Endomorphisms of semigroups of monotone transformations

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Abstract

In this paper, we characterize the monoid of endomorphisms of the semigroup of all monotone full transformations of a finite chain, as well as the monoids of endomorphisms of the semigroup of all monotone partial transformations and of the semigroup of all monotone partial permutations of a finite chain.

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Introduction

For $n \in \mathbb{N}$, let Ω_n be a finite set with n elements. Denote by $\mathcal{PT}_n$ the monoid (under composition) of all partial transformations of Ω_n. The submonoid of $\mathcal{PT}_n$ of all full transformations of Ω_n and the (inverse) submonoid of all partial permutations (i.e. partial injective transformations) of Ω_n are denoted by $\mathcal{T}_n$ and $\mathcal{I}_n$, respectively. Also, denote by $\mathcal{S}_n$ the symmetric group on Ω_n, i.e. the subgroup of $\mathcal{PT}_n$ of all permutations of Ω_n.

Now, suppose that Ω_n is a finite chain with n elements, e.g. $\Omega_n = \{1 < 2 < \cdots < n\}$. We say that a transformation s in $\mathcal{PT}_n$ is *order-preserving* [*order-reversing*] if $x \leqslant y$ implies $xs \leqslant ys$ [$xs \geqslant ys$], for all $x, y \in \mathrm{Dom}(s)$. A transformation that is either order-preserving or order-reversing is also called *monotone*. Observe that the product of two order-preserving transformations or of two order-reversing transformations is order-preserving and the product of an order-preserving transformation by an order-reversing transformation is order-reversing. Moreover, the product of two monotone transformations is monotone.

Denote by $\mathcal{PO}_n$ the submonoid of $\mathcal{PT}_n$ of all partial order-preserving transformations of Ω_n. As usual, $\mathcal{O}_n$ denotes the monoid $\mathcal{PO}_n \cap \mathcal{T}_n$ of all full transformations of Ω_n that preserve the order. This monoid has been largely studied, namely in [1, 24, 26, 30]. The injective counterpart of $\mathcal{O}_n$ is the inverse monoid $\mathcal{POI}_n = \mathcal{PO}_n \cap \mathcal{I}_n$, which is considered, for example, in [4, 9, 10, 12, 13, 14].

Wider classes of monoids are obtained when we take monotone transformations. In this way, we get $\mathcal{POD}_n$, the submonoid of $\mathcal{PT}_n$ of all partial monotone transformations. Naturally, we may also consider $\mathcal{OD}_n = \mathcal{POD}_n \cap \mathcal{T}_n$ and $\mathcal{PODI}_n = \mathcal{POD}_n \cap \mathcal{I}_n$, the monoids of all monotone full transformations and of all monotone partial permutations, respectively. These monoids were studied, for instance, in [3, 5, 7, 8, 15, 16, 17, 18, 19, 21, 22, 25].

The Hasse diagram in Figure 1, with respect to the inclusion relation and where **1** denotes the trivial monoid, clarifies the relationship between these various semigroups.

Describing automorphisms and endomorphisms of transformation semigroups is a classical problem. For instance, they have been determined by Schein and Teclezghi [33, 34] for $\mathcal{I}_n$ in 1997 and for $\mathcal{T}_n$ in 1998, and for the Brauer-type semigroups by Mazorchuk [32] in 2002. Regarding semigroups of order-preserving transformations, in 1962 Aĭzenštat [1] gave a presentation for $\mathcal{O}_n$ from which it can be deduced that $\mathcal{O}_n$ has only one non-trivial automorphism, for $n \geqslant 2$. More recently, in 2010 Fernandes et al. [20] found a description of the endomorphisms of $\mathcal{O}_n$ and in 2019 Fernandes and Santos [23] determined the endomorphisms of $\mathcal{POI}_n$ and of $\mathcal{PO}_n$. Descriptions of automorphisms of semigroups of order-preserving transformations and of some of their

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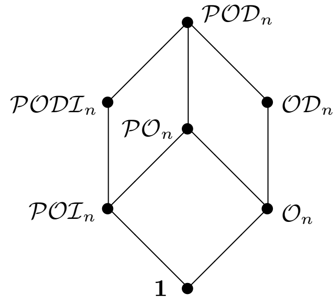

Figure 1: Hasse diagram with respect to the inclusion relation

extensions, such as semigroups of monotone transformations or semigroups of orientation-preserving/reversing transformations, can be found in [3].

In this paper, we give descriptions of the monoids of endomorphisms of the remain semigroups of the above diagram, namely of $\mathcal{OD}_n$, $\mathcal{PODI}_n$ and $\mathcal{POD}_n$, for $n \geqslant 2$. Moreover, we also determine the number of endomorphisms of each of these semigroups.

1 Preliminaries

Let S be a semigroup. For completion, we recall the definition of the Green equivalence relations $\mathcal{R}$, $\mathcal{L}$, $\mathcal{H}$ and $\mathcal{J}$: for all $u, v \in S$,

$u\mathcal{R}v$ if and only if $uS^1 = vS^1$,

$u\mathcal{L}v$ if and only if $S^1u = S^1v$,

$u\mathcal{H}v$ if and only if $u\mathcal{L}v$ and $u\mathcal{R}v$,

$u\mathcal{J}v$ if and only if $S^1uS^1 = S^1vS^1$

(as usual, S^1 denotes S with identity adjoined *if necessary*). Associated to the Green relation $\mathcal{J}$ there is a quasi-order $\leqslant_{\mathcal{J}}$ on S defined by

$$u \leqslant_{\mathcal{J}} v \text{ if and only if } S^1uS^1 \subseteq S^1vS^1,$$

for all $u, v \in S$. Notice that, for every $u, v \in S$, we have $u \,\mathcal{J}\, v$ if and only if $u \leqslant_{\mathcal{J}} v$ and $v \leqslant_{\mathcal{J}} u$. Denote by J_u the $\mathcal{J}$-class of the element $u \in S$. As usual, a partial order relation $\leqslant_{\mathcal{J}}$ is defined on the set $S/\mathcal{J}$ by setting $J_u \leqslant_{\mathcal{J}} J_v$ if and only if $u \leqslant_{\mathcal{J}} v$, for all $u, v \in S$. For $u, v \in S$, we write $u <_{\mathcal{J}} v$ and also $J_u <_{\mathcal{J}} J_v$ if and only if $u \leqslant_{\mathcal{J}} v$ and $(u, v) \notin \mathcal{J}$. Recall that any endomorphism of semigroups preserves Green relations and the quasi-order $\leqslant_{\mathcal{J}}$.

Given a semigroup S, we denote by $E(S)$ the set of its idempotents. An *ideal* of S is a subset I of S such that $S^1IS^1 \subseteq I$. By convenience, we admit the empty set as an ideal. A *Rees congruence* of S is a congruence associated to an ideal of S: if I is an ideal of S, the Rees congruence ρ_I is defined by $(s, t) \in \rho_I$ if and only if $s = t$ or $s, t \in I$, for all $s, t \in S$. The set of congruences of S, the group of automorphisms of S and the monoid of endomorphisms of S are denoted by $\mathrm{Con}(S)$, $\mathrm{Aut}(S)$ and $\mathrm{End}(S)$, respectively.

Let $S \in \{\mathcal{O}_n, \mathcal{POI}_n, \mathcal{PO}_n, \mathcal{OD}_n, \mathcal{PODI}_n, \mathcal{POD}_n\}$. We have the following descriptions of the Green relations in the semigroup S:

$s\mathcal{L}t$ if and only if $\mathrm{Im}(s) = \mathrm{Im}(t)$,

$s\mathcal{R}t$ if and only if $\mathrm{Ker}(s) = \mathrm{Ker}(t)$,

$s\mathcal{J}t$ if and only if $|\,\mathrm{Im}(s)| = |\,\mathrm{Im}(t)|$, and

$s\mathcal{H}t$ if and only if $\mathrm{Ker}(s) = \mathrm{Ker}(t)$ and $\mathrm{Im}(s) = \mathrm{Im}(t)$, for all $s, t \in S$. If $S = \mathcal{POI}_n$ or $\mathcal{PODI}_n$, for the Green relation $\mathcal{R}$, we have, even more simply,

$s\mathcal{R}t$ if and only if $\mathrm{Dom}(s) = \mathrm{Dom}(t)$,

for all $s, t \in S$.

Consider the following order-reversing full transformation

$$\tau = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ n & n-1 & \cdots & 2 & 1 \end{pmatrix}$$

(a permutation of order two).

Recall that $\mathcal{OD}_n = \langle \mathcal{O}_n, \tau \rangle$, $\mathcal{PODI}_n = \langle \mathcal{POI}_n, \tau \rangle$ and $\mathcal{POD}_n = \langle \mathcal{PO}_n, \tau \rangle$.

Next, let $T \in \{\mathcal{O}_n, \mathcal{POI}_n, \mathcal{PO}_n\}$ and $M = \langle T, \tau \rangle$. Then both T and M are regular monoids (moreover, if $T = \mathcal{POI}_n$ then T and M are inverse monoids) and $E(M) = E(T)$.

Remember also that, for the partial order $\leqslant_{\mathcal{J}}$, the quotients $M/\mathcal{J}$ and $T/\mathcal{J}$ are chains (with $n+1$ elements for $T = \mathcal{POI}_n$ and $T = \mathcal{PO}_n$ and with n elements for $T = \mathcal{O}_n$). More precisely, if $S \in \{T, M\}$, then

$$S/\mathcal{J} = \{J_0^S <_{\mathcal{J}} J_1^S <_{\mathcal{J}} \cdots <_{\mathcal{J}} J_n^S\}$$

when $T \in \{\mathcal{POI}_n, \mathcal{PO}_n\}$; and

$$S/\mathcal{J} = \{J_1^S <_{\mathcal{J}} \cdots <_{\mathcal{J}} J_n^S\}$$

when $T = \mathcal{O}_n$. Here

$$J_k^S = \{s \in S \mid |\mathrm{Im}(s)| = k\},$$

with k suitably defined.

For $S \in \{T, M\}$ and $0 \leqslant k \leqslant n$, let $I_k^S = \{s \in S \mid |\mathrm{Im}(s)| \leqslant k\}$. Clearly I_k^S is an ideal of S. Since $S/\mathcal{J}$ is a chain, it follows that

$$\{I_k^S \mid 0 \leqslant k \leqslant n\}$$

is the set of all ideals of S (see [12]).

Observe that T is an aperiodic monoid and that the $\mathcal{H}$-classes of M contained in J_k^M have precisely two elements (one of them belonging to T and the other belonging to $M \setminus T$) when $k \geqslant 2$. If $k = 1$ then such $\mathcal{H}$-classes are trivial.

In [2], Aĭzenštat proved that the congruences of $\mathcal{O}_n$ are exactly the identity and its n Rees congruences. See [30] for another proof. Also, the congruences of $\mathcal{POI}_n$ and $\mathcal{PO}_n$ are exactly their $n+1$ Rees congruences. This has been shown, for $\mathcal{POI}_n$, by Derech [6] and, independently, by Fernandes [12] and, for $\mathcal{PO}_n$, by Fernandes et al. [17]. In short,

$$\{\rho_{I_k^T} \mid 0 \leqslant k \leqslant n\}$$

is the set $\mathrm{Con}(T)$ of all congruences of T.

Concerning the monoid $M \in \{\mathcal{OD}_n, \mathcal{PODI}_n, \mathcal{POD}_n\}$, for $1 \leqslant k \leqslant n$, we can define a congruence π_k on M by: for all $s, t \in M$, $s\,\pi_k\,t$ if and only if

1. $s = t$; or
2. $s, t \in I_{k-1}^M$; or
3. $s, t \in J_k^M$ and $s\,\mathcal{H}\,t$

(see [11, Lemma 4.2]). For $0 \leqslant k \leqslant n$, denote by ρ_k^M the Rees congruence $\rho_{I_k^M}$ associated to the ideal I_k^M of M and by ω the universal congruence of M. Clearly, for $n \geqslant 2$, we have

$$1 = \pi_1 \subsetneq \rho_1^M \subsetneq \pi_2 \subsetneq \rho_2^M \subsetneq \cdots \subsetneq \pi_n \subsetneq \rho_n^M = \omega.$$

Furthermore, Fernandes et al. [17] proved that, for $n \geqslant 2$, these are precisely all congruences of M:

$$\mathrm{Con}(M) = \{1 = \pi_1, \rho_1^M, \pi_2, \rho_2^M, \ldots, \pi_n, \rho_n^M = \omega\}.$$

Let S be a semigroup and let u be a unit of any monoid containing S such that $u^{-1}Su \subseteq S$. Then, it is easy to check that the mapping $\phi^u : S \longrightarrow S$ defined by $s\phi^u = u^{-1}su$, for all $s \in S$, is an automorphism of S, which is called an *inner automorphism* of S. If necessary, in case of ambiguity, we represent ϕ^u by ϕ^u_S.

Let us consider again the permutation τ of Ω_n (recall that $\tau \in \mathcal{POD}_n$) defined above. Then, for every $S \in \{\mathcal{O}_n, \mathcal{POI}_n, \mathcal{PO}_n, \mathcal{OD}_n, \mathcal{PODI}_n, \mathcal{POD}_n\}$, it is easy to verify that $\tau^{-1}S\tau = S$. Therefore, for $n \geqslant 2$, the permutation τ induces a non-trivial automorphism ϕ^τ of S.

Now, let S be a finite monoid and let G be its group of units. Let e and f be two idempotents of S such that $ef = fe = f$. Then, clearly, the mapping $\phi : S \longrightarrow S$ defined by $G\phi = \{e\}$ and $(S \setminus G)\phi = \{f\}$ is an endomorphism (of semigroups) of S. More generally, let f be an idempotent of S, H a subgroup of S such that $fH = \{f\} = Hf$ and $\varphi : G \longrightarrow H$ a homomorphism. Then, it is also clear that the mapping $\phi : S \longrightarrow S$ defined by $\phi|_G = \varphi$ and $(S \setminus G)\phi = \{f\}$ is an endomorphism (of semigroups) of S.

On the other hand, observe that, given a semigroup S, an endomorphism ϕ of S and an idempotent-generated subsemigroup V of S such that $E(V) = E(S)$, as idempotents apply to idempotents, we have $V\phi \subseteq V$ and so we may consider the restriction $\phi|_V$ of ϕ to V as an endomorphism of V.

Recall that the *rank* of a transformation $s \in \mathcal{PT}_n$ is the size of its image, i.e. $|\operatorname{Im}(s)|$.

Next, we are going to construct a certain family of endomorphisms for each of the semigroups $\mathcal{POI}_n$, $\mathcal{PO}_n$, $\mathcal{PODI}_n$ and $\mathcal{POD}_n$.

For $k \in \{2, 3, \ldots, n\}$, consider the following two transformations of $\mathcal{O}_n$ with rank $n-1$:

$$f_k = \begin{pmatrix} 1 & \cdots & k-1 & k & k+1 & \cdots & n \\ 1 & \cdots & k-1 & k-1 & k+1 & \cdots & n \end{pmatrix} \quad \text{and} \quad g_k = \begin{pmatrix} 1 & \cdots & k-2 & k-1 & k & \cdots & n \\ 1 & \cdots & k-2 & k & k & \cdots & n \end{pmatrix}.$$

It is easy to check that each of the $n-1$ $\mathcal{R}$-classes of transformations of rank $n-1$ of $\mathcal{O}_n$ or $\mathcal{OD}_n$ (recall that $E(\mathcal{OD}_n) = E(\mathcal{O}_n)$) has exactly two idempotents, namely f_k and g_k, for some $k \in \{2, 3, \ldots, n\}$. See [23, Lemma 2.4]. On the other hand, for $i \in \{1, 2, \ldots, n\}$, let

$$e_i = \begin{pmatrix} 1 & \cdots & i-1 & i+1 & \cdots & n \\ 1 & \cdots & i-1 & i+1 & \cdots & n \end{pmatrix},$$

which is an idempotent of $\mathcal{POI}_n$ of rank $n-1$. It is clear that each of the n $\mathcal{R}$-classes of transformations of rank $n-1$ of $\mathcal{POI}_n$ or $\mathcal{PODI}_n$ (recall that $E(\mathcal{PODI}_n) = E(\mathcal{POI}_n)$) has exactly one idempotent, namely e_i, for some $i \in \{1, 2, \ldots, n\}$. Therefore, since $J^{\mathcal{POD}_n}_{n-1} = J^{\mathcal{OD}_n}_{n-1} \cup J^{\mathcal{PODI}_n}_{n-1}$ (a disjoint union), by the above observations, $J^{\mathcal{POD}_n}_{n-1}$ contains n $\mathcal{R}$-classes exactly with one idempotent and $n-1$ $\mathcal{R}$-classes exactly with two idempotents.

Now, let us define a mapping $\phi_1 : \mathcal{POD}_n \longrightarrow \mathcal{POD}_n$ by:

1. $1\phi_1 = 1$, $\tau\phi_1 = \tau$;

2. For $s \in J^{\mathcal{PODI}_n}_{n-1}$, let $s\phi_1 = \binom{i}{j}$, where $i, j \in \{1, 2, \ldots, n\}$ are the unique indices such that $e_i \mathcal{R} s \mathcal{L} e_j$;

3. For $s \in J^{\mathcal{OD}_n}_{n-1}$, let $s\phi_1 = \begin{pmatrix} k-1 & k \\ k_s & k_s \end{pmatrix}$, where $\{k_s\} = \Omega_n \setminus \operatorname{Im}(s)$ and $k \in \{2, 3, \ldots, n\}$ is the unique index such that $s\mathcal{R}f_k$ (and $s\mathcal{R}g_k$);

4. $I^{\mathcal{POD}_n}_{n-2}\phi_1 = \{\emptyset\}$.

Clearly, ϕ_1 is a well defined mapping. Moreover, it is a routine matter to show that ϕ is an endomorphism of $\mathcal{POD}_n$ which admits π_{n-1} as kernel. Furthermore, it is clear that, for every $S \in \{\mathcal{POI}_n, \mathcal{PO}_n, \mathcal{PODI}_n\}$, we have $S\phi_1 \subseteq S$ and so the restriction $\phi_1|_S$ of ϕ_1 to S may also be seen as an endomorphism of S.

Next, recall that a subsemigroup S of $\mathcal{PT}_n$ is said to be $\mathcal{S}_n$-*normal* if $\sigma^{-1}S\sigma \subseteq S$, for all $\sigma \in \mathcal{S}_n$. In 1975, Sullivan [35, Theorem 2] proved that $\mathrm{Aut}(S) \simeq \mathcal{S}_n$, for any $\mathcal{S}_n$-normal subsemigroup S of $\mathcal{PT}_n$ containing a constant mapping. Moreover, we obtain an isomorphism $\Phi : \mathcal{S}_n \longrightarrow \mathrm{Aut}(S)$ by defining $\sigma\Phi = \phi_S^\sigma$, where ϕ_S^σ is the inner automorphism of S associated to σ, for all $\sigma \in \mathcal{S}_n$.

Let $S \in \{\mathcal{POI}_n, \mathcal{PO}_n, \mathcal{PODI}_n, \mathcal{POD}_n\}$.

Let $I_1^1 = I_1^S \cup \{1\}$. It is clear that I_1^1 is an $\mathcal{S}_n$-normal subsemigroup of $\mathcal{PT}_n$ containing a constant mapping. Therefore, by Sullivan's Theorem, we have

$$\mathrm{Aut}(I_1^1) = \{\phi_{I_1^1}^\sigma \mid \sigma \in \mathcal{S}_n\} \simeq \mathcal{S}_n.$$

Now, let $I_1^\tau = I_1^S \cup \{1, \tau\}$. Then I_1^τ is a subsemigroup of $\mathcal{PT}_n$ admitting $\{1, \tau\}$ as group of units. Therefore, for any automorphism ϕ of I_1^τ, we must have $\tau\phi = \tau$, whence $I_1^1\phi = I_1^1$ and so the restriction $\phi|_{I_1^1}$ of ϕ to I_1^1, as a mapping from I_1^1 to I_1^1, is an automorphism of I_1^1. Hence $\phi|_{I_1^1} = \phi_{I_1^1}^\sigma$, for some $\sigma \in \mathcal{S}_n$. Let $i, j \in \{1, 2, \ldots, n\}$. Then

$$\tbinom{(n-i+1)\sigma}{j\sigma} = \sigma^{-1}\tau\tbinom{i}{j}\sigma = \left(\tau\tbinom{i}{j}\right)\phi = \tau\phi\tbinom{i}{j}\phi = \tau\sigma^{-1}\tbinom{i}{j}\sigma = \tbinom{n-i\sigma+1}{j\sigma},$$

whence $(n-i+1)\sigma = n - i\sigma + 1$, i.e. $i\tau\sigma = i\sigma\tau$. So

$$\sigma \in \mathrm{C}_{\mathcal{S}_n}(\tau) = \{\xi \in \mathcal{S}_n \mid \xi\tau = \tau\xi\},$$

the centralizer of τ in $\mathcal{S}_n$. Notice that, $\sigma^{-1}\tau\sigma = \tau$ (and so $\sigma^{-1}I_1^\tau\sigma = I_1^\tau$), whence ϕ is the inner automorphism of I_1^τ associated to σ. On the other hand, given $\sigma \in \mathrm{C}_{\mathcal{S}_n}(\tau)$, we clearly have $\sigma^{-1}I_1^\tau\sigma = I_1^\tau$ and so we obtain an inner automorphism $\phi_{I_1^\tau}^\sigma$ of I_1^τ. Moreover, it is easy to conclude now that the mapping $\Phi : \mathrm{C}_{\mathcal{S}_n}(\tau) \longrightarrow \mathrm{Aut}(I_1^\tau)$ defined by $\sigma\Phi = \phi_{I_1^\tau}^\sigma$, for all $\sigma \in \mathrm{C}_{\mathcal{S}_n}(\tau)$, is an isomorphism. Since τ has $\lfloor \frac{n}{2} \rfloor$ non-trivial cycles each of which with length 2, we have

$$|\mathrm{Aut}(I_1^\tau)| = |\mathrm{C}_{\mathcal{S}_n}(\tau)| = \lfloor \tfrac{n}{2} \rfloor! \, 2^{\lfloor \frac{n}{2} \rfloor}$$

(see [31, Proposition 23 (page 133)]).

For $S \in \{\mathcal{POI}_n, \mathcal{PO}_n\}$, let $\phi_\sigma^S = \phi_1|_S\phi_{I_1^1}^\sigma$, considered as a mapping from S to S, for all $\sigma \in \mathcal{S}_n$. Clearly,

$$\{\phi_\sigma^S \mid \sigma \in \mathcal{S}_n\}$$

is a set of $n!$ distinct endomorphisms of S. On the other hand, for $S \in \{\mathcal{PODI}_n, \mathcal{POD}_n\}$, let $\phi_\sigma^S = \phi_1|_S\phi_{I_1^\tau}^\sigma$, considered as a mapping from S to S, for all $\sigma \in \mathrm{C}_{\mathcal{S}_n}(\tau)$. Clearly,

$$\{\phi_\sigma^S \mid \sigma \in \mathrm{C}_{\mathcal{S}_n}(\tau)\}$$

is a set of $\lfloor \frac{n}{2} \rfloor! \, 2^{\lfloor \frac{n}{2} \rfloor}$ distinct endomorphisms of S.

For general background on semigroups and standard notations, we refer the reader to Howie's book [27].

From now on, we consider $n \geqslant 2$, whenever not explicitly mentioned.

2 Endomorphisms of $\mathcal{OD}_n$, $\mathcal{PODI}_n$ and $\mathcal{POD}_n$

We begin this section by recalling the following results by Fernandes et al. [20] and Fernandes and Santos [23] above mentioned.

Theorem 2.1 ([20, Theorem 1.1] and [23, Theorem 3.3])**.** *For $n \geqslant 2$, let $T \in \{\mathcal{O}_n, \mathcal{POI}_n, \mathcal{PO}_n\}$ and $\phi : T \longrightarrow T$ be any mapping. Then ϕ is an endomorphism of the semigroup T if and only if one of the following properties holds:*

1. *ϕ is an automorphism and so ϕ is the identity or $\phi = \phi^\tau$;*
2. *if $T \in \{\mathcal{POI}_n, \mathcal{PO}_n\}$ and $\phi = \phi^T_\sigma$, for some $\sigma \in \mathcal{S}_n$;*
3. *there exist idempotents $e, f \in T$ with $e \neq f$ and $ef = fe = f$ such that $1\phi = e$ and $(T \setminus \{1\})\phi = \{f\}$;*
4. *ϕ is a constant mapping with idempotent value.*

And, their corollaries:

Corollary 2.2 ([20, Theorem 1.2] and [23, Theorems 3.4 and 3.7])**.** *Let $n \geqslant 2$. Then:*

1. *the semigroup $\mathcal{O}_n$ has $2 + \sum_{i=0}^{n-1} \binom{n+i}{2i+1} F_{2i+2}$ endomorphisms, where F_{2i+2} denotes the $(2i+2)$th Fibonacci number;*
2. *the semigroup $\mathcal{POI}_n$ has $2 + n! + 3^n$ endomorphisms;*
3. *the semigroup $\mathcal{PO}_n$ has*

$$3 + n! + (\sqrt{5})^{n-1}\left(\left(\tfrac{\sqrt{5}+1}{2}\right)^n - \left(\tfrac{\sqrt{5}-1}{2}\right)^n\right) + \sum_{k=1}^{n}(\sqrt{5})^{k-1}\left(\left(\tfrac{\sqrt{5}+1}{2}\right)^k - \left(\tfrac{\sqrt{5}-1}{2}\right)^k\right)\sum_{i=k}^{n}\binom{n}{i}\binom{i+k-1}{2k-1}$$

   endomorphisms.

Now, let $T \in \{\mathcal{O}_n, \mathcal{PO}_n\}$ and take $M = \langle T, \tau\rangle$. Since T is an idempotent-generated semigroup (see [1, 24]) and $E(T) = E(M)$, then a restriction to T of any endomorphism of M may also be considered as an endomorphism of T. On the other hand, Lemma 2.3 below establishes that this last statement is also true for $T = \mathcal{POI}_n$ (and $M = \mathcal{PODI}_n$).

Let us consider the following elements of $\mathcal{POI}_n$:

$$x_0 = \begin{pmatrix} 2 & \cdots & n-1 & n \\ 1 & \cdots & n-2 & n-1 \end{pmatrix} \quad \text{and} \quad x_i = \left(\begin{array}{ccc|c|ccc} 1 & \cdots & n-i-1 & n-i & n-i+2 & \cdots & n \\ 1 & \cdots & n-i-1 & n-i+1 & n-i+2 & \cdots & n \end{array}\right),$$

for $i \in \{1, 2, \ldots, n-1\}$. Then $\{1, x_0, x_1, \ldots, x_{n-1}\}$ is a (semigroup) generating set of $\mathcal{POI}_n$. Moreover, these transformations satisfy the following equalities:

1. $x_i x_0 = x_0 x_{i+1}$, for $1 \leqslant i \leqslant n-2$;
2. $x_{i+1} x_i x_{i+1} = x_{i+1} x_i$, for $1 \leqslant i \leqslant n-2$;
3. $x_0 x_1 \cdots x_{n-1} x_0 = x_0$.

See [12]. Observe that $x_{i+1}x_i, x_0x_{i+1} \in J^{\mathcal{POI}_n}_{n-2}$ for all $1 \leqslant i \leqslant n-2$.

Lemma 2.3. *Let $n \geqslant 2$ and ϕ be an endomorphism of $\mathcal{PODI}_n$. Then $\mathcal{POI}_n\phi \subseteq \mathcal{POI}_n$.*

Proof. First, notice that 1ϕ is an idempotent and so $1\phi \in \mathcal{POI}_n$.

Let $n = 2$. Then τ is the only non-order-preserving element of $\mathcal{PODI}_2$. Suppose that $\tau = s\phi$, for some $s \in \mathcal{POI}_2$. Hence $s = \binom{1}{2}$ or $s = \binom{2}{1}$ (these are the only non-idempotents of $\mathcal{POI}_2$) and so $\emptyset\phi = (s^2)\phi = (s\phi)^2 = \tau^2 = 1$ and $\emptyset\phi = (s^3)\phi = (s\phi)^3 = \tau^3 = \tau$, which is a contradiction. Thus $\mathcal{POI}_2\phi \subseteq \mathcal{POI}_2$.

Next, let $n = 3$. In order to obtain a contradiction, suppose there exists $i \in \{0, 1, 2\}$ such that $x_i\phi \notin \mathcal{POI}_3$. Then $J^{\mathcal{PODI}_3}_2\phi \subseteq J^{\mathcal{PODI}_3}_m$, for some $m \geqslant 2$.

If $\pi_2 \subsetneq \mathrm{Ker}(\phi)$ then $\rho_2^{\mathcal{PODI}_3} \subseteq \mathrm{Ker}(\phi)$, whence $|I_2^{\mathcal{PODI}_3}\phi| = 1$ and so $x_i\phi$ is an idempotent, which is a contradiction. Hence $\mathrm{Ker}(\phi) \subseteq \pi_2$ and so ϕ is injective in $J_2^{\mathcal{POI}_3} \cup \{1,\tau\}$. Therefore $m = 2$, $\tau\phi = \tau$ and $I_1^{\mathcal{PODI}_3}\phi \subseteq I_1^{\mathcal{PODI}_3}$.

Suppose $i = 0$. Since $x_0x_1x_2x_0 = x_0$, we have $x_0\phi x_1\phi x_2\phi x_0\phi = x_0\phi \notin \mathcal{POI}_3$, whence $x_1\phi x_2\phi \notin \mathcal{POI}_3$ and so $x_1\phi \in \mathcal{POI}_3$ or $x_2\phi \in \mathcal{POI}_3$. Since $\tau x_0x_1\tau = x_1$ and $\tau x_2x_0\tau = x_2$, we have $\tau(x_0\phi)(x_1\phi)\tau = x_1\phi \in J_2^{\mathcal{PODI}_3}$ and $\tau(x_2\phi)(x_0\phi)\tau = x_2\phi \in J_2^{\mathcal{PODI}_3}$. If $x_1\phi \in \mathcal{POI}_3$ then $x_1\phi = \tau(x_0\phi)(x_1\phi)\tau \notin \mathcal{POI}_3$, which is a contradiction. If $x_2\phi \in \mathcal{POI}_3$ then $x_2\phi = \tau(x_2\phi)(x_0\phi)\tau \notin \mathcal{POI}_3$, which is again a contradiction. Thus $i \neq 0$ and so $x_0\phi \in \mathcal{POI}_3$.

Now, from $x_0\phi x_1\phi x_2\phi x_0\phi = x_0\phi \in J_2^{\mathcal{POI}_3}$, we may conclude that $x_1\phi x_2\phi \in \mathcal{POI}_3$ and so, since $x_1\phi \notin \mathcal{POI}_3$ or $x_2\phi \notin \mathcal{POI}_3$, we have $x_1\phi, x_2\phi \notin \mathcal{POI}_3$.

Since x_1^2 and x_2^2 are idempotents with rank 1, then $(x_1\phi)^2$ and $(x_2\phi)^2$ are also idempotents and their rank must be 1 or 0. On the other hand, we may routinely check that

$$\left\{x \in J_2^{\mathcal{PODI}_3} \setminus \mathcal{POI}_3 \mid x^2 \text{ is an idempotent of rank 1 or 0}\right\} = \left\{\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}\right\},$$

whence $\{x_1\phi, x_2\phi\} = \left\{\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}\right\}$ and so $x_1\phi = (x_2\phi)^{-1} = x_2^{-1}\phi$. Since $\mathrm{Ker}(\phi) \subseteq \pi_2$, it follow that $x_1\mathcal{H}x_2^{-1}$, which is a contradiction.

Thus $x_i\phi \in \mathcal{POI}_3$ for all $i \in \{0,1,2\}$ and so $\mathcal{POI}_3\phi \subseteq \mathcal{POI}_3$.

Now, consider $n \geqslant 4$. Let $\ell, m \in \{1,2,\dots,n\}$ be such that $J_{n-2}^{\mathcal{PODI}_n}\phi \subseteq J_\ell^{\mathcal{PODI}_n}$ and $J_{n-1}^{\mathcal{PODI}_n}\phi \subseteq J_m^{\mathcal{PODI}_n}$. Then $\ell \leqslant m$.

We begin by assuming that $\ell \geqslant 2$.

In order to obtain a contradiction, suppose there exists $i \in \{0,1,\dots,n-1\}$ such that $x_i\phi \notin \mathcal{POI}_n$.

Suppose $i = 0$. Since $x_0x_1\cdots x_{n-1}x_0 = x_0$, we have $x_0\phi(x_1\phi\cdots x_{n-1}\phi)x_0\phi = x_0\phi \notin \mathcal{POI}_n$ and so there must be $j \in \{1,2,\dots,n-1\}$ such $x_j\phi \notin \mathcal{POI}_n$. If $j = n-1$ then $x_{n-2}x_0 = x_0x_{n-1}$ implies $x_{n-2}\phi x_0\phi = x_0\phi x_{n-1}\phi \in J_\ell^{\mathcal{PODI}_n}$ and so $x_{n-2}\phi \notin \mathcal{POI}_n$, since $\ell \geqslant 2$ and $x_0\phi, x_{n-1}\phi \notin \mathcal{POI}_n$. On the other hand, we also have $x_{n-1}x_{n-2}x_{n-1} = x_{n-1}x_{n-2}$, whence $x_{n-1}\phi x_{n-2}\phi x_{n-1}\phi = x_{n-1}\phi x_{n-2}\phi \in J_\ell^{\mathcal{PODI}_n}$, which is a contradiction, since $\ell \geqslant 2$ and $x_{n-2}\phi, x_{n-1}\phi \notin \mathcal{POI}_n$. Therefore $j \in \{1,2,\dots,n-2\}$ and so we have $x_jx_0 = x_0x_{j+1}$. Hence $x_j\phi x_0\phi = x_0\phi x_{j+1}\phi \in J_\ell^{\mathcal{PODI}_n}$. Since $\ell \geqslant 2$ and $x_0\phi, x_j\phi \notin \mathcal{POI}_n$, we deduce that also $x_{j+1}\phi \notin \mathcal{POI}_n$. On the other hand, we also have $x_{j+1}x_jx_{j+1} = x_{j+1}x_j$, whence $x_{j+1}\phi x_j\phi x_{j+1}\phi = x_{j+1}\phi x_j\phi \in J_\ell^{\mathcal{PODI}_n}$, which is again a contradiction, since $\ell \geqslant 2$ and $x_j\phi, x_{j+1}\phi \notin \mathcal{POI}_n$. Thus $i \neq 0$ and so $x_0\phi \in \mathcal{POI}_n$.

Let $i \in \{1,2,\dots,n-1\}$ be the smallest index such that $x_i\phi \notin \mathcal{POI}_n$.

If $i \geqslant 2$ then $x_{i-1}x_0 = x_0x_i$ and so $x_{i-1}\phi x_0\phi = x_0\phi x_i\phi \in J_\ell^{\mathcal{PODI}_n}$, which is a contradiction, since $\ell \geqslant 2$, $x_0\phi, x_{i-1}\phi \in \mathcal{POI}_n$ and $x_i\phi \notin \mathcal{POI}_n$. Hence $i = 1$. From $x_1x_0 = x_0x_2$, we get $x_1\phi x_0\phi = x_0\phi x_2\phi \in J_\ell^{\mathcal{PODI}_n}$ and, as $x_0\phi \in \mathcal{POI}_n$ and $x_1\phi \notin \mathcal{POI}_n$, we deduce that $x_2\phi \notin \mathcal{POI}_n$. On the other hand, we also have $x_2x_1x_2 = x_2x_1$, whence $x_2\phi x_1\phi x_2\phi = x_2\phi x_1\phi \in J_\ell^{\mathcal{PODI}_n}$, which is again a contradiction, since $\ell \geqslant 2$ and $x_1\phi, x_2\phi \notin \mathcal{POI}_n$.

Thus $x_i\phi \in \mathcal{POI}_n$ for all $i \in \{0,1,\dots,n-1\}$ and so $\mathcal{POI}_n\phi \subseteq \mathcal{POI}_n$.

Now, let $\ell \leqslant 1$.

Suppose that $\mathrm{Ker}(\phi) = \pi_k$, with $k \leqslant n-2$, or $\mathrm{Ker}(\phi) = \rho_k^{\mathcal{PODI}_n}$, with $k \leqslant n-3$. Then ϕ is injective in $J_{n-2}^{\mathcal{POI}_n}$ and so $|E(J_{n-2}^{\mathcal{POI}_n})| \leqslant |E(J_\ell^{\mathcal{POI}_n})| \leqslant n$, which is a contradiction, since $|E(J_{n-2}^{\mathcal{POI}_n})| = \binom{n}{n-2} > n$, for $n \geqslant 4$.

Hence $\mathrm{Ker}(\phi) \in \left\{\rho_{n-2}^{\mathcal{PODI}_n}, \pi_{n-1}, \rho_{n-1}^{\mathcal{PODI}_n}, \pi_n, \rho_n^{\mathcal{PODI}_n} = \omega\right\}$. Clearly, if $\mathrm{Ker}(\phi) \in \left\{\rho_{n-1}^{\mathcal{PODI}_n}, \pi_n, \rho_n^{\mathcal{PODI}_n}\right\}$ then $\mathcal{POI}_n\phi \subseteq \mathcal{POI}_n$. Therefore, let us admit that $\mathrm{Ker}(\phi) \in \left\{\rho_{n-2}^{\mathcal{PODI}_n}, \pi_{n-1}\right\}$. Then $I_{n-2}^{\mathcal{PODI}_n}\phi = \{f\}$, for some idempotent $f \in J_\ell^{\mathcal{PODI}_n}$, and $m \geqslant 1$ (recall that $J_{n-1}^{\mathcal{PODI}_n}\phi \subseteq J_m^{\mathcal{PODI}_n}$), since ϕ is injective in $J_{n-1}^{\mathcal{POI}_n}$.

Let $e_1, e_2, \dots, e_n$ be the n distinct idempotents of $J_{n-1}^{\mathcal{PODI}_n}$ considered in Section 1 (i.e. e_i is the partial identity with domain $\Omega_n \setminus \{i\}$, for $i \in \{1,2,\dots,n\}$).

Let $i \in \{1,2,\ldots,n\}$. Then $f = \emptyset\phi = (\emptyset e_i)\phi = \emptyset\phi e_i\phi = f(e_i\phi)$, whence $\mathrm{Dom}(f) \subseteq \mathrm{Dom}(e_i\phi)$. Since $(\emptyset, e_i) \notin \mathrm{Ker}(\phi)$ then $f = \emptyset\phi \neq e_i\phi$ and so $\mathrm{Dom}(f) \subsetneq \mathrm{Dom}(e_i\phi)$.

On the other hand, for $1 \leqslant i < j \leqslant n$, we have $e_ie_j \in I_{n-2}^{\mathcal{PODI}_n}$, whence $f = (e_ie_j)\phi = e_i\phi e_j\phi$ and so $\mathrm{Dom}(f) = \mathrm{Dom}(e_i\phi) \cap \mathrm{Dom}(e_j\phi)$.

Take $\ell_i \in \mathrm{Dom}(e_i\phi) \setminus \mathrm{Dom}(f)$, for $i \in \{1,2,\ldots,n\}$. If there exist $1 \leqslant i < j \leqslant n$ such that $\ell_i = \ell_j$ then $\ell_i \in (\mathrm{Dom}(e_i\phi) \cap \mathrm{Dom}(e_j\phi)) \setminus \mathrm{Dom}(f) = \emptyset$, which is a contradiction. Hence, $\ell_i \neq \ell_j$ for all $1 \leqslant i < j \leqslant n$ and so $\{\ell_1, \ell_2, \ldots, \ell_n\} = \{1,2,\ldots,n\}$. Now, since $\ell_i \notin \mathrm{Dom}(f)$ for all $i \in \{1,2,\ldots,n\}$, we conclude that $\mathrm{Dom}(f) = \emptyset$, i.e. $f = \emptyset$, and $\mathrm{Dom}(e_i\phi) \cap \mathrm{Dom}(e_j\phi) = \emptyset$ for all $1 \leqslant i < j \leqslant n$. Moreover, we have $|\mathrm{Dom}(e_i\phi)| = m \geqslant 1$ for $i \in \{1,2,\ldots,n\}$. Hence $|\mathrm{Dom}(e_i\phi)| = 1$ for $i \in \{1,2,\ldots,n\}$ and so $m = 1$. Thus $J_{n-1}^{\mathcal{PODI}_n}\phi \subseteq J_1^{\mathcal{PODI}_n} \subseteq \mathcal{POI}_n$ and so $\mathcal{POI}_n\phi \subseteq \mathcal{POI}_n$, as required. □

We may now deduce from Theorem 2.1 the following descriptions of the endomorphisms of $\mathcal{OD}_n$, $\mathcal{PODI}_n$ and $\mathcal{POD}_n$:

Theorem 2.4. *For $n \geqslant 2$, let $M \in \{\mathcal{OD}_n, \mathcal{PODI}_n, \mathcal{POD}_n\}$ and $\phi : M \longrightarrow M$ be any mapping. Then ϕ is an endomorphism of the semigroup M if and only if one of the following properties holds:*

1. *ϕ is the identity or $\phi = \phi^\tau$ and so ϕ is an automorphism;*
2. *if $M \in \{\mathcal{PODI}_n, \mathcal{POD}_n\}$ and $\phi = \phi_\sigma^M$, for some $\sigma \in \mathrm{C}_{\mathcal{S}_n}(\tau)$;*
3. *if $M \in \{\mathcal{PODI}_n, \mathcal{POD}_n\}$ and there exists a non-idempotent group element h of M such that $\tau\phi = h$, $1\phi = h^2$ and $(M \setminus \{1,\tau\})\phi = \{\emptyset\}$;*
4. *there exist a non-idempotent group element h of M and an idempotent transformation f of M with rank 1 such that $hf = fh = f$, $\tau\phi = h$, $1\phi = h^2$ and $(M \setminus \{1,\tau\})\phi = \{f\}$;*
5. *there exist idempotents $e, f \in M$ with $e \neq f$ and $ef = fe = f$ such that $\{1,\tau\}\phi = \{e\}$ and $(M \setminus \{1,\tau\})\phi = \{f\}$;*
6. *ϕ is a constant mapping with idempotent value.*

Proof. Obviously, if Property 1, 2 or 6 holds then ϕ is an endomorphism of M. On the other hand, if Property 3, 4 or 5 holds then, as observed above in a more general context, ϕ is an endomorphism of M. In fact, this is immediate for Property 5. On the other hand, in the case of Property 3 or 4, $h^2 \neq h$, h^2 is an idempotent and $h\mathcal{H}h^2$, whence $\phi|_{\{1,\tau\}} : \{1,\tau\} \longrightarrow \{h^2, h\}$ is an isomorphism of groups. Moreover, for Property 3, we have immediately $\emptyset\{h^2, h\} = \{\emptyset\} = \{h^2, h\}\emptyset$. Regarding Property 4, since $hf = fh = f$, we clearly have $f\{h^2, h\} = \{f\} = \{h^2, h\}f$. Therefore, in all cases, ϕ is an endomorphism of M.

Conversely, assume that $\phi : M \longrightarrow M$ is an endomorphism. Let $T \in \{\mathcal{O}_n, \mathcal{POI}_n, \mathcal{PO}_n\}$ be such that $M = \langle T, \tau \rangle$. Then, as mentioned above, $\phi|_T : T \longrightarrow T$ is an endomorphism of T.

We start by supposing that $\phi|_T$ is an automorphism of T. Then $1\phi = 1$ and, since $1\mathcal{H}\tau$ implies $1\phi\mathcal{H}\tau\phi$, it follows that $\tau\phi = \tau$. In fact, if $\tau\phi = 1$ then $(1,\tau) \in \mathrm{Ker}(\phi)$ and so $\pi_n \subseteq \mathrm{Ker}(\phi)$, which contradicts that $\phi|_T$ is an automorphism of T, whence $\tau\phi = \tau$.

Let s be any element of $M \setminus T$. Then $s\tau \in T$. Hence, if $\phi|_T = \mathrm{id}_T$ then

$$s\phi = (s\tau\tau)\phi = (s\tau)\phi\tau\phi = (s\tau)\tau = s$$

and if $\phi|_T = \phi_T^\tau$ then

$$s\phi = (s\tau\tau)\phi = (s\tau)\phi\tau\phi = (\tau(s\tau)\tau)\tau = \tau s\tau = (s)\phi_M^\tau.$$

Thus $\phi = \mathrm{id}_M$ or $\phi = \phi_M^\tau$, and so ϕ is an endomorphism verifying Property 1.

Next, suppose that $T \in \{\mathcal{POI}_n, \mathcal{PO}_n\}$ and $\phi|_T = \phi_\sigma^T$, for some $\sigma \in \mathcal{S}_n$.

Let $s \in J_{n-2}^T$. Then $s\phi = s\phi_\sigma^T = s\phi_1\phi_{I_1^1}^\sigma = \emptyset\phi_{I_1^1}^\sigma = \sigma^{-1}\emptyset\sigma = \emptyset$. Hence $J_{n-2}^T\phi = \{\emptyset\}$. If $n = 2$ then $I_{n-2}^M\phi = J_{n-2}^T\phi = \{\emptyset\}$. On the other hand, if $n \geqslant 3$ then ϕ is not injective in J_{n-2}^T, whence $\operatorname{Ker}(\phi) \nsubseteq \pi_{n-2}$ and so $\rho_{n-2}^M \subseteq \operatorname{Ker}(\phi)$, from which follows again $I_{n-2}^M\phi = \{\emptyset\}$.

Now, notice that ϕ is injective in J_{n-1}^T, whence $\operatorname{Ker}(\phi) \subsetneq \rho_{n-1}^M$. If $n = 2$ then it follows immediately that $\operatorname{Ker}(\phi) = \pi_1 = \pi_{n-1}$. So, take $n \geqslant 3$ and let $s \in J_{n-1}^M \setminus J_{n-1}^T$ and $t \in J_{n-1}^T$ be such that $s\mathcal{H}t$. Then $s\phi\mathcal{H}t\phi$ and so $s\phi = t\phi$, since $t\phi = t\phi_1\phi_{I_1^1}^\sigma \in J_1^T = J_1^M$. Hence $\rho_{n-2}^M \subsetneq \operatorname{Ker}(\phi)$ and so we deduce again that $\operatorname{Ker}(\phi) = \pi_{n-1}$.

Then, we have $\tau\phi \neq 1\phi$ and, since $1\phi = 1\phi_1\phi_{I_1^1}^\sigma = \sigma^{-1}1\sigma = 1 = 1\phi = (\tau^2)\phi = (\tau\phi)^2$, it follows that $\tau\phi = \tau$.

Let $i, j \in \{1, 2, \dots, n\}$ and take $s \in J_{n-1}^{\mathcal{POI}_n}$ such that $e_i\mathcal{R}s\mathcal{L}e_j$. Then $s\phi_1 = \binom{i}{j}$ and, since $e_{n-i+1}\mathcal{R}\tau s\mathcal{L}e_j$, we get $(\tau s)\phi_1 = \binom{n-i+1}{j}$. Hence

$$\binom{(n-i+1)\sigma}{j\sigma} = \sigma^{-1}\binom{n-i+1}{j}\sigma = \binom{n-i+1}{j}\phi_{I_1^1}^\sigma = (\tau s)\phi_1\phi_{I_1^1}^\sigma = (\tau s)\phi =$$

$$\tau\phi s\phi = \tau(s\phi_1\phi_{I_1^1}^\sigma) = \tau\left(\binom{i}{j}\phi_{I_1^1}^\sigma\right) = \tau\sigma^{-1}\binom{i}{j}\sigma = \binom{n-i\sigma+1}{j\sigma}$$

and so $(n-i+1)\sigma = n - i\sigma + 1$, i.e. $i\tau\sigma = i\sigma\tau$. Thus $\sigma \in \mathrm{C}_{\mathcal{S}_n}(\tau)$.

Therefore, we conclude that $\phi = \phi_\sigma^M$ for some $\sigma \in \mathrm{C}_{\mathcal{S}_n}(\tau)$, and so ϕ is an endomorphism verifying Property 2.

Now, assume there exist idempotents $e, f \in T$ with $e \neq f$ and $ef = fe = f$ such that $1\phi = e$ and $(T \setminus \{1\})\phi = \{f\}$.

If $\tau\phi = e$ then, for any $s \in M \setminus (T \cup \{\tau\})$, we have $s = \tau(\tau s)$ and $\tau s \in T \setminus \{1\}$, whence $s\phi = \tau\phi(\tau s)\phi = ef = f$ and so ϕ is an endomorphism verifying Property 5.

On the other hand, admit that $\tau\phi \neq e$. Let $h = \tau\phi$. Then $h \neq e$ and $h^2 = (\tau\phi)^2 = (\tau^2)\phi = 1\phi = e$, whence h is a non-idempotent group element of M and so $h \notin T$.

Let s be a constant transformation of M. Then $s, s\tau, \tau s \in T \setminus \{1\}$ and so $fh = s\phi\tau\phi = (s\tau)\phi = f = (\tau s)\phi = \tau\phi s\phi = hf$. Since $f \in T$ and $h \notin T$, the equality $f = fh$ (or $f = hf$) allows us to conclude that f has rank 1, if $T = \mathcal{O}_n$, and that f has rank 1 or f is the empty transformation, if $T \in \{\mathcal{POI}_n, \mathcal{PO}_n\}$.

Now, let $s \in M \setminus (T \cup \{\tau\})$. Then $s = (s\tau)\tau$ and $s\tau \in T \setminus \{1\}$, whence $s\phi = (s\tau)\phi\tau\phi = fh = f$.

Hence if $M \in \{\mathcal{PODI}_n, \mathcal{POD}_n\}$ and $f = \emptyset$, then $\tau\phi = h$, $1\phi = h^2$ and $(M \setminus \{1, \tau\})\phi = \{\emptyset\}$, and so ϕ is an endomorphism verifying Property 3. If $M \in \{\mathcal{OD}_n, \mathcal{PODI}_n, \mathcal{POD}_n\}$ and f is an idempotent of rank 1, then $\tau\phi = h$, $1\phi = h^2$ and $(M \setminus \{1, \tau\})\phi = \{f\}$, and so ϕ is an endomorphism verifying Property 4.

Finally, we admit that $\phi|_T$ is a constant mapping with idempotent value. Let $s \in T \setminus \{1\}$. Then $(1, s) \notin \pi_n$ and $(1, s) \in \operatorname{Ker}(\phi)$. Hence $\operatorname{Ker}(\phi) = \omega$, i.e. ϕ is also a constant mapping with idempotent value, thus verifying Property 6. □

As an immediate corollary, we have:

Corollary 2.5. *For $n \geqslant 2$, let $M \in \{\mathcal{OD}_n, \mathcal{PODI}_n, \mathcal{POD}_n\}$. Then* $\operatorname{Aut}(M) = \{\mathrm{id}, \phi^\tau\}$.

Let $T \in \{\mathcal{O}_n, \mathcal{POI}_n, \mathcal{PO}_n\}$ and take $M = \langle T, \tau\rangle$. Notice that, at the time when Theorem 2.1 was proved it was already known that $\operatorname{Aut}(T) = \{\mathrm{id}, \phi^\tau\}$, for $n \geqslant 2$. However, for the monoid M, it was known until now that $\operatorname{Aut}(M) = \{\mathrm{id}, \phi^\tau\}$ but only for $n \geq 10$. See [3, Theorem 5.4].

Now, we will count the number of endomorphisms of $\mathcal{OD}_n$, $\mathcal{PODI}_n$ and $\mathcal{POD}_n$.

As above, let $T \in \{\mathcal{O}_n, \mathcal{POI}_n, \mathcal{PO}_n\}$ and $M = \langle T, \tau\rangle$.

We begin by calculating the number of endomorphisms of M satisfying Property 4 of Theorem 2.4.

Let h be a non-idempotent group element of M. Then $|\operatorname{Im}(h)| \geqslant 2$ and h is an order-reversing transformation such that h^2 is an idempotent and $h\mathcal{H}h^2$. Note that h has fixed points if and only if $|\operatorname{Im}(h)|$ is odd and, in this case, it has exactly one.

On the other hand, we may also conclude that, for $2 \leqslant i \leqslant n$, the number of non-idempotent group elements of M belonging to J_i^M is equal to $|E(J_i^T)|$.

Let h be a fixed non-idempotent group element in M and let

$$F(h) = \{f \in E(J_1^M) \mid hf = fh = f\}.$$

Then the number of f in M satisfying $hf = fh = f$ and $f \in E(J_1^M)$ is $|F(h)|$, and so the number of endomorphisms of M satisfying Property 4 of Theorem 2.4 such that $\tau\phi = h$ is $|F(h)|$. Notice that, $fh = f$ if and only if h fixes the image of f. Then, if $|\operatorname{Im}(h)|$ is even, we know that h does not have fixed points and so $|F(h)| = 0$. If $M \in \{\mathcal{OD}_n, \mathcal{PODI}_n\}$ then, for each $j \in \Omega_n$, there exists exactly one idempotent f of M such that $\operatorname{Im}(f) = \{j\}$. Moreover, for $j \in \Omega_n$, if $jh = j$ and f is the idempotent of M such that $\operatorname{Im}(f) = \{j\}$, then $hf = f$. Thus, if $|\operatorname{Im}(h)|$ is odd and $M \in \{\mathcal{OD}_n, \mathcal{PODI}_n\}$, it is also clear that $|F(h)| = 1$.

Now, let $M = \mathcal{POD}_n$ and suppose that $|\operatorname{Im}(h)| = i$ is odd. Then $i \geq 3$. In the following, we will show that $|F(h)| = 2^{\frac{i-1}{2}}$. Put

$$h = \begin{pmatrix} A_1 & \cdots & A_j & \cdots & A_{\frac{i+1}{2}} & \cdots & A_{i-j+1} & \cdots & A_i \\ a_i & \cdots & a_{i-j+1} & \cdots & a_{\frac{i+1}{2}} & \cdots & a_j & \cdots & a_1 \end{pmatrix},$$

where $A_1, A_2, \ldots, A_i$ are the kernel classes of h in order $\max A_r < \max A_{r+1}$ for $r \in \{1, \ldots, i-1\}$, and $j \in \{1, \ldots, \frac{i-1}{2}\}$ and $a_k \in A_k$ for $k \in \{1, \ldots, i\}$.

Clearly, $a_{\frac{i+1}{2}}$ is the fixed point of h. Suppose that $f \in F(h)$. Then $\operatorname{Im}(f) = \{a_{\frac{i+1}{2}}\}$. Since f is an idempotent, $a_{\frac{i+1}{2}} \in \operatorname{Dom}(f)$. From the equality $hf = f$, we deduce that $\operatorname{Dom}(f) \subseteq \operatorname{Dom}(h)$ and $x \in \operatorname{Dom}(f)$ if and only if $xh \in \operatorname{Dom}(f)$, for all $x \in \Omega_n$. Therefore, $A_{\frac{i+1}{2}} \subseteq \operatorname{Dom}(f)$.

Let $j \in \{1, \ldots, \frac{i-1}{2}\}$. Suppose that $A_j \cap \operatorname{Dom}(f) \neq \varnothing$ and let $x \in A_j \cap \operatorname{Dom}(f)$. Then $a_{i-j+1} = (A_j)h = xh \in \operatorname{Dom}(f)$, and so $A_j \subseteq \operatorname{Dom}(f)$. As $a_j \in A_j \subseteq \operatorname{Dom}(f)$, then $(A_{i-j+1})h = a_j \in \operatorname{Dom}(f)$, and so $A_{i-j+1} \subseteq \operatorname{Dom}(f)$. Hence $A_j \cup A_{i-j+1} \subseteq \operatorname{Dom}(f)$. Similarly, we can show that if $A_{i-j+1} \cap \operatorname{Dom}(f) \neq \varnothing$ then we also have $A_j \cup A_{i-j+1} \subseteq \operatorname{Dom}(f)$. Thus, for each pair $(j, i-j+1)$, we have $(A_j \cup A_{i-j+1}) \subseteq \operatorname{Dom}(f)$ or $(A_j \cup A_{i-j+1}) \cap \operatorname{Dom}(f) = \varnothing$. As we have $\frac{i-1}{2}$ of these pairs, it follows that the domain of f can be chosen in $2^{\frac{i-1}{2}}$ ways. Thus $|F(h)| = 2^{\frac{i-1}{2}}$.

Next, for $1 \leqslant i \leqslant n$, recall that:

1. $|E(J_i^{\mathcal{O}_n})| = \binom{n+i-1}{2i-1}$ (see [29, Corollary 4.4]);
2. $|E(J_i^{\mathcal{POI}_n})| = \binom{n}{i}$, i.e. the number of partial identities of rank i;
3. $|E(J_i^{\mathcal{PO}_n})| = \sum_{k=i}^{n} \binom{n}{k}\binom{k+i-1}{2i-1}$ (see [23, Lemma 3.6]).

Therefore, being

$$\partial_i = \begin{cases} 0 & \text{if } i \text{ is even} \\ 1 & \text{if } i \text{ is odd and } M \in \{\mathcal{OD}_n, \mathcal{PODI}_n\} \\ 2^{\frac{i-1}{2}} & \text{if } i \text{ is odd and } M = \mathcal{POD}_n, \end{cases}$$

then the number of endomorphisms of M satisfying Property 4 of Theorem 2.4 is

1. $\sum_{i=2}^{n} \binom{n+i-1}{2i-1}\partial_i = \sum_{i=3}^{n} \binom{n+i-1}{2i-1}\partial_i = \sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n+2i}{4i+1}$, if $M = \mathcal{OD}_n$;
2. $\sum_{i=2}^{n} \binom{n}{i}\partial_i = \sum_{i=3}^{n} \binom{n}{i}\partial_i = \sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2i+1}$, if $M = \mathcal{PODI}_n$;
3. $\sum_{i=2}^{n} \sum_{k=i}^{n} \binom{n}{k}\binom{k+i-1}{2i-1}\partial_i = \sum_{i=3}^{n} \sum_{k=i}^{n} \binom{n}{k}\binom{k+i-1}{2i-1}\partial_i = \sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{k=2i+1}^{n} \binom{n}{k}\binom{k+2i}{4i+1}2^i$, if $M = \mathcal{POD}_n$.

Regarding the number of endomorphisms of $M \in \{\mathcal{PODI}_n, \mathcal{POD}_n\}$ satisfying Property 3 of Theorem 2.4, clearly, it coincides with the number of non-idempotent group elements of M, which coincides with the number of idempotents of T with rank greater than or equal to two. Therefore, the number of endomorphisms of M satisfying Property 3 of Theorem 2.4 is

1. $|E(\mathcal{POI}_n)| - |E(J_1^{\mathcal{POI}_n})| - |E(J_0^{\mathcal{POI}_n})| = 2^n - n - 1$, if $M = \mathcal{PODI}_n$;

2. $|E(\mathcal{PO}_n)| - |E(J_1^{\mathcal{PO}_n})| - |E(J_0^{\mathcal{PO}_n})| = |E(\mathcal{PO}_n)| - \sum_{k=1}^{n}\binom{n}{k}\binom{k}{1} - 1 = e_n - n2^{n-1} - 1$, where e_n is defined by the recurrence relation $e_{n+1} = 1 + 5(e_n - e_{n-1})$, with initial conditions $e_0 = 1$ and $e_1 = 2$, or explicitly by $e_n = 1 + (\sqrt{5})^{n-1}\left(\left(\frac{\sqrt{5}+1}{2}\right)^n - \left(\frac{\sqrt{5}-1}{2}\right)^n\right)$ (see [28]), if $M = \mathcal{POD}_n$.

Thus, by applying also Theorem 2.1 and Theorem 2.4, we obtain:

1. $|\operatorname{End}(\mathcal{OD}_n)| = |\operatorname{End}(\mathcal{O}_n)| + \sum_{i=1}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n+2i}{4i+1}$;

2. $|\operatorname{End}(\mathcal{PODI}_n)| = |\operatorname{End}(\mathcal{POI}_n)| - n! + \lfloor\frac{n}{2}\rfloor!\,2^{\lfloor\frac{n}{2}\rfloor} + (2^n - n - 1) + \sum_{i=1}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n}{2i+1}$;

3. $|\operatorname{End}(\mathcal{POD}_n)| = |\operatorname{End}(\mathcal{PO}_n)| - n! + \lfloor\frac{n}{2}\rfloor!\,2^{\lfloor\frac{n}{2}\rfloor} + (e_n - n2^{n-1} - 1) + \sum_{i=1}^{\lfloor\frac{n-1}{2}\rfloor}\sum_{k=2i+1}^{n}\binom{n}{k}\binom{k+2i}{4i+1}2^i$.

Finally, in view of Corollary 2.2, we have:

Corollary 2.6. *Let $n \geqslant 2$. Then:*

1. *the semigroup $\mathcal{OD}_n$ has*
$$2 + \sum_{i=0}^{n-1}\binom{n+i}{2i+1}F_{2i+2} + \sum_{i=1}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n+2i}{4i+1}$$
endomorphisms, where F_{2i+2} denotes the $(2i+2)$th Fibonacci number;

2. *the semigroup $\mathcal{PODI}_n$ has*
$$1 + 2^n + 3^n - n + \lfloor\tfrac{n}{2}\rfloor!\,2^{\lfloor\frac{n}{2}\rfloor} + \sum_{i=1}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n}{2i+1}$$
endomorphisms;

3. *the semigroup $\mathcal{POD}_n$ has*
$$1 + \lfloor\tfrac{n}{2}\rfloor!\,2^{\lfloor\frac{n}{2}\rfloor} - n2^{n-1} + 2e_n + \sum_{k=1}^{n}(e_k - 1)\sum_{i=k}^{n}\binom{n}{i}\binom{i+k-1}{2k-1} + \sum_{i=1}^{\lfloor\frac{n-1}{2}\rfloor}\sum_{k=2i+1}^{n}\binom{n}{k}\binom{k+2i}{4i+1}2^i$$
endomorphisms.

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