(X)$ the triangulated category of perfect complexes. Note that by construction we have the inclusions $\mathrm{perf}(X) \to \mathcal{D}_{\mathrm{Qcoh}}(X) \to \mathcal{D}(X)$.

Now, consider Definition 2.4.9 and let $\mathcal{E} := \mathrm{Mod}(X)$. We write $\mathcal{D}_{\mathrm{dg}}(X)$ for the dg category $\mathcal{D}_{\mathrm{dg}}(\mathcal{E})$, $\mathcal{D}_{\mathrm{Qcoh,dg}}(X)$ for the full dg category of complexes of $\mathcal{O}_X$-modules with quasi-coherent cohomology, and $\mathrm{perf}_{\mathrm{dg}}(X)$ for its full subcategory of perfect complexes. By construction, we have $\mathrm{perf}_{\mathrm{dg}}(X) \subseteq \mathcal{D}_{\mathrm{Qcoh,dg}}(X) \subseteq \mathcal{D}_{\mathrm{dg}}(X)$.

2.5 The Grothendieck group

To define the Grothendieck group of a dg category we are going to assume some familiarity with triangulated categories. For a book on triangulated categories we suggest the reader to consult Neeman's book [72]. For instance, consult, respectively [72, §1.3, §4.2 and §4.5] for the definition of triangulated category, of its compact objects and for the Grothendieck group K_0 of a triangular category.

Definition 2.5.1. *Given a dg category $\mathcal{A}$ we denote by $\mathcal{D}_c(\mathcal{A})$ the full subcategory of compact objects of $\mathcal{D}(\mathcal{A})$.*

Remark 2.5.2. This category $\mathcal{D}_c(\mathcal{A})$ is the smallest triangulated subcategory of $\mathcal{D}(\mathcal{A})$, closed under direct summands, containing all representable right dg modules $X^\wedge$, for $X \in \mathcal{A}$, consult [79, §1.1.2.].

We can now define the Grothendieck group of a dg category.

Definition 2.5.3. *For $\mathcal{A}$ a dg category, its Grothendieck group is the Grothendieck group of the triangulated category $\mathcal{D}_c(\mathcal{A})$. That is:*

$$K_0(\mathcal{A}) = K_0\big(\mathcal{D}_c(\mathcal{A})\big).$$

2.6 Smoothness and properness

In this section we are going to introduce the concepts of smooth and proper dg categories. These notions were first defined by Kontsevich, see [44, 45, 43].

Definition 2.6.1. *Let $\mathcal{A}, \mathcal{B} \in \mathrm{dgcat}(k)$. A $\mathcal{A}$-$\mathcal{B}$-*bimodule X *is a dg $\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B}$-module. That is, it is a functor:*

$$\mathrm{X} : \mathcal{A} \otimes \mathcal{B}^{\mathrm{op}} \longrightarrow C_{\mathrm{dg}}(k).$$

Equivalently, a $\mathcal{A}$-$\mathcal{B}$-bimodule X is given by complexes $\mathrm{X}(B, A)$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$ together with morphisms of complexes:

$$\mathcal{A}(A, A') \otimes \mathrm{X}(B, A) \otimes \mathcal{B}(B', B) \longrightarrow \mathrm{X}(B', A')$$

Example 2.6.2. Take a $\mathcal{A}, \mathcal{B} \in \mathrm{dgcat}(k)$ and a dg functor $F : \mathcal{A} \to \mathcal{B}$ there exists an associated *$\mathcal{A}$-$\mathcal{B}$-bimodule* defined as follows:

$$\begin{aligned} {}_F\mathcal{B} : \mathcal{A} \otimes \mathcal{B}^{\mathrm{op}} &\longrightarrow C_{\mathrm{dg}}(k), \\ (A, B) &\longmapsto \mathcal{B}\big(B, F(A)\big). \end{aligned}$$

Definition 2.6.3. *Let $\mathcal{A}$, $\mathcal{B}$ and C be dg categories and consider X a $\mathcal{A}$-$\mathcal{B}$-bimodule and Y a $\mathcal{B}$-C-bimodule. One define the* tensor product of bimodules Y *and* X *to be the following functor:*

$$\begin{aligned} \mathrm{X} \otimes \mathrm{Y} : \mathcal{A} \otimes C^{\mathrm{op}} &\longrightarrow C_{\mathrm{dg}}(k) \\ (A, C) &\longmapsto \mathrm{X}(-, A) \otimes_{\mathcal{B}} \mathrm{Y}(C, -). \end{aligned}$$

This is well defined thanks to Definition 2.3.2. Indeed, $\mathrm{X}(-, A)$ is a right dg $\mathcal{B}$-module and $\mathrm{Y}(C, -)$ is a left dg $\mathcal{B}$-module for all $A \in \mathcal{A}$ and $C \in C$.

Hence, we can now define what is meant by a smooth dg category $\mathcal{A}$.

Definition 2.6.4. *A dg category $\mathcal{A}$ is called* smooth *if the dg $\mathcal{A}$-$\mathcal{A}$-bimodule ${}_{\mathrm{id}}\mathcal{A}$ belongs to the category $\mathcal{D}_c(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{A})$.*

Definition 2.6.5. *A dg category $\mathcal{A}$ is called* proper *if $\sum_i \dim \mathrm{H}^i \mathcal{A}(X, Y) < \infty$ for every pair of objects $X, Y \in \mathcal{A}$.*

Notation 2.6.6. We shall write $\mathrm{dgcat}_{\mathrm{sp}}(k) \subseteq \mathrm{dgcat}(k)$ to denote the full subcategory of smooth proper dg categories.

Example 2.6.7. We now introduce two examples of smooth proper dg categories. For more details we suggest the reader to consult [79, Example 1.42].

(i) Assume that k is a perfect field and let A be a finite dimension k-algebra of finite global dimension. Then, the associated dg category of A, see Example 2.1.7(i), is smooth and proper.

(ii) Let X be a smooth proper k-scheme. Then, the associated dg category of perfect complexes $\mathrm{perf}_{\mathrm{dg}}(X)$ is a smooth proper dg category; see Example 2.4.10.

3

Additive invariants

In this chapter we recall the notion of an additive invariant functor. Intuitively speaking, it is a functor on the category of (small) dg categories $\mathrm{dgcat}(k)$ with values in an additive category that sends Morita equivalences to isomorphisms and it also sends semi-orthogonal decompositions, as defined by Bondal and Orlov, to direct sums. Note that the definition of additive invariant is not written in terms of these semi-orthogonal decompositions but in terms of a kind of "upper triangular dg category", this analogy becomes clear once this dg category is built. Here we also recall the universal additive invariant, that will play an important role later in the definition of noncommutative Chow motives in §4.3 and in the "factorization of equivalence relations along semi-orthogonal decomposition", see §5.4.

3.1 Definitions

To introduce the notion of an additive functor we first need to introduce an "upper triangular dg category".

Definition 3.1.1. *Let us consider dg categories $\mathcal{A}$ and $\mathcal{B}$ and a $\mathcal{B}$-$\mathcal{A}$-bimolude* B*. We define the dg category $T(\mathcal{A},\mathcal{B};\mathrm{B})$ to be the category whose set of objects is the disjoint union of the sets of objects of $\mathcal{A}$ and $\mathcal{B}$, and whose complexes of morphisms are defined as:*

$$T(\mathcal{A},\mathcal{B};\mathrm{B})(x,y) := \begin{cases} \mathcal{A}(x,y) & \text{if} \quad x,y \in \mathcal{A} \\ \mathcal{B}(x,y) & \text{if} \quad x,y \in \mathcal{B} \\ \mathrm{B}(y,x) & \text{if} \quad x \in \mathcal{A}, y \in \mathcal{B} \\ 0 & \text{if} \quad y \in \mathcal{A}, x \in \mathcal{B}. \end{cases}$$

By construction, there exists two canonical inclusions

$$\begin{aligned} i_{\mathcal{A}} &: \mathcal{A} \to T(\mathcal{A},\mathcal{B};\mathrm{B}) \\ i_{\mathcal{B}} &: \mathcal{B} \to T(\mathcal{A},\mathcal{B};\mathrm{B}). \end{aligned} \tag{3.1}$$

We can now define additive invariant functors.

Definition 3.1.2. *Let D be an additive category. A functor $E : \mathrm{dgcat}(k) \to D$ is called* additive invariant *if* *(i) it sends Morita equivalences (see Definition 2.4.6) to isomorphisms.*

(ii) given dg categories $\mathcal{A}$, $\mathcal{B}$ and a $\mathcal{B}$-$\mathcal{A}$-bimolude B*, the dg functors $i_{\mathcal{A}}$ and $i_{\mathcal{B}}$, (3.1), induce an isomorphism*

$$E(\mathcal{A}) \oplus E(\mathcal{B}) \longrightarrow E\big(T(\mathcal{A}, \mathcal{B}; \mathrm{B})\big).$$

Example 3.1.3. The following functors are additive invariant: algebraic K-theory, Karoubi-Villamayor K-theory, non-connective algebraic K-theory, homotopy K-theory, cyclic homology and topological Hochschild homology, etc; consult [79, §2.2] for more examples and details.

The following lemma was stated in [74] without a proof.

Lemma 3.1.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be two dg categories and* B *a dg $\mathcal{B}$-$\mathcal{A}$-bimolude. Given a dg category C we have a canonical identification between the dg categories $T(\mathcal{A}, \mathcal{B}; \mathrm{B}) \otimes C$ and $T(\mathcal{A} \otimes C, \mathcal{B} \otimes C; \mathrm{B} \otimes C)$.*

Proof. Note that $\mathrm{B}\otimes C$ is naturally a $\mathcal{B}\otimes C$-$\mathcal{A}\otimes C$-bimolude. For instance, for $(b_1, c_1) \in \mathcal{B}\otimes C$ and $(a_2, c_2) \in \mathcal{A} \otimes C$, we have $\mathrm{B} \otimes C\big((b_1, c_1), (a_2, c_2)\big) = \mathrm{B}(b_1, a_2) \otimes C(c_1, c_2)$ a complex of k-vector spaces. So, the dg category $T(\mathcal{A} \otimes C, \mathcal{B} \otimes C; \mathrm{B} \otimes C)$ is well defined.

The identification of the sets of objects is trivial. The identification of the sets of morphisms follows from the definition of tensor product of dg categories. □

3.2 Universal additive invariant

In this subsection we are going to introduce the universal additive invariant category. We follow [79, §2.3].

Definition 3.2.1. *Let $\mathcal{A}, \mathcal{B} \in \mathrm{dgcat}(k)$. The category $\mathrm{rep}(\mathcal{A}, \mathcal{B})$ is the full subcategory of the derived category $\mathcal{D}(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})$ of $\mathcal{A}$-$\mathcal{B}$-bimodules whose objects are the bimodules X such that for every $A \in \mathcal{A}$ the right dg $\mathcal{B}$-module $X(-, A) : \mathcal{B}^{\mathrm{op}} \to C_{\mathrm{dg}}(k)$ is in $\mathcal{D}_c(\mathcal{B})$ for all $A \in \mathcal{A}$.*

Remark 3.2.2. Thanks to [76, Théorème 5.3], $\mathrm{dgcat}(k)$ carries a cofibrantly generated Quillen model structure where the weak equivalences are the Morita equivalences, see Definition 2.4.6. For details regarding Quillen model structures and homotopy theories consult [28].

Remark 3.2.3. For $\mathcal{A}$, $\mathcal{B} \in \mathrm{dgcat}(k)$ with $\mathcal{A}$ smooth and proper, according to [79, Corollary 1.44] one has that $\mathrm{rep}(\mathcal{A}, \mathcal{B}) \simeq \mathcal{D}_c(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})$.

Definition 3.2.4. *Consider the category $\mathrm{dgcat}(k)$ equipped with the Quillen model structure mentioned in Remark 3.2.2. We define the category $\mathrm{Hmo}(k)$ to be the localization of $\mathrm{dgcat}(k)$ with respect to the Morita equivalences.*

Remark 3.2.5. Thanks to [85, Theorem 4.2] and [81, Proposition 2.14 and Proposition 2.35], for $\mathcal{A}, \mathcal{B} \in \mathrm{dgcat}(k)$, one has

$$\mathrm{Hom}_{\mathrm{Hmo}(k)}(\mathcal{A}, \mathcal{B}) \simeq \mathrm{Iso}\,\mathrm{rep}(\mathcal{A}, \mathcal{B}), \tag{3.2}$$

where Iso stands for isomorphism classes of objects. Consequently, the composition in $\mathrm{Hmo}(k)$ is induced from the composition in $\mathrm{rep}(-,-)$:

$$\begin{aligned} \mathrm{rep}(\mathcal{A}, \mathcal{B}) \times \mathrm{rep}(\mathcal{B}, \mathcal{C}) &\longrightarrow \mathrm{rep}(\mathcal{A}, \mathcal{C}) \\ (M, N) &\longmapsto M \otimes_{\mathcal{B}} N, \end{aligned} \tag{3.3}$$

Remark 3.2.6. The canonical functor from $\mathrm{dgcat}(k)$ to $\mathrm{Hmo}(k)$ is

$$\begin{aligned} \mathrm{dgcat}(k) &\longrightarrow \mathrm{Hmo}(k) \\ \mathcal{A} &\longmapsto \mathcal{A} \\ (F : \mathcal{A} \to \mathcal{B}) &\longmapsto {}_F\mathcal{B}. \end{aligned} \tag{3.4}$$

See Example 2.6.2 to recall the construction of ${}_F\mathcal{B}$. Note that this dg $\mathcal{A}$-$\mathcal{B}$-bimodule is, by definition, in $\mathrm{rep}(\mathcal{A}, \mathcal{B})$ and hence, thanks to (3.2), in $\mathrm{Hom}_{\mathrm{Hmo}(k)}(\mathcal{A}, \mathcal{B})$.

Since $\mathrm{rep}(\mathcal{A}, \mathcal{B})$ is a triangulated category, one can consider its Grothendieck group and one obtains the following bilinear map:

$$\begin{aligned} K_0\big(\mathrm{rep}(\mathcal{A}, \mathcal{B})\big) \times K_0\big(\mathrm{rep}(\mathcal{B}, \mathcal{C})\big) &\longrightarrow K_0\big(\mathrm{rep}(\mathcal{A}, \mathcal{C})\big) \\ \big([M], [N]\big) &\longmapsto [M \otimes_{\mathcal{B}} N]. \end{aligned} \tag{3.5}$$

Definition 3.2.7. *Let us define the category* $\mathrm{Hmo}_0(k)$ *as the category whose object are the (small) dg categories and the class of morphisms* $\mathrm{Hom}_{\mathrm{Hmo}_0(k)}(\mathcal{A}, \mathcal{B}) := K_0\big(\mathrm{rep}(\mathcal{A}, \mathcal{B})\big)$ *and the composition is given by (3.5). For any object* $\mathcal{A}$*, the identity map is* $\big[{}_{\mathrm{id}}\mathcal{A}\big]$.

Remark 3.2.8. Note that by construction of $\mathrm{Hmo}_0(k)$ there exists the following functor:

$$\begin{aligned} \mathrm{Hmo}(k) &\longrightarrow \mathrm{Hmo}_0(k) \\ \mathcal{A} &\longmapsto \mathcal{A} \\ \big(\mathrm{X} : \mathcal{A} \otimes \mathcal{B}^{\mathrm{op}} \longrightarrow \mathcal{C}_{\mathrm{dg}}(k)\big) &\longmapsto [\mathrm{X}]. \end{aligned} \tag{3.6}$$

Remark 3.2.9. The category $\mathrm{Hmo}_0(k)$ is an additive category, see [79, Lemma 2.8].

Thanks to [76, Théorème 6.3] we can define the universal additive invariant.

Definition 3.2.10. *The following map is the* universal additive invariant*:*

$$\begin{aligned} \mathcal{U} : \mathrm{dgcat}(k) &\overset{(3.4)}{\longrightarrow} \mathrm{Hmo}(k) \overset{(3.6)}{\longrightarrow} \mathrm{Hmo}_0(k) \\ \mathcal{A} &\longmapsto \quad \mathcal{A} \quad \longmapsto \mathcal{A} \\ (F : \mathcal{A} \to \mathcal{B}) &\longmapsto \quad {}_F\mathcal{B} \quad \longmapsto \big[{}_F\mathcal{B}\big]. \end{aligned} \tag{3.7}$$

In particular, we have that for any additive invariant $\mathcal{F} : \mathrm{dgcat}(k) \to D$*, for* D *an additive category, there is an induced functor* $\overline{\mathcal{F}} : \mathrm{Hmo}_0(k) \to D$ *such that* $\mathcal{F} = \overline{\mathcal{F}} \circ \mathcal{U}$.

One can have a universal additive category with coefficients in commutative ring. That is, we have the following definition:

Definition 3.2.11. *Let R be a commutative ring. One defines the* universal additive category with coefficients in R, *and it is denoted by* $\mathrm{Hmo}_0(k)_R$, *to be the R-linear additive invariant category obtained from* $\mathrm{Hmo}_0(k)$ *by tensoring each abelian group of morphisms with R.*

Remark 3.2.12. Note that the category $\mathrm{Hmo}_0(k)_R$ is still symmetric monoidal and the following composition is a symmetric monoidal additive invariant:

$$\mathcal{U}(-)_R : \mathrm{dgcat}(k) \xrightarrow{\mathcal{U}} \mathrm{Hmo}_0(k) \xrightarrow{(-)_R} \mathrm{Hmo}_0(k)_R. \tag{3.8}$$

Moreover, given any R-linear additive invariant $\mathcal{F} : \mathrm{dgcat}(k) \to D$, for D an R-linear additive category, there is an induced functor $\overline{\mathcal{F}} : \mathrm{Hmo}_0(k)_R \to D$ such that $\mathcal{F} = \overline{\mathcal{F}} \circ \mathcal{U}(-)_R$.

Remark 3.2.13. We will always be working with coefficients in $\mathbb{Q}$, though this theory applies to other rings.

3.3 Semi-orthogonal decompositions

In this section we are introducing the concept of semi-orthogonal decomposition of triangulated categories and are going to show some properties of additive invariant functors when applied to triangulated categories. The main references are [21] and [55].

Definition 3.3.1. *Let* T *be a triangulated category an* B *a full subcategory of* T. *We define* $\mathrm{B}^\perp$ *to be the full subcategory of* T *whose objects are $A \in \mathrm{T}$ such that for all $B \in \mathrm{B}$ one has* $\mathrm{Hom}_{\mathrm{T}}(B, A) = 0$. *This category* $\mathrm{B}^\perp$ *is called the* right ortoghonal of B *and it is a triangulated category.*

Definition 3.3.2. *Given a triangulated category* T, *a sequence* $(\mathrm{B}_1, \cdots, \mathrm{B}_n)$ *is called* semi-orthogonal *if* $\mathrm{B}_i \subseteq \mathrm{B}_j^\perp$ *for all $1 \leq i < j \leq n$. Moreover, a semi-orthogonal sequence is called a* semi-orthogonal decomposition *of* T *if that sequence generates* T. *We denote it in the following way:* $\mathrm{T} = \langle \mathrm{B}_1, \cdots, \mathrm{B}_n \rangle$.

Definition 3.3.3. *For* T *a triangulated category an object $E \in \mathrm{T}$ is called* exceptional *if* $\mathrm{Hom}_{\mathrm{T}}(E, E) = k$ *and* $\mathrm{Hom}_{\mathrm{T}}\left(E, E[t]\right) = 0$ *for all $t \neq 0$. Moreover, we define an* exceptional collection *to be a sequence of exceptional objects $(E_1, \cdots, E_n)$ such that* $\mathrm{Hom}_{\mathrm{T}}\left(E_j, E_i[t]\right) = 0$ *for all $i < j$ and $t \in \mathbb{Z}$.*

Remark 3.3.4. An exceptional collection $(E_1, \cdots, E_n)$ of a triangulated category T determines a semi-orthogonal decomposition. Indeed, denote by E_i the category generated by E_i. Then, for $\mathrm{A} = \langle \mathrm{E}_1, \cdots \mathrm{E}_n \rangle^\perp$ we have $\mathrm{T} = \langle \mathrm{A}, \mathrm{E}_1, \cdots \mathrm{E}_n \rangle$. If $\mathrm{A} = 0$ then this exceptional collection is called *full*.

Remark 3.3.5. For $F : \mathrm{dgcat} \to D$ an additive invariant functor and $\mathcal{B}, \mathcal{C} \subset \mathcal{A}$ dg categories such that $H^0(\mathcal{A}) = \langle H^0(\mathcal{B}), H^0(\mathcal{C})\rangle$ is a semi-orthogonal decomposition. One has that $F(\mathcal{A}) \simeq F(\mathcal{B}) \oplus F(\mathcal{C})$. See [79, Proposition 2.2] for a proof.

We now define the notion of a Lefschetz decompostion.

Definition 3.3.6. *Take an algebraic variety X and a line bundle $\mathcal{L}$ on X. A* right Lefschetz decomposition *of* $\mathrm{perf}(X)$ *the category of perfect complexes of X with respect to $\mathcal{L}$ is a semi-orthogonal decomposition* $\mathrm{perf}(X) = \langle \mathrm{A}_0, \mathrm{A}_1 \otimes \mathcal{L}, \cdots, \mathrm{A}_{i-1} \otimes \mathcal{L}^{i-1}\rangle$ *with* $0 \subseteq \mathrm{A}_{i-1} \subseteq \cdots \subseteq \mathrm{A}_1 \subseteq \mathrm{A}_0$.

Analogously, a left Lefschetz decomposition of $\mathrm{perf}(X)$ *with respect to $\mathcal{L}$ is a semi-orthogonal decomposition* $\mathrm{perf}(X) = \langle \mathcal{B}_{i-1} \otimes \mathcal{L}^{1-i}, \ldots, \mathcal{B}_1 \otimes \mathcal{L}^{-1}, \mathcal{B}_0\rangle$ *with* $0 \subseteq \mathcal{B}_{i-1} \subseteq \ldots \mathcal{B}_1 \subseteq \mathcal{B}_0$.

Definition 3.3.7. *For X and algebraic variety admitting a right Lefschetz decomposition of* $\mathrm{perf}(X)$ *as in Definition 3.3.6, the right orthogonal to* A_{k+1} *in* A_k *is called a* primitive category *of the Lefschetz decomposition and it is denoted by* $\mathfrak{a}_k$.

Note that by Definition 3.3.7, one has the following semi-orthogonal decomposition $\mathrm{A}_k = \langle \mathfrak{a}_k, \mathfrak{a}_{k+1}, \ldots, \mathfrak{a}_{i-1}\rangle$.

4

Motives

In this chapter we introduce some notions on classical motives and on noncommutative motives. To what concerns this thesis, something very significant about motives is that we can relate them with equivalence relations on algebraic cycles. Similarly, some of the noncommutative motives are related with a generalization of those equivalence relations. Ultimately, it is the relation between motives and noncommutative motives that allows us to generalize the classical setting of questions regarding algebraic cycles on smooth proper schemes to a noncommutative setting. Most of the time we will be following [2, 79].

4.1 Algebraic cycles and equivalences relations

In this section we are going to introduce the concepts of algebraic cycles on k-scheme as well as equivalence relations on those algebraic cycles. Our main reference for this section is [2].

Definition 4.1.1. *Let X be a scheme of finite type over a field k. An* algebraic r-cycle *on X is a formal linear combination of r-dimensional closed integral k-subschemes V_i of X: $\sum n_i[V_i]$.*

Remark 4.1.2. Note that from Definition 4.1.1 above, one can define the following groups:

(i) The set of all r-cycles defines the free abelian group $\mathcal{Z}^r(X) = \bigoplus_{V\subseteq X} \mathbb{Z}[V]$, where $\mathrm{codim} V = r$.

(ii) The *group of algebraic cycles* is defined as $\mathcal{Z}^*(X) = \bigoplus_r \mathcal{Z}^r(X)$.

(iii) The *group of algebraic cycles with coefficient in* $\mathbb{Q}$ is defined as $\mathcal{Z}^*(X)_{\mathbb{Q}} = \mathcal{Z}^*(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.

We now start to define equivalence relations on these algebraic cycles.

Definition 4.1.3. *Let X be a scheme of finite type over a field k. An algebraic cycle $\alpha \in \mathcal{Z}^*(X)_{\mathbb{Q}}$ is called* rationally trivial *if there exists $\beta \in \mathcal{Z}^*(X \times \mathbb{P}^1)_{\mathbb{Q}}$ such that $\alpha = \beta(0) - \beta(\infty)$.*

Definition 4.1.3 gives rise to the *rational equivalence relation* $\sim_{\mathrm{rat}}$ on $\mathcal{Z}^*(X)_{\mathbb{Q}}$. We write $\mathcal{Z}^*(X)_{\mathbb{Q}}/_{\sim_{\mathrm{rat}}}$ for the associated quotient.

Definition 4.1.4. *Let X be a scheme of finite type over a field k. An algebraic cycle $\alpha \in \mathcal{Z}^*(X)_{\mathbb{Q}}$ is called* algebraically trivial *if it is in the image of the homomorphism*

$$\oplus_{(T,p,q)}\mathcal{Z}^*(X \times T)_{\mathbb{Q}} \xrightarrow{\oplus(id\times p^*)-(id\times q^*)} \mathcal{Z}^*(X)_{\mathbb{Q}}, \tag{4.1}$$

where the direct sum ranges over all smooth connected k-schemes T equipped with two k-rational points, $p, q : \mathrm{Spec}(k) \to T$.

Definition 4.1.4 gives rise to the *algebraic equivalence relation* $\sim_{\mathrm{alg}}$ on $\mathcal{Z}^*(X)_{\mathbb{Q}}$. We write $\mathcal{Z}^*(X)_{\mathbb{Q}}/_{\sim_{\mathrm{alg}}}$ for the associated quotient.

Definition 4.1.5. *Let X be a scheme of finite type over a field k. An algebraic cycle $\alpha \in \mathcal{Z}^*(X)_{\mathbb{Q}}$ is called* nilpotently trivial *if there exists a $N \in \mathbb{N}$ such that $\underbrace{\alpha \times \cdots \times \alpha}_{N \text{ times}}$ is rationally trivial in $\mathcal{Z}^*(X^{\times N})_{\mathbb{Q}}$.*

This gives rise to the *nilpotent equivalence relation* $\sim_{\mathrm{nil}}$ on $\mathcal{Z}^*(X)_{\mathbb{Q}}$. We write $\mathcal{Z}^*(X)_{\mathbb{Q}}/_{\sim_{\mathrm{nil}}}$ for the associated quotient.

To define the numerical equivalence relation, let us first note that the degree of a point $[P]$ in a k-scheme is defined to be $\deg\big([P]\big) := [k(P) : k]$ and we extend this definition linearly for linear combinations of points in a k-scheme. Moreover, consider the bilinear pairing:

$$\begin{aligned} - \cdot - : \mathcal{Z}^*(X)_{\mathbb{Q}}/_{\sim_{\mathrm{rat}}} \times \mathcal{Z}^{\dim X - *}(X)_{\mathbb{Q}}/_{\sim_{\mathrm{rat}}} &\longrightarrow \mathbb{Q} \\ (\alpha, \beta) &\longmapsto \deg(\alpha \cdot \beta) = \deg\Big(\sum n_i[P_i]\Big), \end{aligned} \tag{4.2}$$

where $[P_i]$ stands for the intersection points of α and β and n_i stands for their multiplicities. We can now write the definition of numerically trivial cycle

Definition 4.1.6. *Let X be a scheme of finite type over a field k. An algebraic cycle $\alpha \in \mathcal{Z}^*(X)_{\mathbb{Q}}$ is called* numerically trivial *if for all $\beta \in \mathcal{Z}^{\dim X - *}(X)_{\mathbb{Q}}$ the degree of $\alpha \cdot \beta$ is zero.*

This gives rise to the *numerical equivalence relation* $\sim_{\mathrm{num}}$ on $\mathcal{Z}^*(X)_{\mathbb{Q}}$. We write $\mathcal{Z}^*(X)_{\mathbb{Q}}/_{\sim_{\mathrm{num}}}$ for the associated quotient.

Up to this point we have just defined some equivalence relations on algebraic cycles, we now mention some results that compare these equivalence relations.

Theorem 4.1.7. *For X a smooth proper k-scheme and an element $\alpha \in \mathcal{Z}^*(X)_{\mathbb{Q}}$, we have the implications: $\alpha \sim_{\mathrm{rat}} 0 \Rightarrow \alpha \sim_{\mathrm{alg}} 0 \Rightarrow \alpha \sim_{\mathrm{nil}} 0 \Rightarrow \alpha \sim_{\mathrm{num}} 0$.*

Proof. Consult [2, §3.2]. □

Remark 4.1.8. From Theorem 4.1.7 we have that for X a smooth proper k-scheme one has the following quotient maps:

$$\mathcal{Z}^*(X)_{\mathbb{Q}}/_{\sim_{\mathrm{rat}}} \twoheadrightarrow \mathcal{Z}^*(X)_{\mathbb{Q}}/_{\sim_{\mathrm{alg}}} \twoheadrightarrow \mathcal{Z}^*(X)_{\mathbb{Q}}/_{\sim_{\mathrm{nil}}} \twoheadrightarrow \mathcal{Z}^*(X)_{\mathbb{Q}}/_{\sim_{\mathrm{num}}}. \tag{4.3}$$

4.2 Classical motives

In this section we start by introducing the classical theory of motives. We do it via category theory so that we can more easily generalize this approach to the theory of noncommutative motives.

4.2.1 Chow motives

The main reference for the construction of Chow motives over a field is André's book, namely [2, §4.1.3].

We are going to define the Chow motives of a projective k-variety with coefficients in $\mathbb{Q}$. Note that we have to bear in mind Definitions 4.1.1, 4.1.3, and Remark 4.1.2(ii) above.

Definition 4.2.1. *Take X a smooth projective k-variety and an integer d. The* d-codimensional cycle group Chow group with $\mathbb{Q}$-rational coefficients *is defined to be the group:*

$$A^d := \mathcal{Z}^d(X)_{\mathbb{Q}}/_{\sim_{\mathrm{rat}}}.$$

Given an algebraic cycle α of X, we shall denote by $[\alpha]$ its class in $A^d(X)$.

Definition 4.2.2. *Let X and Y be two smooth projective k-varieties, $X = \amalg_i X_i$ the decomposition of X in its connected components and $d_i = \dim X_i$. We define the* space of correspondences of degree r from X to Y *to be*

$$\mathrm{Corr}^r(X, Y) := \oplus_i A^{d_i+r}(X_i \times Y).$$

Given elements f in $\mathrm{Corr}^r(X, Y)$ and g in $\mathrm{Corr}^s(Y, Z)$ its composition is defined as:

$$g \circ f := (\pi_{XZ})_*\big(\pi_{XY}^*(f) \cdot \pi_{YZ}^*(g)\big) \in \mathrm{Corr}^{r+s}(X, Z), \tag{4.4}$$

where π_{XY}, π_{XZ} and π_{YZ} stand for projection from $X \times Y \times Z$ to, respectively, $X \times Y$, $X \times Z$ and $Y \times Z$.

We are now in conditions to define the category of Chow motives with rational coefficients. Note that we could define the Chow motives with coefficients in any other commutative field.

Definition 4.2.3. *The category of* Chow motives over a field k with rational coefficients $\mathrm{Chow}(k)_{\mathbb{Q}}$ *is defined in the following way.*

The objects are (X, p, m), where X is a smooth projective k-variety, m is an integer and p is an idempotent endomorphism $p = p^2 \in \mathrm{Corr}^0(X, X)$.

The class of morphisms between two objects is defined as:

$$\mathrm{Hom}_{\mathrm{Chow}(k)_{\mathbb{Q}}}\big((X, p, m), (Y, q, n)\big) := q \circ \mathrm{Corr}^{n-m}(X, Y) \circ p,$$

where composition is induced by the composition of correspondences (4.4).

Remark 4.2.4. In this remark we want to mention some properties related to Chow motives.

(i) Note that, from construction, we have that the category of Chow motives $\mathrm{Chow}(k)_{\mathbb{Q}}$ is $\mathbb{Q}$-linear, additive and pseudo-abelian (the idempotent morphisms of an additive category admit kernels, consult [40, Definition 6.7]).

(ii) The category of Chow motives $\mathrm{Chow}(k)_{\mathbb{Q}}$ has a monoidal structure. Indeed, for objects (X, p, m) and (Y, q, n) one defines $(X, p, m) \otimes (Y, q, n) := (X \times Y, p \otimes q, m + n)$. For this monoidal structure, the unit object is the Chow motive $\big(\mathrm{Spec}(k), \mathrm{id}, 0\big)$, where id is the class of the diagonal in $\mathrm{Corr}^0\big(\mathrm{Spec}(k), \mathrm{Spec}(k)\big)$.

(iii) There is a natural symmetric monoidal functor defined on the category of smooth projective k-schemes $\mathrm{SmProp}(k)^{\mathrm{op}}$:

$$\begin{aligned} \mathfrak{h}(-)_{\mathbb{Q}} : \mathrm{SmProp}(k)^{\mathrm{op}} &\longrightarrow \mathrm{Chow}(k)_{\mathbb{Q}} \\ X &\longmapsto (X, \mathrm{id}, 0) \\ (f : X \mapsto Y) &\longmapsto [\Gamma_f^t], \end{aligned} \tag{4.5}$$

where Γ_f^t is the transpose of the graph of f: $\Gamma_f = \Big\{\big(x, f(x)\big)\big| x \in X\Big\} \subseteq X \times Y$.

(iv) The *Tate motive* is defined as $\big(\mathrm{Spec}(k), \mathrm{id}, 1\big)$, it is denoted by $\mathbb{Q}(1)$ and it is a $\otimes$-invertible object. Indeed, thanks to [2, §4.1.5], we have that the Chow motive of the projective line $\mathfrak{h}(\mathbb{P}^1)_{\mathbb{Q}}$ decomposes into the direct sum $\mathfrak{h}(\mathrm{Spec}(k))_{\mathbb{Q}} \oplus \mathbb{L}$ where $\mathbb{L}$ is defined as the *Lefschetz motive* and its $\otimes$-inverse is $\mathbb{Q}(1)$, the *Tate motive*.

Remark 4.2.5. In this remark we point out that there is an isomophism relating Chow motives and the algebraic cycles with coefficients in $\mathbb{Q}$ up to the rational equivalence relation. Note that one has

$$\mathrm{Hom}_{\mathrm{Chow}(k)_{\mathbb{Q}}}\left(\mathfrak{h}(X)_{\mathbb{Q}} \otimes \mathbb{Q}(1)^{\otimes i}, \mathfrak{h}(Y)_{\mathbb{Q}} \otimes \mathbb{Q}(1)^{\otimes i'}\right) \simeq \mathcal{Z}^{\dim(X)+i'-i}(X \times Y)_{\mathbb{Q}}/_{\sim_{\mathrm{rat}}};$$

consult [79, §3.0.1] for details.

Later on, Chow motives are going to play an important role to prove Theorem 5.2.13.

4.2.2 Nilpotent motives

Definition 4.2.6. *Consider $(\mathcal{M}, \otimes, \mathrm{id})$ a $\mathbb{Q}$-linear, additive, rigid (consult [29, §2.10]) symmmetric monoidal category. Given $a, b \in \mathcal{M}$ its $\otimes_{\mathrm{nil}}$ is defined to be*

$$\otimes_{\mathrm{nil}}(a, b) := \{f \in \mathrm{Hom}_{\mathcal{M}}(a, b) \mid \exists n > 0 \textit{ such that } f^{\otimes} = 0\}.$$

One can verify that $\otimes_{\mathrm{nil}}$ is a $\otimes$-ideal. That is, it is an ideal with respect to $\otimes$. In addition, note that for all $a \in \mathcal{M}$, the ideal $\otimes_{\mathrm{nil}}(a, a) \subseteq \mathrm{End}_{\mathcal{M}}(a)$ is nilpotent: there is a $m > 0$ such that the product of any m elements of $\otimes_{\mathrm{nil}}(a, a)$ is zero. For a proof of this result consult [3, Lemme 7.4.2. ii)]. As a consequence, not only the quotient functor $\mathcal{M} \to \mathcal{M}/\otimes_{\mathrm{nil}}$ is

$\mathbb{Q}$-linear and symmetric monoidal, it is also a conservative functor. Note also that $\mathcal{M}/\otimes_{\mathrm{nil}}$ is idempotent complete whenever $\mathcal{M}$ is idempotent complete, for idempotents can be lifted along nilpotent ideals. Consult [38, §1] and [66, §4.4.5] for, respectively, the definition of conservative functor and idempotent complete category. We can now define a quotient category:

Definition 4.2.7. *Consider the category of Chow motives* $\mathrm{Chow}(k)_{\mathbb{Q}}$ *over k with coefficients in $\mathbb{Q}$. The* category of $\otimes$-nilpotent motives $\mathrm{Voev}(k)_{\mathbb{Q}}$ *over k with coefficients in $\mathbb{Q}$ is the quotient category of* $\mathrm{Chow}(k)_{\mathbb{Q}}$ *by* $\otimes_{\mathrm{nil}}$.

This notation is motivated by the Voevodsky's nilpotence conjecture that was formulated in [87]. This becomes more clear in §5.

Remark 4.2.8. From Definition 4.2.7 it is clear that there is an induced functor from the Chow motives to the $\otimes$-nilpotent motives $\mathrm{Chow}(k)_{\mathbb{Q}} \twoheadrightarrow \mathrm{Voev}(k)_{\mathbb{Q}}$.

Remark 4.2.9. As in Remark 4.2.5, there exists an isomophism relating $\otimes$-nilpotent motives and the algebraic cycles with coefficients in $\mathbb{Q}$ up to the nilpotence equivalence relation:

$$\mathrm{Hom}_{\mathrm{Voev}(k)_{\mathbb{Q}}}\left(\mathfrak{h}(X)_{\mathbb{Q}} \otimes \mathbb{Q}(1)^{\otimes i}, \mathfrak{h}(Y)_{\mathbb{Q}} \otimes \mathbb{Q}(1)^{\otimes i'}\right) \simeq \mathcal{Z}^{\dim(X)+i'-i}(X \times Y)_{\mathbb{Q}}/_{\sim_{\mathrm{nil}}};$$

consult [79, §3.0.2] for details.

4.2.3 Numerical motives

Definition 4.2.10. *Consider* $(\mathcal{M}, \otimes, \mathrm{id})$ *a $\mathbb{Q}$-linear, additive, rigid symmmetric monoidal category. Given $a, b \in \mathcal{M}$ its $\mathcal{N}$ ideal is defined to be*

$$\mathcal{N}(a,b) := \{f \in \mathrm{Hom}_{\mathcal{N}}(b,a) \,|\, \forall g \in \mathrm{Hom}_{\mathcal{N}}(a,b) \text{ one has } \mathrm{tr}(g \circ f) = 0\}.$$

Consult [79, §1.7.1] for the definition of categorical trace tr.

Thanks to [3, Lemme 7.1.1] we have that $\mathcal{N}$ is indeed a $\otimes$-ideal.

Definition 4.2.11. *Consider the category of Chow motives* $\mathrm{Chow}(k)_{\mathbb{Q}}$ *over k with coefficients in $\mathbb{Q}$. The category of* numerical motives $\mathrm{Num}(k)_{\mathbb{Q}}$ *over k with coefficients in $\mathbb{Q}$ is the idempotent completion of the quotient of* $\mathrm{Chow}(k)_{\mathbb{Q}}$ *by the $\mathcal{N}$-ideal.*

Remark 4.2.12. Note that the category of Numerical motives $\mathrm{Num}(k)_{\mathbb{Q}}$ is $\mathbb{Q}$-linear, additive, rigid symmetric monoidal and idempotent complete. Thanks to [3, Proposition 7.1.4 b], since $\mathrm{End}_{\mathrm{Chow}(k)_{\mathbb{Q}}}\left(\mathfrak{h}\big(\mathrm{Spec}(k)\big)_{\mathbb{Q}}\right) \simeq \mathbb{Q}$, we have that $\mathcal{N}$ is the largest $\otimes$-ideal of $\mathrm{Chow}(k)$, distinct from $\mathrm{Chow}(k)$. Consequently, there is a natural map $\mathrm{Voev}(k)_{\mathbb{Q}} \twoheadrightarrow \mathrm{Num}(k)_{\mathbb{Q}}$.

Remark 4.2.13. Similar to Remarks 4.2.5 and 4.2.9, there exists an isomorphism relating numerical motives and the algebraic cycles with coefficients in $\mathbb{Q}$ up to the numerical equivalence relation:

$$\mathrm{Hom}_{\mathrm{Num}(k)_{\mathbb{Q}}}\left(\mathfrak{h}(X)_{\mathbb{Q}} \otimes \mathbb{Q}(1)^{\otimes i}, \mathfrak{h}(Y)_{\mathbb{Q}} \otimes \mathbb{Q}(1)^{\otimes i'}\right) \simeq \mathcal{Z}^{\dim(X)+i'-i}(X \times Y)_{\mathbb{Q}}/_{\sim_{\mathrm{num}}};$$

consult [79, §3.0.4] for details.

Remark 4.2.14. Bearing in mind both Remarks 4.2.8 and 4.2.12, we have the following quotients maps:

$$\mathrm{Chow}(k)_{\mathbb{Q}} \twoheadrightarrow \mathrm{Voev}(k)_{\mathbb{Q}} \twoheadrightarrow \mathrm{Num}(k)_{\mathbb{Q}}.$$

Note that this is similar to what happened for the algebraic cycles up to the rational, algebraic, nilpotent and numerical equivalence relations, see (4.3).

4.3 Noncommutative motives

In this section we introduce the noncommutative version of the classical motives we have just introduced in §4.2.

4.3.1 Noncommutative Chow motives

We start by introducing the analogue of the Chow motives.

Definition 4.3.1. *We define the category of* noncommutative Chow motives $\mathrm{NChow}(k)_{\mathbb{Q}}$ over k with coefficient in $\mathbb{Q}$ *to be the idempotent completion of the full subcategory of* $Hmo_0(k)_{\mathbb{Q}}$ *whose object are* $\mathcal{U}(\mathcal{A})_{\mathbb{Q}}$ *with* $\mathcal{A}$ *a smooth proper dg category.*

Remark 4.3.2. We make three observations:

(i) The functor $\mathcal{U}(-)_{\mathbb{Q}}$ is defined in Remark 3.2.12, where the ring $R = \mathbb{Q}$.

(ii) We point out that Definition 4.3.1 can be written in a more general setting where the field of coefficients $\mathbb{Q}$ is replaced by any field F. In fact, the field of coefficients can be any commutative ring. For further details consult [79, §4.1].

(iii) The category $\mathrm{NChow}(k)_{\mathbb{Q}}$ is $\mathbb{Q}$-linear, additive, rigid symmetric monoidal category, consult [79, Lemma 2.8 and Theorem 1.43].

Remark 4.3.3. Note that thanks to Definitions 2.5.3, 4.3.1 and to Remark 3.2.3, for $\mathcal{A}$ and $\mathcal{B}$ smooth and proper dg categories, one has

$$\begin{aligned}\mathrm{Hom}_{\mathrm{NChow}(k)_{\mathbb{Q}}}\big(\mathcal{U}(\mathcal{A})_{\mathbb{Q}}, \mathcal{U}(\mathcal{B})_{\mathbb{Q}}\big) &:= K_0\big(\operatorname{rep}(\mathcal{A}, \mathcal{B})\big)_{\mathbb{Q}} \\ &\simeq K_0\big(\mathcal{D}_c(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})\big)_{\mathbb{Q}} =: K_0(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})_{\mathbb{Q}}.\end{aligned} \tag{4.6}$$

4.3.2 Noncommutative nilpotent motives

To define the $\otimes$-nilpotent motives $\mathrm{Voev}(k)_{\mathbb{Q}}$ one just considered the quotient of $\mathrm{Chow}(k)_{\mathbb{Q}}$ by the ideal $\otimes_{\mathrm{nil}}$. For the noncommutative version we do something similar. Indeed, since the category of noncommutative Chow motives $\mathrm{NChow}(k)_{\mathbb{Q}}$ is $\mathbb{Q}$-linear, additive, rigid symmetric monoidal category it has an associated $\otimes$-ideal $\otimes_{\mathrm{nil}}$; see Definition 4.2.6 and Remark 4.3.2 (iii) above. Hence, we can now define the noncommutative nilpotent motives.

Definition 4.3.4. *Consider the category of noncommutative Chow motives* $\mathrm{NChow}(k)_{\mathbb{Q}}$ *over k with coefficients in* $\mathbb{Q}$. *The* category of noncommutative ⊗-nilpotent motives $\mathrm{NVoev}(k)_{\mathbb{Q}}$ *over k with coefficients in* $\mathbb{Q}$ *is the quotient category of* $\mathrm{NChow}(k)_{\mathbb{Q}}$ *by its* $\otimes_{\mathrm{nil}}$ *ideal.*

Remark 4.3.5. By Definition 4.3.4 there is an induced functor from the noncommutative Chow motives to the noncommutative ⊗-nilpotent motives $\mathrm{NChow}(k)_{\mathbb{Q}} \twoheadrightarrow \mathrm{NVoev}(k)_{\mathbb{Q}}$.

Remark 4.3.6. Note that we could have defined noncommutative ⊗-nilpotent motives in more generality by allowing the coefficients to be in a commutative ring R.

4.3.3 Noncommutative numerical motives

We now define the noncommutative version of the numerical motives. We start by noticing that from Remark 4.3.2 (iii) we have that $\mathrm{NChow}(k)_{\mathbb{Q}}$ is a $\mathbb{Q}$-linear, additive and rigid symmetric monoidal category. A direct implication is that one can use Definition 4.2.10 to obtain $\mathcal{N}$, a ⊗-ideal of $\mathrm{NChow}(k)_{\mathbb{Q}}$.

Definition 4.3.7. *Consider the category of noncommutative Chow motives* $\mathrm{NChow}(k)_{\mathbb{Q}}$ *over k with coefficients in* $\mathbb{Q}$. *The* category of noncommutative numerical motives $\mathrm{NNum}(k)_{\mathbb{Q}}$ *over k with coefficients in* $\mathbb{Q}$ *is the quotient category of* $\mathrm{NChow}(k)_{\mathbb{Q}}$ *by its* $\mathcal{N}$ *ideal.*

Remark 4.3.8. It is immediate that there is an induced functor $\mathrm{NChow}(k)_{\mathbb{Q}} \to \mathrm{NNum}(k)_{\mathbb{Q}}$. What is not so trivial is that there is an induced functor $\mathrm{NVoev}(k)_{\mathbb{Q}} \twoheadrightarrow \mathrm{NNum}(k)_{\mathbb{Q}}$.

Indeed, $\mathrm{Hom}_{\mathrm{NChow}(k)_{\mathbb{Q}}}\left(\mathcal{U}(k)_{\mathbb{Q}}\right) \simeq \mathbb{Q}$. So, thanks again to [3, Proposition 7.1.4 b], $\mathcal{N}$ is the largest ⊗-ideal of $\mathrm{NChow}(k)_{\mathbb{Q}}$. Consequently, by considering the construction of both $\mathrm{NVoev}(k)_{\mathbb{Q}}$ and $\mathrm{NNum}(k)_{\mathbb{Q}}$, there is an induced functor $\mathrm{NVoev}(k)_{\mathbb{Q}} \twoheadrightarrow \mathrm{NNum}(k)_{\mathbb{Q}}$. This Remark is the "noncommutative version" of Remark 4.2.12 above.

Remark 4.3.9. As before, Definition 4.3.7 can be written in more general terms: $\mathbb{Q}$ can be replaced by any commutative ring R.

4.4 Relating classical and noncommutative motives

Now that we have introduced both classical and noncommutative motives, we are going to relate both concepts, otherwise one could ask why Definitions 4.3.1, 4.3.4 and 4.3.7 are, respectively, the noncommutative versions of Definitions 4.2.3, 4.2.7 and 4.2.11. We start by introducing the concept of associated orbit category as it was defined in [79, §4.2]:

Definition 4.4.1. *Let* $(\mathcal{M}, \otimes, \mathrm{id})$ *be a F-linear, additive, rigid symmetric monoidal category and* $\mathcal{O} \in \mathcal{M}$ *a ⊗-invertible object. The* associated orbit category $\mathcal{M}/(-\otimes \mathcal{O})$ *is the category whose objects are the same as those of* $\mathcal{M}$ *and the morphisms are defined as follows:*

$$\mathrm{Hom}_{\mathcal{M}/(-\otimes\mathcal{O})}(a, b) = \oplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{M}}(a, b \otimes \mathcal{O}^{\otimes i}),$$

for a and b objects in $\mathcal{M}/(-\otimes \mathcal{O})$. The compositions of morphisms is defined in the following way. For $a, b, c \in \mathcal{M}/-\otimes \mathcal{O}$, and morphisms

$$f = \{f_i\}_{i\in\mathbb{Z}} \in \oplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{M}}(a, b\otimes \mathcal{O}^{\otimes i}) \,,\, g = \{g_i\}_{i\in\mathbb{Z}} \in \oplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{M}}(b, c\otimes \mathcal{O}^{\otimes i}),$$

the j^{th} component of $g \circ f$ is defined as $(g\circ f)_j = \sum_i (g_{j-i} \otimes \mathcal{O}^{\otimes i}) \circ f_i$.

Remark 4.4.2. Thanks to [77, §7], the following functor

$$\begin{aligned} \eta : \mathcal{M} &\longrightarrow \mathcal{M}/(-\otimes \mathcal{O}) \\ A &\longmapsto A \\ f &\longmapsto f = \{f_i\}_{i\in\mathbb{Z}}, \end{aligned} \tag{4.7}$$

with $f_0 = f$ and $f_i = 0$ for all $i \neq 0$. By [77, Lemma 7.1] one has that $\mathcal{M}/(-\otimes \mathcal{O})$ is $\mathbb{Q}$-linear, additive and it inherits a rigid symmetric monoidal structure. Consequently, η is symmetric monoidal.

In the next result a relation between the Chow motives and the noncommutative Chow motives is shown. This result was first suggested by Kontsevich in [45, §4.1].

Theorem 4.4.3 ([77, Theorem 1.1])**.** *There exists a $\mathbb{Q}$-linear, fully faithful, symmetric monoidal functor Φ making the following diagram commutative:*

$$\begin{tikzcd} SmProp(k)^{\mathrm{op}} \arrow[rr, "X\longmapsto \mathrm{perf}_{\mathrm{dg}}(X)"] \arrow[d, "\mathfrak{h}(-)_{\mathbb{Q}}"'] & & \mathrm{dgcat}_{\mathrm{sp}}(k) \arrow[d, "\mathcal{U}"] \\ \mathrm{Chow}(k)_{\mathbb{Q}} \arrow[d, "\eta"'] & & \mathrm{Hmo}_0(k) \arrow[d, "(-)_{\mathbb{Q}}"] \\ \mathrm{Chow}(k)_{\mathbb{Q}}/\big(-\otimes\mathbb{Q}(1)\big) \arrow[rr, "\Phi"'] & & \mathrm{NChow}(k)_{\mathbb{Q}}. \end{tikzcd} \tag{4.8}$$

Remark 4.4.4. (i) Note that Theorem 4.4.3 is still valid when $\mathbb{Q}$ is replaced by any field F of characteristic zero.

(ii) Thanks to (4.8) we have that the commutative Chow motives sit, in some sense, inside the noncommutative Chow motives. In fact, this functor Φ is the link between all the commutative motives and noncommutatives motives defined in §4.2-4.3.

Now that the there is a "bridge" between Chow and noncommutative Chow motives we point out that ϕ descends to the other motives. Indeed, thanks to [17, Proposition 4.1] one has the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Chow}(k)_{\mathbb{Q}}/\big(-\otimes\mathbb{Q}(1)\big) \arrow[r, "\Phi"'] \arrow[d, two heads] & \mathrm{NChow}(k)_{\mathbb{Q}} \arrow[d, two heads] \\
\mathrm{Voev}(k)_{\mathbb{Q}}/\big(-\otimes\mathbb{Q}(1)\big) \arrow[r, "\Phi_{\mathrm{nil}}"'] \arrow[d, two heads] & \mathrm{NVoev}(k)_{\mathbb{Q}} \arrow[d, two heads] \\
\mathrm{Num}(k)_{\mathbb{Q}}/\big(-\otimes\mathbb{Q}(1)\big) \arrow[r, "\Phi_{\mathcal{N}}"'] & \mathrm{NNum}(k)_{\mathbb{Q}}.
\end{tikzcd} \tag{4.9}$$

Note that we can write diagram (4.8) and (4.9) together. Moreover, Φ_{nil} and $\Phi_{\mathcal{N}}$ inherit from Φ the properties of being $\mathbb{Q}$-linear, fully faithful and symmetric monoidal.

5

Improvement of the Voevodsky nilpotence conjecture

This is the key chapter of the first part of this thesis: we are going to refine some results regarding the Voevodsky nilpotence conjecture and, with the tools developed, we compute the quotient between the rational and the algebraic equivalence relations of certain five-folds. We start by writing the Voevodsky nilpotence conjecture which states that the nilpotence and numerical equivalence relations agree.

We then define the noncommutatative version of the algebraic equivalences defined in §4.1. To make sure that we are actually defining the noncommutative counterparts of the classical definitions we need that results on the noncommutative setting translate into results on the classical setting, and vice-versa. For our purposes, this follows from Theorem 5.3.1. After making sure that the classical and the noncommutative setting are related, we start to compare equivalence relations via kernels and this, ultimately, provides us with a tool to show that the algebraic and numerical equivalence relations on algebraic cycles agree on many examples. That is, we refine the Voevodsky nilpotence conjecture on those cases. Note that we obtain these result by proving first that the noncommutative version of these equivalence relations agree.

Apart from §5.1, the main reference for this chapter is [74].

5.1 Classical setting

In 1995, Voevodsky conjectured that the nilpotence equivalence relation, Definition 4.1.5, and the numerical equivalence relation, Definition 4.1.6, agree. More formally:

Conjecture 5.1.1 (The Voevodsky nilpotence conjecture, [87])**.** *Let X be a smooth proper K-scheme. The surjective quotient map $\mathcal{Z}^*(X)_\mathbb{Q}/_{\sim_{\text{nil}}} \twoheadrightarrow \mathcal{Z}^*(X)_\mathbb{Q}/_{\sim_{\text{num}}}$ is injective. Or, equivalently, the equivalence relations $\sim_{\text{nil}}$ and $\sim_{\text{num}}$ agree on $\mathcal{Z}^*(X)_\mathbb{Q}$.*

Remark 5.1.2. We start by noticing that this conjecture is known to hold for certains cases. Indeed, thanks to Kahn-Sebastian [39], Matsusaka [70], Voevodsky [87], Voisin [88] and

André [2, §11.5.2.3], the Voevodsky nilpotence conjecure holds for curves, surfaces, and abelian 3-folds (when k is of characteristic zero).

Remark 5.1.3. Using noncommutative motives, Bernardara-Marcolli-Tabuada proved that Conjecture 5.1.1 holds for certain quadric fibrations, intersection of quadrics, linear sections of Grassmannians, linear sections of determinantal varieties, homological projective duals, and Moishezon manifolds, see [17]. In addition, building on these ideas, Ornaghi-Pertusi [73] proved that Conjecture 5.1.1 holds for Cubic fourfolds and certain Gushel-Mukai fourfolds.

We wrote Remarks 5.1.2 and 5.1.3 separately because in this Thesis our reasoning is inspired by the approach introduced in [17]. Instead of tackling the problem directly one generalizes these conjecture to a noncommutative setting and solve the problem using tools developed in that more general setting.

Note that considering (4.3) it is reasonable to ask if one cannot make a stronger claim than the one in the Voevodsky nilpotence conjecture, Conjecture 5.1.1.

Claim 5.1.4 (Refinement of the Voevodsky nilpotence conjecture)**.** *Let X be a smooth proper K-scheme. The surjective quotient map $\mathcal{Z}^*(X)_{\mathbb{Q}}/_{\sim_{\mathrm{alg}}} \twoheadrightarrow \mathcal{Z}^*(X)_{\mathbb{Q}}/_{\sim_{\mathrm{num}}}$ is injective. Or, equivalently, the equivalence relations $\sim_{\mathrm{alg}}$ and $\sim_{\mathrm{num}}$ agree on $\mathcal{Z}^*(X)_{\mathbb{Q}}$.*

Remark 5.1.5. It is known that this claim does not hold in general! For example, it follows from the pioneering work of Griffiths [34] that if X is a general quintic 3-fold, then the algebraic and numerical equivalence relations on $\mathcal{Z}^*(X)_{\mathbb{Q}}$ do not agree; consult also the later works of Clemens [24] and Voisin [89] for further examples.

Nonetheless, it is important to note that Claim 5.1.4 holds for all cases mentioned in Remark 5.1.3 and some other that we show below in §5.5. Moreover, using the tools developed in §5.2 - §5.4 we are able to compute the quotient between the rational and the algebraic equivalence relations for certain 5-folds. That is, our approach is not only focused on the cases where the quotient between two equivalence relations is trivial (i.e. two equivalence relations agree). It can, in some cases, measure how much different two equivalence relations are. For examples see Corollaries 5.5.36 and 5.5.37 below.

5.2 Noncommutative setting

We now are going to generalize the previous section §5.1 to the context of smooth proper dg categories. The general idea is to translate the definitions from the classical setting to the noncommutative setting.

Let $\mathcal{A}$ be a smooth proper dg category. In this section, we introduce three different equivalence relations on the rational Grothendieck group $K_0(\mathcal{A})_{\mathbb{Q}} := K_0(\mathcal{A}) \otimes \mathbb{Q}$.

5.2.1 Algebraic equivalence relation

We start by introducing the algebraic equivalence relation in the context of dg categories and we give some properties of this equivalence relation. Given a smooth connected k-scheme T and a k-rational point $p : \mathrm{Spec}(k) \to T$, consider the associated pull-back dg functor $p^* : \mathrm{perf}_{\mathrm{dg}}(T) \to \mathrm{perf}_{\mathrm{dg}}\big(\mathrm{Spec}(k)\big)$. Note that $\mathrm{perf}_{\mathrm{dg}}\big(\mathrm{Spec}(k)\big)$ and k are Morita equivalent.

Definition 5.2.1. *An element $\alpha \in K_0(\mathcal{A})_{\mathbb{Q}}$ is called* algebraically trivial *if it belongs to the image of the homomorphism*

$$\oplus_{(T,p,q)} K_0\big(\mathcal{A} \otimes \mathrm{perf}_{\mathrm{dg}}(T)\big)_{\mathbb{Q}} \xrightarrow{\oplus\left(K_0(id\otimes p^*)_{\mathbb{Q}} - K_0(id\otimes q^*)_{\mathbb{Q}}\right)} K_0(\mathcal{A})_{\mathbb{Q}}, \tag{5.1}$$

where the direct sum ranges over all smooth connected k-schemes T equipped with two k-rational points p and q.

Remark 5.2.2. In the particular case where k is algebraically closed, any two k-rational points of T can be joined by a smooth projective k-curve. Hence, in this case, it suffices to consider the triples (C, p, q) where C is a smooth projective k-curve equipped with two k-rational points p and q.

Remark 5.2.3. Note that it is easy to see that Definition 5.2.1 is inspired by Definition 4.1.4. Indeed, it is very similar and we make use of the functorial properties of the construction of the category of perfect complexes and the fact that $K_0(-)_{\mathbb{Q}}$ is a functor.

The above homomorphism (5.1) gives rise to the *algebraic equivalence relation* $\sim_{\mathrm{alg}}$ on $K_0(\mathcal{A})_{\mathbb{Q}}$. In what follows, we will write $K_0(\mathcal{A})_{\mathbb{Q}}/_{\sim_{\mathrm{alg}}}$ for the associated quotient, i.e., for the cokernel of (5.1).

Lemma 5.2.4. *The functor $K_0(-)_{\mathbb{Q}}/_{\sim_{\mathrm{alg}}}$: $\mathrm{dgcat}_{\mathrm{sp}}(k) \to \mathrm{Vect}(\mathbb{Q})$ sends Morita equivalences to isomorphisms.*

Proof. Let $F : \mathcal{A} \to \mathcal{B}$ be a Morita equivalence in $\mathrm{dgcat}_{\mathrm{sp}}(k)$. Since the functor $K_0(-)_{\mathbb{Q}}$ is an additive invariant (consult [79, §2]), $K_0(F)_{\mathbb{Q}}$ is an isomorphism. Moreover, since Morita equivalences are preserved by tensor products over a field (consult [79, §1.6.4]), we have a Morita equivalence $F \otimes id_{\mathrm{perf}_{\mathrm{dg}}(T)}$ and, consequently, an isomorphism $K_0\big(F \otimes id_{\mathrm{perf}_{\mathrm{dg}}(T)}\big)_{\mathbb{Q}}$. We then obtain the following commutative diagram:

$$\begin{tikzcd}
\oplus_{(T,p,q)} K_0\big(\mathcal{A} \otimes \mathrm{perf}_{\mathrm{dg}}(T)\big)_{\mathbb{Q}} \arrow[r, "(5.1)"] \arrow[d, "\simeq", "\oplus K_0(F\otimes id_{\mathrm{perf}_{\mathrm{dg}}(T)})_{\mathbb{Q}}"'] & K_0(\mathcal{A})_{\mathbb{Q}} \arrow[r, two heads] \arrow[d, "\simeq"', "K_0(F)_{\mathbb{Q}}"] & K_0(\mathcal{A})_{\mathbb{Q}}/_{\sim_{\mathrm{alg}}} \arrow[d, "K_0(F)_{\mathbb{Q}}/_{\sim_{\mathrm{alg}}}"] \\
\oplus_{(T,p,q)} K_0\big(\mathcal{B} \otimes \mathrm{perf}_{\mathrm{dg}}(T)\big)_{\mathbb{Q}} \arrow[r, "(5.1)"'] & K_0(\mathcal{B})_{\mathbb{Q}} \arrow[r, two heads] & K_0(\mathcal{B})_{\mathbb{Q}}/_{\sim_{\mathrm{alg}}}.
\end{tikzcd} \tag{5.2}$$

Thanks to Definition 5.2.1, we hence conclude from (5.2) that $K_0(F)_{\mathbb{Q}}/_{\sim_{\mathrm{alg}}}$ is an isomorphism. $\square$

Lemma 5.2.5. *Let $\mathcal{A}$ be a smooth proper dg category such that $\mathrm{H}^0(\mathcal{A}) = \langle \mathfrak{b}, \mathfrak{c} \rangle$ admits a semi-orthogonal decomposition. Let $\mathfrak{b}^{\mathrm{dg}}$ and $\mathfrak{c}^{\mathrm{dg}}$ denote, respectively, the dg enhancement of $\mathfrak{b}$ and $\mathfrak{c}$ induced from $\mathcal{A}$. Under these assumptions, the inclusions of $\mathfrak{b}^{\mathrm{dg}}$ and $\mathfrak{c}^{\mathrm{dg}}$ into $\mathcal{A}$ induce an isomorphism:*

$$K_0(\mathfrak{b}^{\mathrm{dg}})_{\mathbb{Q}}/_{\sim_{\mathrm{alg}}} \oplus K_0(\mathfrak{c}^{\mathrm{dg}})_{\mathbb{Q}}/_{\sim_{\mathrm{alg}}} \xrightarrow{\simeq} K_0(\mathcal{A})_{\mathbb{Q}}/_{\sim_{\mathrm{alg}}}.$$

Proof. Consider the dg category $T(\mathfrak{b}^{\mathrm{dg}}, \mathfrak{c}^{\mathrm{dg}}; {}_{id}\mathcal{A})$, which is known to be smooth and proper; see [57, Proposition 4.9]. Following the proof of [79, Proposition 2.2], the inclusion of $T(\mathfrak{b}^{\mathrm{dg}}, \mathfrak{c}^{\mathrm{dg}}; {}_{id}\mathcal{A})$ into $\mathcal{A}$ is a Morita equivalence. Therefore, since the functor $K_0(-)_{\mathbb{Q}}$ is an additive invariant (see Example 3.1.3), by Definition 3.1.2 we obtain an induced isomorphism between $K_0(T(\mathfrak{b}^{\mathrm{dg}}, \mathfrak{c}^{\mathrm{dg}}; {}_{id}\mathcal{A}))_{\mathbb{Q}}$ and $K_0(\mathcal{A})_{\mathbb{Q}}$. Consequently, in order to finish the proof, it suffices to show that the inclusions of $\mathfrak{b}^{\mathrm{dg}}$ and $\mathfrak{c}^{\mathrm{dg}}$ into $T(\mathfrak{b}^{\mathrm{dg}}, \mathfrak{c}^{\mathrm{dg}}; {}_{id}\mathcal{A})$ induce an isomorphism

$$K_0(\mathfrak{b}^{\mathrm{dg}})_{\mathbb{Q}}/_{\sim_{\mathrm{alg}}} \oplus K_0(\mathfrak{c}^{\mathrm{dg}})_{\mathbb{Q}}/_{\sim_{\mathrm{alg}}} \longrightarrow K_0(T(\mathfrak{b}^{\mathrm{dg}}, \mathfrak{c}^{\mathrm{dg}}; {}_{id}\mathcal{A}))_{\mathbb{Q}}/_{\sim_{\mathrm{alg}}}. \tag{5.3}$$

Consider the following commutative diagram:

$$\begin{tikzcd}
\begin{array}{c}\oplus_{(T,p,q)} K_0(\mathfrak{b}^{\mathrm{dg}} \otimes \mathrm{perf}_{\mathrm{dg}}(T))_{\mathbb{Q}} \\ \oplus \\ \oplus_{(T,p,q)} K_0(\mathfrak{c}^{\mathrm{dg}} \otimes \mathrm{perf}_{\mathrm{dg}}(T))_{\mathbb{Q}}\end{array} \arrow[r, "(5.1)\oplus(5.1)"] \arrow[d] & \begin{array}{c}K_0(\mathfrak{b}^{\mathrm{dg}})_{\mathbb{Q}} \\ \oplus \\ K_0(\mathfrak{c}^{\mathrm{dg}})_{\mathbb{Q}}\end{array} \arrow[r, two heads] \arrow[d, "\simeq"'] & \begin{array}{c}K_0(\mathfrak{b}^{\mathrm{dg}})_{\mathbb{Q}}/_{\sim_{\mathrm{alg}}} \\ \oplus \\ K_0(\mathfrak{c}^{\mathrm{dg}})_{\mathbb{Q}}/_{\sim_{\mathrm{alg}}}\end{array} \arrow[d, "(5.3)"] \\
\oplus_{(T,p,q)} K_0\Big(T(\mathfrak{b}^{\mathrm{dg}}, \mathfrak{c}^{\mathrm{dg}}; {}_{id}\mathcal{A}) \otimes \mathrm{perf}_{\mathrm{dg}}(T)\Big)_{\mathbb{Q}} \arrow[r, "(5.1)"'] & K_0\Big(T(\mathfrak{b}^{\mathrm{dg}}, \mathfrak{c}^{\mathrm{dg}}; {}_{id}\mathcal{A})\Big)_{\mathbb{Q}} \arrow[r, two heads] & K_0\Big(T(\mathfrak{b}^{\mathrm{dg}}, \mathfrak{c}^{\mathrm{dg}}; {}_{id}\mathcal{A})\Big)_{\mathbb{Q}}/_{\sim_{\mathrm{alg}}}
\end{tikzcd}$$

with exact rows and whose vertical arrows are induced by the inclusions of $\mathfrak{b}^{\mathrm{dg}}$ and $\mathfrak{c}^{\mathrm{dg}}$ into $T(\mathfrak{b}^{\mathrm{dg}}, \mathfrak{c}^{\mathrm{dg}}; {}_{id}\mathcal{A})$. We need to prove that (5.3) is an isomorphism. Note that the middle vertical map is an isomorphism because $K_0(-)_{\mathbb{Q}}$ is an additive invariant, see Example 3.1.3. By diagram chasing, we observe that (5.3) is surjective. Since the middle vertical map of the above diagram is an isomorphism, Lemma 3.1.4 implies that, for a fixed (T, p, q), the corresponding map

$$K_0(\mathfrak{b}^{\mathrm{dg}} \otimes \mathrm{perf}_{\mathrm{dg}}(T))_{\mathbb{Q}} \oplus K_0(\mathfrak{c}^{\mathrm{dg}} \otimes \mathrm{perf}_{\mathrm{dg}}(T))_{\mathbb{Q}} \longrightarrow K_0(T(\mathfrak{b}^{\mathrm{dg}}, \mathfrak{c}^{\mathrm{dg}}; {}_{id}\mathcal{A}) \otimes \mathrm{perf}_{\mathrm{dg}}(T))_{\mathbb{Q}}$$

is an isomorphism as well. This implies that the left-hand-side vertical map of the above diagram is surjective. Consequently, we conclude (once again by diagram chasing) that (5.3) is moreover injective and hence an isomorphim. □

Lemma 5.2.6. *Let X be a smooth proper k-scheme and $\mathbb{B}_0$ an Azumaya algebra over X, (consult [82, §1]). The canonical dg functor $i_{X,\mathbb{B}_0} : \mathrm{perf}_{\mathrm{dg}}(X) \to \mathrm{perf}_{\mathrm{dg}}(X, \mathbb{B}_0)$ induces an isomorphism:*

$$K_0(i_{X,\mathbb{B}_0})_{\mathbb{Q}}/_{\sim_{\mathrm{alg}}} \colon K_0(\mathrm{perf}_{\mathrm{dg}}(X))_{\mathbb{Q}}/_{\sim_{\mathrm{alg}}} \xrightarrow{\simeq} K_0(\mathrm{perf}_{\mathrm{dg}}(X, \mathbb{B}_0))_{\mathbb{Q}}/_{\sim_{\mathrm{alg}}}.$$

Proof. Thanks to [82, Propositions 8.3 and 8.17], the canonical dg functor $i_{X,\mathbb{B}_0}$ induces an isomorphism $\mathcal{U}(i_{X,\mathbb{B}_0})_{\mathbb{Q}}$ in the category of noncommutative Chow motives $\mathrm{NChow}(k)_{\mathbb{Q}}$; see Definitions 3.2.10 and 4.3.1 and Remark 3.2.12. As a consequence, since the functor $\mathcal{U}(-)_{\mathbb{Q}}$ is symmetric monoidal, the morphism $\mathcal{U}\big(i_{X,\mathbb{B}_0}\otimes id_{\mathrm{perf}_{\mathrm{dg}}(T)}\big)_{\mathbb{Q}}$ is also an isomorphism. Thanks to the computation (4.6), this implies that the following two maps are invertible:

$$K_0(i_{X,\mathbb{B}_0})_{\mathbb{Q}} : K_0\big(\mathrm{perf}_{\mathrm{dg}}(X)\big)_{\mathbb{Q}} \xrightarrow{\simeq} K_0\big(\mathrm{perf}_{\mathrm{dg}}(X,\mathbb{B}_0)\big)_{\mathbb{Q}};$$
$$K_0\big(i_{X,\mathbb{B}_0}\otimes id_{\mathrm{perf}_{\mathrm{dg}}(T)}\big)_{\mathbb{Q}} : K_0\big(\mathrm{perf}_{\mathrm{dg}}(X)\otimes\mathrm{perf}_{\mathrm{dg}}(T)\big)_{\mathbb{Q}} \xrightarrow{\simeq} K_0\big(\mathrm{perf}_{\mathrm{dg}}(X,\mathbb{B}_0)\otimes\mathrm{perf}_{\mathrm{dg}}(T)\big)_{\mathbb{Q}}.$$

Therefore, we obtain the following commutative diagram:

$$\begin{tikzcd}
\oplus_{(T,p,q)}K_0\big(\mathrm{perf}_{\mathrm{dg}}(X)\otimes\mathrm{perf}_{\mathrm{dg}}(T)\big)_{\mathbb{Q}} \arrow[r, "(5.1)"] \arrow[d, "\oplus K_0(i_{X,\mathbb{B}_0}\otimes id_{\mathrm{perf}_{\mathrm{dg}}(T)})_{\mathbb{Q}}", "\simeq"'] & K_0\big(\mathrm{perf}_{\mathrm{dg}}(X)\big)_{\mathbb{Q}} \arrow[r, two heads] \arrow[d, "\simeq", "K_0(i_{X,\mathbb{B}_0})_{\mathbb{Q}}"'] & K_0\big(\mathrm{perf}_{\mathrm{dg}}(X)\big)_{\mathbb{Q}}/_{\sim_{\mathrm{alg}}} \arrow[d, "K_0(i_{X,\mathbb{B}_0})_{\mathbb{Q}}/_{\sim_{\mathrm{alg}}}"'] \\
\oplus_{(T,p,q)}K_0\big(\mathrm{perf}_{\mathrm{dg}}(X,\mathbb{B}_0)\otimes\mathrm{perf}_{\mathrm{dg}}(T)\big)_{\mathbb{Q}} \arrow[r, "(5.1)"'] & K_0\big(\mathrm{perf}_{\mathrm{dg}}(X,\mathbb{B}_0)\big)_{\mathbb{Q}} \arrow[r, two heads] & K_0\big(\mathrm{perf}_{\mathrm{dg}}(X,\mathbb{B}_0)\big)_{\mathbb{Q}}/_{\sim_{\mathrm{alg}}}.
\end{tikzcd} \tag{5.4}$$

Thanks to Definition 5.2.1, we hence conclude from (5.4) that $K_0(i_{X,\mathbb{B}_0})_{\mathbb{Q}}/_{\sim_{\mathrm{alg}}}$ is an isomorphism. □

5.2.2 Nilpotence equivalence relation

In this subsection we are going to introduce the nilpotence equivalence relation.

Given an integer $m\geq 1$ consider the following functor:

$$\prod_{i=1}^{m}\mathcal{D}_c(\mathcal{A})\longrightarrow\mathcal{D}_c\big(\mathcal{A}^{\otimes m}\big)\qquad \{M_i\}_{1\leq i\leq m}\longmapsto\otimes_{i=1}^{m}M_i. \tag{5.5}$$

The functor (5.5) is triangulated in each one of its variables. Hence, it gives rise to the following multilinear pairing:

$$\prod_{i=1}^{m}K_0(\mathcal{A})_{\mathbb{Q}}\longrightarrow K_0\big(\mathcal{A}^{\otimes m}\big)_{\mathbb{Q}}\qquad \big\{[M_i]\big\}_{1\leq i\leq m}\longmapsto[\otimes_{i=1}^{m}M_i]. \tag{5.6}$$

Definition 5.2.7. *An element $\alpha\in K_0(\mathcal{A})_{\mathbb{Q}}$ is called* nilpotently trivial *if there is an integer $m\geq 1$ such that the image of $\underbrace{(\alpha,\ldots,\alpha)}_{m\text{-copies}}$ under the multilinear pairing (5.6) is equal to* 0.

The above multilinear pairing (5.6) gives rise to the *nilpotence equivalence relation* $\sim_{\mathrm{nil}}$ on $K_0(\mathcal{A})_{\mathbb{Q}}$. In what follows, we will write $K_0(\mathcal{A})_{\mathbb{Q}}/_{\sim_{\mathrm{nil}}}$ for the associated quotient.

Remark 5.2.8. Just like what happened with the algebraic equivalence relations, for the nilpotence equivalence relations both Definitions 4.1.5 and 5.2.7 are very similar.

Remark 5.2.9. Thanks to (4.6), one has the following computation in the category of noncommutative Chow motives:

$$\mathrm{Hom}_{\mathrm{NChow}(k)_{\mathbb{Q}}}\left(\mathcal{U}(k)_{\mathbb{Q}}, \mathcal{U}(\mathcal{A})_{\mathbb{Q}}\right) = K_0\left(\mathcal{D}_c(k^{\mathrm{op}} \otimes \mathcal{A})\right)_{\mathbb{Q}} = K_0(\mathcal{A})_{\mathbb{Q}}.$$

This enables the following re-phrasing of the nilpotence equivalence relation: an element $\alpha \in K_0(\mathcal{A})_{\mathbb{Q}}$ is nilpotently trivial if and only if $\alpha : \mathcal{U}(k)_{\mathbb{Q}} \to \mathcal{U}(\mathcal{A})_{\mathbb{Q}}$, the associated morphism in $\mathrm{NChow}(k)_{\mathbb{Q}}$, is $\otimes$-nilpotent, or, equivalently, if and only if the associated morphism $\alpha : \mathcal{U}(\mathcal{A}^{\mathrm{op}})_{\mathbb{Q}} \to \mathcal{U}(k)_{\mathbb{Q}}$ in $\mathrm{NChow}(k)_{\mathbb{Q}}$ is $\otimes$-nilpotent.

5.2.3 Numerical equivalence relation

In order to define the numerical equivalene relation consider the following Euler bilinear pairing:

$$\begin{aligned} \chi : K_0(\mathcal{A})_{\mathbb{Q}} \times K_0(\mathcal{A})_{\mathbb{Q}} &\longrightarrow \mathbb{Q} \\ ([M],[N]) &\longmapsto \sum_n (-1)^n \dim_k \mathrm{Hom}_{\mathcal{D}_c(\mathcal{A})}\left(M, N[n]\right). \end{aligned} \tag{5.7}$$

Since $\mathcal{A}$ is a proper dg category, the bilinear pairing (5.7) is well-defined. Moreover, although it is not symmetric nor skew-symmetric, its left and right kernels agree; consult [79, §4]. Let us then write $\ker(\chi)$ for this (unique) kernel.

Definition 5.2.10. *An element $\alpha \in K_0(\mathcal{A})_{\mathbb{Q}}$ is called* numerically trivial *if it belongs to* $\ker(\chi)$*. In other words, we have $\chi(\alpha, \beta) = 0$ (or, equivalently, $\chi(\beta, \alpha) = 0$) for every $\beta \in K_0(\mathcal{A})_{\mathbb{Q}}$.*

Remark 5.2.11. Definitions 4.1.6 and 5.2.10 are very similar. Note that, in some sense, (5.7) has in Definition 5.2.10 the role that (4.2) has in Definition 4.1.6.

Definition 5.2.10 gives rise to the *numerical equivalence relation* $\sim_{\mathrm{num}}$ on $K_0(\mathcal{A})_{\mathbb{Q}}$. In what follows we will write $K_0(\mathcal{A})_{\mathbb{Q}}/_{\sim_{\mathrm{num}}}$ for the associated quotient.

Remark 5.2.12. Thanks to [68, §5 and Proposition 6.2] we have that an element $\alpha \in K_0(\mathcal{A})_{\mathbb{Q}}$ is numerically trivial if and only if for every $\beta \in K_0(\mathcal{A}^{\mathrm{op}})_{\mathbb{Q}}$ the associated composition

$$\mathcal{U}(k)_{\mathbb{Q}} \xrightarrow{\alpha} \mathcal{U}(\mathcal{A})_{\mathbb{Q}} \xrightarrow{\beta} \mathcal{U}(k)_{\mathbb{Q}}$$

is equal to zero.

5.2.4 Comparison between the different equivalence relations

In this subsection we compare the different equivalence relations introduced in §5.2.1-§5.2.3.

Theorem 5.2.13. *Given an element $\alpha \in K_0(\mathcal{A})_{\mathbb{Q}}$, we have the implications*

$$\alpha \sim_{\mathrm{alg}} 0 \Rightarrow \alpha \sim_{\mathrm{nil}} 0 \Rightarrow \alpha \sim_{\mathrm{num}} 0.$$

Proof. We start by proving the implication $\alpha \sim_{\mathrm{alg}} 0 \Rightarrow \alpha \sim_{\mathrm{nil}} 0$. Thanks to Proposition 5.2.14 below, we can assume without loss of generality that k is algebraically closed. Take $\alpha \in K_0(\mathcal{A})_{\mathbb{Q}}$ an algebraically trivial element. Recall from Remark 5.2.9 that it is enough to show that the associated morphism $\alpha : \mathcal{U}(\mathcal{A}^{\mathrm{op}})_{\mathbb{Q}} \to \mathcal{U}(k)_{\mathbb{Q}}$ in the category $\mathrm{NChow}(k)_{\mathbb{Q}}$ is $\otimes$-nilpotent. Since k is algebraically closed and the nilpotently trivial elements form a $\mathbb{Q}$-linear subspace of $K_0(\mathcal{A})_{\mathbb{Q}}$, we can assume that there exists a single smooth projective curve C, equipped with two k-rational points p and q, and a element $\beta \in K_0\big(\mathcal{A} \otimes \mathrm{perf}_{\mathrm{dg}}(C)\big)_{\mathbb{Q}}$ such that $\alpha = K_0(id_{\mathcal{A}} \otimes p^*)_{\mathbb{Q}}(\beta) - K_0(id_{\mathcal{A}} \otimes q^*)_{\mathbb{Q}}(\beta)$; see Remark 5.2.2. Making use of the following computations

$$\mathrm{Hom}_{\mathrm{NChow}(k)_{\mathbb{Q}}}\Big(\mathcal{U}(\mathcal{A}^{\mathrm{op}})_{\mathbb{Q}}, \mathcal{U}\big(\mathrm{perf}_{\mathrm{dg}}(C)\big)_{\mathbb{Q}}\Big) \simeq K_0\big(\mathcal{A} \otimes_k \mathrm{perf}_{\mathrm{dg}}(C)\big)_{\mathbb{Q}},$$
$$\mathrm{Hom}_{\mathrm{NChow}(k)_{\mathbb{Q}}}\big(\mathcal{U}(\mathcal{A}^{\mathrm{op}})_{\mathbb{Q}}, \mathcal{U}(k)_{\mathbb{Q}}\big) \simeq K_0(\mathcal{A}),$$

we observe that the associated morphism $\alpha : \mathcal{U}(\mathcal{A}^{\mathrm{op}})_{\mathbb{Q}} \to \mathcal{U}(k)_{\mathbb{Q}}$ can be written as the following composition

$$\alpha : \mathcal{U}(\mathcal{A}^{\mathrm{op}})_{\mathbb{Q}} \xrightarrow{\beta} \mathcal{U}\big(\mathrm{perf}_{\mathrm{dg}}(C)\big)_{\mathbb{Q}} \xrightarrow{\mathcal{U}(p^*)_{\mathbb{Q}} - \mathcal{U}(q^*)_{\mathbb{Q}}} \mathcal{U}(k)_{\mathbb{Q}},$$

where β is the morphism associated to the element $\beta \in K_0\big(\mathcal{A} \otimes \mathrm{perf}_{\mathrm{dg}}(C)\big)_{\mathbb{Q}}$. Consequently, once we show that $\mathcal{U}(p^*)_{\mathbb{Q}} - \mathcal{U}(q^*)_{\mathbb{Q}}$ is $\otimes$-nilpotent, we conclude that α is $\otimes$-nilpotent too. Recall from Remark 4.2.4(iii) that there exists a symmmetric monoidal functor $\mathfrak{h}(-)_{\mathbb{Q}} : \mathrm{SmProp}(k)^{\mathrm{op}} \to \mathrm{Chow}(k)_{\mathbb{Q}}$ defined on smooth proper k-schemes, consult [67] for more details on Chow motives. Following [79, Theorem 4.3], there exists a $\mathbb{Q}$-linear, fully-faithfull, symmmetric monoidal functor Φ making the following diagram commutative:

$$\begin{tikzcd}
\mathrm{SmProp}(k)^{\mathrm{op}} \arrow[r, "X \mapsto \mathrm{perf}_{\mathrm{dg}}(X)"] \arrow[d, "\mathfrak{h}(-)_{\mathbb{Q}}"'] & \mathrm{dgcat}_{\mathrm{sp}}(k) \arrow[dd, "\mathcal{U}(-)_{\mathbb{Q}}"] \\
\mathrm{Chow}(k)_{\mathbb{Q}} \arrow[d, "\tau"'] & \\
\mathrm{Chow}(k)_{\mathbb{Q}}/\big(-\otimes\mathbb{Q}(1)\big) \arrow[r, "\Phi"'] & \mathrm{NChow}(k)_{\mathbb{Q}},
\end{tikzcd} \tag{5.8}$$

where we denote by $\mathrm{Chow}(k)_{\mathbb{Q}}/\big(-\otimes\mathbb{Q}(1)\big)$ the orbit category with respect to the Tate motive $\mathbb{Q}(1)$, recall Definition 4.4.1. In [87, Proposition 3.1] it is proved that the morphism $\mathfrak{h}(p)_{\mathbb{Q}} - \mathfrak{h}(q)_{\mathbb{Q}} : \mathfrak{h}(C)_{\mathbb{Q}} \to \mathfrak{h}\big(\mathrm{Spec}(k)\big)_{\mathbb{Q}}$, corresponding to the degree zero cycle $p - q$ on C, is $\otimes$-nilpotent. Since both functors τ and Φ are symmetric monoidal, the commutativity of diagram (5.8) hence implies that $\mathcal{U}(p^*)_{\mathbb{Q}} - \mathcal{U}(q^*)_{\mathbb{Q}}$ is $\otimes$-nilpotent. This concludes the proof.

We now prove the implication $\alpha \sim_{\mathrm{nil}} 0 \Rightarrow \alpha \sim_{\mathrm{num}} 0$. Recall from Remark 5.2.12 that $\alpha \in K_0(\mathcal{A})_{\mathbb{Q}}$ is numerically trivial if and only if for every $\beta \in K_0(\mathcal{A}^{\mathrm{op}})_{\mathbb{Q}}$ the associated composition

$$\mathcal{U}(k)_{\mathbb{Q}} \xrightarrow{\alpha} \mathcal{U}(\mathcal{A})_{\mathbb{Q}} \xrightarrow{\beta} \mathcal{U}(k)_{\mathbb{Q}}$$

is equal to zero. Let $\alpha \in K_0(\mathcal{A})_{\mathbb{Q}}$ be a nilpotently trivial element. Bearing in mind Remark 5.2.9, the associated morphism $\alpha : \mathcal{U}(k)_{\mathbb{Q}} \to \mathcal{U}(\mathcal{A})_{\mathbb{Q}}$ is $\otimes$-nilpotent. Let us assume by absurd that α is not numerically trivial. In this case, there would exist an element $\beta \in K_0(\mathcal{A}^{\mathrm{op}})_{\mathbb{Q}}$ such that the associated morphism $\beta : \mathcal{U}(\mathcal{A})_{\mathbb{Q}} \to \mathcal{U}(k)_{\mathbb{Q}}$ composed with α is different from zero. Using the fact that $\beta \circ \alpha \in \mathrm{Hom}_{\mathrm{NChow}(k)_{\mathbb{Q}}}(\mathcal{U}(k)_{\mathbb{Q}}, \mathcal{U}(k)_{\mathbb{Q}}) \simeq \mathbb{Q}$, we can further assume without loss of generality that $\beta \circ \alpha$ is the identity. But, this would then imply that $\beta \circ \alpha$ is not $\otimes$-nilpotent, which contradicts the assumption that α is $\otimes$-nilpotent. In conclusion, α is also numerically trivial. □

Proposition 5.2.14. *Let $\overline{k}/k$ be a fixed algebraic closure of k. Given an element $\alpha \in K_0(\mathcal{A})_{\mathbb{Q}}$, the following holds:*

$$\alpha \sim_{\mathrm{alg}} 0 \Rightarrow (\alpha \otimes_k \overline{k}) \sim_{\mathrm{alg}} 0 \qquad \textit{and} \qquad \alpha \sim_{\mathrm{nil}} 0 \Leftrightarrow (\alpha \otimes_k \overline{k}) \sim_{\mathrm{nil}} 0.$$

Proof. It follows from [69, Proposition 7.2] that if $\mathcal{A} \in \mathrm{dgcat}_{\mathrm{sp}}(k)$, then $\mathcal{A} \otimes_k \overline{k} \in \mathrm{dgcat}_{\mathrm{sp}}(\overline{k})$. Note also that given a smooth connected k-scheme T, $\mathrm{perf}_{\mathrm{dg}}(T) \otimes_k \overline{k}$ is a smooth $\overline{k}$-linear dg category which is canonically Morita equivalent to $\mathrm{perf}_{\mathrm{dg}}(\overline{T})$, where $\overline{T} := T \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k})$. Hence, the implication $\alpha \sim_{\mathrm{alg}} 0 \Rightarrow \alpha \otimes_k \overline{k} \sim_{\mathrm{alg}} 0$ follows from diagram (5.9) below (where T' is a smooth connected $\overline{k}$-scheme equipped with two $\overline{k}$-rational points p' and q'):

$$\begin{tikzcd}
\oplus_{(T',p',q')} K_0\big((\mathcal{A} \otimes_k \overline{k}) \otimes_{\overline{k}} \mathrm{perf}_{\mathrm{dg}}(T')\big)_{\mathbb{Q}} \arrow[r, "(5.1)"] & K_0(\mathcal{A} \otimes_k \overline{k})_{\mathbb{Q}} \\
\oplus_{(T,p,q)} K_0\big(\mathcal{A} \otimes_k \mathrm{perf}_{\mathrm{dg}}(T)\big)_{\mathbb{Q}} \arrow[r, "(5.1)"'] \arrow[u, "K_0(-\otimes_k \overline{k})_{\mathbb{Q}}"] & K_0(\mathcal{A})_{\mathbb{Q}}. \arrow[u, "K_0(-\otimes_k \overline{k})_{\mathbb{Q}}"']
\end{tikzcd} \tag{5.9}$$

We now prove the equivalence $\alpha \sim_{\mathrm{nil}} 0 \Leftrightarrow (\alpha \otimes_k \overline{k}) \sim_{\mathrm{nil}} 0$. For any integer $m \geq 1$, we have the commutative diagram:

$$\begin{tikzcd}
\prod_{i=1}^m K_0(\mathcal{A} \otimes_k \overline{k})_{\mathbb{Q}} \arrow[r, "(5.6)"] & K_0\big((\mathcal{A} \otimes_k \overline{k})^{\otimes_{\overline{k}} m}\big)_{\mathbb{Q}} \\
\prod_{i=1}^m K_0(\mathcal{A})_{\mathbb{Q}} \arrow[r, "(5.6)"'] \arrow[u, "\prod_{i=1}^m K_0(-\otimes_k \overline{k})_{\mathbb{Q}}"] & K_0\big(\mathcal{A}^{\otimes_k m}\big)_{\mathbb{Q}}. \arrow[u, "K_0(-\otimes_k \overline{k})_{\mathbb{Q}}"']
\end{tikzcd} \tag{5.10}$$

Moreover, we have a canonical identification $\mathcal{A}^{\otimes_k m} \otimes_k \overline{k} \simeq (\mathcal{A} \otimes_k \overline{k})^{\otimes_{\overline{k}} m}$. Therefore, by applying Lemma 5.2.15 below to the dg category $\mathcal{A}^{\otimes_k m}$, we conclude that the homomorphism $K_0(- \otimes_k \overline{k})_{\mathbb{Q}}$ is injective. Consequently, the proof follows now from (5.10). □

Lemma 5.2.15. *Given an algebraic closure $\overline{k}/k$, the induced homomorphism*

$$K_0(-\otimes_k \overline{k})_{\mathbb{Q}} : K_0(\mathcal{A})_{\mathbb{Q}} \to K_0(\mathcal{A}\otimes_k \overline{k})_{\mathbb{Q}}$$

is injective.

Proof. Assume that $\overline{k}/k$ is a finite field extension of degree d. In this case, the restriction along the field extension $k \to \overline{k}$ yields an homomorphism $\mathrm{Res}_{\overline{k}/k} : K_0(\mathcal{A}\otimes_k \overline{k})_{\mathbb{Q}} \to K_0(\mathcal{A})_{\mathbb{Q}}$ such that $\mathrm{Res}_{\overline{k}/k} \circ K_0(-\otimes_k \overline{k})_{\mathbb{Q}}$ is equal to multiplication by d. Therefore, since $K_0(\mathcal{A})_{\mathbb{Q}}$ is a $\mathbb{Q}$-vector space, the induced homomorphism $K_0(-\otimes_k \overline{k})_{\mathbb{Q}}$ is injective. In the case where the field extension $\overline{k}/k$ is infinite, $\overline{k}$ identifies with the colimit of the filtrant diagram $\{k_i\}_{i\in I}$ of all the intermediate field extensions $\overline{k}/k_i/k$ which are finite over k. Since both functors $-\otimes_k \overline{k}$ and $K_0(-)_{\mathbb{Q}}$ preserve filtrant (homotopy) colimits, we have that $K_0(\mathcal{A}\otimes_k \overline{k})_{\mathbb{Q}} \simeq \mathrm{colim}_{i\in I} K_0(\mathcal{A}\otimes_k k_i)_{\mathbb{Q}}$ and injectivity of $K_0(-\otimes_k \overline{k})_{\mathbb{Q}}$ follows now from the finite field extension case. $\square$

5.3 Relating classical and noncommutative settings

In this section we are going to relate the classical equivalence relations on algebraic cycles, see §4.1, with the equivalences relations we have defined in §5.2.1-§5.2.3. This section is important to be able to use tools from the context of dg categories in order to solve problems related to equivalence relations on algebraic cycles.

Given a smooth proper k-scheme X, recall from [31, Corollary 18.3.2] that the following map is invertible

$$\begin{aligned} K_0\big(\mathrm{perf}_{\mathrm{dg}}(X)\big)_{\mathbb{Q}} &\xrightarrow{\simeq} \mathcal{Z}^*(X)_{\mathbb{Q}}/_{\sim_{\mathrm{rat}}} \\ [\mathcal{F}] &\longmapsto \mathrm{ch}(\mathcal{F})\cdot\sqrt{\mathrm{td}_X}, \end{aligned} \tag{5.11}$$

where $\mathrm{ch}(\mathcal{F})$ stands for the Chern character of $\mathcal{F}$ and td_X for the Todd class of X.

Theorem 5.3.1. *The above map (5.11) induces the following isomorphisms:*

$$\begin{aligned} K_0\big(\mathrm{perf}_{\mathrm{dg}}(X)\big)_{\mathbb{Q}}/_{\sim_{\mathrm{alg}}} &\xrightarrow{\simeq} \mathcal{Z}^*(X)_{\mathbb{Q}}/_{\sim_{\mathrm{alg}}} \\ K_0\big(\mathrm{perf}_{\mathrm{dg}}(X)\big)_{\mathbb{Q}}/_{\sim_{\mathrm{nil}}} &\xrightarrow{\simeq} \mathcal{Z}^*(X)_{\mathbb{Q}}/_{\sim_{\mathrm{nil}}} \\ K_0\big(\mathrm{perf}_{\mathrm{dg}}(X)\big)_{\mathbb{Q}}/_{\sim_{\mathrm{num}}} &\xrightarrow{\simeq} \mathcal{Z}^*(X)_{\mathbb{Q}}/_{\sim_{\mathrm{num}}}. \end{aligned} \tag{5.12}$$

Proof. The upper and middle isomorphisms in (5.12) follow from Lemmas 5.3.2 and 5.3.3 below. In order to prove the lower morphism in (5.12) is an isomorphism, note that, given any two perfect complexes $\mathcal{F}, \mathcal{G} \in \mathrm{perf}_{\mathrm{dg}}(X)$, we have the following equalities

$$\chi([\mathcal{F}],[\mathcal{G}]) = K_0\Big(\pi_*\Big(\big[\mathbf{R}\underline{\mathrm{Hom}}(\mathcal{F},\mathcal{G})\big]\Big)\Big)_{\mathbb{Q}} = K_0\Big(\pi_*\big([\mathcal{F}^*\otimes^{\mathbf{L}}_X \mathcal{G}]\big)\Big)_{\mathbb{Q}}, \tag{5.13}$$

where $\pi : X \to \mathrm{Spec}(k)$ stands for the structure map of X and $\mathbf{R}\underline{\mathrm{Hom}}(-,-)$, respectively $(-)^*$, stands for the internal Hom, respectively (contravariant) duality, of the category $\mathrm{perf}_{\mathrm{dg}}(X)$. Hence, we obtain the following commmutative diagram:

$$
\begin{tikzcd}
K_0\big(\mathrm{perf}_{\mathrm{dg}}(X)\big)_{\mathbb{Q}} \times K_0\big(\mathrm{perf}_{\mathrm{dg}}(X)\big)_{\mathbb{Q}} \arrow[rr, "\chi"] \arrow[d, equal] \arrow[drr, phantom, "①"] & & \mathbb{Q} \arrow[d, equal] \\
K_0\big(\mathrm{perf}_{\mathrm{dg}}(X)\big)_{\mathbb{Q}} \times K_0\big(\mathrm{perf}_{\mathrm{dg}}(X)\big)_{\mathbb{Q}} \arrow[r, "K_0\left((-)^*\otimes^{\mathbf{L}}_X -\right)_{\mathbb{Q}}"'] \arrow[d, "K_0\left((-)^*\right)_{\mathbb{Q}}\times id"', "\simeq"] \arrow[dr, phantom, "②"] & K_0\big(\mathrm{perf}_{\mathrm{dg}}(X)\big)_{\mathbb{Q}} \arrow[r, "K_0(\pi_*)_{\mathbb{Q}}"'] \arrow[d, equal] \arrow[dr, phantom, "③"] & \mathbb{Q} \arrow[d, equal] \\
K_0\big(\mathrm{perf}_{\mathrm{dg}}(X)\big)_{\mathbb{Q}} \times K_0\big(\mathrm{perf}_{\mathrm{dg}}(X)\big)_{\mathbb{Q}} \arrow[r, "-\cdot-"] \arrow[d, "\left(\mathrm{ch}(-)\cdot\sqrt{\mathrm{td}_X}\right)\times\left(\mathrm{ch}(-)\cdot\sqrt{\mathrm{td}_X}\right)"', "\simeq"] \arrow[dr, phantom, "④"] & K_0\big(\mathrm{perf}_{\mathrm{dg}}(X)\big)_{\mathbb{Q}} \arrow[r, "K_0(\pi_*)_{\mathbb{Q}}"] \arrow[d, "\simeq"', "\mathrm{ch}(-)\cdot\mathrm{td}_X"] \arrow[dr, phantom, "⑤"] & \mathbb{Q} \arrow[d, equal] \\
\mathcal{Z}^*(X)_{\mathbb{Q}}/_{\sim_{\mathrm{rat}}} \times \mathcal{Z}^*(X)_{\mathbb{Q}}/_{\sim_{\mathrm{rat}}} \arrow[r, "-\cdot-"'] & \mathcal{Z}^*(X)_{\mathbb{Q}}/_{\sim_{\mathrm{rat}}} \arrow[r, "\pi_*"'] & \mathbb{Q}.
\end{tikzcd}
\qquad (5.14)
$$

The square ① is commutative, thanks to (5.13). The commutativity of the squares ② and ③ is obvious; note that $\mathbb{Q} = K_0\big(\mathrm{Spec}(k)\big)_{\mathbb{Q}}$. The square ④ is commutative since $\mathrm{ch}(-)$ is multiplicative. Finally, the square ⑤ is commutative because of the Grothendieck-Riemann-Roch formula; see [36, Theorem 5.26]. Note that the right vertical map of the square ⑤, i.e., $\mathrm{ch}(-) \cdot \mathrm{td}_{\mathrm{Spec}(k)}$, is the identity map. Indeed, since $K_0\big(\mathrm{Spec}(k)\big)_{\mathbb{Q}} = \mathbb{Q}$ is a graded ring concentrated in degree zero, we have $\mathrm{td}_{\mathrm{Spec}(k)} = 1$ and the ring map $\mathrm{ch}(-)$ from $K_0\big(\mathrm{Spec}(k)\big)_{\mathbb{Q}}$ to $\mathcal{Z}^*\big(\mathrm{Spec}(k)\big)_{\mathbb{Q}}/_{\sim_{\mathrm{rat}}} = \mathbb{Q}$ is the identity. We already know that $\mathrm{ch}(-) \cdot \sqrt{\mathrm{td}_X}$ is an isomorphism and that $K_0\big((-)^*\big)_{\mathbb{Q}} \times id$ is an isomorphism because both $(-)^*$ and id are isomorphisms. This explains diagram (5.14).

Finally, note that $\pi_* : \mathcal{Z}^*(X) \to \mathcal{Z}^*\big(\mathrm{Spec}(k)\big)$ identifies with the degree map (4.2). Hence an algebraic cycle $\mu \in \mathcal{Z}^*(X)_{\mathbb{Q}}/_{\sim_{\mathrm{rat}}}$ is called numerically trivial if $\pi_*(\nu \cdot \mu) = 0$ for all $\nu \in \mathcal{Z}^*(X)_{\mathbb{Q}}/_{\sim_{\mathrm{rat}}}$. Therefore, given an element $[\mathcal{F}] \in K_0\big(\mathrm{perf}_{\mathrm{dg}}(X)\big)_{\mathbb{Q}}$, we conclude from the outer commutative square of diagram (5.14) that $[\mathcal{F}]$ is numerically trivial if $\mathrm{ch}(\mathcal{F}) \cdot \sqrt{\mathrm{td}_X}$ is numerically trivial. This concludes the proof of Theorem 5.3.1. □

Lemma 5.3.2. *The Chern character* $\mathrm{ch}(-) : K_0\big(\mathrm{perf}_{\mathrm{dg}}(X)\big)_{\mathbb{Q}} \xrightarrow{\simeq} \mathcal{Z}^*(X)_{\mathbb{Q}}/_{\sim_{\mathrm{rat}}}$ *induces the following isomorphisms:*

$$K_0\big(\mathrm{perf}_{\mathrm{dg}}(X)\big)_{\mathbb{Q}}/_{\sim_{\mathrm{alg}}} \xrightarrow{\simeq} \mathcal{Z}^*(X)_{\mathbb{Q}}/_{\sim_{\mathrm{alg}}} \qquad K_0\big(\mathrm{perf}_{\mathrm{dg}}(X)\big)_{\mathbb{Q}}/_{\sim_{\mathrm{nil}}} \xrightarrow{\simeq} \mathcal{Z}^*(X)_{\mathbb{Q}}/_{\sim_{\mathrm{nil}}}. \qquad (5.15)$$

Proof. From [83, Lemma 4.26] we have that, for T a smooth connected k-scheme, the following dg functor is a Morita equivalence:

$$-\boxtimes - : \mathrm{perf}_{\mathrm{dg}}(X)\otimes \mathrm{perf}_{\mathrm{dg}}(T)\longrightarrow \mathrm{perf}_{\mathrm{dg}}(X\times T).$$

Moreover, by the properties of Chern character (see [31, page 282, (ii)]), one obtains the commutative diagram:

$$\begin{tikzcd}
\oplus_{(T,p,q)}K_0\big(\mathrm{perf}_{\mathrm{dg}}(X)\otimes\mathrm{perf}_{\mathrm{dg}}(T)\big)_{\mathbb{Q}} \arrow[rr, "\oplus\left(K_0(id\otimes p^*)_{\mathbb{Q}}-K_0(id\otimes q^*)_{\mathbb{Q}}\right)"] \arrow[d, "\oplus K_0(-\boxtimes-)_{\mathbb{Q}}"', "\simeq"] & & K_0\big(\mathrm{perf}_{\mathrm{dg}}(X)\big)_{\mathbb{Q}} \arrow[dd, "\simeq"', "\mathrm{ch}(-)"] \\
\oplus_{(T,p,q)}K_0\big(\mathrm{perf}_{\mathrm{dg}}(X\times T)\big)_{\mathbb{Q}} \arrow[d, "\oplus\,\mathrm{ch}(-)"', "\simeq"] & & \\
\oplus_{(T,p,q)}\mathcal{Z}^*(X\times T)_{\mathbb{Q}}/_{\sim_{\mathrm{rat}}} \arrow[rr, "\oplus\left((id\times p)^*-(id\times q)^*\right)"'] & & \mathcal{Z}^*(X)_{\mathbb{Q}}/_{\sim_{\mathrm{rat}}},
\end{tikzcd} \tag{5.16}$$

where p and q stand for k-rational points of a smooth connected k-scheme T. Recall that an algebraic cycle $\mu\in\mathcal{Z}^*(X)_{\mathbb{Q}}/_{\sim_{\mathrm{rat}}}$ is called algebraically trivial if it belongs to the image of $\oplus_{(T,p,q)}\big((id\otimes p)^*-(id\otimes q)^*\big)$, see Definition 4.1.4. Therefore, the left-hand-side isomorphism in (5.15) follows from the above commutative diagram (5.16).

In order to prove the right-hand-side isomorphism in (5.15), note that a repeated use of [83, Lemma 4.26] shows that the following dg functor

$$-\boxtimes-\cdots-\boxtimes- : \mathrm{perf}_{\mathrm{dg}}(X)^{\otimes m}\longrightarrow \mathrm{perf}_{\mathrm{dg}}\left(X^{\otimes m}\right)$$

is a Morita equivalence. Therefore, for any integer $m\geq 1$ we obtain the following commutative diagram:

$$\begin{tikzcd}
\prod_{i=1}^m K_0\big(\mathrm{perf}_{\mathrm{dg}}(X)\big)_{\mathbb{Q}} \arrow[rr, "\left([\mathcal{F}_1],\ldots,[\mathcal{F}_m]\right)\longmapsto[\mathcal{F}_1\otimes\ldots\otimes\mathcal{F}_m]"] \arrow[dd, "\prod\mathrm{ch}(-)"', "\simeq"] & & K_0\big(\mathrm{perf}_{\mathrm{dg}}(X)^{\otimes m}\big)_{\mathbb{Q}} \arrow[d, "\simeq"', "K_0(-\boxtimes-\ldots-\boxtimes-)_{\mathbb{Q}}"] \\
 & & K_0\big(\mathrm{perf}_{\mathrm{dg}}(X^{\times m})\big)_{\mathbb{Q}} \arrow[d, "\simeq"', "\mathrm{ch}(-)"] \\
\prod_{i=1}^m \mathcal{Z}^*(X)_{\mathbb{Q}}/_{\sim_{\mathrm{rat}}} \arrow[rr, "(\mu_1,\ldots,\mu_m)\longmapsto\mu_1\times\ldots\times\mu_m"'] & & \mathcal{Z}^*\left(X^{\times m}\right)/_{\sim_{\mathrm{rat}}}.
\end{tikzcd} \tag{5.17}$$

Recall that an algebraic cycle $\mu\in\mathcal{Z}^*(X)_{\mathbb{Q}}/_{\sim_{\mathrm{rat}}}$ is called nilpotently trivial if there is a positive integer m such that μ^m is equal to 0, see Definition 4.1.5. Therefore, the right-hand-side isomorphism in (5.15) follows from the above commutative diagram (5.17). □

Lemma 5.3.3. *The homomorphism* $- \cdot \sqrt{\mathrm{td}_X} : \mathcal{Z}^*(X)_{\mathbb{Q}}/_{\sim_{\mathrm{rat}}} \xrightarrow{\simeq} \mathcal{Z}^*(X)_{\mathbb{Q}}/_{\sim_{\mathrm{rat}}}$ *induces the following isomorphisms:*

$$\mathcal{Z}^*(X)_{\mathbb{Q}}/_{\sim_{\mathrm{alg}}} \xrightarrow{\simeq} \mathcal{Z}^*(X)_{\mathbb{Q}}/_{\sim_{\mathrm{alg}}} \qquad \mathcal{Z}^*(X)_{\mathbb{Q}}/_{\sim_{\mathrm{nil}}} \xrightarrow{\simeq} \mathcal{Z}^*(X)_{\mathbb{Q}}/_{\sim_{\mathrm{nil}}}. \tag{5.18}$$

Proof. Note that in order to prove the left-hand-side of (5.18) it suffices to show that the following square is commutative:

$$\begin{tikzcd}
\oplus_{(T,p,q)}\mathcal{Z}^*(X\times T)_{\mathbb{Q}}/_{\sim_{\mathrm{rat}}} \arrow[r, "\oplus\left((id\times p)^*-(id\times q)^*\right)"] \arrow[d, "\oplus\left(-\cdot\sqrt{\mathrm{td}_{X\times T}}\right)"', "\simeq"] & \mathcal{Z}^*(X)_{\mathbb{Q}}/_{\sim_{\mathrm{rat}}} \arrow[d, "\simeq"', "-\cdot\sqrt{\mathrm{td}_X}"] \\
\oplus_{(T,p,q)}\mathcal{Z}^*(X\times T)_{\mathbb{Q}}/_{\sim_{\mathrm{rat}}} \arrow[r, "\oplus\left((id\times p)^*-(id\times q)^*\right)"'] & \mathcal{Z}^*(X)_{\mathbb{Q}}/_{\sim_{\mathrm{rat}}},
\end{tikzcd} \tag{5.19}$$

where p and q stand for k-rational points of a smooth connected k-scheme T. Let π_X and π_T be, respectively, the projections from $X \times T$ to X and T. Consider the pull-backs $\pi_X^* : \mathcal{Z}^*(X)_{\mathbb{Q}}/_{\sim_{\mathrm{rat}}} \to \mathcal{Z}^*(X\times T)_{\mathbb{Q}}/_{\sim_{\mathrm{rat}}}$ and $\pi_T^* : \mathcal{Z}^*(T)_{\mathbb{Q}}/_{\sim_{\mathrm{rat}}} \to \mathcal{Z}^*(X\times T)_{\mathbb{Q}}/_{\sim_{\mathrm{rat}}}$.

It is known, by the proof of [36, Lemma 10.6], that $\sqrt{\mathrm{td}_{X\times T}} = \pi_X^*\big(\sqrt{\mathrm{td}_X}\big)\cdot\pi_T^*\big(\sqrt{\mathrm{td}_T}\big)$. Consequently, we conclude that

$$\begin{aligned}
(id\times p)^*(\sqrt{\mathrm{td}_{X\times T}}) &= (id\times p)^*\Big(\pi_X^*\big(\sqrt{\mathrm{td}_X}\big)\cdot\pi_T^*\big(\sqrt{\mathrm{td}_T}\big)\Big)\\
&= (id\times p)^*\Big(\pi_X^*\big(\sqrt{\mathrm{td}_X}\big)\Big)\cdot(id\times p)^*\Big(\pi_T^*\big(\sqrt{\mathrm{td}_T}\big)\Big)\\
&= \big(\pi_X\circ(id\times p)\big)^*\big(\sqrt{\mathrm{td}_X}\big)\cdot\big(\pi_T\circ(id\times p)\big)^*\big(\sqrt{\mathrm{td}_T}\big)\\
&= id^*\big(\sqrt{\mathrm{td}_X}\big)\cdot\big(\pi_T\circ(id\times p)\big)^*\big(\sqrt{\mathrm{td}_T}\big)\\
&= \sqrt{\mathrm{td}_X}\cdot\big(\pi_T\circ(id\times p)\big)^*\big(\sqrt{\mathrm{td}_T}\big).
\end{aligned}$$

Recall from [36, §5.2] that the degree zero term of td_T, in $\mathrm{H}^0(T;\mathbb{Q})$, is 1. Therefore, choose for $\sqrt{\mathrm{td}_T}$ the cohomology class whose degree zero term is 1. Since p^* is a graded ring morphism and $\mathcal{Z}^*\big(\mathrm{Spec}(k)\big)_{\mathbb{Q}}/_{\sim_{\mathrm{rat}}} = \mathbb{Q}$, we have that the following equalities hold for every $\nu \in \mathcal{Z}^*(X\times T)_{\mathbb{Q}}/_{\sim_{\mathrm{rat}}}$.

$$\begin{aligned}
(id\times p)^*\big(\nu\cdot\sqrt{\mathrm{td}_{X\times T}}\big) &= (id\times p)^*(\nu)\cdot(id\times p)^*\big(\sqrt{\mathrm{td}_{X\times T}}\big)\\
&= (id\times p)^*(\nu)\cdot\sqrt{\mathrm{td}_X}\cdot(\pi_T\circ(id\times p))^*\big(\sqrt{\mathrm{td}_T}\big)\\
&= (id\times p)^*(\nu)\cdot\sqrt{\mathrm{td}_X}\cdot(p\circ\pi_{\mathrm{Spec}(k)})^*\big(\sqrt{\mathrm{td}_T}\big)\\
&= (id\times p)^*(\nu)\cdot\sqrt{\mathrm{td}_X}\cdot(\pi_{\mathrm{Spec}(k)}^*\circ p^*)\big(\sqrt{\mathrm{td}_T}\big)\\
&= (id\times p)^*(\nu)\cdot\sqrt{\mathrm{td}_X}\cdot\pi_{\mathrm{Spec}(k)}^*(1)\\
&= (id\times p)^*(\nu)\cdot\sqrt{\mathrm{td}_X}.
\end{aligned}$$

This implies that the above diagram (5.19) is commutative.

Now, in order to prove the right-hand-side of (5.18), note that the homomorphism $\{\mu_i\}_{1\leq i\leq m} \longmapsto \mu_1 \times \ldots \times \mu_m$ can be written as the following composition:

$$\prod_{i=1}^{m} \mathcal{Z}^*(X)_{\mathbb{Q}}/_{\sim_{\mathrm{rat}}} \xrightarrow{\prod \pi_i^*} \prod_{i=1}^{m} \mathcal{Z}^*\big(X^{\times m}\big)_{\mathbb{Q}}/_{\sim_{\mathrm{rat}}} \xrightarrow{\mathrm{mult}} \mathcal{Z}^*\big(X^{\times m}\big)_{\mathbb{Q}}/_{\sim_{\mathrm{rat}}},$$

where π_i stands for the i^{th} projection map from $X^{\times m}$ to X. Consequently, since the pull-back homomorphism π_i^* is multiplicative, if an algebraic cycle $\mu \in \mathcal{Z}^*(X)_{\mathbb{Q}}/_{\sim_{\mathrm{rat}}}$ is nilpotently trivial, then so is $\mu \cdot \sqrt{\mathrm{td}_X}$. Conversely, since the homomorphism π_i^* is multiplicative and $\sqrt{\mathrm{td}_X}$ is invertible, if $\mu \cdot \sqrt{\mathrm{td}_X}$ is nilpotently trivial, then μ is also necessarily nilpotently trivial. $\square$

5.4 Comparing equivalence relations via kernels

In this section, we consider the quotients between the different equivalence relations introduced in §5.2. We start by fixing some useful notations that will be used in the remainder of the paper.

Notation 5.4.1. Let X be a smooth proper k-scheme and $\mathcal{A}$ a smooth proper dg category (over k).

(i) Given equivalence relations $\sim_1$ and $\sim_2$ on a set S, we will write $\sim_1 \geq \sim_2$ if, for all a, $b \in S$, $a \sim_1 b$ implies $a \sim_2 b$.

(ii) Given an equivalence relation $\sim$ on $\mathcal{Z}^*(X)_{\mathbb{Q}}$, respectively on $K_0(\mathcal{A})_{\mathbb{Q}}$, let us write

$$\mathcal{Z}^*_{\sim}(X)_{\mathbb{Q}} := \{\alpha \in \mathcal{Z}^*(X)_{\mathbb{Q}} | \alpha \sim 0\} \text{ respectively } K_{0,\sim}(\mathcal{A})_{\mathbb{Q}} := \{\alpha \in K_0(\mathcal{A})_{\mathbb{Q}} | \alpha \sim 0\}$$

for the $\mathbb{Q}$-subspace of $\sim$-trivial elements.

(iii) Given equivalence relations $\sim_1 \geq \sim_2$ on $\mathcal{Z}^*(X)_{\mathbb{Q}}$, respectively on $K_0(\mathcal{A})_{\mathbb{Q}}$, let us write

$$\mathcal{Z}^*_{\sim_2/\sim_1}(X)_{\mathbb{Q}} := \frac{\mathcal{Z}^*_{\sim_2}(X)_{\mathbb{Q}}}{\mathcal{Z}^*_{\sim_1}(X)_{\mathbb{Q}}} \text{ respectively } K_{0,\sim_2/\sim_1}(\mathcal{A})_{\mathbb{Q}} := \frac{K_{0,\sim_2}(\mathcal{A})_{\mathbb{Q}}}{K_{0,\sim_1}(\mathcal{A})_{\mathbb{Q}}}$$

for the associated quotient.

(iv) Given equivalence relations $\sim_1 \geq \sim_2$ on $\mathcal{Z}^*(X)_{\mathbb{Q}}$, respectively on $K_0(\mathcal{A})_{\mathbb{Q}}$, let us write

$$q^X_{\sim_2/\sim_1} : \mathcal{Z}^*(X)_{\mathbb{Q}}/_{\sim_1} \twoheadrightarrow \mathcal{Z}^*(X)_{\mathbb{Q}}/_{\sim_2} \text{ respectively } q^{\mathcal{A},\mathrm{nc}}_{\sim_2/\sim_1} : K_0(\mathcal{A})_{\mathbb{Q}}/_{\sim_1} \twoheadrightarrow K_0(\mathcal{A})_{\mathbb{Q}}/_{\sim_2}$$

for the quotient map.

(v) Given equivalence relations $\sim_1 \geq \sim_2$ on $K_0\big(\mathrm{perf}_{\mathrm{dg}}(X)\big)_{\mathbb{Q}}$, let

$$q^{X,\mathrm{nc}}_{\sim_2/\sim_1} : K_0\big(\mathrm{perf}_{\mathrm{dg}}(X)\big)_{\mathbb{Q}}/_{\sim_1} \twoheadrightarrow K_0\big(\mathrm{perf}_{\mathrm{dg}}(X)\big)_{\mathbb{Q}}/_{\sim_2}$$

be the quotient map.

Remark 5.4.2. (i) The equivalence relations $\sim_{\mathrm{alg}}$, $\sim_{\mathrm{nil}}$ and $\sim_{\mathrm{num}}$ agree for smooth proper k-schemes X of dim ≤ 2; see [2, §3.2.7] and [31, §19.3.5]. Moreover, $\sim_{\mathrm{rat}}$, $\sim_{\mathrm{alg}}$, $\sim_{\mathrm{nil}}$ and $\sim_{\mathrm{num}}$ agree for 0-dimensional k-schemes.

(ii) In general, we have $\ker\left(q^X_{\sim_2/\sim_1}\right) \neq 0$; consult Remark 5.1.5.

(iii) For every smooth projective complex curve X of positive genus g, we have that $\mathcal{Z}^*_{\sim_{\mathrm{alg}}/\sim_{\mathrm{rat}}}(X)_{\mathbb{Q}} \simeq (\mathbb{Q}/\mathbb{Z})^{\oplus 2g}$. Since X is a curve, we have $\mathcal{Z}^*_{\sim_{\mathrm{alg}}/\sim_{\mathrm{rat}}}(X)_{\mathbb{Q}} = \mathcal{Z}^1_{\sim_{\mathrm{alg}}/\sim_{\mathrm{rat}}}(X)_{\mathbb{Q}}$ which, thanks to [31, p. 19.3.5], is isomorphic to the $\mathbb{Q}$-vector space of torsion points of the Albanese variety $\mathrm{Alb}(X)$. Since X is a curve, we have $\mathrm{Alb}(X) \simeq \mathrm{Jac}(X)$ (the Jacobian variety of X) and it is well-known that the $\mathbb{Q}$-vector space of torsion points of $\mathrm{Jac}(X)$ is isomorphic to $(\mathbb{Q}/\mathbb{Z})^{\oplus 2g}$.

(iv) The Voevodsky nilpotence conjecture asserts that $\ker\left(q^X_{\sim_{\mathrm{num}}/\sim_{\mathrm{nil}}}\right) = 0$.

Remark 5.4.3. Let $\sim_1, \sim_2 \in \{\sim_{\mathrm{rat}}, \sim_{\mathrm{alg}}, \sim_{\mathrm{nil}}, \sim_{\mathrm{num}}\}$.

(i) Given a smooth proper k-scheme X, we have $\mathcal{Z}^*_{\sim_2/\sim_1}(X)_{\mathbb{Q}} \simeq \ker\left(q^X_{\sim_2/\sim_1}\right)$. Indeed, the elements of $\ker\left(q^X_{\sim_2/\sim_1}\right)$ are algebraic cycles up to $\sim_1$-trivial elements that are $\sim_2$-trivial. That is, they are $\sim_2$-trivial elements up to the equivalence relation $\sim_1$. Hence they are the elements of $\mathcal{Z}^*_{\sim_2/\sim_1}(X)_{\mathbb{Q}}$.

(ii) Given a smooth proper dg category $\mathcal{A}$ (over k), we have $K_{0,\sim_2/\sim_1}(\mathcal{A})_{\mathbb{Q}} \simeq \ker\left(q^{\mathcal{A},\mathrm{nc}}_{\sim_2/\sim_1}\right)$. The same reasoning applied in (i) explains why this is true as well.

Corollary 5.4.4 (of Theorem 5.3.1). *Given a smooth proper k-scheme X and $\sim_1, \sim_2 \in \{\sim_{\mathrm{rat}}, \sim_{\mathrm{alg}}, \sim_{\mathrm{nil}}, \sim_{\mathrm{num}}\}$, we have an isomorphism* $\ker\left(q^X_{\sim_2/\sim_1}\right) \simeq \ker\left(q^{X,\mathrm{nc}}_{\sim_2/\sim_1}\right)$. *Equivalently, we have an isomorphism* $\mathcal{Z}^*_{\sim_2/\sim_1}(X)_{\mathbb{Q}} \simeq K_{0,\sim_2/\sim_1}\left(\mathrm{perf}_{\mathrm{dg}}(X)\right)_{\mathbb{Q}}$.

Proof. Thanks to Theorem 5.3.1, we have that the map (5.11) induces an isomorphism $K_0\left(\mathrm{perf}_{\mathrm{dg}}(X)\right)_{\mathbb{Q}}/_{\sim} \xrightarrow{\simeq} \mathcal{Z}^*(X)_{\mathbb{Q}}/_{\sim}$, where $\sim \in \{\sim_{\mathrm{rat}}, \sim_{\mathrm{alg}}, \sim_{\mathrm{nil}}, \sim_{\mathrm{num}}\}$. Consequently, we have the following commutative diagram:

$$\begin{tikzcd}
\ker\left(q^{X,\mathrm{nc}}_{\sim_2/\sim_1}\right) \arrow[r] \arrow[d] & K_0\left(\mathrm{perf}_{\mathrm{dg}}(X)\right)_{\mathbb{Q}}/_{\sim_1} \arrow[r, two heads] \arrow[d, "\simeq"', "(5.11)"] & K_0\left(\mathrm{perf}_{\mathrm{dg}}(X)\right)_{\mathbb{Q}}/_{\sim_2} \arrow[d, "\simeq"', "(5.11)"] \\
\ker\left(q^{X}_{\sim_2/\sim_1}\right) \arrow[r] & \mathcal{Z}^*(X)_{\mathbb{Q}}/_{\sim_1} \arrow[r, two heads] & \mathcal{Z}^*(X)_{\mathbb{Q}}/_{\sim_2}.
\end{tikzcd} \tag{5.20}$$

Since both rows in (5.20) are exact, we hence conclude that the left-hand-side vertical homomorphism in (5.20) is also invertible. □

Proposition 5.4.5. *Let $\mathcal{A}$ be a smooth proper dg category such that $\mathrm{H}^0(\mathcal{A}) = \langle \mathfrak{b}, \mathfrak{c} \rangle$ admits a semi-orthogonal decomposition, see Definition 3.3.2. Let $\mathfrak{b}^{\mathrm{dg}}$ and $\mathfrak{c}^{\mathrm{dg}}$ denote, respectively, the dg* *enhancement of* $\mathfrak{b}$ *and* $\mathfrak{c}$ *induced from* $\mathcal{A}$. *Under these assumptions, the inclusions of* $\mathfrak{b}^{\mathrm{dg}}$ *and* $\mathfrak{c}^{\mathrm{dg}}$ *into* $\mathcal{A}$ *induce an isomorphism*

$$\ker\left(q^{\mathfrak{b}^{\mathrm{dg}},\mathrm{nc}}_{\sim_2/\sim_1}\right) \oplus \ker\left(q^{\mathfrak{c}^{\mathrm{dg}},\mathrm{nc}}_{\sim_2/\sim_1}\right) \xrightarrow{\simeq} \ker\left(q^{\mathcal{A},\mathrm{nc}}_{\sim_2/\sim_1}\right) \text{ with } \sim_1, \sim_2 \in \{\sim_{\mathrm{rat}}, \sim_{\mathrm{alg}}, \sim_{\mathrm{nil}}, \sim_{\mathrm{num}}\}.$$

Equivalently, we have an induced isomorphism $K_{0,\sim_2/\sim_1}\left(\mathfrak{b}^{\mathrm{dg}}\right)_{\mathbb{Q}} \oplus K_{0,\sim_2/\sim_1}\left(\mathfrak{c}^{\mathrm{dg}}\right)_{\mathbb{Q}} \xrightarrow{\simeq} K_{0,\sim_2/\sim_1}(\mathcal{A})_{\mathbb{Q}}$.

Proof. Since the dg category $\mathcal{A}$ is smooth and proper, it follows from the proof of [17, Lemma 2.1] that the dg categories $\mathfrak{b}^{\mathrm{dg}}$ and $\mathfrak{c}^{\mathrm{dg}}$ are also smooth and proper. Moreover, as explained in [80, §3.2], we have the following computations:

$$\begin{aligned} \mathrm{Hom}_{\mathrm{NVoev}(k)_{\mathbb{Q}}}\left(\mathcal{U}(k)_{\mathbb{Q}}, \mathcal{U}(\mathcal{A})_{\mathbb{Q}}\right) &\simeq K_0(\mathcal{A})_{\mathbb{Q}}/_{\sim_{\mathrm{nil}}}, \\ \mathrm{Hom}_{\mathrm{NNum}(k)_{\mathbb{Q}}}\left(\mathcal{U}(k)_{\mathbb{Q}}, \mathcal{U}(\mathcal{A})_{\mathbb{Q}}\right) &\simeq K_0(\mathcal{A})_{\mathbb{Q}}/_{\sim_{\mathrm{num}}}. \end{aligned} \tag{5.21}$$

As $\mathrm{H}^0(\mathcal{A}) = \langle \mathfrak{b}, \mathfrak{c} \rangle$, we deduce from [79, Proposition 2.2 and Theorem 2.9] that the induced morphism $\mathcal{U}\left(\mathfrak{b}^{\mathrm{dg}}\right)_{\mathbb{Q}} \oplus \mathcal{U}\left(\mathfrak{c}^{\mathrm{dg}}\right)_{\mathbb{Q}} \to \mathcal{U}(\mathcal{A})_{\mathbb{Q}}$ is invertible. Therefore, making use of the computations (5.21), we obtain induced isomorphisms:

$$\begin{aligned} K_0\left(\mathfrak{b}^{\mathrm{dg}}\right)_{\mathbb{Q}}/_{\sim_{\mathrm{nil}}} \oplus K_0\left(\mathfrak{c}^{\mathrm{dg}}\right)_{\mathbb{Q}}/_{\sim_{\mathrm{nil}}} &\xrightarrow{\simeq} K_0(\mathcal{A})_{\mathbb{Q}}/_{\sim_{\mathrm{nil}}} \\ K_0\left(\mathfrak{b}^{\mathrm{dg}}\right)_{\mathbb{Q}}/_{\sim_{\mathrm{num}}} \oplus K_0\left(\mathfrak{c}^{\mathrm{dg}}\right)_{\mathbb{Q}}/_{\sim_{\mathrm{num}}} &\xrightarrow{\simeq} K_0(\mathcal{A})_{\mathbb{Q}}/_{\sim_{\mathrm{num}}}. \end{aligned}$$

By computation (4.6), we have an induced isomorphism $K_0\left(\mathfrak{b}^{\mathrm{dg}}\right)_{\mathbb{Q}} \oplus K_0\left(\mathfrak{c}^{\mathrm{dg}}\right)_{\mathbb{Q}} \xrightarrow{\simeq} K_0(\mathcal{A})_{\mathbb{Q}}$. Note that thanks to Lemma 5.2.5, the inclusions of $\mathfrak{b}^{\mathrm{dg}}$ and $\mathfrak{c}^{\mathrm{dg}}$ into $\mathcal{A}$ also induce an isomorphism $K_0\left(\mathfrak{b}^{\mathrm{dg}}\right)_{\mathbb{Q}}/_{\sim_{\mathrm{alg}}} \oplus K_0\left(\mathfrak{c}^{\mathrm{dg}}\right)_{\mathbb{Q}}/_{\sim_{\mathrm{alg}}} \xrightarrow{\simeq} K_0(\mathcal{A})_{\mathbb{Q}}/_{\sim_{\mathrm{alg}}}$. Consequently, we have the following commutative diagram:

$$\begin{tikzcd} \ker\left(q^{\mathfrak{b}^{\mathrm{dg}},\mathrm{nc}}_{\sim_2/\sim_1}\right) \oplus \ker\left(q^{\mathfrak{c}^{\mathrm{dg}},\mathrm{nc}}_{\sim_2/\sim_1}\right) \arrow[r] \arrow[d] & K_0\left(\mathfrak{b}^{\mathrm{dg}}\right)_{\mathbb{Q}}/_{\sim_1} \oplus K_0\left(\mathfrak{c}^{\mathrm{dg}}\right)_{\mathbb{Q}}/_{\sim_1} \arrow[r, two heads] \arrow[d, "\simeq"] & K_0\left(\mathfrak{b}^{\mathrm{dg}}\right)_{\mathbb{Q}}/_{\sim_2} \oplus K_0\left(\mathfrak{c}^{\mathrm{dg}}\right)_{\mathbb{Q}}/_{\sim_2} \arrow[d, "\simeq"] \\ \ker\left(q^{\mathcal{A},\mathrm{nc}}_{\sim_2/\sim_1}\right) \arrow[r] & K_0(\mathcal{A})_{\mathbb{Q}}/_{\sim_1} \arrow[r, two heads] & K_0(\mathcal{A})_{\mathbb{Q}}/_{\sim_2}, \end{tikzcd} \tag{5.22}$$

with $\sim_1, \sim_2 \in \{\sim_{\mathrm{rat}}, \sim_{\mathrm{alg}}, \sim_{\mathrm{nil}}, \sim_{\mathrm{num}}\}$. Since both rows in (5.22) are exact, we conclude that the left-hand-side vertical homomorphism in (5.22) is also invertible. □

Proposition 5.4.6. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be two smooth proper dg categories.*

(i) Given $F : \mathcal{A} \to \mathcal{B}$ *a Morita equivalence, the homomorphism* $K_0(F)_{\mathbb{Q}}/_{\sim}$ *is invertible when* $\sim \in \{\sim_{\mathrm{rat}}, \sim_{\mathrm{alg}}, \sim_{\mathrm{nil}}, \sim_{\mathrm{num}}\}$.

(ii) For $\mathcal{A}$ *Morita equivalent to* $\mathcal{B}$, *we have* $\ker\left(q^{\mathcal{A},\mathrm{nc}}_{\sim_2/\sim_1}\right) \simeq \ker\left(q^{\mathcal{B},\mathrm{nc}}_{\sim_2/\sim_1}\right)$, *or, equivalently,* $K_{0,\sim_2/\sim_1}(\mathcal{A})_{\mathbb{Q}} \simeq K_{0,\sim_2/\sim_1}(\mathcal{B})_{\mathbb{Q}}$, *when* $\sim_1, \sim_2 \in \{\sim_{\mathrm{rat}}, \sim_{\mathrm{alg}}, \sim_{\mathrm{nil}}, \sim_{\mathrm{num}}\}$.

Proof. We start by proving item (i). Since $K_0(-)_{\mathbb{Q}}$ is an additive invariant, we have that $K_0(F)_{\mathbb{Q}}$ is an isomorphism. Thanks to Lemma 5.2.4, $K_0(F)_{\mathbb{Q}}/_{\sim_{\mathrm{alg}}}$ is an isomorphism. Following [79, Proposition 2.2 and Theorem 2.9], we have that $\mathcal{U}(-)$ sends Morita equivalences to isomorphisms. This together with the following computations (see [80, §3.2])

$$\mathrm{Hom}_{\mathrm{NVoev}(k)_{\mathbb{Q}}}\left(\mathcal{U}(k)_{\mathbb{Q}}, \mathcal{U}(-)_{\mathbb{Q}}\right) \simeq K_0(-)_{\mathbb{Q}}/_{\sim_{\mathrm{nil}}},$$
$$\mathrm{Hom}_{\mathrm{NNum}(k)_{\mathbb{Q}}}\left(\mathcal{U}(k)_{\mathbb{Q}}, \mathcal{U}(-)_{\mathbb{Q}}\right) \simeq K_0(-)_{\mathbb{Q}}/_{\sim_{\mathrm{num}}},$$

is enough to conclude that both $K_0(F)_{\mathbb{Q}}/_{\sim_{\mathrm{nil}}}$, and $K_0(F)_{\mathbb{Q}}/_{\sim_{\mathrm{num}}}$ are isomorphisms.

Now, note that item (ii) follows from item (i). Indeed, from (i) we obtain the following commutative diagram:

$$\begin{tikzcd}
\ker\left(q^{\mathcal{A},\mathrm{nc}}_{\sim_2/\sim_1}\right) \arrow[r] \arrow[d] & K_0(\mathcal{A})_{\mathbb{Q}}/_{\sim_1} \arrow[r, two heads] \arrow[d, "\simeq"] & K_0(\mathcal{A})_{\mathbb{Q}}/_{\sim_2} \arrow[d, "\simeq"] \\
\ker\left(q^{\mathcal{B},\mathrm{nc}}_{\sim_2/\sim_1}\right) \arrow[r] & K_0(\mathcal{B})_{\mathbb{Q}}/_{\sim_1} \arrow[r, two heads] & K_0(\mathcal{B})_{\mathbb{Q}}/_{\sim_2}.
\end{tikzcd} \tag{5.23}$$

Since both rows in (5.23) are exact, the left-hand-side vertical homomorphism in (5.23) is also invertible. ☐

Proposition 5.4.7. *Let X be a smooth proper k-scheme and $\mathbb{B}_0$ an Azumaya algebra over X.*

(i) The canonical dg functor $i_{X,\mathbb{B}_0} : \mathrm{perf}_{\mathrm{dg}}(X) \to \mathrm{perf}_{\mathrm{dg}}(X,\mathbb{B}_0)$ induces an isomorphism

$$K_0(i_{X,\mathbb{B}_0})_{\mathbb{Q}}/_{\sim} \colon K_0\big(\mathrm{perf}_{\mathrm{dg}}(X)\big)_{\mathbb{Q}}/_{\sim} \longrightarrow K_0\big(\mathrm{perf}_{\mathrm{dg}}(X,\mathbb{B}_0)\big)_{\mathbb{Q}}/_{\sim}$$

with $\sim \in \{\sim_{\mathrm{rat}}, \sim_{\mathrm{alg}}, \sim_{\mathrm{nil}}, \sim_{\mathrm{num}}\}$.

(ii) For $\sim_1, \sim_2 \in \{\sim_{\mathrm{rat}}, \sim_{\mathrm{alg}}, \sim_{\mathrm{nil}}, \sim_{\mathrm{num}}\}$, we have $\ker\left(q^{X,\mathrm{nc}}_{\sim_2/\sim_1}\right) \simeq \ker\left(q^{\mathrm{perf}_{\mathrm{dg}}(X,\mathbb{B}_0),\mathrm{nc}}_{\sim_2/\sim_1}\right)$, or, equivalently, an isomorphism

$$K_{0,\sim_2/\sim_1}\big(\mathrm{perf}_{\mathrm{dg}}(X)\big)_{\mathbb{Q}} \simeq K_{0,\sim_2/\sim_1}\big(\mathrm{perf}_{\mathrm{dg}}(X,\mathbb{B}_0)\big)_{\mathbb{Q}}.$$

Proof. To prove item (i), we start by noting that, following [82, Corollary 3.1], we have $K_0\big(\mathrm{perf}_{\mathrm{dg}}(X)\big)_{\mathbb{Q}} \simeq K_0\big(\mathrm{perf}_{\mathrm{dg}}(X,\mathbb{B}_0)\big)_{\mathbb{Q}}$. Moreover, thanks to Lemma 5.2.6, we have that $K_0\big(\mathrm{perf}_{\mathrm{dg}}(X)\big)_{\mathbb{Q}}/_{\sim_{\mathrm{alg}}} \simeq K_0\big(\mathrm{perf}_{\mathrm{dg}}(X,\mathbb{B}_0)\big)_{\mathbb{Q}}/_{\sim_{\mathrm{alg}}}$. Following [82, Propositions 8.3 and 8.7], the canonical dg functor $i_{X,\mathbb{B}_0}$ induces an isomorphism $\mathcal{U}\big(\mathrm{perf}_{\mathrm{dg}}(X)\big)_{\mathbb{Q}} \simeq \mathcal{U}\big(\mathrm{perf}_{\mathrm{dg}}(X,\mathbb{B}_0)\big)_{\mathbb{Q}}$. So, the same reasoning in Proposition 5.4.6 allows us to conclude that

$$K_0\big(\mathrm{perf}_{\mathrm{dg}}(X)\big)_{\mathbb{Q}}/_{\sim_{\mathrm{nil}}} \simeq K_0\big(\mathrm{perf}_{\mathrm{dg}}(X,\mathbb{B}_0)\big)_{\mathbb{Q}}/_{\sim_{\mathrm{nil}}},$$
$$K_0\big(\mathrm{perf}_{\mathrm{dg}}(X)\big)_{\mathbb{Q}}/_{\sim_{\mathrm{num}}} \simeq K_0\big(\mathrm{perf}_{\mathrm{dg}}(X,\mathbb{B}_0)\big)_{\mathbb{Q}}/_{\sim_{\mathrm{num}}}.$$

This finishes the proof of item (i).

To prove item (ii), note that item (i) yields the following commutative diagram:

$$\begin{tikzcd}
\ker\left(q^{X,\mathrm{nc}}_{\sim_2/\sim_1}\right) \arrow[r] \arrow[d] & K_0\big(\mathrm{perf}_{\mathrm{dg}}(X)\big)_{\mathbb{Q}}/_{\sim_1} \arrow[r, two heads] \arrow[d, "\simeq"] & K_0\big(\mathrm{perf}_{\mathrm{dg}}(X)\big)_{\mathbb{Q}}/_{\sim_2} \arrow[d, "\simeq"] \\
\ker\left(q^{\mathrm{perf}_{\mathrm{dg}}(X,\mathbb{B}_0),\mathrm{nc}}_{\sim_2/\sim_1}\right) \arrow[r] & K_0\big(\mathrm{perf}_{\mathrm{dg}}(X,\mathbb{B}_0)\big)_{\mathbb{Q}}/_{\sim_1} \arrow[r, two heads] & K_0\big(\mathrm{perf}_{\mathrm{dg}}(X,\mathbb{B}_0)\big)_{\mathbb{Q}}/_{\sim_2}.
\end{tikzcd} \tag{5.24}$$

Since both rows in (5.24) are exact, the left-hand-side vertical homomorphism in (5.24) is also invertible. □

5.5 Applications

In this section, making use of the mathematical tools developed in §5.2-§5.4, we prove the following Theorem.

Theorem 5.5.1 (Refinement)**.** *The algebraic and numerical equivalence relations agree for certain quadric fibrations, intersections of quadrics, linear sections of Grassmannians, linear sections of determinantal varieties, Moishezon manifolds, cubic fourfolds and Gushel-Mukai fourfolds. In addition, they also agree for certain Küchle fourfolds, families of Sextic del Pezzo surfaces and families of Fano fourfolds of K3 type.*

We will break the proof of this theorem into different cases and that is done throughout subsections §5.5.1-§5.5.11, though some sections are only instrumental. The proofs are, in some cases, adaptations of the ones given in [17] and [73].

5.5.1 Full exceptional collection

Corollary 5.5.2 (of Proposition 5.4.5)**.** *Let X be a smooth proper k-scheme such that* $\mathrm{perf}(X)$ *admits a full exceptional collection, cf. Remark 3.3.4 above. The equivalence relations $\sim_{\mathrm{alg}}$ and $\sim_{\mathrm{num}}$ agree on $\mathcal{Z}^*(X)_{\mathbb{Q}}$.*

Proof. By assumption, let $\mathrm{perf}(X) = \langle \mathrm{E}_1, \dots, \mathrm{E}_n \rangle$ be a full exceptional collection of $\mathrm{perf}(X)$. By repeated use of Proposition 5.4.5 we have:

$$K_{0,\sim_{\mathrm{num}}/\sim_{\mathrm{alg}}}\big(\mathrm{perf}_{\mathrm{dg}}(X)\big)_{\mathbb{Q}} \simeq K_{0,\sim_{\mathrm{num}}/\sim_{\mathrm{alg}}}\big({\mathrm{E}_1}^{\mathrm{dg}}\big)_{\mathbb{Q}} \oplus \cdots \oplus K_{0,\sim_{\mathrm{num}}/\sim_{\mathrm{alg}}}\big({\mathrm{E}_\mathrm{n}}^{\mathrm{dg}}\big)_{\mathbb{Q}}.$$

Thanks to Remark 5.4.2(i) and Corollary 5.4.4, we have that $K_{0,\sim_{\mathrm{num}}/\sim_{\mathrm{alg}}}\big({\mathrm{E}_\mathrm{i}}^{\mathrm{dg}}\big)_{\mathbb{Q}}$ is the trivial group for all $i \in \{1, \dots, n\}$. Consequently, $K_{0,\sim_{\mathrm{num}}/\sim_{\mathrm{alg}}}\big(\mathrm{perf}_{\mathrm{dg}}(X)\big)_{\mathbb{Q}}$ is trivial and Corollary 5.4.4 implies that $\mathcal{Z}^*_{\sim_{\mathrm{num}}/\sim_{\mathrm{alg}}}(X)_{\mathbb{Q}}$ is trivial. This finishes the proof. □

5.5.2 Severi-Brauer varieties

Let us recall the definitions of central simple algebra and of a Severi-Brauer variety.

Definition 5.5.3. *A finite dimensional algebra A over a field k is called a* a central simple algebra over k *if it is simple as a ring and if has centre k.*

Remark 5.5.4. Note that an central simple algebra over k is a particular case of an Azumaya algebra over k.

Definition 5.5.5. *A k-variety X is called a* Severi-Brauer variety *if there is a field extension L of k such that $X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(L) \simeq \mathbb{P}^n_L$.*

Definition 5.5.6. *Let A be a central simple algebra over k of dimension n^2. Denote by* $\mathrm{SB}(A)$ *the collection of right ideals of A that are n dimensional over k. This collection is called the* Severi-Brauer variety of *A.*

Remark 5.5.7. Note that one can show that the Severi-Brauer variety of a central simple algebra defined in Definition 5.5.6 is indeed a Severi-Brauer variety. Moreover, given a Severi-Brauer variety X there is a unique central simple algebra A such that $X = \mathrm{SB}(A)$.

Theorem 5.5.8. *Let X be a Severi-Brauer variety. The equivalence relations $\sim_{\mathrm{alg}}$ and $\sim_{\mathrm{num}}$ agree on $\mathcal{Z}^*(X)_{\mathbb{Q}}$.*

Proof. Let A be the unique central simple k-algebra such that $X = \mathrm{SB}(A)$. Following [15, Theorem 4.1], we have the semi-orthogonal decomposition:

$$\mathrm{perf}(X) = \left\langle \mathrm{perf}(k), \mathrm{perf}(A), \dots, \mathrm{perf}\left(A^{\otimes n}\right) \right\rangle,$$

where n stands for the dimension of X. Thanks to Proposition 5.4.5 we have

$$\begin{aligned} K_{0,\sim_2/\sim_1}\big(\mathrm{perf}_{\mathrm{dg}}(X)\big)_{\mathbb{Q}} \simeq{}& K_{0,\sim_2/\sim_1}\big(\mathrm{perf}_{\mathrm{dg}}(k)\big)_{\mathbb{Q}} \oplus \\ &\oplus K_{0,\sim_2/\sim_1}\big(\mathrm{perf}_{\mathrm{dg}}(A)\big)_{\mathbb{Q}} \oplus \dots \oplus K_{0,\sim_2/\sim_1}\big(\mathrm{perf}_{\mathrm{dg}}(A^{\otimes n})\big)_{\mathbb{Q}}. \end{aligned} \tag{5.25}$$

Since the tensor product of central simple k-algebras is still a central simple k-algebra, and thanks to Remark 5.5.4, we can rewrite (5.25) as follows:

$$\begin{aligned} K_{0,\sim_2/\sim_1}\big(\mathrm{perf}_{\mathrm{dg}}(X)\big)_{\mathbb{Q}} \simeq{}& K_{0,\sim_2/\sim_1}\big(\mathrm{perf}_{\mathrm{dg}}(k)\big)_{\mathbb{Q}} \oplus \\ &\oplus K_{0,\sim_2/\sim_1}\big(\mathrm{perf}_{\mathrm{dg}}(k, \mathbb{B}_1)\big)_{\mathbb{Q}} \oplus \dots \oplus K_{0,\sim_2/\sim_1}\big(\mathrm{perf}_{\mathrm{dg}}(k, \mathbb{B}_n)\big)_{\mathbb{Q}}, \end{aligned}$$

where $\mathbb{B}_1, ..., \mathbb{B}_n$ are Azumaya algebras over k. Consequently, applying Proposition 5.4.7(ii) one obtains $K_{0,\sim_2/\sim_1}\big(\mathrm{perf}_{\mathrm{dg}}(X)\big)_{\mathbb{Q}} \simeq K_{0,\sim_2/\sim_1}\big(\mathrm{perf}_{\mathrm{dg}}(k)\big)_{\mathbb{Q}}^{\oplus m}$, for a certain m. Remark 5.4.2(i) implies that $K_{0,\sim_{\mathrm{num}}/\sim_{\mathrm{alg}}}\big(\mathrm{perf}_{\mathrm{dg}}(X)\big)_{\mathbb{Q}}$ is trivial. Therefore, from Corollary 5.4.4 we deduce that $\sim_{\mathrm{alg}}$ and $\sim_{\mathrm{num}}$ agree on $\mathcal{Z}^*(X)_{\mathbb{Q}}$. □

5.5.3 Quadric fibrations

Take S a smooth projective k-scheme and $q : Q \to S$ a flat quadric fibration of relative dimension n with Q smooth. Let C_0 be the sheaf of even parts of the Clifford algebra associated to q; see [6, §1] and [50, §3]. Recall from [50, §3.5-§3.6] that when the discriminant divisor of q is smooth and n is even (respectively, odd) we have a discriminant double cover $\widetilde{S} \to S$ (respectively, a square root stack $\widehat{S}$) equipped with an Azumaya algebra $\mathbb{B}_0$.

Theorem 5.5.9. *Under the above assumptions, with* $\sim_1, \sim_2 \in \{\sim_{\mathrm{rat}}, \sim_{\mathrm{alg}}, \sim_{\mathrm{nil}}, \sim_{\mathrm{num}}\}$, *the following holds:*

(i) We have

$$\mathcal{Z}^*_{\sim_2/\sim_1}(Q)_{\mathbb{Q}} \simeq K_{0,\sim_2/\sim_1}\big(\mathrm{perf}_{\mathrm{dg}}(S, C_0)\big)_{\mathbb{Q}} \oplus \mathcal{Z}^*_{\sim_2/\sim_1}(S)^{\oplus n}_{\mathbb{Q}}.$$

(ii) If the discriminant divisor of q is smooth and n is even, then

$$\mathcal{Z}^*_{\sim_2/\sim_1}(Q)_{\mathbb{Q}} \simeq \mathcal{Z}^*_{\sim_2/\sim_1}\big(\widetilde{S}\big)_{\mathbb{Q}} \oplus \mathcal{Z}^*_{\sim_2/\sim_1}(S)^{\oplus n}_{\mathbb{Q}}.$$

(iii) If the discriminant divisor of q is smooth and n is odd, then

$$\mathcal{Z}^*_{\sim_2/\sim_1}(Q)_{\mathbb{Q}} \simeq K_{0,\sim_2/\sim_1}\Big(\mathrm{perf}_{\mathrm{dg}}\big(\widehat{S}, \mathbb{B}_0\big)\Big)_{\mathbb{Q}} \oplus \mathcal{Z}^*_{\sim_2/\sim_1}(S)^{\oplus n}_{\mathbb{Q}}.$$

Proof. We adapt the reasoning of the proof of [17, Theorem 1.2]. Indeed, from [50, Theorem 4.2], $\mathrm{perf}(Q) = \big\langle \mathrm{perf}(S, C_0), \mathrm{perf}(S)_1, \ldots, \mathrm{perf}(S)_n \big\rangle$ is a semi-orthogonal decomposition, where $\mathrm{perf}(S)_i = q^* \mathrm{perf}(S) \otimes \mathcal{O}_{Q/S}(i)$. Note that $\mathrm{perf}(S)_i \simeq \mathrm{perf}(S)$ for all i. Thanks to Proposition 5.4.5 we have

$$K_{0,\sim_2/\sim_1}\big(\mathrm{perf}_{\mathrm{dg}}(Q)\big)_{\mathbb{Q}} \simeq K_{0,\sim_2/\sim_1}\big(\mathrm{perf}^{\mathrm{dg}}(S, C_0)\big)_{\mathbb{Q}} \oplus \Big(K_{0,\sim_2/\sim_1}\big(\mathrm{perf}_{\mathrm{dg}}(S)\big)\Big)^{\oplus n}.$$

where $\mathrm{perf}^{\mathrm{dg}}(S, C_0)$ is the dg enhancement of $\mathrm{perf}(S, C_0)$ induced from $\mathrm{perf}_{\mathrm{dg}}(Q)$. This category is smooth and proper, cf. [17, Lemma 2.1]. Note that thanks to [50, Proposition 4.9], the inclusion $\mathrm{perf}(S, C_0) \hookrightarrow \mathrm{perf}(Q)$ is of Fourier-Mukai type; consult [36, §5]. Consequently, its kernel in $\mathrm{perf}(S \times Q, C_0^{\mathrm{op}} \boxtimes \mathcal{O}_X)$ induces a Fourier-Mukai Morita equivalence $\mathrm{perf}_{\mathrm{dg}}(S, C_0) \to \mathrm{perf}^{\mathrm{dg}}(S, C_0)$, where $\mathrm{perf}_{\mathrm{dg}}(S, C_0)$ is the dg enhancement of $\mathrm{perf}(S, C_0)$ induced from the inclusion $\mathrm{perf}(S, C_0) \hookrightarrow \mathrm{perf}(Q)$. So, thanks to Proposition 5.4.6 we have

$$K_{0,\sim_2/\sim_1}\big(\mathrm{perf}_{\mathrm{dg}}(Q)\big)_{\mathbb{Q}} \simeq K_{0,\sim_2/\sim_1}\big(\mathrm{perf}_{\mathrm{dg}}(S, C_0)\big)_{\mathbb{Q}} \oplus \Big(K_{0,\sim_2/\sim_1}\big(\mathrm{perf}_{\mathrm{dg}}(S)\big)\Big)^{\oplus n}.$$

Using Corollary 5.4.4 we conclude that

$$\mathcal{Z}^*_{\sim_2/\sim_1}(Q)_{\mathbb{Q}} \simeq K_{0,\sim_2/\sim_1}\big(\mathrm{perf}_{\mathrm{dg}}(S, C_0)\big)_{\mathbb{Q}} \oplus \mathcal{Z}^*_{\sim_2/\sim_1}(S)^{\oplus n}_{\mathbb{Q}}.$$

We now prove item (ii). Note that from [50, Propositionn 3.13] we have that $\mathrm{perf}(S, C_0)$ and $\mathrm{perf}(\widetilde{S}, \mathbb{B}_0)$ are Fourier-Mukai equivalent. This, together with item (i), implies $\mathcal{Z}^*_{\sim_2/\sim_1}(Q)_\mathbb{Q} \simeq K_{0,\sim_2/\sim_1}\big(\mathrm{perf}_{\mathrm{dg}}(\widetilde{S}, \mathbb{B}_0)\big)_\mathbb{Q} \oplus \mathcal{Z}^*_{\sim_2/\sim_1}(S)^{\oplus n}_\mathbb{Q}$. Using Proposition 5.4.7 we obtain

$$\mathcal{Z}^*_{\sim_2/\sim_1}(Q)_\mathbb{Q} \simeq K_{0,\sim_2/\sim_1}\big(\mathrm{perf}_{\mathrm{dg}}(S)\big)_\mathbb{Q} \oplus \mathcal{Z}^*_{\sim_2/\sim_1}(S)^{\oplus n}_\mathbb{Q}.$$

To prove item (iii), note that, thanks to [50, Propositionn 3.15], $\mathrm{perf}(S, C_0)$ and $\mathrm{perf}(\widehat{S}, \mathbb{B}_0)$ are Fourier-Mukai equivalent. So, from item (i) we obtain

$$\mathcal{Z}^*_{\sim_2/\sim_1}(Q)_\mathbb{Q} \simeq K_{0,\sim_2/\sim_1}\big(\mathrm{perf}_{\mathrm{dg}}(\widehat{S}, \mathbb{B}_0)\big)_\mathbb{Q} \oplus \mathcal{Z}^*_{\sim_2/\sim_1}(S)^{\oplus n}_\mathbb{Q}.$$

□

Corollary 5.5.10. *When* $\dim(S) \leq 2$, *the following holds:*

(i) If the discriminant divisor of q *is smooth and* n *is even, then the equivalence relations* $\sim_{\mathrm{alg}}$ *and* $\sim_{\mathrm{num}}$ *on* $\mathcal{Z}^*(Q)_\mathbb{Q}$ *agree.*

(ii) If the discriminant divisor of q *is smooth and* n *is odd, then*

$$\mathcal{Z}^*_{\sim_{\mathrm{num}}/\sim_{\mathrm{alg}}}(Q)_\mathbb{Q} \simeq K_{0,\sim_{\mathrm{num}}/\sim_{\mathrm{alg}}}\Big(\mathrm{perf}_{\mathrm{dg}}\,(\widehat{S}, \mathbb{B}_0)\Big)_\mathbb{Q}.$$

Proof. Since $\dim(\widetilde{S}) = \dim(S)$, the proof follows from Remark 5.4.2(i). □

5.5.4 Intersection of quadrics

Let X be a smooth complete intersection of r quadric hypersurfaces in $\mathbb{P}^m$. The linear span of these r quadrics gives rise to a hypersurface $Q \subseteq \mathbb{P}^{r-1} \times \mathbb{P}^m$, and the projection into the first factor to a flat quadric fibration $q : Q \to \mathbb{P}^{r-1}$ of relative dimension $m-1$.

Theorem 5.5.11. *Under the above assumptions, with* $\sim_1, \sim_2 \in \{\sim_{\mathrm{rat}}, \sim_{\mathrm{alg}}, \sim_{\mathrm{nil}}, \sim_{\mathrm{num}}\}$, *the following holds:*

(i) We have

$$\mathcal{Z}^*_{\sim_2/\sim_1}(X)_\mathbb{Q} \simeq K_{0,\sim_2/\sim_1}\big(\mathrm{perf}_{\mathrm{dg}}(\mathbb{P}^{r-1}, C_0)\big)_\mathbb{Q}.$$

(ii) If the discriminant divisor of q *is smooth and* m *is odd, then*

$$\mathcal{Z}^*_{\sim_2/\sim_1}(X)_\mathbb{Q} \simeq \mathcal{Z}^*_{\sim_2/\sim_1}(\widetilde{\mathbb{P}^{r-1}})_\mathbb{Q}.$$

(iii) If the discriminant divisor of q *is smooth and* m *is even, then*

$$\mathcal{Z}^*_{\sim_2/\sim_1}(X)_\mathbb{Q} \simeq K_{0,\sim_2/\sim_1}\Big(\mathrm{perf}_{\mathrm{dg}}\,(\widehat{\mathbb{P}^{r-1}}, \mathbb{B}_0)\Big)_\mathbb{Q}.$$

Proof. The proof is similar to the one given in [17, §7] with the some modifications. Thanks to [50, Theorem 5.5], depending on the value of $m-2r+1$ there are three different cases.

If $m-2r+1=0$, then $\mathrm{perf}(X)$ and $\mathrm{perf}(\mathbb{P}^{r-1}, C_0)$ are Fourier-Mukai equivalent. Consequently, we obtain $K_{0,\sim_2/\sim_1}\big(\mathrm{perf}_{\mathrm{dg}}(X)\big)_{\mathbb{Q}} \simeq K_{0,\sim_2/\sim_1}\big(\mathrm{perf}_{\mathrm{dg}}(\mathbb{P}^{r-1}, C_0)\big)_{\mathbb{Q}}$ and an application of Corollary 5.4.4 gives us the statement of item (i).

If $m-2r+1>0$, then one has this semi-orthogonal decomposition

$$\mathrm{perf}(X) = \big\langle \mathrm{perf}(\mathbb{P}^{r-1}, C_0), \mathcal{O}_X(1), \ldots, \mathcal{O}_X(m-2r+1)\big\rangle. \tag{5.26}$$

Consequently, Proposition 5.4.5 implies

$$K_{0,\sim_2/\sim_1}\big(\mathrm{perf}_{\mathrm{dg}}(X)\big)_{\mathbb{Q}} \simeq K_{0,\sim_2/\sim_1}\big(\mathrm{perf}_{\mathrm{dg}}(\mathbb{P}^{r-1}, C_0)\big)_{\mathbb{Q}} \oplus K_{0,\sim_2/\sim_1}\Big(\mathrm{perf}_{\mathrm{dg}}\left(\mathcal{O}_X(1)\right)\Big)_{\mathbb{Q}} \oplus \cdots \oplus$$
$$\oplus K_{0,\sim_2/\sim_1}\Big(\mathrm{perf}_{\mathrm{dg}}\left(\mathcal{O}_X(m-2r+1)\right)\Big)_{\mathbb{Q}}.$$

Thanks to Remark 5.4.2(i) and Corollary 5.4.4, the previous direct sum decomposition becomes

$$K_{0,\sim_2/\sim_1}\big(\mathrm{perf}_{\mathrm{dg}}(X)\big)_{\mathbb{Q}} \simeq K_{0,\sim_2/\sim_1}\big(\mathrm{perf}_{\mathrm{dg}}(\mathbb{P}^{r-1}, C_0)\big)_{\mathbb{Q}}.$$

Again, using Corollary 5.4.4 concludes the proof of item (i) for this case.

If $m-2r+1<0$, one has the following semi-orthogonal decomposition

$$\mathrm{perf}(\mathbb{P}^{r-1}, C_0) = \big\langle \mathrm{E}_{m-2r+1}, \ldots, \mathrm{E}_1, \mathrm{perf}(X)\big\rangle. \tag{5.27}$$

where the E_i are exceptional objects. Note that the semi-orthogonal decomposition (5.27) is a dual semi-orthogonal decomposition of (5.26). The proof now follows the exact same reasoning of the previous case.

The proofs of items (ii)-(iii) follow a similar reasoning to the proofs of Theorem 5.5.9(ii)-(iii). Indeed, from [50, Propositionn 3.13] we have that $\mathrm{perf}(\mathbb{P}^{r-1}, C_0)$ and $\mathrm{perf}(\widetilde{\mathbb{P}^{r-1}}, \mathbb{B}_0)$ are Fourier-Mukai equivalent, where $\mathbb{B}_0$ is an Azumaya algebra over $\widetilde{\mathbb{P}^{r-1}}$. So, using item (i), we obtain $\mathcal{Z}^*_{\sim_2/\sim_1}(X)_{\mathbb{Q}} \simeq K_{0,\sim_2/\sim_1}\big(\mathrm{perf}_{\mathrm{dg}}(\widetilde{\mathbb{P}^{r-1}}, \mathbb{B}_0)\big)_{\mathbb{Q}}$ and Proposition 5.4.7 implies item (ii).

To prove item (iii) note that from [50, Proposition 3.15], $\mathrm{perf}(\mathbb{P}^{r-1}, C_0)$ and $\mathrm{perf}(\widehat{\mathbb{P}^{r-1}}, \mathbb{B}_0)$ are Fourier-Mukai equivalent, with $\mathbb{B}_0$ and Azumaya algebra over $\widehat{\mathbb{P}^{r-1}}$. Hence, item (i) implies $\mathcal{Z}^*_{\sim_2/\sim_1}(X)_{\mathbb{Q}} \simeq K_{0,\sim_2/\sim_1}\big(\mathrm{perf}_{\mathrm{dg}}(\widehat{\mathbb{P}^{r-1}}, \mathbb{B}_0)\big)_{\mathbb{Q}}$ and this finishes the proof. □

Corollary 5.5.12. *When $r \leq 3$, the following holds:*

(i) If the discriminant divisor of q is smooth and m is odd, then the equivalence relations $\sim_{\mathrm{alg}}$ and $\sim_{\mathrm{num}}$ on $\mathcal{Z}^*(X)_{\mathbb{Q}}$ agree.

(ii) If the discriminant divisor of q is smooth, m is even and k is algebraically closed, then the equivalence relations $\sim_{\mathrm{alg}}$ and $\sim_{\mathrm{num}}$ on $\mathcal{Z}^*(X)_{\mathbb{Q}}$ agree.

Proof. To prove item (i) it is enough to consider Theorem 5.5.11(ii) and Remark 5.4.2(i). To prove item (ii) consider Theorem 5.5.11(iii) and Remark 5.4.2(i). □

5.5.5 Moishezon manifolds

A *Moishezon manifold* X is a compact complex manifold such that the field of meromorphic functions on each component of X has transcendence degree equal to the dimension of the component. As proved in [71], X is a smooth projective $\mathbb{C}$-scheme if and only if it admits a Kähler metric. In the remaining cases, it is shown in [4] that X is a proper algebraic space over $\mathbb{C}$. Let $Y \to \mathbb{P}^2$ be one of the non-rational conic bundles described by Artin and Mumford in [5], and $X \to Y$ a small resolution. In this case, X is a smooth (not necessarily projective) Moishezon manifold.

Theorem 5.5.13. *The equivalence relations $\sim_{\mathrm{alg}}$ and $\sim_{\mathrm{num}}$ on $K_0\big(\mathrm{perf}_{\mathrm{dg}}(X)\big)_{\mathbb{Q}}$ agree.*

Proof. The proof of [17, Theorem 1.14] can be adapted as follows: replace the reference to the proof of [17, Thereom 1.2(i)] by the proof of Theorem 5.5.9(i), consider Remark 5.4.2(i), and use Proposition 5.4.5 instead of [17, Theorem 1.2]. □

Remark 5.5.14. We cannot apply Corollary 5.4.4 in the previous proof because X is not a proper k-scheme.

5.5.6 Cubic fourfolds and Gushel-Mukai fourfolds

We start by recalling the definitions of Cubic fourfold and of Gushel-Mukai n-fold, we took these definitions from [73, §2].

Definition 5.5.15. *A* cubic fourfold *is a smooth complex hypersurface of degree 3 in $\mathbb{P}^5$.*

To define a Gushel-Mukai n-fold start by considering a k-vector space V of dimension 5. Thanks to the Plücker embedding, we have that $\mathrm{Cone}\big(\mathrm{Gr}(2,V)\big) \subseteq \mathbb{P}\big(k \oplus (\wedge^2 V)\big)$. Let us denote by W a linear subspace of dimension $n+5$ of $k \oplus (\wedge^2 V)$, where $2 \leq n \leq 6$.

Definition 5.5.16. *Let V and W be as above. A* Gushel-Mukai n-fold *X is a smooth and transverse intersection of the form $X = \mathrm{Cone}\big(\mathrm{Gr}(2,V)\big) \cap Q$, where Q is a quadric hypersurface in $\mathbb{P}(W)$.*

Moreover, X is called ordinary *if it is isomorphic to a linear section of $\mathrm{Gr}(2,V) \subseteq \mathbb{P}^9$ and it is called* special *if it is isomorphic to a double cover of a linear section of $\mathrm{Gr}(2,V)$ branched along a quadric section.*

In what follows, we adapt [73, Theorems (A)-(D)] to our context.

Theorem 5.5.17. *The equivalence relations $\sim_{\mathrm{alg}}$ and $\sim_{\mathrm{num}}$ on $\mathcal{Z}^*(X)_{\mathbb{Q}}$ agree when X is a cubic fourfold or an ordinary generic Gushel-Mukai fourfold.*

Proof. We start by noticing that the proof of [73, Theorem (A)] works here with two minor changes. Indeed, their reasoning is the classical setting and the only thing that one needs to do is replace the equivalence relation $\sim_{\mathrm{nil}}$ by the equivalence relation $\sim_{\mathrm{alg}}$, and then consider Remark 5.4.2(i). Nonetheless, we present a detailed proof.

Let us assume that X is a cubic fourfold. We note that X contains a line l, see [73, §2.1]. Thanks to [73, Lemma 2.1] the linear projection from the line l induces a smooth flat quadric fibration from the blow-up $\mathrm{Bl}_l(X) \to \mathbb{P}^3$. But then, [73, §1.3], or see [86, Theorem 4.2 and Corollary 4.4], and Beilinson's full exceptional collection of $\mathrm{perf}(\mathbb{P}^3)$, consult [14], imply that the Chow motive of the blow up decomposes as follows:

$$\mathfrak{h}\big(\mathrm{Bl}_l(X)\big) \simeq \oplus_{j=0}^{1}\mathfrak{h}(\mathbb{P}^3)(j) \oplus (Z, r, 1) \simeq \oplus_{j=0}^{1}\big(\oplus_{i=0}^{3} \mathbf{1}(-1)\big) \oplus (Z, r, 1), \tag{5.28}$$

where $(Z, r, 1)$ is an object of $\mathrm{Chow}(k)_{\mathbb{Q}}$. That is, we have Z a smooth projective k-scheme of dimension 2 and r an endomorphism of $\mathfrak{h}(Z)$. Note that $\mathbf{1} = \big(\mathrm{Spec}(k), \mathrm{id}, 0\big)$ is the unit of $\mathrm{Chow}(k)$.

Now, thanks to (5.28) and Remark 5.4.2(i), we have that $\mathcal{Z}^*_{\sim_{\mathrm{num}}/\sim_{\mathrm{alg}}}\big(\mathrm{Bl}_l(X)\big)$ is trivial. Hence, [73, §1.2] (or [67]) implies that $\mathcal{Z}^*_{\sim_{\mathrm{num}}/\sim_{\mathrm{alg}}}(X)$ is trivial.

Let us now assume that X is an ordinary Gushel-Mukai fourfold. Thanks to [73, Corollary 2.3], for $\mathrm{Bl}_E(X)$ a blow-up of X in a surface E, consult [73, §2.2] for the details, there is a conic smooth flat fibration $\tilde{\rho} : \mathrm{Bl}_E(X) \to \mathbb{P}(V')$, for V' a k-vector space of dimension 4. Now, as $\mathbb{P}(V') \simeq \mathbb{P}^3$ and $\dim\big(\mathrm{Bl}_E(X)\big) = 4$, [73, §1.3] (or [86, Theorem 4.2 and Corollary 4.4]) implies $\mathfrak{h}\big(\mathrm{Bl}_E(X)\big) \simeq \oplus_{k=0}^{1}\mathfrak{h}(\mathbb{P}^3)(k) \oplus (Z, r, 1)$, where $\dim Z = 3 - 1 = 2$. Thanks to [14] we have $\mathfrak{h}\big(\mathrm{Bl}_l(X)\big) \simeq \oplus_{k=0}^{1}\big(\oplus_{i=0}^{3} \mathbf{1}(-1)\big) \oplus (Z, r, 1)$. Remark 5.4.2(i) implies that $\mathcal{Z}^*_{\sim_{\mathrm{num}}/\sim_{\mathrm{alg}}}\big(\mathrm{Bl}_l(X)\big)$ is trivial. Finally, [73, §1.2] (see also [67]) implies that $\mathcal{Z}^*_{\sim_{\mathrm{num}}/\sim_{\mathrm{alg}}}(X)$ is trivial. □

Remark 5.5.18. (i) Let X be a cubic fourfold. Recall from [51, §4.1] that the Kuznetsov category $\mathcal{A}_X$ of X is defined as a certain semi-orthogonal component of the following semi-orthogonal decomposition

$$\mathrm{perf}(X) = \big\langle \mathcal{A}_X, \mathcal{O}_X(-2H), \mathcal{O}_X(-H), \mathcal{O}_X \big\rangle$$

where H is the class of a hyperplane in $\mathbb{P}^5$. Thanks to Proposition 5.4.5 we have

$$\begin{aligned} K_{0,\sim_2/\sim_1}\big(\mathrm{perf}_{\mathrm{dg}}(X)\big) \simeq {} & K_{0,\sim_2/\sim_1}\big(\mathcal{A}_X^{\mathrm{dg}}\big) \oplus K_{0,\sim_2/\sim_1}\Big(\mathrm{perf}_{\mathrm{dg}}\left(\mathcal{O}_X(-2H)\right)\Big) \\ & \oplus K_{0,\sim_2/\sim_1}\Big(\mathrm{perf}_{\mathrm{dg}}\left(\mathcal{O}_X(-H)\right)\Big) \oplus K_{0,\sim_2/\sim_1}\big(\mathrm{perf}_{\mathrm{dg}}(\mathcal{O}_X)\big), \end{aligned}$$

where $\mathcal{A}_X^{\mathrm{dg}}$ denotes the dg enhancement of $\mathcal{A}_X$ induced from $\mathrm{perf}_{\mathrm{dg}}(X)$. Now, Corollary 5.4.4 implies

$$\begin{aligned} \mathcal{Z}^*_{\sim_2/\sim_1}(X) \simeq {} & K_{0,\sim_2/\sim_1}\big(\mathcal{A}_X^{\mathrm{dg}}\big) \oplus \mathcal{Z}^*_{\sim_2/\sim_1}\big(\mathcal{O}_X(-2H)\big) \\ & \oplus \mathcal{Z}^*_{\sim_2/\sim_1}\big(\mathcal{O}_X(-H)\big) \oplus \mathcal{Z}^*_{\sim_2/\sim_1}\big((\mathcal{O}_X)\big). \end{aligned}$$

Remark 5.4.2(i) allows us to obtain $\mathcal{Z}^*_{\sim_2/\sim_1}(X) \simeq K_{0,\sim_2/\sim_1}\big(\mathcal{A}_X^{\mathrm{dg}}\big)$. Specifying the equivalence relations to be $\sim_{\mathrm{alg}}$ and $\sim_{\mathrm{num}}$, using Theorem 5.5.17 we conclude that $K_{0,\sim_{\mathrm{num}}/\sim_{\mathrm{alg}}}\big(\mathcal{A}_X^{\mathrm{dg}}\big) = 0$, or equivalently $\ker\left(q^{\mathcal{A}_X^{\mathrm{dg}},\mathrm{nc}}_{\sim_{\mathrm{num}}/\sim_{\mathrm{alg}}}\right) = 0$. That is, the equivalence relations $\sim_{\mathrm{alg}}$ and $\sim_{\mathrm{num}}$ on $\mathcal{A}_X^{\mathrm{dg}}$ agree.

(ii) Let X be a Gushel-Mukai n-fold. Recall from [58] that the Gushel-Mukai category $\mathcal{A}_X$ of X is defined as a certain semi-orthogonal component of $\mathrm{perf}(X)$. Therefore, for X a Gushel-Mukai fourfold, Theorem 5.5.17, Corollary 5.4.4, and Proposition 5.4.5, imply that $\ker\left(q^{\mathcal{A}_X^{\mathrm{dg}},\mathrm{nc}}_{\sim_{\mathrm{num}}/\sim_{\mathrm{alg}}}\right) = 0$, i.e., the equivalence relations $\sim_{\mathrm{alg}}$ and $\sim_{\mathrm{num}}$ on $\mathcal{A}_X^{\mathrm{dg}}$ agree.

Under the above notations, we obtain the analogous of [73, Theorem (B)]:

Theorem 5.5.19. *The equivalence relations $\sim_{\mathrm{alg}}$ and $\sim_{\mathrm{num}}$ on $K_0(\mathcal{A}_X^{\mathrm{dg}})_{\mathbb{Q}}$ agree when X is a cubic fourfold or an ordinary generic Gushel-Mukai fourfold.*

We now adapt [73, Theorems (C) and (D)] to our context:

Theorem 5.5.20. *The equivalence relations $\sim_{\mathrm{alg}}$ and $\sim_{\mathrm{num}}$ on $\mathcal{Z}^*(X)_{\mathbb{Q}}$ agree when X is a generic Gushel-Mukai fourfold containing a plane P of type* $\mathrm{Gr}(2,3)$.

Proof. The proof is similar to the one given in [73, §5]. We follow the reasoning of that proof and make some adaptations. Firstly, we have a semi-orthogonal decomposition of $\mathrm{perf}(X)$ consisting of $\mathcal{A}_X$ and four exceptional objects,

Secondly, obtain $K_{0,\sim_2/\sim_1}(\mathrm{perf}_{\mathrm{dg}}(X)) \simeq K_{0,\sim_2/\sim_1}(\mathcal{A}_X^{\mathrm{dg}})$ by using Remark 5.4.2(i) and Proposition 5.4.5.

Thirdly, note that this Gushel-Mukai category $\mathcal{A}_X$ is Fourier-Mukai equivalent to Kuznetsov category $\mathcal{A}_{X'}$ of a cubic fourfold X', cf. [58, Theorem 1.3]. So, Proposition 5.4.6 implies $K_{0,\sim_{\mathrm{num}}/\sim_{\mathrm{alg}}}(\mathcal{A}_X^{\mathrm{dg}}) \simeq K_{0,\sim_{\mathrm{num}}/\sim_{\mathrm{alg}}}(\mathcal{A}_{X'}^{\mathrm{dg}})$, which is trivial, thanks to Theorem 5.5.19.

Finally, an application of Corollary 5.4.4 concludes the proof. □

Theorem 5.5.21. *The equivalence relations $\sim_{\mathrm{alg}}$ and $\sim_{\mathrm{num}}$ on $\mathcal{Z}^*(X)_{\mathbb{Q}}$ agree when X is an ordinary Gushel-Mukai fourfold containing a quintic del Pezzo surface.*

Proof. Thanks to [58, Proposition 2.3], we obtain a semi-orthogonal decomposition of $\mathrm{perf}(X)$ consisting only of $\mathcal{A}_X$ and exceptional objects. So, Proposition 5.4.5 and Remark 5.4.2(i) imply $K_{0,\sim_{\mathrm{num}}/\sim_{\mathrm{alg}}}(\mathrm{perf}_{\mathrm{dg}}(X)) \simeq K_{0,\sim_{\mathrm{num}}/\sim_{\mathrm{alg}}}(\mathcal{A}_X^{\mathrm{dg}})$. From [58, Theorem 1.2], we have that there is a K3 surface Y such that $\mathcal{A}_X \simeq \mathrm{perf}(Y)$. Hence, Proposition 5.4.6 implies $K_{0,\sim_{\mathrm{num}}/\sim_{\mathrm{alg}}}(\mathcal{A}_X^{\mathrm{dg}}) \simeq K_{0,\sim_{\mathrm{num}}/\sim_{\mathrm{alg}}}(\mathrm{perf}_{\mathrm{dg}}(Y))$, which is trivial because of Remark 5.4.2(i). Consequently, from Corollary 5.4.4 we have that $\mathcal{Z}^*_{\sim_{\mathrm{num}}/\sim_{\mathrm{alg}}}(X) \simeq K_{0,\sim_{\mathrm{num}}/\sim_{\mathrm{alg}}}(\mathcal{A}_X^{\mathrm{dg}})$ is trivial. □

Remark 5.5.22. Consult [37, §1] for the notion of K3 surfaces and K3 categories.

We are in conditions to generalize Theorem 5.5.19.

Theorem 5.5.23. *The equivalence relations $\sim_{\mathrm{alg}}$ and $\sim_{\mathrm{num}}$ on $K_0(\mathcal{A}_X^{\mathrm{dg}})_{\mathbb{Q}}$ agree when X is a cubic fourfold, an ordinary generic Gushel-Mukai fourfold, an ordinary Gushel-Mukai fourfold containing a plane of type* $\mathrm{Gr}(2,3)$ *or an ordinary Gushel-Mukai fourfold containing a quintic del Pezzo surface.*

5.5.7 Küchle fourfolds

A Küchle fourfold is the zero locus of a global section of a certain vector bundle on a specific Grassmannian. In particular, a *Küchle fourfold of type* c_7, denoted by X_{c_7}, is the zero locus of a global section of the vector bundle $\Lambda^2 U^{\perp}(1) \oplus \mathcal{O}(1)$ on $\mathrm{Gr}(3,8)$, where $U^{\perp}$ is the tautological vector subbundle of rank 5 on the Grassmannian $\mathrm{Gr}(3,8)$ and $\mathcal{O}(1)$ stands for the ample generator of its Picard group; consult [46, 54] for further details.

Theorem 5.5.24. *The equivalence relations $\sim_{\mathrm{alg}}$ and $\sim_{\mathrm{num}}$ on $\mathcal{Z}^*(X_{c_7})_{\mathbb{Q}}$ agree.*

Proof. Thanks to [54, Corollary 4.12], the category $\mathrm{perf}(X_{c_7})$ admits a semi-orthogonal decomposition with 6 exceptional line bundles and a noncommutative K3 category $\mathcal{A}_X$. So, we have $K_{0,\sim_{\mathrm{num}}/\sim_{\mathrm{alg}}}\big(\mathrm{perf}_{\mathrm{dg}}(X_{c7})\big) \simeq K_{0,\sim_{\mathrm{num}}/\sim_{\mathrm{alg}}}(\mathcal{A}_X^{\mathrm{dg}})$, just apply Proposition 5.4.5 and Remark 5.4.2(i) to the semi-orthogonal decomposition we just mentioned. Also from [54, Corollary 4.12] we have that the category $\mathcal{A}_X$ is (Fourier-Mukai) equivalent to the nontrivial part of the derived category of a cubic fourfold Z. This means that $K_{0,\sim_{\mathrm{num}}/\sim_{\mathrm{alg}}}(\mathcal{A}_X^{\mathrm{dg}})$ is trivial, just consider Proposition 5.4.6 and Theorem 5.5.19. So, using Corollary 5.4.4 we conclude, that $\mathcal{Z}^*_{\sim_{\mathrm{num}}/\sim_{\mathrm{alg}}}(X_{c_7})_{\mathbb{Q}}$ is trivial. That is, the equivalence relations $\sim_{\mathrm{alg}}$ and $\sim_{\mathrm{num}}$ on $\mathcal{Z}^*(X_{c_7})_{\mathbb{Q}}$ agree. □

Remark 5.5.25. A consequence of Theorem 5.5.24 is that the Voevodsky nilpotence conjecture holds for X_{c_7}. To the best of the author's knowledge, this proves the Voevodsky nilpotence conjecture in new cases. Note that X_{c_7} is not a Gushel-Mukai fourfold because it has Picard number greater than 1; see [26, §1].

5.5.8 Family of sextic del Pezzo surfaces

In this section we follow the definitions and notations used in [48]. A *sextic du Val del Pezzo surface* is a normal integral projective surface X with at worst du Val singularities and ample anticanonical class such that $K_X^2 = 6$. Take S and T smooth projective k-schemes and $f : T \to S$ a *du Val family of sextic del Pezzo surfaces*, i.e., f is a flat morphism such that for every geometric point $s \in S$ the fiber T_s of T over S is a sextic du Val del Pezzo surface. Following [48, §5], with $d = 2,3$, let $\mathcal{M}_d$ denote the relative moduli stack of semi-stable sheaves on fibers of T over S with Hilbert polynomial $h_d(t) := (3t+d)(t+1)$ and Z_d the coarse moduli space of $\mathcal{M}_d$. Consequently, there are finite flat morphisms $Z_2 \to S$ and $Z_3 \to S$ with degree 2 and 3, respectively.

Theorem 5.5.26. *Let $f : T \to S$ be a du Val family of sextic del Pezzo surfaces, and assume that the characteristic of k is not* 2 *neither* 3. *Under these conditions, with $\sim_1, \sim_2 \in \{\sim_{\mathrm{rat}}, \sim_{\mathrm{alg}}, \sim_{\mathrm{nil}}, \sim_{\mathrm{num}}\}$, we have an isomorphism:*

$$\mathcal{Z}^*_{\sim_2/\sim_1}(T)_{\mathbb{Q}} \simeq \mathcal{Z}^*_{\sim_2/\sim_1}(S)_{\mathbb{Q}} \oplus \mathcal{Z}^*_{\sim_2/\sim_1}(Z_2)_{\mathbb{Q}} \oplus \mathcal{Z}^*_{\sim_2/\sim_1}(Z_3)_{\mathbb{Q}}.$$

Proof. Following [48, Theorem 5.2 and Proposition 5.11], the category $\mathrm{perf}(T)$ admits a semi-orthogonal decomposition

$$\mathrm{perf}(T) = \big\langle\, \mathrm{perf}(S), \mathrm{perf}(Z_2, \mathbb{B}_2), \mathrm{perf}(Z_3, \mathbb{B}_3)\big\rangle,$$

where $\mathcal{B}_2$ and $\mathcal{B}_3$ are, respectively, certains sheafs of Azumaya algebras over Z_2, and Z_3, of order 2, and 3, respectively. By considering Propositions 5.4.5 and 5.4.7, we hence conclude that

$$K_{0,\sim 2/\sim 1}\big(\,\mathrm{perf}_{\mathrm{dg}}(T)\big)_{\mathbb{Q}} \simeq K_{0,\sim 2/\sim 1}\big(\,\mathrm{perf}_{\mathrm{dg}}(S)\big)_{\mathbb{Q}} \oplus K_{0,\sim 2/\sim 1}\big(\,\mathrm{perf}_{\mathrm{dg}}(Z_2)\big)_{\mathbb{Q}} \oplus \\ \oplus K_{0,\sim 2/\sim 1}\big(\,\mathrm{perf}_{\mathrm{dg}}(Z_3)\big)_{\mathbb{Q}}.$$

Now, an application of Corollary 5.4.4 finishes the proof. □

Corollary 5.5.27. *Let $f : T \to S$ be a du Val family of sextic del Pezzo surface and assume that the characteristic of k is not* 2 *neither* 3*. If* $\dim(S) \leq 2$*, then the equivalence relations $\sim_{\mathrm{num}}$ and $\sim_{\mathrm{alg}}$ on $\mathcal{Z}^*(T)_{\mathbb{Q}}$ agree.*

Proof. Since $\dim(S) \leq 2$, we have that $\dim(Z_2), \dim(Z_3) \leq 2$. Consequently, an application of Theorem 5.5.26 and Remark 5.4.2(i) finishes the proof. □

Remark 5.5.28. Note that, in the conditions of Corollary 5.5.27, the Voevodsky nilpotence conjecture holds for T.

5.5.9 Homological projective duality

We now introduce a powerful theory developed by Kuznetsov [52]: Homological projective duality. Let X be a smooth projective k-scheme equipped with a line bundle $\mathcal{L}_X(1)$ and let us write $X \to \mathbb{P}(V)$ for the associated morphism where $V := \mathrm{H}^0\big(X, \mathcal{L}_X(1)\big)^*$. Assume that the triangulated category $\mathrm{perf}(X)$ admits a Lefschetz decomposition:

$$\mathrm{perf}(X) : \big\langle \mathbb{A}_0, \mathbb{A}_1(1), \ldots, \mathbb{A}_{i-1}(i-1)\big\rangle, \tag{5.29}$$

with respect to $\mathcal{L}_X(1)$, where $\mathbb{A}_r(r) := \mathbb{A}_r \otimes \mathcal{L}_X(r)$, recall Definition 3.3.6. Note that $\mathbb{A}_r(r) \simeq \mathbb{A}_r$ and let us assume that $\dim(V) = N > i$, the number of terms in the Lefschetz decomposition.

Consider also the dual projective space $\mathbb{P}(V^\vee)$. Given (5.29) and thanks to [55, Theorem 1.8], or consult [47], $\mathrm{perf}\,\big(X \times \mathbb{P}(V^\vee)\big)$ has a semi-orthogonal decomposition:

$$\mathrm{perf}\,\big((X \times \mathbb{P}(V^\vee)\big) = \big\langle \mathbb{A}'_0, \mathbb{A}'_1(1), \ldots, \mathbb{A}'_{i-1}(i-1)\big\rangle,$$

over $\mathbb{P}(V^\vee)$, with $\mathbb{A}'_j$ certain triangulated categories. We denote the universal hyperplane section of X by $\chi := X \times_{\mathbb{P}(V)} Q \subseteq X \times \mathbb{P}(V^\vee)$, where $Q \subseteq \mathbb{P}(V) \times \mathbb{P}(V^\vee)$ is the incidence quadric and denote by $\alpha : \chi \to X \times \mathbb{P}(V^\vee)$ the natural embedding. Thanks to [55, Lemma 2.3], one has the following semi-orthogonal decomposition

$$\mathrm{perf}(\chi) = \big\langle C, \mathbb{A}'_1(1), \ldots, \mathbb{A}'_{i-1}(i-1)\big\rangle, \tag{5.30}$$

where C is called the *Homological projective dual* category of X. If C is equivalent to $\mathrm{perf}(Y)$ for Y some algebraic variety, we have the following definiton:

Definition 5.5.29. *Let Y be an algebraic variety equipped with a morphism $g : Y \to \mathbb{P}(V^\vee)$. Y is called* homologically projectively dual *to $f : X \to \mathbb{P}(V)$ with respect to a given Lefschetz decomposition* $\mathrm{perf}(X) = \langle \mathbb{A}_0, \mathbb{A}_1(1), \dots, \mathbb{A}_{i-1}(i-1)\rangle$ *if exists an $\mathcal{E} \in \mathrm{perf}\,(Q(X,Y))$, with $Q(X,Y) = (X \times Y) \times_{\mathbb{P}(V)\times\mathbb{P}(V^\vee)} Q = \chi \times_{\mathbb{P}(V^\vee)} Y$, such that the Fourier-Mukai functor $\Phi_{\mathcal{E}}$:* $\mathrm{perf}(Y) \to \mathrm{perf}(\chi)$ *is an equivalence onto C, the homological projective dual defined in (5.30).*

In order to be able to state an important result by Kuznetsov regarding linear sections take $L \subseteq V^\vee$ and consider its orthogonal complement in V, which is $L^\perp : \ker(V \to L^\vee)$. Moreover, consider the linear sections $X_L := X \times_{\mathbb{P}(V)} \mathbb{P}(L^\perp)$ and $Y_L := Y \times_{\mathbb{P}(V^\vee)} \mathbb{P}(L)$. We will say that these linear sections have the expected dimensions if $\dim(X_L) = \dim(X) - \dim(L)$ and $\dim(Y_L) = \dim(Y) - \dim(L^\perp) = \dim(Y) - (N - \dim(L))$, where $N = \dim(V)$.

We are now in condition to write Kuznetsov theorem on Homological projective duality.

Theorem 5.5.30 (Homological projective duality [52, Theorem 6.3])**.** *In the above conditions, and assuming that (Y, g) is homologically projectively dual to (X, f) with respect to the semi-orthogonal decomposition* $\mathrm{perf}(X) = \langle \mathbb{A}_0, \mathbb{A}_1(1), \dots, \mathbb{A}_{i-1}(i-1)\rangle$ *we have:*

(i) Y is a smooth algebraic variety and $\mathrm{perf}(Y)$ *has an admissible subcategory $\mathbb{B}_0$, cf. [52, Definition 2.2], that is equivalent to $\mathbb{A}_0$ and admits a left Lefschetz decomposition:*

$$\mathrm{perf}(Y) = \langle \mathbb{B}_{j-1}(1-j), \dots, \mathbb{B}_1(-1), \mathbb{B}_0\rangle, \tag{5.31}$$

where $\mathbb{B}_{j-1} \subseteq \ldots \subseteq \mathbb{B}_1 \subseteq \mathbb{B}_0$, $j = N - 1 - \max\{k | \mathbb{A}_k = \mathbb{A}_0\}$ and the primitive categories for (5.31) are the same as the primitive categories for (5.29). See Definition 3.3.7 to recall the definition of primitive category.

(ii) For X_L and Y_L defined as above, and with expected dimension, one has the following semi-orthogonal decompositions:

$$\begin{aligned} \mathrm{perf}(X_L) &= \langle C_L, \mathbb{A}_r(r), \dots, \mathbb{A}_{i-1}(i-1)\rangle \\ \mathrm{perf}(Y_L) &= \langle \mathbb{B}_{j-1}(1-j), \dots, \mathbb{B}_{N-\dim(L)}(\dim(L)-N), C_L\rangle. \end{aligned}$$

Remark 5.5.31. For more details on Theorem 5.5.30 consult [52, 55].

Theorem 5.5.32 (HPD invariance)**.** *Let X and Y be as above and assume that X_L and Y_L are both smooth and both have the expected dimensions. Consider the equivalence relations $\sim_1, \sim_2 \in \{\sim_{\mathrm{rat}}, \sim_{\mathrm{alg}}, \sim_{\mathrm{nil}}, \sim_{\mathrm{num}}\}$ and assume moreover that* $\ker\left(q^{\mathbb{A}_0^{\mathrm{dg}},\mathrm{nc}}_{\sim_2/\sim_1}\right) = 0$ *(or, equivalently, that $K_{0,\sim_2/\sim_1}(\mathbb{A}_0^{\mathrm{dg}})_{\mathbb{Q}} = 0$), where $\mathbb{A}_0^{\mathrm{dg}}$ stands for the dg enhancement of $\mathbb{A}_0$ induced from* $\mathrm{perf}_{\mathrm{dg}}(X)$. *Under these assumptions, we have an isomorphism $\mathcal{Z}^*_{\sim_2/\sim_1}(X_L)_{\mathbb{Q}} \simeq \mathcal{Z}^*_{\sim_2/\sim_1}(Y_L)_{\mathbb{Q}}$.*

Remark 5.5.33. Given a *generic* subspace $L \subseteq V^\vee$, the sections X_L and Y_L are smooth, and we have $\dim(X_L) = \dim(X) - \dim(L)$ and $\dim(Y_L) = \dim(Y) - \dim(L^\perp)$. Moreover, by inductive use of Proposition 5.4.5, we have $\ker\left(q^{\mathbb{A}^{\mathrm{dg}},\mathrm{nc}}_{\sim_2/\sim_1}\right) = 0$ whenever $\mathbb{A}$ admits a full exceptional collection; see [55, §1.1] and Remark 5.4.2(i). This shows that the assumptions of Theorem 5.5.32 are quite mild.

Proof. The proof is very similar to the proof given in [17, §9], we follow their reasoning with some modifications. Considering the above notation, let $\mathfrak{a}_i$ be the orthogonal complement of $\mathbb{A}_{i+1}$ in $\mathbb{A}_i$. Given the Lefschetz decomposition (5.29) of $\mathrm{perf}(X)$ and bearing in mind the definition of primitives subcategories, one has the following semi-orthogonal decompositions $\mathbb{A}_k = \langle \mathfrak{a}_k, \mathfrak{a}_{k+1} \dots, \mathfrak{a}_{i-1}\rangle$. Since we are assuming $K_{0,\sim_2/\sim_1}(\mathbb{A}_0^{\mathrm{dg}})_{\mathbb{Q}} = 0$, Proposition 5.4.5 implies that $K_{0,\sim_2/\sim_1}(\mathfrak{a}_s^{\mathrm{dg}})_{\mathbb{Q}}$ and $K_{0,\sim_2/\sim_1}(\mathbb{A}_s^{\mathrm{dg}})_{\mathbb{Q}}$ are trivial for all s.

Thanks to Theorem 5.5.30, $\mathrm{perf}(Y) = \langle \mathbb{B}_{j-1}(1-j), \dots, \mathbb{B}_0\rangle$ admits a Lefschetz decomposition with respect to $\mathcal{O}_Y(1)$ such that their primitive subcategories $\mathfrak{b}_s$ are Fourier-Mukai equivalent to the primitive subcategories $\mathfrak{a}_s$. Applying Proposition 5.4.6 we obtain that $K_{0,\sim_2/\sim_1}(\mathfrak{b}_s^{\mathrm{dg}})_{\mathbb{Q}}$ is trivial for all s. Moreover, a successive application of Proposition 5.4.5, starting with $\mathbb{B}_{j-1} = \mathfrak{b}_{j-1}$, allows us to conclude that $K_{0,\sim_2/\sim_1}(\mathbb{B}_s^{\mathrm{dg}})_{\mathbb{Q}}$ is trivial for all s.

Bearing in mind Theorem 5.5.30, we have the following semi-orthogonal decompositions:

$$\mathrm{perf}(X_L) = \Big\langle \mathbb{C}_L, \mathbb{A}_{\dim(L)}(1), \dots, \mathbb{A}_n\big(j - \dim(L)\big)\Big\rangle$$
$$\mathrm{perf}(Y_L) = \Big\langle \mathbb{B}_i\big(\dim(L^\perp - i), \dots, \mathbb{B}_{\dim(L^\perp)}(-1), \mathbb{C}_L\Big\rangle.$$

Hence, Proposition 5.4.5 together with Corollary 5.4.4 imply $\mathcal{Z}^*_{\sim_2/\sim_1}(X_L)_{\mathbb{Q}} \simeq \mathcal{Z}^*_{\sim_2/\sim_1}(Y_L)_{\mathbb{Q}}$. □

Example 5.5.34 (Linear sections of Grassmannians). Let us apply Theorem 5.5.32 to the case of linear sections of Grassmannians:

(i) For $W = k^{\oplus 6}$, let X_L be a generic linear section of codimension r of the Grassmannian $X = \mathrm{Gr}(2, W)$ under the Plücker embedding, and Y_L the corresponding dual linear section of the cubic Pfaffian $\mathrm{Pf}(4, W^*)$ in $\mathbb{P}(\Lambda^2 W^*)$. Note that X_L and Y_L are smooth and that $\dim(X_L) = 8 - r$ and $\dim(Y_L) = r - 2$ when $r \leq 6$.

(ii) For $W = k^{\oplus 7}$, let X_L be a generic linear section of codimension r of the Grassmannian $X = \mathrm{Gr}(2, W)$ under the Plücker embedding, and Y_L the corresponding dual linear section of the cubic Pfaffian $\mathrm{Pf}(4, W^*)$ in $\mathbb{P}(\Lambda^2 W^*)$. Note that X_L and Y_L are smooth and that $\dim(X_L) = 10 - r$ and $\dim(Y_L) = r - 4$ when $r \leq 10$.

Note also that from [53, (11) and (12)] we obtain the following semi-orthogonal decompositions, where E_0, E_1, E_2 stand for certain exceptional objects and H_X is the divisor class of a hyperplane section of X.

For class (i):

$$\mathrm{perf}(X) = \big\langle \mathbb{A}_0, \mathbb{A}_1(H_X), \mathbb{A}_2(2H_X), \mathbb{A}_3(3H_X), \mathbb{A}_4(4H_X), \mathbb{A}_5(5H_X)\big\rangle,$$

with $\mathbb{A}_0 = \mathbb{A}_1 = \mathbb{A}_2 = \langle E_2, E_1, E_0\rangle$ and $\mathbb{A}_3 = \mathbb{A}_4 = \mathbb{A}_5 = \langle E_1, E_0\rangle$.

For class (ii):

$$\mathrm{perf}(X) = \big\langle \mathbb{A}_0, \mathbb{A}_1(H_X), \mathbb{A}_2(2H_X), \mathbb{A}_3(3H_X), \mathbb{A}_4(4H_X), \mathbb{A}_5(5H_X), \mathbb{A}_6(6H_X)\big\rangle,$$

with $\mathbb{A}_0 = \mathbb{A}_1 = \mathbb{A}_2 = \mathbb{A}_3 = \mathbb{A}_4 = \mathbb{A}_5 = \mathbb{A}_6 = \langle E_2, E_1, E_0\rangle$.

Consequently, Proposition 5.4.5 and Remark 5.4.2 imply that for both classes (i)-(ii) there is a Lefschetz decomposition of $\mathrm{perf}\big(\mathrm{Gr}(2,W)\big)$ and that $\ker\Big(q^{\mathbb{A}_0^{\mathrm{dg}},\mathrm{nc}}_{\sim_2/\sim_1}\Big) = 0$, where $\mathbb{A}_0$ is the first component of the Lefschetz decomposition of $\mathrm{perf}\big(\mathrm{Gr}(2,W)\big)$.

Corollary 5.5.35. *Let X_L and Y_L be as in the above classes (i)-(ii) and $\sim_1, \sim_2 \in \{\sim_{\mathrm{rat}}, \sim_{\mathrm{alg}}, \sim_{\mathrm{nil}}, \sim_{\mathrm{num}}\}$. Under the assumption that X_L and Y_L are smooth, we have $\mathcal{Z}^*_{\sim_2/\sim_1}(X_L)_{\mathbb{Q}} \simeq \mathcal{Z}^*_{\sim_2/\sim_1}(Y_L)_{\mathbb{Q}}$. Moreover, the equivalence relations $\sim_{\mathrm{num}}$ and $\sim_{\mathrm{alg}}$ on $\mathcal{Z}^*(X_L)$ agree when $r \leq 6$ (class (i)), and when $r \leq 6$ and $8 \leq r \leq 10$ (class (ii)).*

Proof. The first statement is immediate from Theorem 5.5.32. The second statement follows from Remark 5.4.2(i), except the case where X_L and Y_L are as in class (i) and $r = 5$.

For this latter case, thanks to [33, §5] , X_L is a Fano 3-fold and the Chow motive of X_L admits a decomposition into Lefschetz motives and submotives of curves. Hence, an application of Remark 5.4.2(i), Proposition 5.4.5 and Corollary 5.4.4 imply that the equivalence relations $\sim_{\mathrm{num}}$ and $\sim_{\mathrm{alg}}$ agree on $\mathcal{Z}^*(X_L)$. □

Corollary 5.5.36. *For $k = \mathbb{C}$, let X_L and Y_L be as above in class (i) and let $\dim L = r = 3$. Then $\mathcal{Z}^*_{\sim_{\mathrm{alg}}/\sim_{\mathrm{rat}}}(X_L)_{\mathbb{Q}} \simeq (\mathbb{Q}/\mathbb{Z})^{\oplus 2}$.*

Proof. In this case, X_L is a 5-fold and Y_L is an elliptic curve; see [53, §10]. Therefore, since Y_L is of genus 1, $\mathcal{Z}^*_{\sim_{\mathrm{alg}}/\sim_{\mathrm{rat}}}(Y_L)_{\mathbb{Q}}$ is isomorphic to $(\mathbb{Q}/\mathbb{Z})^{\oplus 2\times 1}$; see Remark 5.4.2(iii). Consequently, Theorem 5.5.32 implies $\mathcal{Z}^*_{\sim_{\mathrm{alg}}/\sim_{\mathrm{rat}}}(X_L)_{\mathbb{Q}}$ isomorphic to $(\mathbb{Q}/\mathbb{Z})^{\oplus 2}$. □

Corollary 5.5.37. *For $k = \mathbb{C}$, let X_L and Y_L be as in the above class (ii) and let $\dim L = r$.*

(i) If $r = 5$, then $\mathcal{Z}^*_{\sim_{\mathrm{alg}}/\sim_{\mathrm{rat}}}(X_L)_{\mathbb{Q}} \simeq (\mathbb{Q}/\mathbb{Z})^{\oplus 86}$.

(ii) If $r = 9$, then $\mathcal{Z}^*_{\sim_{\mathrm{alg}}/\sim_{\mathrm{rat}}}(Y_L)_{\mathbb{Q}} \simeq (\mathbb{Q}/\mathbb{Z})^{\oplus 30}$.

Proof. When $r = 5$, X_L is a Fano 5-fold of index 2 and Y_L is a curve of genus 43; see [53, §11]. Therefore, we deduce that $\mathcal{Z}^*_{\sim_{\mathrm{alg}}/\sim_{\mathrm{rat}}}(Y_L)_{\mathbb{Q}}$ is isomorphic to $(\mathbb{Q}/\mathbb{Z})^{\oplus 2\times 43}$; see Remark 5.4.2(iii). Consequently, Theorem 5.5.32 implies that $\mathcal{Z}^*_{\sim_{\mathrm{alg}}/\sim_{\mathrm{rat}}}(X_L)_{\mathbb{Q}}$ and $(\mathbb{Q}/\mathbb{Z})^{\oplus 86}$ are isomorphic.

When $r = 9$, X_L is a curve of genus 15 and Y_L is a Fano 5-fold of index 2; see [53, §11]. Hence, by combining Theorem 5.5.32 with Remark 5.4.2(iii), we conclude that $\mathcal{Z}^*_{\sim_{\mathrm{alg}}/\sim_{\mathrm{rat}}}(Y_L)_{\mathbb{Q}}$, $\mathcal{Z}^*_{\sim_{\mathrm{alg}}/\sim_{\mathrm{rat}}}(X_L)_{\mathbb{Q}}$ and $(\mathbb{Q}/\mathbb{Z})^{\oplus 2\times 15}$ are all isomorphic to each other. □

Example 5.5.38 (Linear sections of determinantal varieties). Let U and V be two k-vector spaces of dimensions m and n, respectively, with $m \leq n$ and r an integer such that $0 < r < m$. As in [16], consider the determinantal variety $Z^r_{m,n} \subseteq \mathbb{P}(U \otimes V)$ defined as the locus of those matrices $M : V \to U^*$ with rank $\leq r$. It is known that $Z^r_{m,n}$ admits a canonical Springer resolution of singularities $\mathcal{X}^r_{m,n} := \mathbb{P}(\mathcal{Q} \otimes U) \to \mathrm{Gr}(r, U)$, where $\mathcal{Q}$ stands for the tautological quotient on $\mathrm{Gr}(r, U)$. Under these notations, let X_L be a generic linear section of codimension c of $\mathcal{X}^r_{m,n}$ under the map $\mathcal{X}^r_{m,n} \to \mathbb{P}(U \otimes V)$, and Y_L the corresponding dual linear section of $\mathcal{X}^{m-r}_{m,n}$ under the map $\mathcal{X}^{m-r}_{m,n} \to \mathbb{P}(U^* \otimes V^*)$. From [16, §3] we have that X_L and Y_L are both smooth and, from [78, §3], we have $\dim(X_L) = r(m+n-r) - 1 - \dim(L)$ and $\dim(Y_L) = r(m-n-r) - 1 + \dim(L)$. Moreover, by [16, §3], Proposition 5.4.5, and Remark 5.4.2(i), there is a Lefschetz decomposition of $\mathrm{perf}(\mathcal{X}^r_{m,n})$ and $\ker\left(q^{\mathbb{A}^{\mathrm{dg}}_0,\mathrm{nc}}_{\sim_2/\sim_1}\right) = 0$.

Corollary 5.5.39. *Let X_L and Y_L be as in Example 5.5.38 and $\sim_1, \sim_2 \in \{\sim_{\mathrm{rat}}, \sim_{\mathrm{alg}}, \sim_{\mathrm{nil}}, \sim_{\mathrm{num}}\}$. Under these assumptions, we have $\mathcal{Z}^*_{\sim_2/\sim_1}(X_L)_{\mathbb{Q}} \simeq \mathcal{Z}^*_{\sim_2/\sim_1}(Y_L)_{\mathbb{Q}}$. Moreover, the equivalence relations $\sim_{\mathrm{num}}$ and $\sim_{\mathrm{alg}}$ on $\mathcal{Z}^*(X_L)$ (and on $\mathcal{Z}^*(Y_L)$) agree whenever* $\dim(L) \geq r(m+n-r) - 3$ *or* $\dim(L) \leq 3 + r(n-m+r)$.

5.5.10 Prime Fano threefolds and del Pezzo threefolds

A *Fano variety* is a smooth proper connected algebraic variety whose anticanonical class is ample. Following [56, §5.4], we have that a *prime Fano threefold* is a Fano threefold X with $\mathrm{Pic}(X) = \mathbb{Z}K_X$ whose genus $\mathrm{g}(X)$ is defined from $(-K_X)^3 = 2\mathrm{g}(X) - 2$ and it is known that $1 \leq \mathrm{g}(X) \leq 12$ and $\mathrm{g}(X) \neq 11$. Moreover, for prime Fano threefolds of even genus there is a semi-orthogonal decomposition of $\mathrm{perf}(X)$ with a nontrivial component $\mathcal{A}_X$.

In the same way, following [56, §5.4], a *del Pezzo threefold* is a Fano threefold Y with $-K_Y = 2H$, for a primitive Cartier divisor class H and its degree is defined as $\mathrm{d}(Y) = H^3$. It is known that $1 \leq \mathrm{d}(Y) \leq 5$ for del Pezzo threefolds of Picard rank 1. In addition, we have that a del Pezzo threefold admits a semi-orthogonal decomposition with a non-trivial component $\mathcal{B}_Y$; consult [49, §3] for further details.

Let X be a prime Fano threefold with $\mathrm{g}(X) \in \{8, 10, 12\}$. Following [49, Theorem 3.8], there exists a unique del Pezzo threefold Y with degree $\mathrm{d}(Y) = \dfrac{\mathrm{g}(X)}{2} - 1 \in \{3, 4, 5\}$ such that $\mathcal{A}_X \simeq \mathcal{B}_Y$.

Proposition 5.5.40. *Let X be a prime Fano threefold and Y a del Pezzo threefold in the above conditions. Then, we have an isomorphism $\mathcal{Z}^*_{\sim_{\mathrm{alg}}/\sim_{\mathrm{rat}}}(X)_{\mathbb{Q}} \simeq \mathcal{Z}^*_{\sim_{\mathrm{alg}}/\sim_{\mathrm{rat}}}(Y)_{\mathbb{Q}}$.*

Proof. Proposition 5.4.5 and Remark 5.4.2(i) imply that $K_{0,\sim_{\mathrm{alg}}/\sim_{\mathrm{rat}}}(X)_{\mathbb{Q}}$ is isomorphic to $K_{0,\sim_{\mathrm{alg}}/\sim_{\mathrm{rat}}}(\mathcal{A}^{\mathrm{dg}}_X)_{\mathbb{Q}}$. Similarly, $K_{0,\sim_{\mathrm{alg}}/\sim_{\mathrm{rat}}}(Y)_{\mathbb{Q}}$ is isomorphic to $K_{0,\sim_{\mathrm{alg}}/\sim_{\mathrm{rat}}}(\mathcal{B}^{\mathrm{dg}}_Y)_{\mathbb{Q}}$. Since the categories $\mathcal{A}_X$ and $\mathcal{B}_Y$ are (Fourier-Mukai) equivalent, we have, moreover, an isomorphism between $K_{0,\sim_{\mathrm{alg}}/\sim_{\mathrm{rat}}}(\mathcal{A}^{\mathrm{dg}}_X)_{\mathbb{Q}}$ and $K_{0,\sim_{\mathrm{alg}}/\sim_{\mathrm{rat}}}(\mathcal{B}^{\mathrm{dg}}_Y)_{\mathbb{Q}}$, see Proposition 5.4.6. This implies that $K_{0,\sim_{\mathrm{alg}}/\sim_{\mathrm{rat}}}(X)_{\mathbb{Q}}$ and $K_{0,\sim_{\mathrm{alg}}/\sim_{\mathrm{rat}}}(Y)_{\mathbb{Q}}$ are isomorphic. Consequently, Corollary 5.4.4 allows us to conclude that $\mathcal{Z}^*_{\sim_{\mathrm{alg}}/\sim_{\mathrm{rat}}}(X)_{\mathbb{Q}}$ is isomorphic to $\mathcal{Z}^*_{\sim_{\mathrm{alg}}/\sim_{\mathrm{rat}}}(Y)_{\mathbb{Q}}$. $\square$

5.5.11 Fano fourfolds of K3 type

In [18] a list of 64 Fano fourfolds of K3 type is given. By combining Remark 5.4.2(i), Corollary 5.4.4, Propositions 5.4.5, 5.4.7 and Theorem 5.5.23 one obtains the following theorem, where we use the notation in [18].

Theorem 5.5.41. *The equivalence relations $\sim_{\mathrm{alg}}$ and $\sim_{\mathrm{num}}$ agree when X belongs to one of the following 59 families Fano fourfolds of type K3:*

(i) 16 families of Fano fourfolds of K3 type obtained inside products of flag manifolds, where at least one projection is the blow up of a cubic fourfold: from C-2 to C-17.

(ii) 4 families of Fano fourfolds of K3 type obtained inside products of flag manifolds, where at least one projection is a blow up of a Gushel–Mukai fourfold: from GM-18 to GM-20 and GM-22.

(iii) 36 families of Fano fourfolds of K3 type that we obtained inside products of flag manifolds, where at least one projection is a blow up with center birational to a K3 surface, and no cubic or Gushel–Mukai fourfold is involved: from K3-24 to K3-35 and from K3-37 to K3-60.

(iv) 3 other families of Fano fourfolds of K3 type: from R-61 to R-63.

Remark 5.5.42. Theorem 5.5.41 is still valid for three more families of Fano fourfolds that are not of K3 type. Namely: A-65, A-67 and A-68; consult [18, Appendix A].

Remark 5.5.43. Note that Theorem 5.5.41 and Remark 5.5.42 imply that the Voevodsky nilpotence conjecture holds for all those families of Fano fourfolds.

Part II

Farrell-Jones isomorphism conjecture

6

Infinity categories

In this chapter we are going to recall definitions on ∞-categories and some related concepts that are going to be useful to obtain some results regarding the Farrell-Jones conjecture.

6.1 Definitions

We start by fixing some notation.

Notation 6.1.1. (i) We shall define $\Delta^n := \{(x_0, \ldots, x_n) \in [0,1]^{n+1} | x_0 + \ldots + x_n = 1\}$ to be the *standard n-simplex*, see [66, page 8].

(ii) Given Δ^n, we define the *i^{th}-horn* Λ_i^n to be obtained from Δ^n by deleting the interior and the face opposite to the i^{th} vertex, see [66, Definition 1.1.2.1].

Definition 6.1.2 ([66, Definition 1.1.2.4.]). *A category M is called an* ∞-category *if it is a simplicial set such that for any $0 < i < n$ and any map $f_0 : \Lambda_i^n \to M$ admits an extension $f : \Delta^n \to M$.*

Definition 6.1.3 ([22, §1]). *An ∞-category is called a* left-exact ∞-category *if it has a zero object and if it contains all finite limits. We shall denote the (large) ∞-category of small left-exact ∞-categories and finite limit preserving functors by* $\mathrm{Cat}_{\infty,*}^{\mathrm{Lex}}$.

Definition 6.1.4 ([65, Definition 1.1.1.9]). *An ∞-category is called a* stable ∞-category *if it has a zero object, every morphism admits a fiber and a cofiber, and a triangle is a fiber sequence (see [65, Definition 1.1.1.4]) if and only if it is a cofiber sequence, see [65, Definition 1.1.1.4]. We shall denote the (large) ∞-category of small stable ∞-categories by* $\mathrm{Cat}_{\infty,*}^{\mathrm{ex}}$.

Remark 6.1.5. Let $\mathrm{stab} : \mathrm{Cat}_{\infty,*}^{\mathrm{Lex}} \to \mathrm{Cat}_{\infty,*}^{\mathrm{ex}}$ be the stabilization functor defined in [23, §7.4] and consider the inclusion functor $\mathrm{inc} : \mathrm{Cat}_{\infty,*}^{\mathrm{ex}} \to \mathrm{Cat}_{\infty,*}^{\mathrm{Lex}}$. Thanks to [23, Lemma 7.41], we have the following adjunction:

$$\begin{tikzcd} \mathrm{Cat}_{\infty,*}^{\mathrm{Lex}} \arrow[d, shift right, "\mathrm{stab}"'] \\ \mathrm{Cat}_{\infty,*}^{\mathrm{ex}} \arrow[u, shift right, "\mathrm{inc}"'] \end{tikzcd}.$$

Definition 6.1.6. *A category is called* cocomplete *if it contains all small colimits.*

Definition 6.1.7 ([65, Notation and Terminology])**.** *A stable ∞-category is called* presentable *if it is cocomplete and it is generated, under colimits, by a set of small objects.*

6.2 Phantom objects and phantom equivalences

In the main result of [22], the notions of phantom objects and of phantom equivalences play a very important role. As we improve that result, we now introduce these notions as well as some of their properties.

Remark 6.2.1. Recall that an object K in an ∞-category M is called *compact* if the functor $\mathrm{Hom}_M(K, -) : M \to \mathrm{Spc}$ preserves filtered colimits, where Spc stands for the ∞-category of spaces; consult [22, §2].

In the next definitions we follow [22, §2].

Definition 6.2.2 ([22, Definitions 2.4 and 2.5])**.** *Let M a cocomplete ∞-category.*

(i) *An object m of M is called a* phantom object *if* $\mathrm{Hom}_M(K, m) \simeq *$ *for every compact object K of M.*

(ii) *A morphism $f : m \to m'$ in M is called a* phantom equivalence *if* $\mathrm{Hom}_M(K, f)$ *is an equivalence of spaces for every compact object K of M.*

Remark 6.2.3. We call the attention to four important consequences of Definition 6.2.2:

(i) A final object is necessarily a phantom object and that an equivalence is a phantom equivalence,

(ii) In case the ∞-category M is compactly generated, then a phantom object is a final object and a phantom equivalence is an equivalence,

(iii) If M is not compactly generated, then the class of phantom equivalences is strictly bigger than the class of equivalences!

(iv) Finally, in a stable ∞-category one has that a morphism is a phantom equivalence if and only if its cofiber is a phantom object.

Definition 6.2.4 ([22, §2]). *An ∞-category M is called* semi-additive *if it has a zero object, admits finite coproducts and products, and the canonical morphism from the coproduct to the product is an equivalence.*

Note that for M a semi-additive cocomplete ∞-category admitting countable products for every object m there is a canonical morphism $\oplus_{\mathbb{N}} m \to \prod_{\mathbb{N}} m$. With this in mind:

Definition 6.2.5 ([22, Definition 2.7]). *Let M be a semi-additive cocomplete ∞-category admitting countable products. Then for $m \in M$ we define the following object:*

$$\prod/\oplus(m) := \mathrm{cofib}\Big(\bigoplus_{\mathbb{N}} m \to \prod_{\mathbb{N}} m\Big).$$

The next Lemma is important because it allows us to conclude, in some cases, whether an object is a phantom object.

Lemma 6.2.6 ([22, Lemma 2.8]). *For M a semi-additive cocomplete ∞-category admitting countable products and $m \in M$, if the morphism*

$$m \xrightarrow{\mathrm{diag}} \prod_{\mathbb{N}} m \to \prod/\oplus(m)$$

is trivial, then m is a phantom object.

Remark 6.2.7. We point out that the assumption of Lemma 6.2.6 implies that the map $\mathrm{diag} : m \to \prod_{\mathbb{N}} m$ factors through the canonical map $\oplus_{\mathbb{N}} m \to \prod_{\mathbb{N}} m$. In Lemma 6.2.8 we show that this property is preserved under colimit preserving functors.

Lemma 6.2.8 ([75, Lemma 3.1]). *Let o be an object of an ∞-category $\mathcal{E}$ such that the diagonal map $\Delta : o \to \Pi_{\mathbb{N}} o$ factors through the canonical map $\oplus_{\mathbb{N}} o \to \Pi_{\mathbb{N}} o$. Given a colimit preserving functor $F : \mathcal{E} \to \mathcal{D}$, the diagonal map $\Delta : F(o) \to \Pi_{\mathbb{N}} F(o)$ factors through the canonical map $\oplus_{\mathbb{N}} F(o) \to \Pi_{\mathbb{N}} F(o)$.*

Proof. By assumption, there is a map $h : o \to \oplus_{\mathbb{N}} o$ such that the following diagram is commutative:

$$\begin{tikzcd} & \oplus_{\mathbb{N}} o \arrow[d, "\mathrm{can}"] \\ o \arrow[ur, "h"] \arrow[r, "\mathrm{diag}"'] & \prod_{\mathbb{N}} o. \end{tikzcd} \tag{6.1}$$

Consequently, applying the functor F to the factorization (6.1) and bearing in mind the definition of product, one obtains:

$$\begin{tikzcd}
& F\big(\bigoplus_{\mathbb{N}} o\big) = \bigoplus\big(F(o)\big) \arrow[d, "\mathrm{can}"] \arrow[dr, "\mathrm{can}"] & \\
F(o) \arrow[ur, "F(h)"] \arrow[r, "F(\mathrm{diag})"'] \arrow[rr, bend right, "\mathrm{diag}"'] & F\big(\prod_{\mathbb{N}} o\big) \arrow[r, "\mathrm{can}"'] & \prod_{\mathbb{N}} F(o).
\end{tikzcd} \tag{6.2}$$

The proof of the lemma ends once we have that (6.2) is commutative.

The equality $\mathrm{can} \circ F(h) = F(\mathrm{diag})$ is an immediate consequence of (6.1) and the fact that F preserves colimits. To prove that $\mathrm{diag} = \mathrm{can} \circ F(\mathrm{diag})$ just consider the following commutative diagram, where p_i is the canonical map from $\prod_{\mathbb{N}} F(o)$ to its i^{th} component with $i \in \mathbb{N}$, and have in mind the universal property of products.

$$\begin{tikzcd}
& & F(o) \arrow[d, "F(\mathrm{diag})"] \arrow[dddll, bend right, "\mathrm{id}=F(\mathrm{id})"'] \arrow[dddrr, bend left, "\mathrm{id}=F(\mathrm{id})"] & & & \\
& & F(\prod_{\mathbb{N}} o) \arrow[d, "\mathrm{can}"] \arrow[ddll, "F(p_1)"'] \arrow[ddrr, "F(p_n)"] & & & \\
& & \prod_{\mathbb{N}} F(o) \arrow[dll] \arrow[drr] & & & \\
F(o) & & \cdots & & F(o) & \cdots .
\end{tikzcd}$$

To show that $\mathrm{can} = \mathrm{can} \circ \mathrm{can}$ in (6.2) it is sufficient to consider the next commutative diagram:

$$\begin{tikzcd}
& & F(\oplus o) = \oplus F(o) \arrow[d, "\mathrm{can}"] \arrow[dddll, bend right] \arrow[dddrr, bend left] & & & \\
& & F(\prod_{\mathbb{N}} o) \arrow[d, "\mathrm{can}"] \arrow[ddll, "F(p_1)"'] \arrow[ddrr, "F(p_n)"] & & & \\
& & \prod_{\mathbb{N}} F(o) \arrow[dll] \arrow[drr] & & & \\
F(o) & & \cdots & & F(o) & \cdots
\end{tikzcd}$$

Therefore, (6.2) is commutative.

□

6.3 Localising invariants

In this section we are going to follow [19, 20, 23, 22] in order to define the notion of localising invariant in two different contexts and also to relate those definitions.

Definition 6.3.1. *Let $\mathcal{D}$ be a stable presentable ∞-category. A functor*

$$E : \mathrm{Cat}^{\mathrm{ex}}_{\infty} \longrightarrow \mathcal{D}$$

is called a stable finitary localising invariant *of small stable ∞-categories if it inverts Morita equivalences (consult [19, Definition 2.14]), preserves filtered colimits, and satisfies localization, that is: sends exact sequences*

$$\mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C}$$

of small stable ∞-categories to cofiber sequences

$$E(\mathcal{A}) \longrightarrow E(\mathcal{B}) \longrightarrow E(\mathcal{C})$$

in $\mathcal{D}$.

Remark 6.3.2. In the original definition, see [19, Definition 8.1], this invariant is referred to as a localising invariant. We add the words stable finitary because that is the way this invariant is referred to in [23, 22].

Remark 6.3.3. Thanks to [19, §8], there is a universal stable finitary localising invariant $\mathcal{U}_{\mathrm{loc}} : \mathrm{Cat}^{\mathrm{ex}}_{\infty,*} \to \mathcal{M}_{\mathrm{loc}}$. Note that due to this universal property, $\mathcal{M}_{\mathrm{loc}}$ is called the *(large) ∞-category of noncommutative motives*. A consequence of the existence of $\mathcal{U}_{\mathrm{loc}}$ is that given a stable finitary localising invariant $E : \mathrm{Cat}^{\mathrm{ex}}_{\infty} \longrightarrow M$ there is an unique induced functor $\overline{E} : \mathcal{M}_{\mathrm{loc}} \longrightarrow M$ such that the following diagram is commutative:

$$\begin{tikzcd} \mathrm{Cat}^{\mathrm{ex}}_{\infty,*} \arrow[r, "E"] \arrow[d, "\mathcal{U}_{\mathrm{loc}}"'] & M \\ \mathcal{M}_{\mathrm{loc}} \arrow[ur, dashed, "\overline{E}"'] & \end{tikzcd}.$$

Moreover, $\overline{E}$ is a colimit preserving functor. For more details consult [19, Theorem 8.7].

Remark 6.3.4. An important result regarding the universal stable finitary localising invariant $\mathcal{U}_{\mathrm{loc}}$ is that it is symmetric monoidal, see [20, Theorem 5.8].

The next two notions are taken from [23].

Definition 6.3.5. *Consider the following square in* $\mathrm{Cat}^{\mathrm{Lex}}_{\infty,*}$*:*

$$\begin{tikzcd} D \arrow[r, "\phi"] \arrow[d, "\psi'"'] & C \arrow[d, "\psi"] \\ D' \arrow[r, "\phi'"'] & C'. \end{tikzcd} \tag{6.3}$$

Then, (6.3) is called an excisive square *if* *(i) The functor ϕ and ϕ' are fully faithful,*

(ii) The induced functor on the cofibers of ϕ and ϕ' is an equivalence.

Definition 6.3.6. *Let M be a stable cocomplete ∞-category. A functor*

$$E : \mathrm{Cat}_{\infty,*}^{\mathrm{Lex}} \longrightarrow M$$

is called a finitary localising invariant *if it inverts Morita equivalences, preserves filtered colimits, and sends excisive squares in* $\mathrm{Cat}_{\infty,*}^{\mathrm{Lex}}$ *to pushout squares.*

We can now state some results obtained in [23] that allow us relate these two notions of localising invariants. The following statements were proved in [23, §6].

Remark 6.3.7. Take $H : \mathrm{Cat}_{\infty,*}^{\mathrm{Lex}} \to M$ a finitary localising invariant and $E : \mathrm{Cat}_{\infty,*}^{\mathrm{ex}} \to M$ a stable finitary localising invariant. Then, the following holds:

(i) The composition $H \circ \mathrm{inc}$ is a stable finitary localising invariant,

(ii) The natural transformation $H \to H \circ \mathrm{inc} \circ \mathrm{stab}$ is an equivalence,

(iii) The composition $E \circ \mathrm{stab}$ is a finitary localising invariant,

(iv) The natural transformation $E \circ \mathrm{stab} \circ \mathrm{inc} \to E$ is an equivalence.

Note that these four statements will play a crucial role in the proof of Theorem 7.2.1 below because they allow us to make use of the theory of noncommutative motives.

7

Improvement of the Farrell-Jones isomorphism conjecture

This is the key chapter of Part II: we improve a result regarding the Farrell-Jones Isomorphism conjecture. We are going to introduce the Farrell-Jones conjecture and incremental generalizations that have been made to this conjecture since 1993. The main result of this chapter, see Theorem 7.2.1 is that (a generalized) Farrell-Jones conjecture holds, under certain conditions, for finitary localising invariants.

7.1 Generalizations of the Farrell-Jones conjecture

In this section we are going to follow much of the introduction in [75]. To state the Farrell-Jones conjecture it is important that we recall the following definition:

Definition 7.1.1. *A group V is called* virtually cyclic *if it contains a cyclic subgroup of finite index.*

In [30], Farrell and Jones conjectured that the algebraic K-theory of the group ring RG, for R a commutative ring and G a group, is determined by the algebraic K-theory of the group rings RV, with V a virtually cyclic subgroup of G. Note that originally the ring R was assumed to be the ring of integers.

In the late 90's Davis and Lück [25] proposed a general setting for stating the Farrell-Jones conjecture. To do it let us remember some definitions.

Definition 7.1.2. *Let G be a group, we define the* orbit category *$Or(G)$ of G to be the category whose objects are the left G-sets G/V, with V a subgroup of G, and the morphisms are the maps of G-sets $G/V \to G/V'$.*

Definition 7.1.3 ([23, Definition 6.44])**.** *Let G be a group and $\mathcal{F}$ a set of subgroups of G. The set $\mathcal{F}$ is called a* family of subgroups of *G if it is closed under taking subgroups and conjugation in G.*

Definition 7.1.4. *Let G be a group and let $\mathcal{F}$ be a family of subgroups of G. We denote by $Or_{\mathcal{F}}(G)$ the full subcategory of $Or(G)$ whose objects are the left G-sets with stabilizers in $\mathcal{F}$.*

Definition 7.1.5 (Assembly maps as in [25])**.** *Let G be a group, $\mathcal{F}$ a family of subgroups of G and $\mathbf{E} : Or(G) \longrightarrow Spt$ be a functor with values in the category of Spectra, consult [35, Definition 1.2.1]. The* **E**-assembly map *is the naturally induced map from the homotopy colimit of* **E** *over $Or_{\mathcal{F}}(G)$ to the homotopy colimit of* **E** *over $Or(G)$:*

$$\textit{(ho)}\,\mathrm{colim}_{Or_{\mathcal{F}}(G)}\,\mathbf{E} \longrightarrow \textit{(ho)}\,\mathrm{colim}_{Or(G)}\,\mathbf{E} = \mathbf{E}(G/G).$$

Conjecture 7.1.6 ([25, Definition 5.1])**.** *Under the conditions of Definition 7.1.5, the* **E**-isomorphism conjecture *is the claim that the* **E***-assembly map is a weak equivalence.*

In addition to the formulation of Conjecture 7.1.6, Davis and Lück proved, in [25, §5.3], that the classical Farrell-Jones isomorphism conjecture is equivalent to the **E**-isomorphism conjecture when $\mathcal{F}$ is the family of virtually cyclic subgroups and **E** is the following composition

$$\mathbf{E} : \mathrm{Or}(G) \xrightarrow{\overline{?}} \mathrm{Grp} \xrightarrow{R[-]} \mathrm{Cat}_R \xrightarrow{\mathbb{K}} \mathrm{Spt}, \tag{7.1}$$

where Grp stands for the category of groupoids, Cat_R for the category of R-linear categories, $\overline{?}$ is the functor that sends a left G-set G/V to the associated groupoid $\overline{G/V}$, $R[-]$ stands for the R-linearization functor, and $\mathbb{K}$ is the non-connective K-theory functor.

Remark 7.1.7.
(i) One of the advantages of this general setting is that by simply replacing $\mathbb{K}$ by any other functor H, one immediately obtains a variant of the classical Farrell-Jones conjecture. Examples of such functors H include homotopy K-theory [9], Hochschild and Cyclic Homology [63], topological Hochschild Homology [62], etc.

(ii) Given all these variations of the Farrell-Jones conjecture, Balmer and Tabuada, making use of the theory of noncommutative motives, explicitly described in [7] the Fundamental Isomorphism Conjecture which implies all these variants of the Farrell-Jones conjecture.

In the late 00's, Bartels and Reich [12] introduced the Farrell-Jones conjecture with coefficients in a R-linear category C equipped with a right G-action. Similarly to the setting of Davis and Lück, they considered the following composition

$$\mathbf{E} : \mathrm{Or}(G) \xrightarrow{C *_G -} \mathrm{Cat}_R \xrightarrow{\mathbb{K}} \mathrm{Spt}, \tag{7.2}$$

where $C *_G -$ is the functor that sends a left G-set G/V to its "tensor product over G with C", consult [12, §2] for details.

In the particular case where C is the R-linear category $R_\oplus$ obtained from R by formally adding finite direct sums, equipped with the trivial G-action, the above composition (7.2) reduces to (7.1), see [12, Remark 3.3]. Hence, the setting of Bartels and Reich generalizes the one of Davis and Lück. Other choices of C lead, for example, to the Farrell-Jones conjecture with coefficients in a twisted group ring or, more generally, with coefficients in a crossed product ring.

Recently, Bunke, Kasprowski and Winges in [22] generalized the setting of Bartels and Reich to the realm of ∞-categories. The main result of this second part of the thesis is obtained in this setting. To state it, we still need to recall some concepts and introduce some notation.

Definition 7.1.8 ([1, Definition A.17]). *Consider categories X,Y and Z, and functors $F : X \longrightarrow Y$ and $G : X \longrightarrow Z$. The* left Kan extension *of G along F consists of a functor $J : Y \longrightarrow Z$ and a natural transformation $\eta : G \longrightarrow J \circ F$ that are inicial among those pairs. That is, that pair satisfies the following universal property: if exists a functor $L : Y \longrightarrow Z$ and a natural transformation $\alpha : G \longrightarrow L \circ F$, then there exists a unique natural transformation $\sigma : J \longrightarrow L$ such that $\alpha = \sigma \circ F \circ \eta$. In diagrams:*

$$\begin{tikzcd} & Y \arrow[dr, dashed, "J"] & \\ X \arrow[ur, "F"] \arrow[rr, "G"'] & & Z \end{tikzcd} \qquad \begin{tikzcd} & J \circ F \arrow[dr, dashed, "\sigma\circ F"] & \\ G \arrow[ur, "\eta"] \arrow[rr, "\alpha"'] & & L \circ F. \end{tikzcd} \tag{7.3}$$

Notation 7.1.9. For this setting, let us fix some notation:

(i) By BG we denote be the groupoid with one object and group of automorphisms G.

(ii) We denote by $j^G : BG \longrightarrow \mathrm{Or}(G)$ the canonical inclusion sending the unique object of BG to the left G-set $G/\{1\}$.

(iii) By C we mean a small left-exact ∞-category equipped with a right G-action, i.e., an object of $\mathrm{Fun}(BG, \mathrm{Cat}^{\mathrm{Lex}}_{\infty,*})$, the ∞-category of functors between those ∞-categories, see [19, page 14].

(iv) We will denote the left Kan-extension of C along j^G by $C_G : \mathrm{Or}(G) \to \mathrm{Cat}^{\mathrm{Lex}}_{\infty,*}$.

(v) At last, let $H : \mathrm{Cat}^{\mathrm{Lex}}_{\infty,*} \longrightarrow \mathrm{M}$ be a functor with values in a cocomplete stable ∞-category.

Under Notation 7.1.9, Bunke, Kasprowski and Winges considered the following composition:

$$\mathbf{E} : \mathrm{Or}(G) \xrightarrow{C_G} \mathrm{Cat}^{\mathrm{Lex}}_{\infty,*} \xrightarrow{H} \mathrm{M}. \tag{7.4}$$

As explained in [23, Example 1.6], in the particular case when the ∞-category C is R-linear, the functor C_G reduces to $C *_G -$. Hence, the setting of Bunke, Kasprowski, and Winges generalizes the setting of Bartels and Reich. Note that the setting of Bunke, Kasprowski and Winges is very general: one can vary the group G, the family of subgroups $\mathcal{F}$, the ∞-category of coefficients C and also the functor H. Nevertheless, they were able to prove that the $\mathbf{E}$-isomorphism conjecture (with respect to (7.4)) holds for a large class of lax monoidal functors (consult [1, Definition 3.1]) H:

Theorem 7.1.10 ([22, Theorem 1.4])**.** *Let G be a group, $\mathcal{F}$ a family of subgroups of G, C a small left-exact ∞-category equipped with a right G-action, and $H : \mathrm{Cat}^{\mathrm{Lex}}_{\infty,*} \to \mathrm{M}$ a functor with values in a cocomplete stable ∞-category. Assume that G is a Dress-Farrell-Hsiang-Jones (DFHJ) group relative to $\mathcal{F}$ (consult [22, Definition 7.1]) and that H is a lax monoidal finitary localising invariant. Under these assumptions, the assembly map*

$$A_{\mathcal{F},H\circ C_G} : \operatorname{colim}_{Or_{\mathcal{F}}(G)}(H \circ C_G) \longrightarrow H(\operatorname{colim}_{BG} C) \tag{7.5}$$

is a phantom equivalence.

Remark 7.1.11. Note that G/G is the terminal object of the orbit category $\mathrm{Or}(G)$. Hence, the colimit of the composition $H \circ C_G$ over $\mathrm{Or}(G)$ is given by $(H \circ C_G)(G/G)$. We have the computation $\operatorname{colim}_{BG} C \simeq \operatorname{colim}_{BG} \mathrm{Res}^G_G(C) \simeq C_G(G/G)$, see [22, (1.3)]. Consequently, the assembly map (7.5) agrees with the **E**-assembly map of the above composition (7.4).

Remark 7.1.12. Theorem 7.1.10 (together with Remark 6.2.3(ii)) proves the Farrell-Jones conjecture for a very large class of groups G, of coefficients C, and of functors H.

For a precise notion of Dress-Farrell-Hsiang-Jones (DFHJ) groups relative to a family $\mathcal{F}$ we invite the reader to consult [41, §2]. Note that this notion, DFHJ groups, is built upon other notions and generalizes some types of groups. We now give examples of this kind of groups.

Example 7.1.13 (Finitely homotopy $\mathcal{F}$-amenable groups). This notion was already used in [11, 10, 90] to prove some cases of the Farrell-Jones conjecture with coefficients in additive categories. For a precise definition consult [8, Definition 2.11] and [22, Definition 5.4].

(i) Let $\mathcal{F}$ be the family of virtually cyclic subgroups. It is known that CAT(0)-groups are $\mathcal{F}$-amenable groups, see [8, Example 2.14] .

(ii) Another example is provided by the general linear group $\mathrm{GL}_n(\mathbb{Z})$. Let $\mathcal{F}$ be the family of subgroups generated by the virtually cyclic subgroups and the proper parabolic subgroups of $\mathrm{GL}_n(\mathbb{Z})$. One has that $\mathrm{GL}_n(\mathbb{Z})$ is $\mathcal{F}$-amenable, consult [8, Theorem 4.1].

Example 7.1.14 (Farrell-Hsiang groups). The notion of Farrell-Hsiang groups can be seen in [13, Definition 1.14]. From [13, Lemma 2.8] one has that $\mathbb{Z}^2 \rtimes_{\mathrm{id}} \mathbb{Z}/2$ is a Farrell-Hsiang group relative to the family of virtually cyclic subgroups of $\mathbb{Z}^2 \rtimes_{\mathrm{id}} \mathbb{Z}/2$, where $\mathbb{Z}^2 \rtimes_{\mathrm{id}} \mathbb{Z}/2$ is the semidirect product relative to $\mathrm{id} : \mathbb{Z}^2 \to \mathbb{Z}^2$.

Example 7.1.15 (Dress-Farrell-Hsiang-Jones groups). At last, we give an example of DFHJ groups relative to a family of subgroups. Take a non-zero algebraic number w which is not a root of unity and consider $\mathbb{Z}[w, w^{-1}]$ the subring of $\mathbb{C}$ generated by $\mathbb{Z}$, w and w^{-1}. Moveover, $\mathbb{Z}$ acts on $\mathbb{Z}[w, w^{-1}]$ via multiplication with w. Then, thanks to [41, Proposition 3.3], $\mathbb{Z}[w, w^{-1}] \rtimes \mathbb{Z}$ is a DFHJ group with respect to the family of virtually abelian subgroups.

7.2 An improvement of Theorem 7.1.10

In this section we are going to show that one can drop the lax monoidal assumption on the finitary localising invariant H of Theorem 7.1.10 and still obtain a phantom equivalence.

Theorem 7.2.1 ([75, Theorem 2.6]). *Let G be a group, $\mathcal{F}$ a family of subgroups of G, C a small left-exact ∞-category equipped with a right G-action, and $H : \mathrm{Cat}^{\mathrm{Lex}}_{\infty,*} \to \mathrm{M}$ a functor with values in a cocomplete stable ∞-category. Assume that G is a Dress-Farrell-Hsiang-Jones (DFHJ) group relative to $\mathcal{F}$ and that H is a finitary localising invariant (the lax monoidal assumption is not necessary). Under these assumptions, the assembly map*

$$A_{\mathcal{F},H\circ C_G} : \mathrm{colim}_{Or_{\mathcal{F}}(G)}(H \circ C_G) \longrightarrow H(\mathrm{colim}_{BG}\, C) \tag{7.6}$$

is a phantom equivalence.

Proof. Since we are assuming that H is a finitary localising invariant, Remark 6.3.7 (i) implies that the functor $H \circ \mathrm{inc}$ is a stable finitary localising invariant. Consequently, $H \circ \mathrm{inc}$ factors through the universal stable localising invariant $\mathcal{U}_{\mathrm{loc}}$, see Remark 6.3.3. We denote the induced functor by $\overline{H \circ \mathrm{inc}}$.

Thanks to Remark 6.3.7(ii) the functors H and $H \circ \mathrm{inc} \circ \mathrm{stab}$ are naturally equivalent. Hence, under these notations, we have the following commutative diagram (up to equivalence):

$$\begin{tikzcd}
\mathrm{Or}(G) \arrow[dr, "C_G"] \arrow[drr, bend left, "H\circ C_G"] & & \\
BG \arrow[u, "j^G"] \arrow[r, "C"'] & \mathrm{Cat}^{\mathrm{Lex}}_{\infty,*} \arrow[r, "H"] \arrow[d, shift right, "\mathrm{stab}"'] & \mathrm{M} \\
& \mathrm{Cat}^{\mathrm{ex}}_{\infty,*} \arrow[u, shift right, "\mathrm{inc}"'] \arrow[ur, "H\circ\mathrm{inc}"'] \arrow[d, "\mathcal{U}_{\mathrm{loc}}"'] & \\
& \mathcal{M}_{\mathrm{loc}} \arrow[uur, bend right, "\overline{H\circ\mathrm{inc}}"'] &
\end{tikzcd} \tag{7.7}$$

Considering, again, Remark 6.3.7(iii) we have that $\mathcal{U}_{\mathrm{loc}} \circ \mathrm{stab}$ is a finitary localising. In addition to this, as from Remark 6.3.4 and [22, §1] we have, respectively, that the functors $\mathcal{U}_{\mathrm{loc}}$ and stab are symmetric monoidal, $\mathcal{U}_{\mathrm{loc}} \circ \mathrm{stab}$ is a a lax monoidal finitary localising invariant. Consequently, Theorem 7.1.10 (with H replaced by $\mathcal{U}_{\mathrm{loc}} \circ \mathrm{stab}$) implies that the assembly map

$$A_{\mathcal{F},(\mathcal{U}_{\mathrm{loc}}\circ\mathrm{stab}\,\circ C_G)} : \mathrm{colim}_{\mathrm{Or}_{\mathcal{F}}(G)}(\mathcal{U}_{\mathrm{loc}} \circ \mathrm{stab} \circ C_G) \longrightarrow (\mathcal{U}_{\mathrm{loc}} \circ \mathrm{stab})(\mathrm{colim}_{BG}\, C) \tag{7.8}$$

is a phantom equivalence. We denote by cof the cofiber of (7.8). As explained in the proofs of [22, Theorem 1.4, Proposition 2.31], the diagonal map $\Delta : \mathrm{cof} \to \Pi_{\mathbb{N}}\mathrm{cof}$ factors through the canonical map $\oplus_{\mathbb{N}}\mathrm{cof} \to \Pi_{\mathbb{N}}\mathrm{cof}$. Therefore, since the functor $\overline{H \circ \mathrm{inc}}$ is colimit preserving, consult Remark 6.3.3, it follows from Lemma 6.2.8 above that the morphism $\Delta : (\overline{H \circ \mathrm{inc}})(\mathrm{cof}) \to \Pi_{\mathbb{N}}(\overline{H \circ \mathrm{inc}})(\mathrm{cof})$ factors through the canonical morphism $\oplus_{\mathbb{N}}(\overline{H \circ \mathrm{inc}})(\mathrm{cof}) \to \Pi_{\mathbb{N}}(\overline{H \circ \mathrm{inc}})(\mathrm{cof})$. Following Lemma 6.2.6, this implies, in particular, that $(\overline{H \circ \mathrm{inc}})(\mathrm{cof})$ is a phantom object.

Now, let us apply the functor $\overline{H \circ \mathrm{inc}}$ to the above assembly map (7.8). Since the functor $\overline{H \circ \mathrm{inc}}$ is colimit preserving, it follows from the above commutative diagram (7.7) that $(\overline{H \circ \mathrm{inc}})(7.8)$ identifies with the assembly map (7.6). Moreover, $(\overline{H \circ \mathrm{inc}})(\mathrm{cof})$ identifies with the cofiber of (7.6). Consequently, since $(\overline{H \circ \mathrm{inc}})(\mathrm{cof})$ is a phantom object and thanks to Remark 6.2.3(iv), we conclude that the assembly map (7.6) is a phantom equivalence. □

Remark 7.2.2. It is relevant to point out that Theorem 7.2.1 was incorporated in the latest version of [22]; consult [22, Remark 1.5].

Remark 7.2.3 (Lax monoidal). Note that being lax monoidal is an extra structure on a functor and not a property of a functor[1]. Hence, Theorem 7.2.1 shows that a finitary localising invariant H does not need to be equipped with this extra structure in order for the assembly map (7.5) to be a phantom equivalence.

Remark 7.2.4 (Full Farrell-Jones conjecture). There is a variant of the Farrell-Jones conjecture called the *Full Farrell-Jones conjecture*, consult [22, Definition 1.6]. In addition to Theorem 7.1.10, Bunke, Kasprowski and Winges also proved in [22] that the class of groups which satisfy the Full Farrell-Jones conjecture for $H : \mathrm{Cat}^{\mathrm{Lex}}_{\infty,*} \to \mathrm{M}$ a lax monoidal finitary localising invariant is quite large and closed under several constructions; consult [22, Theorem 1.7]. Similarly to Theorem 7.1.10, this result also holds ipsis verbis with H a finitary localising invariant (the lax monoidal assumption is not necessary). Simply follow the same proof of [22, Theorem 1.7] and replace the references to [22, Theorems 5.1, 6.1 and 7.1] by the reference to Theorem 7.2.1 above.

[1]Let C and D be two symmetric monoidal ∞-categories such that D admits all colimits. As proved in [32, Proposition 2.12], to give a lax monoidal structure on a functor $H : C \to D$ is equivalent to give a E^{∞}-monoid structure on the object H of the symmetric monoidal ∞-category $\mathrm{Fun}(C, D)$ (equipped with the Day convolution product).

Bibliography

[1] M. Aguiar and S. Mahajan. *Monoidal functors, species and Hopf algebras*. English. Vol. **29**. CRM Monogr. Ser. Providence, RI: American Mathematical Society (AMS), 2010. ISBN: 978-0-8218-4776-3 (cit. on p. 77).

[2] Y. André. *Une introduction aux motifs. Motifs purs, motifs mixtes, périodes*. French. Vol. **17**. Panoramas et Synthèses. Société Mathématique de France (SMF), Paris, 2004, pp. xi + 261. ISBN: 2-85629-164-3/pbk (cit. on pp. 25–28, 36, 48).

[3] Y. André and B. Kahn. "Nilpotence, radicals and monoidal structures. With an appendix by Peter O'Sullivan." French. In: *Rend. Semin. Mat. Univ. Padova* **108** (2002), pp. 107–291. ISSN: 0041-8994 (cit. on pp. 28, 29, 31).

[4] M. Artin. "Algebraization of formal moduli. II: Existence of modifications". English. In: *Annals of Mathematics. Second Series* **91** (1970), pp. 88–135. ISSN: 0003-486X. DOI: 10.2307/1970602 (cit. on p. 56).

[5] M. Artin and D. Mumford. "Some elementary examples of unirational varieties which are not rational". English. In: *Proceedings of the London Mathematical Society. Third Series* **25** (1972), pp. 75–95. ISSN: 0024-6115. DOI: 10.1112/plms/s3-25.1.75 (cit. on p. 56).

[6] A. Auel, M. Bernardara, and M. Bolognesi. "Fibrations in complete intersections of quadrics, Clifford algebras, derived categories, and rationality problems". English. In: *Journal de Mathématiques Pures et Appliquées. Neuvième Série* **102**.1 (2014), pp. 249–291. ISSN: 0021-7824. DOI: 10.1016/j.matpur.2013.11.009 (cit. on p. 53).

[7] P. Balmer and G. Tabuada. "Fundamental isomorphism conjecture via non-commutative motives". English. In: *Math. Nachr.* **286**.8-9 (2013), pp. 791–798. ISSN: 0025-584X. DOI: 10.1002/mana.201200057 (cit. on p. 76).

[8] A. Bartels. "K-theory and actions on Euclidean retracts". English. In: *Proceedings of the international congress of mathematicians, ICM 2018, Rio de Janeiro, Brazil, August 1–9, 2018. Volume II. Invited lectures*. Hackensack, NJ: World Scientific; Rio de Janeiro:

Sociedade Brasileira de Matemática (SBM), 2018, pp. 1041–1062. ISBN: 978-981-3272-91-0; 978-981-327-287-3; 978-981-3272-89-7. DOI: 10.1142/9789813272880_0087 (cit. on p. 78).

[9] A. Bartels and W. Lück. "Isomorphism conjecture for homotopy K-theory and groups acting on trees". English. In: *J. Pure Appl. Algebra* **205**.3 (2006), pp. 660–696. ISSN: 0022-4049. DOI: 10.1016/j.jpaa.2005.07.020 (cit. on p. 76).

[10] A. Bartels and W. Lück. "The Borel conjecture for hyperbolic and CAT(0)-groups". English. In: *Ann. Math. (2)* **175**.2 (2012), pp. 631–689. ISSN: 0003-486X. DOI: 10.4007/annals.2012.175.2.5 (cit. on p. 78).

[11] A. Bartels, W. Lück, and H. Reich. "The K-theoretic Farrell-Jones conjecture for hyperbolic groups". English. In: *Invent. Math.* **172**.1 (2008), pp. 29–70. ISSN: 0020-9910. DOI: 10.1007/s00222-007-0093-7 (cit. on p. 78).

[12] A. Bartels and H. Reich. "Coefficients for the Farrell-Jones conjecture". English. In: *Adv. Math.* **209**.1 (2007), pp. 337–362. ISSN: 0001-8708. DOI: 10.1016/j.aim.2006.05.005 (cit. on pp. 3, 76).

[13] A. C. Bartels, F. T. Farrell, and W. Lück. "The Farrell-Jones conjecture for cocompact lattices in virtually connected Lie groups". English. In: *J. Am. Math. Soc.* **27**.2 (2014), pp. 339–388. ISSN: 0894-0347. DOI: 10.1090/S0894-0347-2014-00782-7 (cit. on p. 78).

[14] A. A. Beilinson. "Coherent sheaves on $\mathbb{P}^n$ and problems of linear algebra". English. In: *Funct. Anal. Appl.* **12** (1979), pp. 214–216. ISSN: 0016-2663. DOI: 10.1007/BF01681436 (cit. on p. 57).

[15] M. Bernardara. "A semiorthogonal decomposition for Brauer-Severi schemes". English. In: *Math. Nachr.* **282**.10 (2009), pp. 1406–1413. ISSN: 0025-584X. DOI: 10.1002/mana.200610826 (cit. on p. 52).

[16] M. Bernardara, M. Bolognesi, and D. Faenzi. "Homological projective duality for determinantal varieties". English. In: *Advances in Mathematics* **296** (2016), pp. 181–209. ISSN: 0001-8708. DOI: 10.1016/j.aim.2016.04.003 (cit. on p. 64).

[17] M. Bernardara, M. Marcolli, and G. Tabuada. "Some remarks concerning Voevodsky's nilpotence conjecture". English. In: *Journal für die Reine und Angewandte Mathematik* **738** (2018), pp. 299–312. ISSN: 0075-4102; 1435-5345/e (cit. on pp. 1, 2, 32, 36, 49, 51, 53, 55, 56, 62).

[18] M. Bernardara et al. "Fano fourfolds of K3 type". In: (2021). DOI: 10.48550/ARXIV.2111.13030. URL: https://arxiv.org/abs/2111.13030 (cit. on p. 65).

[19] A. J. Blumberg, D. Gepner, and G. Tabuada. "A universal characterization of higher algebraic K-theory". English. In: *Geom. Topol.* **17**.2 (2013), pp. 733–838. ISSN: 1465-3060. DOI: 10.2140/gt.2013.17.733 (cit. on pp. 73, 77).

[20] A. J. Blumberg, D. Gepner, and G. Tabuada. "Uniqueness of the multiplicative cyclotomic trace". English. In: *Adv. Math.* **260** (2014), pp. 191–232. ISSN: 0001-8708. DOI: 10.1016/j.aim.2014.02.004 (cit. on p. 73).

[21] A. Bondal and D. Orlov. "Derived categories of coherent sheaves". English. In: *Proceedings of the international congress of mathematicians, ICM 2002, Beijing, China, August 20–28, 2002. Vol. II: Invited lectures*. Beijing: Higher Education Press; Singapore: World Scientific/distributor, 2002, pp. 47–56. ISBN: 7-04-008690-5 (cit. on p. 22).

[22] U. Bunke, D. Kasprowski, and C. Winges. "On the Farrell-Jones conjecture for localising invariants". In: (2021). DOI: 10.48550/ARXIV.2111.02490. URL: https://arxiv.org/abs/2111.02490 (cit. on pp. 3, 69–71, 73, 77–80).

[23] U. Bunke et al. "Controlled objects in left-exact ∞-categories and the Novikov conjecture". In: (2019). DOI: 10.48550/ARXIV.1911.02338. URL: https://arxiv.org/abs/1911.02338 (cit. on pp. 69, 73–75, 77).

[24] H. Clemens. "Homological equivalence, modulo algebraic equivalence, is not finitely generated". English. In: *Publ. Math., Inst. Hautes Étud. Sci.* **58** (1983), pp. 231–250. ISSN: 0073-8301 (cit. on pp. 2, 36).

[25] J. F. Davis and W. Lück. "Spaces over a category and assembly maps in isomorphism conjectures in K- and L-theory". English. In: *K-Theory* **15**.3 (1998), pp. 201–252. ISSN: 0920-3036. DOI: 10.1023/A:1007784106877 (cit. on pp. 3, 75, 76).

[26] O. Debarre and A. Kuznetsov. "Gushel-Mukai varieties: classification and birationalities". English. In: *Algebr. Geom.* **5**.1 (2018), pp. 15–76. ISSN: 2313-1691. DOI: 10.14231/AG-2018-002 (cit. on p. 59).

[27] V. Drinfeld. "DG quotients of DG categories." English. In: *J. Algebra* **272**.2 (2004), pp. 643–691. ISSN: 0021-8693. DOI: 10.1016/j.jalgebra.2003.05.001 (cit. on pp. 12–14).

[28] W. G. Dwyer and J. Spalinski. "Homotopy theories and model categories". English. In: *Handbook of algebraic topology*. Amsterdam: North-Holland, 1995, pp. 73–126. ISBN: 0-444-81779-4 (cit. on p. 20).

[29] P. Etingof et al. *Tensor categories*. English. Vol. **205**. Math. Surv. Monogr. Providence, RI: American Mathematical Society (AMS), 2015. ISBN: 978-1-4704-2024-6. DOI: 10.1090/surv/205 (cit. on pp. 11, 28).

[30] F. T. Farrell and L. E. Jones. "Isomorphism conjectures in algebraic K-theory". English. In: *J. Am. Math. Soc.* **6**.2 (1993), pp. 249–297. ISSN: 0894-0347. DOI: 10.2307/2152801 (cit. on pp. 3, 75).

[31] W. Fulton. *Intersection theory*. English. Vol. **2**. Berlin: Springer, 1998, pp. xiii + 470. ISBN: 3-540-62046-X (cit. on pp. 43, 45, 48).

[32] S. Glasman. "Day convolution for ∞-categories". English. In: *Math. Res. Lett.* **23**.5 (2016), pp. 1369–1385. ISSN: 1073-2780. DOI: `10.4310/MRL.2016.v23.n5.a6` (cit. on p. 80).

[33] S. Gorchinskiy and V. Guletskiuı. "Motives and representability of algebraic cycles on threefolds over a field". English. In: *J. Algebr. Geom.* **21**.2 (2012), pp. 347–373. ISSN: 1056-3911. DOI: `10.1090/S1056-3911-2011-00548-1` (cit. on p. 63).

[34] P. A. Griffiths. "On the periods of certain rational integrals. I, II". English. In: *Ann. Math. (2)* **90** (1969), pp. 460–495, 496–541. ISSN: 0003-486X. DOI: `10.2307/1970746` (cit. on pp. 2, 36).

[35] M. Hovey, B. Shipley, and J. Smith. "Symmetric spectra". English. In: *J. Am. Math. Soc.* **13**.1 (2000), pp. 149–208. ISSN: 0894-0347. DOI: `10.1090/S0894-0347-99-00320-3` (cit. on p. 76).

[36] D. Huybrechts. *Fourier-Mukai transforms in algebraic geometry*. English. Oxford: Clarendon Press, 2006, pp. viii + 307. ISBN: 0-19-929686-3 (cit. on pp. 44, 46, 53).

[37] D. Huybrechts. *Lectures on K3 surfaces*. English. Vol. 158. Camb. Stud. Adv. Math. Cambridge: Cambridge University Press, 2016. ISBN: 978-1-107-15304-2; 978-1-316-59419-3. DOI: `10.1017/CBO9781316594193` (cit. on p. 58).

[38] G. B. Im and G. M. Kelly. "Some remarks on conservative functors with left adjoints". English. In: *J. Korean Math. Soc.* **23** (1986), pp. 19–33. ISSN: 0304-9914 (cit. on p. 29).

[39] B. Kahn and R. Sebastian. "Smash-nilpotent cycles on abelian 3-folds". English. In: *Math. Res. Lett.* **16**.5-6 (2009), pp. 1007–1010. ISSN: 1073-2780. DOI: `10.4310/MRL.2009.v16.n6.a8` (cit. on pp. 1, 35).

[40] M. Karoubi. *K-theory. An introduction*. English. Vol. **226**. Grundlehren Math. Wiss. Springer, Cham, 1978 (cit. on p. 28).

[41] D. Kasprowski et al. "The A-theoretic Farrell-Jones conjecture for virtually solvable groups". English. In: *Bull. Lond. Math. Soc.* **50**.2 (2018), pp. 219–228. ISSN: 0024-6093. DOI: `10.1112/blms.12131` (cit. on p. 78).

[42] B. Keller. "On differential graded categories". English. In: *Proceedings of the International Congress of Mathematicians (ICM), Madrid, Spain, August 22–30, 2006. Volume II: Invited lectures*. Zürich: European Mathematical Society (EMS), 2006, pp. 151–190. ISBN: 978-3-03719-022-7/hbk (cit. on pp. 7, 15).

[43] M. Kontsevich. "Mixed noncommutative motives". English. In: *Talk at the Workshop on Homological Mirror Symmetry, Miami* (2010). Notes available at `https://people.math.harvard.edu/~auroux/frg/miami10-notes/` (cit. on p. 16).

[44] M. Kontsevich. "Noncommutative motives". English. In: *Talk at the IAS on the occasion of the 61st birthday of Pierre Deligne* (2005). Video available at `http://video.ias.edu/Geometry-and-Arithmetic` (cit. on p. 16).

[45] M. Kontsevich. "Notes on motives in finite characteristic". English. In: *Algebra, arithmetic, and geometry. In honor of Y. I. Manin on the occasion of his 70th birthday. Vol. II.* Boston, MA: Birkhäuser, 2009, pp. 213–247. ISBN: 978-0-8176-4746-9; 978-0-8176-4747-6. DOI: 10.1007/978-0-8176-4747-6_7 (cit. on pp. 16, 32).

[46] O. Küchle. "On Fano 4-folds of index 1 and homogeneous vector bundles over Grassmannians". English. In: *Math. Z.* **218**.4 (1995), pp. 563–575. ISSN: 0025-5874. DOI: 10.1007/BF02571923 (cit. on p. 59).

[47] A. Kuznetsov. "Base change for semiorthogonal decompositions". English. In: *Compos. Math.* **147**.3 (2011), pp. 852–876. ISSN: 0010-437X. DOI: 10.1112/S0010437X10005166 (cit. on p. 60).

[48] A. Kuznetsov. "Derived categories of families of sextic del Pezzo surfaces". English. In: *Int. Math. Res. Not.* **2021**.12 (2021), pp. 9262–9339. ISSN: 1073-7928. DOI: 10.1093/imrn/rnz081 (cit. on pp. 59, 60).

[49] A. Kuznetsov. "Derived categories of Fano threefolds". English. In: *Proc. Steklov Inst. Math.* **264** (2009), pp. 110–122. ISSN: 0081-5438. DOI: 10.1134/S0081543809010143 (cit. on p. 64).

[50] A. Kuznetsov. "Derived categories of quadric fibrations and intersections of quadrics". English. In: *Advances in Mathematics* **218**.5 (2008), pp. 1340–1369. ISSN: 0001-8708. DOI: 10.1016/j.aim.2008.03.007 (cit. on pp. 53–55).

[51] A. Kuznetsov. "Derived categories view on rationality problems". English. In: *Rationality problems in algebraic geometry. Levico Terme, Italy, June 22-27 2015. Lectures of the CIME-CIRM course.* Cham: Springer; Florence: Fondazione CIME, 2016, pp. 67–104. ISBN: 978-3-319-46208-0; 978-3-319-46209-7. DOI: 10.1007/978-3-319-46209-7_3 (cit. on p. 57).

[52] A. Kuznetsov. "Homological projective duality". en. In: *Publications Mathématiques de l'IHÉS* **105** (2007), pp. 157–220. DOI: 10.1007/s10240-007-0006-8. URL: http://www.numdam.org/articles/10.1007/s10240-007-0006-8/ (cit. on pp. 60, 61).

[53] A. Kuznetsov. "Homological Projective Duality for Grassmannians of Lines". English. In: (2006). Available at https://arxiv.org/abs/math/0610957 (cit. on pp. 62, 63).

[54] A. Kuznetsov. "On Küchle varieties with Picard number greater than 1". English. In: *Izv. Math.* **79**.4 (2015), pp. 698–709. ISSN: 1064-5632. DOI: 10.1070/IM2015v079n04ABEH002758 (cit. on p. 59).

[55] A. Kuznetsov. "Semiorthogonal decompositions in algebraic geometry". English. In: *Proceedings of the International Congress of Mathematicians (ICM 2014), Seoul, Korea, August 13-21, 2014. Vol. II: Invited lectures*. Seoul: KM Kyung Moon Sa, 2014, pp. 635–660. ISBN: 978-89-6105-805-6/hbk; 978-89-6105-803-2/set (cit. on pp. 22, 60–62).

[56] A. Kuznetsov. "Semiorthogonal decompositions in families". In: (2021). Available at https://arxiv.org/abs/2111.00527, *to appear in the Proceedings of the ICM 2022* (cit. on p. 64).

[57] A. Kuznetsov and V. A. Lunts. "Categorical resolutions of irrational singularities". English. In: *IMRN. International Mathematics Research Notices* **2015**.13 (2015), pp. 4536–4625. ISSN: 1073-7928. DOI: 10.1093/imrn/rnu072 (cit. on p. 38).

[58] A. Kuznetsov and A. Perry. "Derived categories of Gushel-Mukai varieties". English. In: *Compositio Mathematica* **154**.7 (2018), pp. 1362–1406. ISSN: 0010-437X. DOI: 10.1112/S0010437X18007091 (cit. on p. 58).

[59] J. M. Lourenço. *The NOVAthesis LaTeX Template User's Manual*. NOVA University Lisbon. 2021. URL: https://github.com/joaomlourenco/novathesis/raw/master/template.pdf (cit. on p. iii).

[60] W. Lück. "K- and L-theory of group rings". English. In: *Proceedings of the international congress of mathematicians (ICM 2010), Hyderabad, India, August 19–27, 2010. Vol. II: Invited lectures*. Hackensack, NJ: World Scientific; New Delhi: Hindustan Book Agency, 2011, pp. 1071–1098. ISBN: 978-981-4324-32-8; 978-81-85931-08-3; 978-981-4324-30-4; 978-981-4324-35-9. URL: ebooks.worldscinet.com/ISBN/9789814324359/9789814324359_0087.html (cit. on p. 3).

[61] W. Lück. "Assembly maps". English. In: *Handbook of homotopy theory*. Boca Raton, FL: CRC Press, 2020, pp. 851–890. ISBN: 978-0-8153-6970-7; 978-1-351-25162-4. DOI: 10.1201/9781351251624-20 (cit. on p. 3).

[62] W. Lück. "On the Farrell-Jones and related conjectures". English. In: *Cohomology of groups and algebraic K-theory. Selected papers of the international summer school on cohomology of groups and algebraic K-theory, Hangzhou, China, July 1–3, 2007*. Somerville, MA: International Press; Beijing: Higher Education Press, 2010, pp. 269–341. ISBN: 978-1-57146-144-5 (cit. on p. 76).

[63] W. Lück and H. Reich. "Detecting K-theory by cyclic homology". In: *Proc. London Math. Soc. (3)* 93.3 (2006), pp. 593–634. ISSN: 0024-6115. DOI: 10.1017/S0024611506015954. URL: https://doi.org/10.1017/S0024611506015954 (cit. on p. 76).

[64] W. Lück and H. Reich. "The Baum-Connes and the Farrell-Jones conjectures in K- and L-theory". English. In: *Handbook of K-theory. Vol. 1 and 2*. Berlin: Springer, 2005, pp. 703–842. ISBN: 3-540-23019-X (cit. on p. 3).

[65] J. Lurie. “Higher algebra”. In: (2017). Notes available at `https://www.math.ias.edu/~lurie/papers/HA.pdf` (cit. on pp. 69, 70).

[66] J. Lurie. *Higher topos theory*. English. Vol. **170**. Ann. Math. Stud. Princeton, NJ: Princeton University Press, 2009. ISBN: 978-0-691-14049-0; 978-0-691-14048-3. DOI: `10.1515/9781400830558` (cit. on pp. 29, 69).

[67] Y. I. Manin. “Correspondences, motifs and monoidal transformations”. English. In: *Mat. Sb., Nov. Ser.* **77** (1968), pp. 475–507 (cit. on pp. 41, 57).

[68] M. Marcolli and G. Tabuada. “Kontsevich’s noncommutative numerical motives”. English. In: *Compositio Mathematica* **148**.6 (2012), pp. 1811–1820. ISSN: 0010-437X; 1570-5846/e (cit. on p. 40).

[69] M. Marcolli and G. Tabuada. “Noncommutative Artin motives”. English. In: *Selecta Mathematica. New Series* **20**.1 (2014), pp. 315–358. ISSN: 1022-1824; 1420-9020/e (cit. on p. 42).

[70] T. Matsusaka. “The criteria for linear equivalence and the torsion group”. English. In: *Am. J. Math.* **79** (1957), pp. 53–66. ISSN: 0002-9327. DOI: `10.2307/2372383` (cit. on pp. 1, 35).

[71] B. G. Moıshezon. “On n-dimensional compact varieties with n algebraically independent meromorphic functions”. Russian. In: *Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya* **30** (1966), pp. 133–174. ISSN: 0373-2436 (cit. on p. 56).

[72] A. Neeman. *Triangulated categories*. English. Vol. **148**. Ann. Math. Stud. Princeton, NJ: Princeton University Press, 2001. ISBN: 0-691-08685-0; 0-691-08686-9. DOI: `10.1515/9781400837212` (cit. on p. 15).

[73] M. Ornaghi and L. Pertusi. “Voevodsky’s conjecture for cubic fourfolds and Gushel-Mukai fourfolds via noncommutative $K3$ surfaces”. English. In: *Journal of Noncommutative Geometry* **13**.2 (2019), pp. 499–515. ISSN: 1661-6952; 1661-6960/e (cit. on pp. 1, 36, 51, 56–58).

[74] J. F. Reis. “A refinement of some previous results of Bernardara-Marcolli-Tabuada and Ornaghi-Pertusi”. In: (2022). DOI: `10.48550/ARXIV.2206.08893`. URL: `https://arxiv.org/abs/2206.08893` (cit. on pp. xi, xiii, 2, 20, 35).

[75] J. F. Reis. “An improvement of the Farrell-Jones conjecture for localising invariants”. In: (2023). To appear in Proceedings of the AMS. DOI: `10.48550/ARXIV.2211.15523`. URL: `https://arxiv.org/abs/2211.15523` (cit. on pp. xi, xiii, 3, 71, 75, 79).

[76] G. Tabuada. “Additive invariants of dg-categories”. French. In: *Int. Math. Res. Not.* **2005**.53 (2005), pp. 3309–3339. ISSN: 1073-7928. DOI: `10.1155/IMRN.2005.3309` (cit. on pp. 20, 21).

[77] G. Tabuada. “Chow motives versus noncommutative motives”. English. In: *J. Noncommut. Geom.* **7**.3 (2013), pp. 767–786. ISSN: 1661-6952. DOI: `10.4171/JNCG/134` (cit. on p. 32).

[78] G. Tabuada. "Noncommutative counterparts of celebrated conjectures". English. In: *K-theory in algebra, analysis and topology. ICM 2018 satellite school and workshop, La Plata and Buenos Aires, Argentina, July 16-20 and July 23-27, 2018*. Providence, RI: American Mathematical Society (AMS), 2020, pp. 353–377. ISBN: 978-1-4704-5026-7; 978-1-4704-5594-1. DOI: 10.1090/conm/749/15078 (cit. on pp. 7, 64).

[79] G. Tabuada. *Noncommutative motives* (with a preface by Yuri I. Manin). English. Vol. **63**. University Lecture Series. Providence, RI: American Mathematical Society (AMS), 2015, pp. x + 114. ISBN: 978-1-4704-2397-1/pbk; 978-1-4704-2627-9/ebook (cit. on pp. 15, 17, 20, 21, 23, 25, 28–31, 37, 38, 40, 41, 49, 50).

[80] G. Tabuada. "Recent developments on noncommutative motives". English. In: *New directions in homotopy theory. Second Mid-Atlantic Topology Conference, Johns Hopkins University, Baltimore, MD, USA, March 12-13, 2016. Proceedings*. Providence, RI: American Mathematical Society (AMS), 2018, pp. 143–173. ISBN: 978-1-4704-3774-9; 978-1-4704-4772-4. DOI: 10.1090/conm/707/14258 (cit. on pp. 49, 50).

[81] G. Tabuada. "Theorie homotopique des DG-categories". PhD thesis. 2007. DOI: 10.48550/ARXIV.0710.4303. URL: https://arxiv.org/abs/0710.4303 (cit. on p. 21).

[82] G. Tabuada and M. Van den Bergh. "Noncommutative motives of Azumaya algebras". English. In: *J. Inst. Math. Jussieu* **14**.2 (2015), pp. 379–403. ISSN: 1474-7480. DOI: 10.1017/S147474801400005X (cit. on pp. 38, 39, 50).

[83] G. Tabuada and M. Van den Bergh. "The Gysin triangle via localization and A^1-homotopy invariance". English. In: *Transactions of the American Mathematical Society* **370**.1 (2018), pp. 421–446. ISSN: 0002-9947. DOI: 10.1090/tran/6956 (cit. on p. 45).

[84] B. Toën. "Lectures on dg-categories". In: *Topics in algebraic and topological K-theory*. Vol. 2008. Lecture Notes in Math. Springer, Berlin, 2011, pp. 243–302. DOI: 10.1007/978-3-642-15708-0. URL: https://doi.org/10.1007/978-3-642-15708-0 (cit. on p. 14).

[85] B. Toën. "The homotopy theory of dg-categories and derived Morita theory". English. In: *Invent. Math.* **167**.3 (2007), pp. 615–667. ISSN: 0020-9910. DOI: 10.1007/s00222-006-0025-y (cit. on p. 21).

[86] C. Vial. "Algebraic cycles and fibrations". English. In: *Doc. Math.* **18** (2013), pp. 1521–1553. ISSN: 1431-0635 (cit. on p. 57).

[87] V. Voevodsky. "A nilpotence theorem for cycles algebraically equivalent to zero". English. In: *International Mathematics Research Notices* **1995**.4 (1995), pp. 187–199. ISSN: 1073-7928; 1687-0247/e (cit. on pp. 1, 29, 35, 41).

[88] C. Voisin. "Remarks on zero-cycles of self-products of varieties". English. In: *Moduli of vector bundles. Papers of the 35th Taniguchi symposium, Sanda, Japan and a symposium held in Kyoto, Japan, 1994*. New York, NY: Marcel Dekker, 1996, pp. 265–285. ISBN: 0-8247-9738-8 (cit. on pp. 1, 35).

[89] C. Voisin. "The Griffiths group of a general Calabi-Yau threefold is not finitely generated". English. In: *Duke Mathematical Journal* **102**.1 (2000), pp. 151–186. ISSN: 0012-7094. DOI: `10.1215/S0012-7094-00-10216-5` (cit. on pp. 2, 36).

[90] C. Wegner. "The *K*-theoretic Farrell-Jones conjecture for CAT(0)-groups". English. In: *Proc. Am. Math. Soc.* **140**.3 (2012), pp. 779–793. ISSN: 0002-9939. DOI: `10.1090/S0002-9939-2011-11150-X` (cit. on p. 78).

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