DEPARTMENT OF MATHEMATICS

NOVA SCHOOL OF SCIENCE & TECHNOLOGY

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Bachelor Degree in Mathematics

RELATIONS BETWEEN UNIFORMLY ALMOST PERIODIC FUNCTIONS AND THE FOURIER TRANSFORM

MASTER IN MATHEMATICS AND APPLICATIONS

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Relations Between Uniformly Almost Periodic Functions and the Fourier Transform

I dedicate this thesis to my dog Guilmon, who passed away in March 2022. He will always be in my heart.

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Abstract

With this work, we intend to study the relation between uniformly almost periodic functions and the Fourier transform. To this end, we start by defining the concept of a uniformly almost periodic function and we study several important algebraic and topological properties of these functions.

Afterwards, we define a new class of functions, which we will call normal functions, and we will show that this class of functions is precisely equal to the set of uniformly almost periodic functions. We then define another class of functions, which we shall denote by $AP(\mathbb{R})$, and we will define it as the closure, on $L^\infty(\mathbb{R})$, of trigonometric polynomial functions, and we prove that this set also coincides with the set of uniformly almost periodic functions. We are then left with three equivalent definitions established.

We then define the Fourier transform of a function belonging to $L^1(\mathbb{R})$ and, after studying some of its most important properties, we extend this concept to functions that belong to $L^2(\mathbb{R})$.

After analyzing significant properties concerning Banach algebras, maximal ideals and multiplicative linear functionals, we define the algebra, $AP_p(\mathbb{R})$ as the closure, in the norm of the Fourier multipliers, of trigonometric polynomial functions, and we conclude this paper by proving that the algebra $AP_p(\mathbb{R})$ is inverse-closed in $AP(\mathbb{R})$.

Keywords: Uniformly Almost Periodic Function; Fourier Transform; Banach Algebra; Inverse-Closed Algebra.

Resumo

Com a realização deste trabalho pretendemos estudar a relação que existe entre as funções uniformemente quase periódicas e a transformada de Fourier. Com esse intuito, começamos por definir o conceito de uma função uniformemente quase periódica e estudamos várias propriedades algébricas e topológicas das mesmas.

Posteriormente, definimos uma nova classe de funções, que iremos designar por funções normais, e demonstraremos que esta classe de funções será mesmo igual ao conjunto das funções uniformemente quase periódicas. Seguidamente, definimos outra classe de funções, que iremos denotar por $AP(\mathbb{R})$ e que será o fecho em $L^\infty(\mathbb{R})$ das funções polinomiais trigonométricas, e provamos que este conjunto também coincide com o conjunto das funções uniformemente quase periódicas. Ficamos então com três definições equivalentes estabelecidas.

Em seguida, definimos a transformada de Fourier de uma função pertencente a $L^1(\mathbb{R})$ e, após estudarmos algumas das suas mais importantes propriedades, estendemos este conceito para as funções de $L^2(\mathbb{R})$.

Depois de analisarmos propriedades significativas relativas a álgebras de Banach, ideais maximais e funcionais lineares multiplicativos, definimos a álgebra, $AP_p(\mathbb{R})$ como sendo o fecho, na norma dos multiplicadores de Fourier, das funções polinomiais trigonométricas, e concluímos este trabalho ao provar que a álgebra $AP_p(\mathbb{R})$ é inversamente fechada em $AP(\mathbb{R})$.

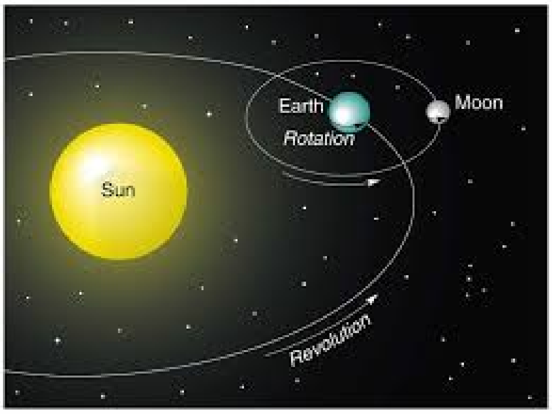
Palavras-chave: Função Uniformemente Quase Periódica; Transformada de Fourier; Álgebra de Banach; Álgebra Inversamente Fechada.

Contents

1 Introduction

We know that periodic functions are really important in Mathematics. However, there exist some functions that are not periodic but satisfy some special properties that make them really similar to periodic functions, and we call them uniformly almost periodic functions. For instance, if we consider the function $f(x) := \cos(2\pi x) + \cos(2\pi\sqrt{2}x)$ we know that $\cos(2\pi x)$ and $\cos(2\pi\sqrt{2}x)$ are periodic functions, but f will not be periodic because $f(x) = 2$ has only one solution, when $x = 0$, as we will see in Example 2.1.6. We will prove in this work that periodic functions are uniformly almost periodic and that the sum of two uniformly almost periodic functions is uniformly almost periodic, consequently f is a uniformly almost periodic function. We can find the behaviour of these functions in our lives, for example, if we consider Earth's revolution around the Sun at the same time that we consider Moon's revolution around the Earth as we can see in the following picture.

Figure 1.1: Earth's and Moon's Revolution



Our main goal is to understand Theorem 5.8.2. With that in mind, we start by analysing and study uniformly almost periodic functions, which were introduced and studied by H. Bohr. We introduce some basic definitions and examples that are going to help us to understand the advanced concepts. After that we see some properties of uniformly almost periodic functions to help us understand why these functions are so important. Following that, we study the behaviour of a sequence of uniformly almost periodic functions, its derivatives and integrals.

Afterwards, in Chapter 3, we examine a new class of so-called normal functions, introduced by S. Bochner. We start by stating its definition and we will be able to establish a relation between those functions and uniformly almost periodic functions, that is, the definitions of uniformly almost periodic and normal functions, given by Bohr and Bochner respectively, are indeed equivalent. Then we analyse the mean value of a uniformly almost periodic function and some consequences about it, which will be really important for the main result about uniformly almost periodic functions. Following that we study Fourier series for uniformly almost periodic functions and we are going to observe similarities with the original definition of Fourier series of periodic functions. We finish Chapter 3 by proving that a function is uniformly almost periodic if and only if it belongs to $AP(\mathbb{R})$, that is, the smallest closed subset of $L^\infty(\mathbb{R})$ that contains the set of trigonometric polynomial functions.

In Chapter 4, we start by defining some spaces and functions that are really important, the $L^p(\mathbb{R})$ spaces and step functions respectively, which will have a crucial role in this work. Succeeding that we define the Fourier transform in $L^1(\mathbb{R})$ and we establish some properties that will help us to understand advanced concepts. Then we examine the definition of convolution and its applications in the Fourier transform. We finish Chapter 4 by generalizing the concept of the Fourier transform to $L^2(\mathbb{R})$, using the fact that, as we will see in this work, the space $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is dense in $L^2(\mathbb{R})$.

In Chapter 5, we start by giving some basic definitions regarding Functional Analysis and, after that, we prove a theorem regarding Banach algebras that is going to aid us during the remain part of the work. Following that we recall the definition of a maximal ideal and, with that being done, we establish some properties of maximal ideals. Moreover, we prove results that relate maximal ideals and invertible elements of a unital commutative Banach algebra. Then we study the multiplicative linear functionals of a unital commutative Banach algebra which, taking into account Gelfand's theory, are related in a special way to each other as we will see in Theorem 5.3.5. Then we define the concept of an algebra embedded densely into another algebra and then we establish two important theorems regarding extensions of multiplicative linear functionals. Following that, we define the concept of a character in the unit circle and we recall the definition of the Banach algebra $l^1(\mathbb{R})$. With that being done, we will prove that, in fact, $l^1(\mathbb{R})$ is homeomorphic to the space of all characters of the unit circle.

Following that, we will be able to prove that the set $AP(\mathbb{R})$ is a unital commutative C^*−subalgebra of $L^\infty(\mathbb{R})$. Afterwards, using the things learned from the Fourier transform in the previous chapter, we define the set $M_p(\mathbb{R})$ as the set of every Fourier multiplier in $L^p(\mathbb{R})$, which are certain functions that satisfy some properties, and then we will be able to give the definition of the set $AP_p(\mathbb{R})$ as the closure of the set of trigonometric polynomial functions in the norm of $M_p(\mathbb{R})$. Following that, we will be able to prove that the algebra $AP_p(\mathbb{R})$ is embedded densely into the algebra $AP(\mathbb{R})$, for each $1 < p < \infty$.

Then we define the Banach algebra $APW(\mathbb{R})$ and we prove that $APW(\mathbb{R})$ is embedded densely into $AP_p(\mathbb{R})$ and also embedded densely into $AP(\mathbb{R})$. Moreover, we see that, in fact, $APW(\mathbb{R})$ is isometrically isomorphic to $l^1(\mathbb{R})$ and also that the Gelfand space of $APW(\mathbb{R})$ is homeomorphic to the Gelfand space of $AP(\mathbb{R})$. After proving that the Gelfand space of $AP_p(\mathbb{R})$ is homeomorphic to the Gelfand space of $AP(\mathbb{R})$ and after characterizing the invertible elements of $AP_p(\mathbb{R})$, we finish this work by proving that the set $AP_p(\mathbb{R})$ is inverse-closed in $AP(\mathbb{R})$.

2

Uniformly Almost Periodic Functions

In this chapter we will start by presenting some simple definitions that will guide us to define a uniformly almost periodic function. Following that, we will establish several important properties of these functions.

2.1 First Definitions

In this section, we will always consider $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

Definition 2.1.1. Let X be a subset of $\mathbb{R}$, that is, $X \subseteq \mathbb{R}$. We say that X is relatively dense in $\mathbb{R}$ if and only if there exists $l > 0$ such that for any open interval $]a, b[$ with length l,

$$X \cap \,]a, b[\, \neq \emptyset.$$

If we analyse the previous definition we can conclude that if a set A is dense in $\mathbb{R}$, then A is relatively dense in $\mathbb{R}$ and if a set A is relatively dense in $\mathbb{R}$ and $A \subseteq X$, then X is also relatively dense in $\mathbb{R}$.

Example 2.1.2. We know that $\mathbb{Q}$ is dense in $\mathbb{R}$ and therefore $\mathbb{Q}$ is relatively dense in $\mathbb{R}$. If we consider the set of integer numbers, $\mathbb{Z}$, we know that $\mathbb{Z}$ is not dense in $\mathbb{R}$. However, it is indeed relatively dense because if we choose $l = 2 > 0$ and choose any interval with length $l = 2$, for example $]a, a+2[$, we can guarantee that

$$\mathbb{Z} \cap \,]a, a+2[\, \neq \emptyset,$$

and therefore $\mathbb{Z}$ is relatively dense in $\mathbb{R}$.

On the other hand, the set of natural numbers, $\mathbb{N}$, it is not relatively dense in $\mathbb{R}$ because for every $x > 0$, there is an open interval $]-x, 0[$, with length x, such that

$$\mathbb{N} \cap \,]-x, 0[\, = \emptyset,$$

which means that, by definition, $\mathbb{N}$ is not a relatively dense set in $\mathbb{R}$.

Definition 2.1.3. Let $f : \mathbb{R} \to \mathbb{K}$ be a function. We say that $\tau \in \mathbb{R}$ is a translation number of f belonging to $\epsilon \geq 0$ if and only if

$$\sup_{x\in\mathbb{R}} |f(x+\tau) - f(x)| \leq \epsilon.$$

From now on we are going to denote the set of all translation numbers of a function f belonging to $\epsilon \geq 0$ by $E_{\epsilon,f}$. With this definition we can deduce some properties of translation numbers.

1. If $\tau \in E_{\epsilon,f}$ then for all $\delta \geq \epsilon$, $\tau \in E_{\delta,f}$.
2. If $\tau \in E_{\epsilon,f}$ then $-\tau \in E_{\epsilon,f}$.
3. If $\tau_1 \in E_{\epsilon_1,f}$ and $\tau_2 \in E_{\epsilon_2,f}$ then $\tau_1 \pm \tau_2 \in E_{\epsilon_1+\epsilon_2,f}$.

Remark: It is important to observe that given $\epsilon > 0$, if the set $E_{\epsilon,f}$ is relatively dense in $\mathbb{R}$, then for each $a, x \in \mathbb{R}$ there exists $\tau_x \in [-x+a, -x+a+l_\epsilon] \cap E_{\epsilon,f}$ such that $x+\tau_x \in [a, a+l_\epsilon]$, where $l_\epsilon > 0$ verifies the condition for which any interval with length l_ϵ intersects $E_{\epsilon,f}$. We are going to use this observation in some proofs established in this work.

Now, we have everything that we need in order to define a uniformly almost periodic function.

Definition 2.1.4. Let $f : \mathbb{R} \to \mathbb{K}$ be a continuous function. We say that f is uniformly almost periodic (u.a.p.) if and only if the set $E_{\epsilon,f}$ is relatively dense for every $\epsilon > 0$. In this work we will denote the set of all uniformly almost periodic functions by $U(\mathbb{R})$.

As our intuition would tell us, in the following result we see that every periodic function is also a u.a.p. function.

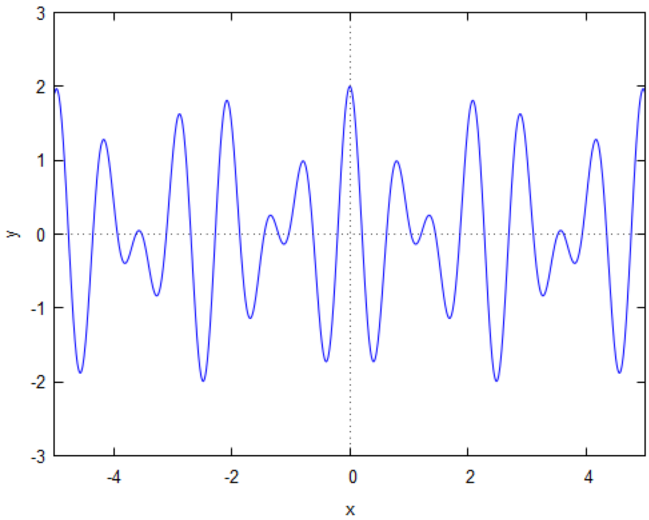
Lemma 2.1.5. *If f is a continuous periodic function, then f is u.a.p.*

Proof. If f is a continuous periodic function with period $T > 0$, then for each $\epsilon > 0$ the set $E_{\epsilon,f}$ contains all numbers of the form nT, with $n \in \mathbb{Z}$. Therefore for any $\epsilon > 0$ the set $E_{\epsilon,f}$ is relatively dense in $\mathbb{R}$, and we conclude that f is u.a.p. as we wanted to prove. □

The following example show us the behaviour of a u.a.p. function that is not periodic.

Example 2.1.6. If we consider the function $f(x) = \cos(2\pi x) + \cos(2\pi\sqrt{2}x)$ for every $x \in \mathbb{R}$, f is not a periodic function because the only solution for $f(x) = 2$ is $x = 0$. In fact, if we have $\cos(2\pi x) = 1$ and $\cos(2\pi\sqrt{2}x) = 1$, then $2\pi x = 2\pi k_1$ and $2\pi\sqrt{2}x = 2\pi k_2$, with $k_1, k_2 \in \mathbb{Z}$. Since k_1 and k_2 are integer numbers, it follows that the equation $k_1 = \frac{k_2}{\sqrt{2}}$ is only satisfied if $k_1 = k_2 = 0$, therefore f is not a periodic function. However f is a uniformly almost periodic function because we will prove in Theorem 2.3.5 that the sum of two u.a.p. functions is a u.a.p. function.

Figure 2.1: $f(x) = \cos(2\pi x) + \cos(2\pi\sqrt{2}x), \quad x \in [-5, 5]$.



Definition 2.1.7. Given a function $f : \mathbb{R} \to \mathbb{K}$ and $a \in \mathbb{R}$, we define the translation function $T_a f$ by

$$(T_a f)(x) := f(x + a),$$

for every $x \in \mathbb{R}$.

We finish this section by checking that the translation function is always u.a.p. supposing that our given function is u.a.p.

Lemma 2.1.8. *If f is a u.a.p. function, then $T_a f$ is also u.a.p. for every $a \in \mathbb{R}$.*

Proof. Let $a, \tau \in \mathbb{R}$ and f be a u.a.p. function. Then

$$\sup_{y \in \mathbb{R}} |f(y + \tau) - f(y)| = \sup_{x \in \mathbb{R}} |f(x + a + \tau) - f(x + a)| = \sup_{x \in \mathbb{R}} |T_a f(x + \tau) - T_a f(x)|,$$

consequently, τ is a translation number of f if and only if τ is a translation number of $T_a f$ for every $a \in \mathbb{R}$. Therefore the set $E_{\epsilon, T_a f}$ is relatively dense in $\mathbb{R}$ for each $a \in \mathbb{R}$ and $\epsilon > 0$, and we conclude that $T_a f$ is u.a.p. as we wanted to prove. □

2.2 Boundedness, Uniform Continuity, and Inverse Closedness

In this sections we will analyse some topological properties of u.a.p. functions. Regarding boundedness, we have the following result.

Theorem 2.2.1 ([4, Chapter 1, Section 1, Theorem 4])**.** *If a function $f : \mathbb{R} \to \mathbb{K}$ is uniformly almost periodic, then f is bounded.*

Proof. Since f is a uniformly almost periodic function, it follows that $E_{\epsilon, f}$ is relatively dense in $\mathbb{R}$ for every $\epsilon > 0$. Let $\epsilon = 1$. In these conditions $E_{1, f}$ is relatively dense and therefore there exists a positive number l_1 such that for any open interval $]a,b[$ with length l_1,

$$E_{1,f} \cap]a,b[\neq \emptyset.$$

Consider the interval $L = [0, l_1]$, with length l_1, and let

$$\max_{x \in L} |f(x)| = M.$$

For each $x \in \mathbb{R}$, we can find a number $\tau_x \in E_{1,f}$ such that $x + \tau_x \in L$. In fact if $y \in \mathbb{R}$, then there is $\tau_y \in [-y, -y + l_1] \cap E_{1,f}$ that verifies $0 \leq y + \tau_y \leq l_1$ and thus $y + \tau_y \in L$. Consequently

$$|f(x + \tau_x)| \leq M.$$

On the other hand, since $\tau_x \in E_{1,f}$, we can say, by definition of translation number of f, that

$$|f(x + \tau_x) - f(x)| \leq 1.$$

Adding both of these inequalities we conclude that for every $x \in \mathbb{R}$,

$$|f(x)| = |f(x) + f(x + \tau_x) - f(x + \tau_x)| \leq |f(x + \tau_x)| + |f(x + \tau_x) - f(x)| \leq M + 1.$$

Since x is an arbitrary real number and M does not depend on x, we have proved the theorem. □

Now we will see that every u.a.p. function is, in fact, uniformly continuous.

Theorem 2.2.2 ([4, Chapter 1, Section 1, Theorem 5])**.** *If a function $f : \mathbb{R} \to \mathbb{K}$ is uniformly almost periodic, then f is uniformly continuous.*

Proof. Let $\epsilon > 0$. Since f is a uniformly almost periodic function, the set $E_{\frac{\epsilon}{3},f}$ is relatively dense. Therefore there exists $l_{\frac{\epsilon}{3}} > 0$ such that for any open interval $]a,b[$ with length $l_{\frac{\epsilon}{3}}$,

$$E_{\frac{\epsilon}{3},f} \cap]a,b[\neq \emptyset.$$

Since f is continuous, for each $x_1 \in]0, l_{\frac{\epsilon}{3}} + 1[$ there exists $\delta \in]0,1[$ such that for any $x_2 \in]0, l_{\frac{\epsilon}{3}} + 1[$,

$$|x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \frac{\epsilon}{3}.$$

Let x, y be two real numbers satisfying $|x - y| < \delta$. In these conditions there is a number $\tau \in E_{\frac{\epsilon}{3},f} \cap]-\min\{x,y\}, -\min\{x,y\} + l_{\frac{\epsilon}{3}}[$ such that $x + \tau \in]0, l_{\frac{\epsilon}{3}} + 1[$ and $y + \tau \in]0, l_{\frac{\epsilon}{3}} + 1[$. Thus the inequality

$$|f(x + \tau) - f(y + \tau)| < \frac{\epsilon}{3}$$

is indeed true. On the other hand, since $\tau \in E_{\frac{\epsilon}{3},f}$, for any $x \in \mathbb{R}$ we have

$$|f(x + \tau) - f(x)| \leq \frac{\epsilon}{3}.$$

Consequently, it follows that

$$\begin{aligned}|f(x)-f(y)| &= |f(x)-f(y)+f(x+\tau)-f(x+\tau)+f(y+\tau)-f(y+\tau)| \\ &\leq |f(x+\tau)-f(x)|+|f(y+\tau)-f(y)|+|f(x+\tau)-f(y+\tau)| \\ &< \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3}=\epsilon,\end{aligned}$$

and we can conclude that f is uniformly continuous. □

The following result is an immediate consequence of the fact that every u.a.p. function is uniformly continuous.

Corollary 2.2.3 ([4, Chapter 1, Section 1, Corollary 5]). *Let f be a u.a.p. function and let $\epsilon > 0$. Then there exists $\delta_\epsilon > 0$ such that*

$$]-\delta_\epsilon, \delta_\epsilon[\; \subseteq \; E_{\epsilon,f}.$$

Proof. Let f be a u.a.p. function and $\epsilon > 0$. Using Theorem 2.2.2 we can assure that f is uniformly continuous and therefore there exists a $\delta_\epsilon > 0$ such that for every $x, y \in \mathbb{R}$,

$$|x-y| < \delta_\epsilon \;\Rightarrow\; |f(x)-f(y)| < \epsilon.$$

Let $\tau \in]-\delta_\epsilon, \delta_\epsilon[$ and consider $x \in \mathbb{R}$. It follows that

$$|x+\tau-x| = |\tau| < \delta_\epsilon \;\Rightarrow\; |f(x+\tau)-f(x)| < \epsilon,$$

and thus we have that

$$\sup_{x\in\mathbb{R}} |f(x+\tau)-f(x)| \leq \epsilon.$$

That is, $\tau \in E_{\epsilon,f}$ and we can conclude that $]-\delta_\epsilon, \delta_\epsilon[\; \subseteq \; E_{\epsilon,f}$. □

We finish this section by verifying the inverse closedness of u.a.p. functions, that is, if $f \in U(\mathbb{R})$ and if the function $\frac{1}{f}$ is well defined, then $\frac{1}{f} \in U(\mathbb{R})$.

Theorem 2.2.4 ([4, Chapter 1, Section 1, Theorem 7]). *If f is a uniformly almost periodic function and if*

$$\inf_{x\in\mathbb{R}} |f(x)| = m \; > \; 0,$$

then the function $\frac{1}{f}$ is also uniformly almost periodic.

Proof. Let $\epsilon > 0$ and let $\tau \in E_{\epsilon,f}$. In these conditions

$$\sup_{x\in\mathbb{R}} |f(x+\tau)-f(x)| \; \leq \; \epsilon.$$

Since

$$\inf_{x\in\mathbb{R}} |f(x)| = m,$$

it follows that

$$\forall x \in \mathbb{R}, \; \left|\frac{1}{f(x)}\right| \leq \frac{1}{m},$$

and therefore
$$\left|\frac{1}{f(x+\tau)} - \frac{1}{f(x)}\right| = \left|\frac{f(x+\tau)-f(x)}{f(x+\tau)\cdot f(x)}\right| \leq \frac{\epsilon}{m^2}.$$
Under these circumstances the set $E_{\epsilon,f}$ is contained in the set $E_{\frac{\epsilon}{m^2},\frac{1}{f}}$ and, consequently, the latter set is relatively dense because the former one is relatively dense by our hypothesis.

□

2.3 Algebraic Properties of u.a.p. Functions

We start this section by seeing simple, but useful, algebraic properties of u.a.p. functions.

Theorem 2.3.1 ([4, Chapter 1, Section 1, Theorem 6])**.** *Let $\lambda \in \mathbb{C}$ and let f be a u.a.p. function. Then the functions λf, $\overline{f}$ and f^2 are also u.a.p. functions.*

Proof. Since f is a u.a.p. function, by Theorem 2.2.1, f is bounded and therefore there is a number $M > 0$ such that
$$\sup_{x\in\mathbb{R}} |f(x)| \leq M.$$
Let $\epsilon > 0$ and let $\tau \in E_{\epsilon,f}$. In these conditions we know that
$$\sup_{x\in\mathbb{R}} |f(x+\tau) - f(x)| \ \leq \ \epsilon.$$
Using this condition and the fact that f is bounded we obtain
$$\sup_{x\in\mathbb{R}} |\overline{f}(x+\tau) - \overline{f}(x)| \ = \ \sup_{x\in\mathbb{R}} |f(x+\tau) - f(x)| \ \leq \ \epsilon,$$
$$\sup_{x\in\mathbb{R}} |\lambda f(x+\tau) - \lambda f(x)| \ = \ |\lambda| \cdot \sup_{x\in\mathbb{R}} |f(x+\tau) - f(x)| \ \leq \ |\lambda| \cdot \epsilon =: \epsilon_1,$$
$$\begin{aligned}\sup_{x\in\mathbb{R}} |f^2(x+\tau) - f^2(x)| &= \sup_{x\in\mathbb{R}} |f(x+\tau) - f(x)| \cdot |f(x+\tau) + f(x)| \\ &\leq \epsilon \cdot (M+M) = 2M\epsilon =: \epsilon_2.\end{aligned}$$
Therefore we can conclude that the set $E_{\epsilon,f}$ is contained in the sets $E_{\epsilon,\overline{f}}, E_{\epsilon_1,\lambda f}$ and E_{ϵ_2,f^2}, that is, these 3 sets are relatively dense in $\mathbb{R}$. Thus $\overline{f}, \lambda f$ and f^2 are uniformly almost periodic. □

Given $x \in \mathbb{K}$ and $A \subseteq \mathbb{K}$, as usual, we define the distance from x to A as
$$d(x,A) := \inf\{|x-y| : y \in A\}.$$

Despite the fact that the previous theorem was not hard to prove, that does not happen if one try to prove that the sum of two u.a.p. functions is u.a.p. From now on, until Theorem 2.3.4, we are going to establish some lemmas that will aid us to prove that fact.

Lemma 2.3.2 ([4, Chapter 1, Section 1, Lemma 9]). *For every $\epsilon_1, \epsilon_2 > 0$, with $\epsilon_2 > \epsilon_1$, there exists a $\delta_{\epsilon_1,\epsilon_2} > 0$ such that $E_{\epsilon_2,f}$ contains any number τ that satisfies*

$$d(\tau, E_{\epsilon_1,f}) < \delta_{\epsilon_1,\epsilon_2}.$$

Proof. Since $\epsilon_2 > \epsilon_1$, it follows that

$$\epsilon_3 := \epsilon_2 - \epsilon_1 > 0,$$

and applying Corollary 2.2.3, we can see that there exists a $\delta_{\epsilon_3} > 0$ such that

$$]-\delta_{\epsilon_3}, \delta_{\epsilon_3}[\ \subseteq\ E_{\epsilon_3,f}.$$

Let $\tau_1 \in E_{\epsilon_1,f}$ and consider $\tau_3 \in E_{\epsilon_3,f}$. Using the property of the sum of translation numbers, we have that

$$\tau_2 := \tau_1 + \tau_3 \in E_{\epsilon_1+\epsilon_3,f} = E_{\epsilon_2,f},$$

and since $]-\delta_{\epsilon_3}, \delta_{\epsilon_3}[$ is contained in $E_{\epsilon_3,f}$, it follows that if $a \in \mathbb{R}$ satisfies $d(a, E_{\epsilon_1,f}) < \delta_{\epsilon_3}$, then $a = b + c$ where $b \in E_{\epsilon_1,f}$ and $c \in]-\delta_{\epsilon_3}, \delta_{\epsilon_3}[\subseteq E_{\epsilon_3,f}$ and thus $a \in E_{\epsilon_2,f}$. Therefore we can conclude that $E_{\epsilon_2,f}$ contains any number τ that verifies $d(\tau, E_{\epsilon_1,f}) < \delta_{\epsilon_3}$ as we wanted to prove. □

Lemma 2.3.3 ([4, Chapter 1, Section 1, Lemma 10]). *Let $\epsilon, \delta > 0$ and f_1, f_2 be uniformly almost periodic functions. Then the set*

$$\{\tau \in E_{\epsilon,f_1} :\ d(\tau, E_{\epsilon,f_2}) < \delta\}$$

is relatively dense.

Proof. Since f_1 and f_2 are u.a.p. functions, we can assure that the sets $E_{\frac{\epsilon}{2},f_1}$ and $E_{\frac{\epsilon}{2},f_2}$ are relatively dense, therefore there exist $l_1, l_2 > 0$ such that any interval with length l_1 intersects $E_{\frac{\epsilon}{2},f_1}$ and every interval with length l_2 intersects $E_{\frac{\epsilon}{2},f_2}$. Then there is $k \in \mathbb{N}$ that satisfies

$$l := k \cdot \delta > \max\{l_1, l_2\}.$$

For every $n \in \mathbb{Z}$, consider the intervals $[(n-1)l, nl]$. It is obvious that these intervals have length l and

$$\bigcup_{n\in\mathbb{Z}} [(n-1)l, nl] = \mathbb{R}.$$

Since $l > \max\{l_1, l_2\}$, it follows that for any $n \in \mathbb{Z}$, there exist $\tau_1^{(n)}, \tau_2^{(n)} \in](n-1)l, nl[$ such that

$$\tau_1^{(n)} \in E_{\frac{\epsilon}{2},f_1} \ \wedge \tau_2^{(n)} \in E_{\frac{\epsilon}{2},f_2},$$

and consequently

$$-l < \tau_1^{(n)} - \tau_2^{(n)} < l.$$

Let

$$I_i = [(i-1)\delta, i\delta[,$$

for any $i \in \{-k+1,\dots,k\}$. In these conditions for every $n \in \mathbb{Z}$, there is $i \in \{-k+1,\dots,k\}$ such that

$$\tau_1^{(n)} - \tau_2^{(n)} \in I_i.$$

It is not hard to see that there exists $n_0 \in \mathbb{N}$ and there is $i \in \{-k+1,\dots,k\}$ such that for any $n \in \mathbb{Z}$, there corresponds an integer number $n_1 \in [-n_0, n_0]$ satisfying

$$\tau_1^{(n)} - \tau_2^{(n)} \in I_i \ \wedge \ \tau_1^{(n_1)} - \tau_2^{(n_1)} \in I_i.$$

That is, $\tau_1^{(n)} - \tau_2^{(n)}$ and $\tau_1^{(n_1)} - \tau_2^{(n_1)}$ belong to the same interval I_i, hence there is $\lambda \in]-1,1[$ verifying

$$\tau_1^{(n)} - \tau_2^{(n)} = \tau_1^{(n_1)} - \tau_2^{(n_1)} + \lambda \cdot \delta \ \Leftrightarrow \ \tau_1^{(n)} - \tau_1^{(n_1)} = \tau_2^{(n)} - \tau_2^{(n_1)} + \lambda \cdot \delta.$$

Since $\tau_1^{(n)}, \tau_1^{(n_1)} \in E_{\frac{\epsilon}{2}, f_1}$ and $\tau_2^{(n)}, \tau_2^{(n_1)} \in E_{\frac{\epsilon}{2}, f_2}$, applying proprieties of translation numbers, we get that $\tau_1^{(n)} - \tau_1^{(n_1)} \in E_{\epsilon, f_1}$ and $\tau_2^{(n)} - \tau_2^{(n_1)} \in E_{\epsilon, f_2}$, consequently we have that

$$|(\tau_1^{(n)} - \tau_1^{(n_1)}) - (\tau_2^{(n)} - \tau_2^{(n_1)})| = |\lambda\delta| < \delta,$$

and thus we can guarantee that

$$d\left(\tau_1^{(n)} - \tau_1^{(n_1)}, E_{\epsilon, f_2}\right) < \delta.$$

We can see that for every $n \in \mathbb{Z}$, $\tau_1^{(n)} \in](n-1)l, nl[$ and $\tau_1^{(n+1)} \in]nl, (n+1)l[$ which implies that

$$|\tau_1^{(n)} - \tau_1^{(n+1)}| < (n+1)l - (n-1)l = 2l,$$

and due to the fact that $n_1, (n+1)_1 \in [-n_0, n_0]$ for every $n \in \mathbb{Z}$, we can also see that $\tau_1^{(n)_1}, \tau_1^{(n+1)_1} \in](-n_0-1)l, n_0 l[$ which implies that

$$|\tau_1^{(n+1)_1} - \tau_1^{(n)_1}| < n_0 l - (-n_0 - 1)l = 2n_0 l + l.$$

Then for each $n \in \mathbb{Z}$,

$$\begin{aligned} |(\tau_1^{(n)} - \tau_1^{(n_1)}) - (\tau_1^{(n+1)} - \tau_1^{(n+1)_1})| &= |(\tau_1^{(n)} - \tau_1^{(n+1)}) + (\tau_1^{(n+1)_1} - \tau_1^{(n)_1})| \\ &\leq |\tau_1^{(n)} - \tau_1^{(n+1)}| + |\tau_1^{(n+1)_1} - \tau_1^{(n)_1}| \\ &< 2l + (2ln_0 + l) = (2n_0 + 3)l =: l_3, \end{aligned}$$

hence we can assure that the set $W := \{\tau_1^{(n)} - \tau_1^{(n)_1} : n \in \mathbb{Z}\}$ is relatively dense, therefore we conclude that

$$\{\tau \in E_{\epsilon, f_1} : \ d(\tau, E_{\epsilon, f_2}) < \delta\}$$

is also relatively dense in $\mathbb{R}$ because it contains the set W. □

Theorem 2.3.4 ([4, Chapter 1, Section 1, Theorem 11]). *If $\epsilon > 0$ and if f_1 and f_2 are uniformly almost periodic functions, then the set $E_{\epsilon, f_1} \cap E_{\epsilon, f_2}$ is relatively dense.*

Proof. Let $\epsilon > \epsilon_1 > 0$. Applying Lemma 2.3.2 there exists a $\delta_{\epsilon,\epsilon_1} > 0$ such that E_{ϵ,f_1} contains any number τ that satisfies

$$d(\tau, E_{\epsilon_1,f_1}) < \delta_{\epsilon,\epsilon_1}.$$

Since f_1 and f_2 are u.a.p. functions, using Lemma 2.3.3 we can assure that the set

$$\{\tau \in E_{\epsilon_1,f_2} : \ d(\tau, E_{\epsilon_1,f_1}) < \delta_{\epsilon,\epsilon_1}\},$$

is relatively dense. Taking into account that $\epsilon_1 < \epsilon$ and considering the previous statements, it follows that

$$\{\tau \in E_{\epsilon_1,f_2} : \ d(\tau, E_{\epsilon_1,f_1}) < \delta_{\epsilon,\epsilon_1}\} \subseteq \{\tau \in E_{\epsilon_1,f_2} : \ \tau \in E_{\epsilon,f_1}\} = E_{\epsilon_1,f_2} \cap E_{\epsilon,f_1},$$

and due to the fact that the former set is relatively dense, $E_{\epsilon,f_1} \cap E_{\epsilon_1,f_2}$ is also relatively dense. Since $\epsilon_1 < \epsilon$ we have that $E_{\epsilon_1,f_2} \subseteq E_{\epsilon,f_2}$ and we conclude that $E_{\epsilon,f_1} \cap E_{\epsilon,f_2}$ is relatively dense. ☐

With that being said, we are now in conditions to prove that the sum of two u.a.p. functions is a u.a.p. function.

Theorem 2.3.5 ([4, Chapter 1, Section 1, Theorem 12]). *If f_1 and f_2 are uniformly almost periodic functions, then $f_1 + f_2$ is uniformly almost periodic function as well.*

Proof. Let $\epsilon > 0$ and let $\tau \in E_{\frac{\epsilon}{2},f_1} \cap E_{\frac{\epsilon}{2},f_2}$. Then $\tau \in E_{\frac{\epsilon}{2},f_1}$ and $\tau \in E_{\frac{\epsilon}{2},f_2}$, therefore it follows that

$$\sup_{x\in\mathbb{R}} |f_1(x+\tau) - f_1(x)| \leq \frac{\epsilon}{2}, \quad \sup_{x\in\mathbb{R}} |f_2(x+\tau) - f_2(x)| \leq \frac{\epsilon}{2}.$$

In these conditions we can guarantee that

$$\begin{aligned}\sup_{x\in\mathbb{R}} |(f_1+f_2)(x+\tau) - (f_1+f_2)(x)| &= \sup_{x\in\mathbb{R}} |f_1(x+\tau) + f_2(x+\tau) - f_1(x) - f_2(x)| \\ &\leq \sup_{x\in\mathbb{R}} |f_1(x+\tau) - f_1(x)| + \sup_{x\in\mathbb{R}} |f_2(x+\tau) - f_2(x)| \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

We have proved that if $\tau \in E_{\frac{\epsilon}{2},f_1} \cap E_{\frac{\epsilon}{2},f_2}$ then $\tau \in E_{\epsilon,f_1+f_2}$, that is

$$E_{\frac{\epsilon}{2},f_1} \cap E_{\frac{\epsilon}{2},f_2} \subseteq E_{\epsilon,f_1+f_2}. \tag{2.1}$$

Applying Theorem 2.3.4 we know that $E_{\frac{\epsilon}{2},f_1} \cap E_{\frac{\epsilon}{2},f_2}$ is relatively dense in $\mathbb{R}$ and, by (2.1), we conclude that E_{ϵ,f_1+f_2} is relatively dense in $\mathbb{R}$ for each $\epsilon > 0$, that is, $f_1 + f_2$ is a uniformly almost periodic function. ☐

Applying the previous results of this section regarding algebraic properties of u.a.p. functions, we have that the product of two u.a.p. functions is also a u.a.p. function.

Theorem 2.3.6 ([4, Chapter 1, Section 1, Theorem 13]). *If f_1 and f_2 are uniformly almost periodic functions then $f_1 \cdot f_2$ is also uniformly almost periodic.*

Proof. Firstly let us observe that for any $x \in \mathbb{R}$,

$$f_1(x) \cdot f_2(x) = \frac{1}{4}(f_1(x) + f_2(x))^2 - \frac{1}{4}(f_1(x) - f_2(x))^2.$$

Since f_1 and f_2 are u.a.p. functions, applying Theorems 2.3.1 and 2.3.5, we conclude that $f_1 \cdot f_2$ is u.a.p. as we wanted to prove. □

We finish this section by seeing that the quotient of two u.a.p. functions is also a u.a.p. function, as a consequence of the previous theorem and the inverse closedness of u.a.p. functions.

Corollary 2.3.7 ([4, Chapter 1, Section 1, Corollary 13]). *If f_1 and f_2 are uniformly almost periodic functions and*

$$\inf_{x \in \mathbb{R}} |f_2(x)| > 0,$$

then $\frac{f_1}{f_2}$ is uniformly almost periodic.

Proof. We know that $\frac{f_1}{f_2} = f_1 \cdot \frac{1}{f_2}$. Applying Theorem 2.2.4, we conclude that $\frac{1}{f_2}$ is uniformly almost periodic and thus, by using Theorem 2.3.6, we deduce that $f_1 \cdot \frac{1}{f_2} = \frac{f_1}{f_2}$ is uniformly almost periodic as we wanted. □

2.4 Limits of Sequences of u.a.p. Functions

In this section we will analyse the behaviour of a sequence of u.a.p. functions that converge uniformly to a certain function.

Definition 2.4.1. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of functions such that $f_n : \mathbb{R} \to \mathbb{K}$ for each $n \in \mathbb{N}$. We say that $(f_n)_{n \in \mathbb{N}}$ is uniformly convergent to a function $f : \mathbb{R} \to \mathbb{K}$ if and only if for each $\epsilon > 0$ there exists $p \in \mathbb{N}$ such that for any $x \in \mathbb{R}$ if $n > p$, then

$$|f_n(x) - f(x)| < \epsilon.$$

Theorem 2.4.2 ([4, Chapter 1, Section 1, Theorem 8]). *Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of u.a.p functions such that $(f_n)_{n \in \mathbb{N}}$ converges uniformly in $\mathbb{R}$ to a function f. Then f is also a u.a.p. function.*

Proof. Let $\epsilon > 0$. Since (f_n) converges uniformly to f, there exists $p \in \mathbb{N}$ such that for every $x \in \mathbb{R}$,

$$|f(x) - f_p(x)| < \frac{\epsilon}{3}.$$

Let $\tau \in E_{\frac{\epsilon}{3}, f_p}$ and $x \in \mathbb{R}$. Then

$$\sup_{x \in \mathbb{R}} |f_p(x + \tau) - f_p(x)| \leq \frac{\epsilon}{3},$$

therefore we have

$$|f(x + \tau) - f(x)| = |f(x + \tau) - f(x) + f_p(x + \tau) - f_p(x + \tau) + f_p(x) - f_p(x)|$$

$$\le |f(x+\tau) - f_p(x+\tau)| + |f_p(x) - f(x)| + |f_p(x+\tau) - f_p(x)|$$
$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,$$

and thus $\tau \in E_{\epsilon,f}$, that is, $E_{\frac{\epsilon}{3},f_p} \subseteq E_{\epsilon,f}$. Since the former set is relatively dense, it follows that $E_{\epsilon,f}$ is also relatively dense, and therefore f is uniformly almost periodic. □

We finish this section by verifying that the sum of any uniformly convergent trigonometric series is also a u.a.p. function.

Corollary 2.4.3 ([4, Chapter 1, Section 1, Corollary 12]). *Let $c_n \in \mathbb{C}$ and $\lambda_n \in \mathbb{R}$ for every $n \in \mathbb{N}$. If the series $\sum_{n=1}^{\infty} c_n e^{i\lambda_n x}$ is uniformly convergent, then its sum is u.a.p.*

Proof. We know that $c_n e^{i\lambda_n x}$ is a purely periodic function and therefore, applying Lemma 2.1.5, is a u.a.p. function, thus if we consider a sequence $f_n(x) = c_n e^{i\lambda_n x}$, f_n is a u.a.p. function for every $n \in \mathbb{N}$ and using Theorem 2.3.5 we get that $\sum_{k=1}^{n} f_k$ is also uniformly almost periodic. Since the series is uniformly convergent, we just need to apply Theorem 2.4.2 and the proof is done. □

2.5 Derivatives and Integrals of u.a.p. Functions

We start this section by seeing that if a real function is u.a.p. and if its derivative is uniformly continuous in $\mathbb{R}$, then it is also a u.a.p. function.

Theorem 2.5.1 ([4, Chapter 1, Section 1, Theorem 14]). *Let f be a real u.a.p. function. If the derivative of f, f', is uniformly continuous in $\mathbb{R}$, then it is also uniformly almost periodic.*

Proof. Let $\epsilon > 0$ and let $(h_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that $h_n \neq 0$ for every $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} h_n = 0.$$

Since f is differentiable, applying Lagrange's mean value theorem, it follows that for every $n \in \mathbb{N}$ and $x \in \mathbb{R}$, there exists $\lambda_{n,x} \in \,]0,1[$ such that

$$(T_{\lambda_{n,x}h_n} f')(x) = \frac{(T_{h_n} f)(x) - f(x)}{h_n}.$$

Due to the fact that f and $T_a f$ are u.a.p. functions for every $a \in \mathbb{R}$, applying Theorem 2.3.1 and Theorem 2.3.5 we have that $T_{\lambda_{n,x}h_n} f'$ is u.a.p. for every $n \in \mathbb{N}$. Since f' is uniformly continuous in $\mathbb{R}$, there exists $\delta_\epsilon > 0$ such that for each $x, y \in \mathbb{R}$,

$$|x - y| < \delta_\epsilon \Rightarrow |f'(x) - f'(y)| < \epsilon. \tag{2.2}$$

Since $\lambda_{n,x} \in]0,1[$ for each $n \in \mathbb{N}$ and for any $x \in \mathbb{R}$, it follows that for every $x \in \mathbb{R}$,

$$\lim_{n\to\infty} |x + \lambda_{n,x} h_n - x| \le \lim_{n\to\infty} |h_n| = 0.$$

Consequently there exists $p \in \mathbb{N}$, which does not depend on x, such that if $n > p$, then

$$|x + \lambda_{n,x} h_n - x| \le |h_n| < \delta_\epsilon.$$

Therefore if $n > p$, applying the previous inequality to the condition (2.2), we have that

$$|f'(x + \lambda_{n,x} h_n) - f'(x)| = |(T_{\lambda_{n,x} h_n} f')(x) - f'(x)| < \epsilon$$

for each $x \in \mathbb{R}$. Consequently $(T_{\lambda_{n,x} h_n} f')_{n\in\mathbb{N}}$ is a sequence of u.a.p. functions that converge uniformly in $\mathbb{R}$ to f' and we conclude, by Theorem 2.4.2, that f' is uniformly almost periodic. □

We finish this chapter by checking that any bounded indefinite integral of a u.a.p. function is also a u.a.p. function.

Theorem 2.5.2 ([4, Chapter 1, Section 1, Theorem 15]). *If an indefinite integral of a u.a.p. function f is bounded, then it is a uniformly almost periodic function.*

Proof. Let us consider, without loss of generality, that f is a real function and let $a \in \mathbb{R}$. Let us assume that

$$g(x) = \int_a^x f(y)\, dy$$

is bounded, that is, there are $k_1, k_2 \in \mathbb{R}$ such that

$$k_1 = \inf_{x\in\mathbb{R}} g(x), \quad k_2 = \sup_{x\in\mathbb{R}} g(x).$$

Let $\eta > 0$. In these conditions we can assure that there are $x_1, x_2 \in \mathbb{R}$ such that

$$g(x_1) < k_1 + \eta, \quad g(x_2) > k_2 - \eta.$$

Let $\epsilon_1 > 0$, $\tau_1 \in E_{\epsilon_1, f}$ and $d = |x_2 - x_1|$. In these circumstances it follows that

$$\left| \int_{x_1+\tau_1}^{x_2+\tau_1} f(x)\, dx - \int_{x_1}^{x_2} f(x)\, dx \right| = \left| \int_{x_1}^{x_2} [f(x+\tau_1) - f(x)]\, dx \right| \le \epsilon_1 d,$$

that is

$$|g(x_2+\tau_1) + g(x_1) - g(x_1+\tau_1) - g(x_2)| \le \epsilon_1 d.$$

Therefore we have

$$g(x_1+\tau_1) \le g(x_2+\tau_1) - g(x_2) + g(x_1) + \epsilon_1 d, \tag{2.3}$$

and since

$$k_2 = \sup_{x\in\mathbb{R}} g(x), \quad g(x_1) < k_1 + \eta, \quad g(x_2) > k_2 - \eta,$$

we have that

$$g(x_2+\tau_1) \le k_2, \quad g(x_2) - g(x_1) > k_2 - k_1 - 2\eta, \tag{2.4}$$

and consequently, using the inequalities given in (2.3) and (2.4), we have

$$g(x_1+\tau_1) < k_1 + 2\eta + \epsilon_1 d. \tag{2.5}$$

Let $\epsilon_2 > 0$ which will be defined later on and let $\tau_2 \in E_{\epsilon_2,f}$. Repeating the same argument and justifications as we did previously and observing that $\tau_1 + \tau_2 \in E_{\epsilon_1+\epsilon_2,f}$, we conclude that

$$g(x_1 + \tau_1 + \tau_2) < k_1 + 2\eta + (\epsilon_1 + \epsilon_2)d. \tag{2.6}$$

Using the fact that $E_{\epsilon_1,f}$ is relatively dense, there is $l_{\epsilon_1} > 0$ such that any interval with length l_{ϵ_1} intersects $E_{\epsilon_1,f}$. Given $x \in \mathbb{R}$, we can choose $\tau_3 \in E_{\epsilon_1,f}$ satisfying the inequalities

$$x < x_1 + \tau_3, \quad x_1 + \tau_3 < x + l_{\epsilon_1}$$

thus we have

$$\int_x^{x+\tau_2} f(y)\,dy = \int_{x_1+\tau_3}^{x_1+\tau_2+\tau_3} f(y)\,dy + \int_x^{x_1+\tau_3} [f(y) - f(y+\tau_2)]\,dy. \tag{2.7}$$

Taking into account that $g(x) \geq k_1$ for each $x \in \mathbb{R}$, it follows that

$$-g(x_1 + \tau_3) \leq -k_1, \quad g(x_1 + \tau_2 + \tau_3) \geq k_1$$

and applying inequality (2.6) with $\tau_1 = \tau_3$ and the fact that

$$-g(x_1 + \tau_3) \leq -k_1,$$

we get

$$\left| \int_{x_1+\tau_3}^{x_1+\tau_2+\tau_3} f(x)\,dx \right| = |g(x_1 + \tau_2 + \tau_3) - g(x_1 + \tau_3)| < 2\eta + (\epsilon_1 + \epsilon_2)d. \tag{2.8}$$

Since

$$x < x_1 + \tau_3, \quad x_1 + \tau_3 < x + l_{\epsilon_1},$$

we see that

$$\left| \int_x^{x_1+\tau_3} [f(y) - f(y+\tau_2)]\,dy \right| < \int_x^{x+l_{\epsilon_1}} |f(y+\tau_2) - f(y)|\,dy \leq \epsilon_2 l_{\epsilon_1}. \tag{2.9}$$

Consequently using equation (2.7) and inequalities (2.8) and (2.9), we have

$$\left| \int_x^{x+\tau_2} f(y)\,dy \right| < 2\eta + (\epsilon_1 + \epsilon_2)d + \epsilon_2 l_{\epsilon_1}.$$

Given $\epsilon > 0$, if we consider

$$\eta = \frac{\epsilon}{6},\ \epsilon_1 = \frac{\epsilon}{6d},\ \epsilon_2 = \min\left\{\epsilon_1, \frac{\epsilon}{3l_{\epsilon_1}}\right\},$$

we obtain that

$$|g(x+\tau_2) - g(x)| = \left| \int_x^{x+\tau_2} f(y)\,dy \right| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,$$

for every $x \in \mathbb{R}$ and for any $\tau_2 \in E_{\epsilon_2,f}$, therefore we can conclude that $E_{\epsilon_2,f} \subseteq E_{\epsilon,g}$, that is, $E_{\epsilon,g}$ is relatively dense as we wanted to prove. □

3

Relations Between Normal, U.A.P. and Trigonometric Polynomial Functions

In this chapter we will start by introducing the class of normal functions and, in the same section, we are going to prove that this new class coincides with the set $U(\mathbb{R})$. After defining the mean value of a u.a.p. function, we will verify that, in fact, the mean value of a u.a.p. function always exists. Following that, we are going to give an alternative definition of the mean value and we will present some properties of it. Afterwards, we will not only construct the Fourier series for a u.a.p. function, by defining its Fourier coefficients as the mean value of product between that function and the function $e_{-\lambda}$, where $e_\lambda(x) := e^{i\lambda x}$ for each $x \in \mathbb{R}$, but also establish Bessel's inequality. Moreover, we will study some properties of the Fourier series regarding sequences of u.a.p. functions and also the uniqueness of the Fourier series for these functions. We finish this chapter by verifying that the closure, on $L^\infty(\mathbb{R})$, of the set of trigonometric polynomial functions, is equal to the set $U(\mathbb{R})$.

3.1 Normal Functions

We start this section by recalling a well known definition from Functional Analysis.

Definition 3.1.1. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions such that $f_n : \mathbb{R} \to \mathbb{K}$ for each $n \in \mathbb{N}$. We say that $(f_n)_{\mathbb{N}}$ is uniformly Cauchy if and only if for each $\epsilon > 0$ there exists $p \in \mathbb{N}$ such that for any $x \in \mathbb{R}$ if $m, n > p$, then

$$|f_n(x) - f_m(x)| < \epsilon.$$

Definition 3.1.2. Let $f : \mathbb{R} \to \mathbb{K}$ be a continuous function. We say that f is normal if and only if for any sequence $(h_n)_{n\in\mathbb{N}}$ of real numbers, there exists a subsequence $(h_{n_k})_{k\in\mathbb{N}}$ such that $(T_{h_{n_k}} f)_{k\in\mathbb{N}}$ is a uniformly convergent sequence of functions. In this work we will denote the set of all normal functions by $N(\mathbb{R})$.

In order to understand the relation between u.a.p. functions and normal functions, we will need the following result.

Theorem 3.1.3 ([4, Chapter 1, Section 2, Lemma 2]). *Let f be a u.a.p. function and $(h_n)_{n\in\mathbb{N}}$ a sequence of real numbers. Then for any $\epsilon > 0$ there corresponds a subsequence $(h_{n_k})_{k\in\mathbb{N}}$ of $(h_n)_{n\in\mathbb{N}}$ such that*

$$\sup_{x\in\mathbb{R}} |f(x+h_{n_i}) - f(x+h_{n_j})| < \epsilon,$$

for any $i, j \in \mathbb{N}$.

Proof. Let $\epsilon > 0$. Taking into account that f is a u.a.p. function, there exists $l_{\frac{\epsilon}{4}} > 0$ such that any interval with length $l_{\frac{\epsilon}{4}}$ intersects $E_{\frac{\epsilon}{4},f}$. For each $n \in \mathbb{N}$ we can say that

$$h_n = \tau_n + r_n,$$

where $\tau_n \in E_{\frac{\epsilon}{4},f}$ and r_n is a real number satisfying the inequalities $0 \le r_n \le l_{\frac{\epsilon}{4}}$. Let r be the limit of some convergent subsequence of $(r_n)_{n\in\mathbb{N}}$, which indeed exists because every bounded sequence admits a convergent subsequence, and consider $\delta > 0$ such that for each $x_1, x_2 \in \mathbb{R}$,

$$|x_2 - x_1| < 2\delta \Rightarrow |f(x_2) - f(x_1)| < \frac{\epsilon}{3}.$$

In fact, this δ exists because, by Theorem 2.2.2, f is uniformly continuous. Consider the subsequence $(h_{n_k})_{k\in\mathbb{N}}$ formed by every h_n that verifies

$$r - \delta < r_n < r + \delta.$$

Since $\tau_{n_i} - \tau_{n_j} \in E_{\frac{\epsilon}{2},f}$ and $|r_{n_i} - r_{n_j}| < 2\delta$ for each $i, j \in \mathbb{N}$, it follows that

$$\sup_{x\in\mathbb{R}} |f(x+\tau_{n_i} - \tau_{n_j} + r_{n_i} - r_{n_j}) - f(x + r_{n_i} - r_{n_j})| \le \frac{\epsilon}{2}, \quad \sup_{x\in\mathbb{R}} |f(x + r_{n_i} - r_{n_j}) - f(x)| \le \frac{\epsilon}{3} < \frac{\epsilon}{2}.$$

Consequently we have

$$\begin{aligned}
\sup_{x\in\mathbb{R}} |f(x+h_{n_i}) - f(x+h_{n_j})| &= \sup_{y\in\mathbb{R}} |f(y - h_{n_j} + h_{n_i}) - f(y)| \\
&= \sup_{x\in\mathbb{R}} |f(x + \tau_{n_i} + r_{n_i} - \tau_{n_j} - r_{n_j}) - f(x)| \\
&\le \sup_{x\in\mathbb{R}} |f(x+\tau_{n_i} - \tau_{n_j} + r_{n_i} - r_{n_j}) - f(x + r_{n_i} - r_{n_j})| \\
&\quad + \sup_{x\in\mathbb{R}} |f(x + r_{n_i} - r_{n_j}) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

□

Now we shall see that a function is u.a.p. if and only if it is normal, that is, the set of all u.a.p. functions coincides with the set of all normal functions. Having that in mind, we will start to prove that the set of all uniformly almost periodic functions is contained in the set of all normal functions.

Theorem 3.1.4 ([4, Chapter 1, Section 2, Theorem 3]). *If a function f is uniformly almost periodic, then f is normal.*

Proof. Let $(h_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Since f is a u.a.p. function, we can apply Theorem 3.1.3 and assure that for any $\epsilon > 0$ there exists a subsequence of $(h_n)_{n\in\mathbb{N}}$, for example $(h_{n_k}^{(\epsilon)})_{k\in\mathbb{N}}$, such that for any $i, j \in \mathbb{N}$,

$$\sup_{x\in\mathbb{R}} |f(x + h_{n_i}^{(\epsilon)}) - f(x + h_{n_j}^{(\epsilon)})| < \epsilon.$$

Consider $\epsilon = 1$. In these conditions there is a subsequence $(h_{n_k}^{(1)})_{k\in\mathbb{N}}$ of the sequence $(h_n)_{n\in\mathbb{N}}$ such that

$$\sup_{x\in\mathbb{R}} |f(x + h_{n_i}^{(1)}) - f(x + h_{n_j}^{(1)})| < 1.$$

Put $\epsilon = \frac{1}{2}$. In these circumstances there exists a subsequence $(h_{n_k}^{(\frac{1}{2})})_{k\in\mathbb{N}}$ of the sequence $(h_{n_k}^{(1)})_{k\in\mathbb{N}}$, that verifies

$$\sup_{x\in\mathbb{R}} |f(x + h_{n_i}^{(\frac{1}{2})}) - f(x + h_{n_j}^{(\frac{1}{2})})| < \frac{1}{2}.$$

Let $\epsilon = \frac{1}{3}$. In this case there is a subsequence $(h_{n_k}^{(\frac{1}{3})})_{k\in\mathbb{N}}$ of the sequence $(h_{n_k}^{(\frac{1}{2})})_{k\in\mathbb{N}}$ that satisfies the inequality

$$\sup_{x\in\mathbb{R}} |f(x + h_{n_i}^{(\frac{1}{3})}) - f(x + h_{n_j}^{(\frac{1}{3})})| < \frac{1}{3}.$$

Repeating this reasoning, we can assure that the sequence $(T_{h_{n_k}^{(\frac{1}{k})}} f)_{k\in\mathbb{N}}$, verifies for every $i, j \in \mathbb{N}$ with $i < j$,

$$\sup_{x\in\mathbb{R}} |f(x + h_{n_i}^{(\frac{1}{i})}) - f(x + h_{n_j}^{(\frac{1}{j})})| < \frac{1}{i},$$

due to the fact that $(h_{n_k}^{(\frac{1}{j})})_{k\in\mathbb{N}}$ is a subsequence of $(h_{n_k}^{(\frac{1}{i})})_{k\in\mathbb{N}}$. Consequently $(T_{h_{n_k}^{(\frac{1}{k})}} f)_{k\in\mathbb{N}}$ is a uniformly Cauchy sequence, which implies that this sequence is also uniformly convergent because $\mathbb{K}$ is complete, and we conclude that f is normal. □

We finish this section by proving that $N(\mathbb{R}) \subseteq U(\mathbb{R})$, therefore, taking into account the previous theorem, we have, in fact, the equality $N(\mathbb{R}) = U(\mathbb{R})$.

Theorem 3.1.5 ([4, Chapter 1, Section 2, Theorem 4]). *If a function f is normal, then f is uniformly almost periodic.*

Proof. Suppose, by contradiction, that f is not uniformly almost periodic. Then we can assure that there exists an $\epsilon > 0$ such that $E_{\epsilon,f}$ is not relatively dense. Let $h_1 \in \mathbb{R}$ and consider an interval $]a_2, b_2[$ with length greater than $2|h_1|$, where the interval $]a_2, b_2[$ does not contain any number of $E_{\epsilon,f}$. Let

$$h_2 = \frac{a_2 + b_2}{2}.$$

In these conditions it follows that $h_2-h_1 \in]a_2, b_2[$ and consequently $h_2-h_1 \notin E_{\epsilon,f}$. Consider now an interval $]a_3, b_3[\subseteq \mathbb{R}$ with length greater than $2(|h_1|+|h_2|)$ and which does not contain any number of $E_{\epsilon,f}$. Put

$$h_3 = \frac{a_3+b_3}{2}.$$

In these circumstances we have that $h_3-h_1, h_3-h_2 \in]a_3, b_3[$, hence $h_3-h_1, h_3-h_2 \notin E_{\epsilon,f}$. Repeating this reasoning, we can find a sequence $(h_n)_{n\in\mathbb{N}}$, such that for any $i, j \in \mathbb{N}$,

$$h_i - h_j \notin E_{\epsilon,f},$$

and therefore we have

$$\sup_{x\in\mathbb{R}} |f(x+h_i) - f(x+h_j)| > \epsilon.$$

Consequently, given a sequence $(h_n)_{n\in\mathbb{N}}$, the corresponding sequence $(T_{h_n}f)_{n\in\mathbb{N}}$ does not have any subsequence which is uniformly Cauchy. Therefore, due to the fact that $\mathbb{K}$ is a Banach space, the sequence $(T_{h_n}f)_{n\in\mathbb{N}}$ does not have any uniformly convergent subsequence. But this is a contradiction because f is normal by our hypothesis, thus we have that f is uniformly almost periodic as we wanted to prove. □

3.2 Mean Value of a u.a.p. Function

We start this section by defining the mean value of a u.a.p. function.

Definition 3.2.1. Let f be a real u.a.p. function. We define the upper (respectively lower) mean value of the function f, and we denote by $\overline{M}_f$, (respectively $\underline{M}_f$) as being

$$\overline{M}_f = \limsup_{y\to+\infty}\left(\frac{1}{y}\int_0^y f(x)\,dx\right), \quad \underline{M}_f = \liminf_{y\to+\infty}\left(\frac{1}{y}\int_0^y f(x)\,dx\right).$$

If $\overline{M}_f = \underline{M}_f$ then we denote their common value by M_f and we say that M_f is the mean value of the function f. On the other hand, if f is a complex function, then we only define the mean value M_f as

$$\lim_{y\to+\infty}\left(\frac{1}{y}\int_0^y f(x)\,dx\right).$$

If we have a function f of n variables, then we indicate the variable with respect to which the mean is being calculated, for example,

$$M_{x_1,f} = \lim_{y\to+\infty}\left(\frac{1}{y}\int_0^y f(x_1,\ldots,x_n)\,dx_1\right).$$

The following lemma shows that if the mean value of a u.a.p. function exists, then it coincides precisely with the mean value of the translation function.

Lemma 3.2.2. *Let $a\in\mathbb{R}$. If f is a u.a.p. function and if M_f exists, then M_{T_af} exists and is equal to M_f.*

Proof. Since f is a u.a.p. function, it follows, by Theorem 2.2.1, that f is bounded and thus there exists $\delta \in \mathbb{R}^+$ such that for every $x \in \mathbb{R}$ we have $|f(x)| \leq \delta$, therefore we get

$$\lim_{y\to+\infty}\left|\frac{1}{y}\int_0^a f(x)\,dx\right| \leq \lim_{y\to+\infty}\frac{1}{y}\int_0^a |f(x)|\,dx \leq \lim_{y\to+\infty}\frac{1}{y}\int_0^a \delta\,dx = \frac{a\delta}{+\infty} = 0,$$

which implies that

$$\lim_{y\to+\infty}\frac{1}{y}\int_0^a f(x)\,dx = 0. \tag{3.1}$$

And we also have

$$\lim_{y\to+\infty}\left|\frac{1}{y}\int_y^{y+a} f(x)\,dx\right| \leq \lim_{y\to+\infty}\frac{1}{y}\int_y^{y+a} |f(x)|\,dx \leq \lim_{y\to+\infty}\frac{1}{y}\int_y^{y+a} \delta\,dx = \frac{a\delta}{+\infty} = 0,$$

which implies that

$$\lim_{y\to+\infty}\frac{1}{y}\int_y^{y+a} f(x)\,dx = 0. \tag{3.2}$$

Consequently, taking into account the equations (3.1) and (3.2), we have that

$$\begin{aligned}
M_{T_a f} &= \lim_{y\to+\infty}\frac{1}{y}\int_0^y (T_a f)(x)\,dx \\
&= \lim_{y\to+\infty}\frac{1}{y}\int_0^y f(x+a)\,dx \\
&= \lim_{y\to+\infty}\frac{1}{y}\int_a^{y+a} f(t)\,dt \\
&= \lim_{y\to+\infty}\frac{1}{y}\int_0^{y+a} f(x)\,dx - \lim_{y\to+\infty}\frac{1}{y}\int_0^a f(x)\,dx \\
&= \lim_{y\to+\infty}\frac{1}{y}\int_0^y f(x)\,dx + \lim_{y\to+\infty}\frac{1}{y}\int_y^{y+a} f(x)\,dx - 0 \\
&= \lim_{y\to+\infty}\frac{1}{y}\int_0^y f(x)\,dx + 0 = M_f.
\end{aligned}$$

□

The next result is a well-known fact from a basic analysis course (see, e.g., [16, Section 3.2.4, Theorem 4]), which is similar to Cauchy's criterion for sequences. For the convenience of the reader, we give its proof below.

Lemma 3.2.3. *Let $f : \mathbb{R} \to \mathbb{K}$ be a function. If for every $\epsilon > 0$ there exists $M > 0$ such that*

$$|f(x) - f(y)| < \epsilon$$

for each $x, y > M$, then the finite limit $\lim_{x\to+\infty} f(x)$ exists.

Proof. Suppose, without loss of generality, that f is a real function and let $(x_n)_{n\in\mathbb{N}}$ a sequence of real numbers tending to $+\infty$. Then we know that the sequence $(f(x_n))_{n\in\mathbb{N}}$ has a monotone subsequence $(f(x_{n_j}))_{j\in\mathbb{N}}$. Let $\epsilon = 1$. Then, by our hypothesis, there is $M_1 > 0$ such that

$$|f(x) - f(y)| < 1$$

for each $x, y > M_1$. Consequently,

$$f(M_1+1) - 1 < f(x) < 1 + f(M_1+1)$$

for every $x > M_1$. Since

$$\lim_{j\to\infty} x_{n_j} = +\infty,$$

there exists $p \in \mathbb{N}$ such that if $j > p$, then $x_{n_j} > M_1$. Consider

$$a := \min\{f(M_1+1)-1, f(x_{n_1}), \dots, f(x_{n_p})\}, \quad b := \max\{f(M_1+1)+1, f(x_{n_1}), \dots, f(x_{n_p})\}.$$

In these conditions we have that

$$a \le f(x_{n_j}) \le b$$

for each $j \in \mathbb{N}$, therefore the sequence $(f(x_{n_j}))_{j\in\mathbb{N}}$ is a bounded sequence. Taking into account that $(f(x_{n_j}))_{j\in\mathbb{N}}$ is also monotone, we get that the sequence $(f(x_{n_j}))_{j\in\mathbb{N}}$ is convergent to some $L \in \mathbb{R}$. Let $\epsilon > 0$. Then there is $M > 0$ such that

$$|f(x) - f(y)| < \frac{\epsilon}{2}$$

for each $x, y > M$. Due to the fact that

$$\lim_{n\to\infty} x_n = +\infty,$$

there exists $N \in \mathbb{N}$ such that if $n > N$, then $x_n > M$. Since

$$\lim_{j\to\infty} x_{n_j} = +\infty,$$

there is $J_1 \in \mathbb{N}$ such that if $j > J_1$, then $x_{n_j} > M$. On the other hand, since

$$\lim_{j\to\infty} f(x_{n_j}) = L,$$

there exists $J_2 \in \mathbb{N}$ such that if $j > J_2$, then

$$|f(x_{n_j}) - L| < \frac{\epsilon}{2}.$$

Consider $K := \max\{J_1, J_2\} + 1$. In these conditions if $n > N$, then

$$|f(x_n) - L| \le |f(x_n) - f(x_{n_K})| + |f(x_{n_K}) - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

and, consequently, we have that

$$\lim_{n\to\infty} f(x_n) = L.$$

We have yet to prove that the sequence $(f(x_n))_{n\in\mathbb{N}}$ converge to the same finite limit, regardless of the choice of the sequence $(x_n)_{n\in\mathbb{N}}$ tending to $+\infty$. Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be sequences that tend to $+\infty$. Then, the sequences $(f(u_n))_{n\in\mathbb{N}}$ and $(f(v_n))_{n\in\mathbb{N}}$ converge to some $L, K \in \mathbb{R}$, respectively. Let us define the sequence $(w_n)_{n\in\mathbb{N}}$ by

$$w_n := \begin{cases} u_n, & \text{if } n \text{ is even,} \\ v_n, & \text{if } n \text{ is odd.} \end{cases}$$

In these conditions we have that $(w_n)_{n\in\mathbb{N}}$ tends to $+\infty$, hence the sequence $(f(w_n))_{n\in\mathbb{N}}$ converges to some $P \in \mathbb{R}$. But, by definition, the sequences $(f(u_n))_{n\in\mathbb{N}}$ and $(f(v_n))_{n\in\mathbb{N}}$ are subsequences of the sequence $(f(w_n))_{n\in\mathbb{N}}$, therefore they also converge to P. Hence, by the uniqueness of the limit in $\mathbb{R}$, we have that $L = K$. Due to the fact that Cauchy's limit definition and Heine's limit definition are equivalent, we conclude that the finite limit $\lim_{x\to\infty} f(x)$ exists as we wanted to prove. □

As we will check, the mean value of a u.a.p. function always exists.

Theorem 3.2.4 ([4, Chapter 1, Section 3, Theorem 2]). *If f is a uniformly almost periodic function, then M_f exists.*

Proof. Since

$$[0, nz] = \bigcup_{k=0}^{n-1} [kz, (k+1)z]$$

for each $n \in \mathbb{N}$ and $z > 0$, it follows from the properties of integrals that

$$\frac{1}{nz}\int_0^{nz} f(x)\,dx = \sum_{k=0}^{n-1}\left(\frac{1}{nz}\int_{kz}^{(k+1)z} f(x)\,dx\right)$$

for every $n \in \mathbb{N}$ and $z > 0$. Let $\epsilon > 0$. Due to the fact that f is a u.a.p. function, we know that $E_{\epsilon,f}$ is relatively dense and therefore there exists $l_\epsilon > 0$ such that any interval with length l_ϵ intersects $E_{\epsilon,f}$. Let $n \in \mathbb{N}$ and $z > 0$. Given $k \in \{0, 1, \ldots, n-1\}$, consider the interval $]kz, kz + l_\epsilon[$. In these circumstances the interval $]kz, kz + l_\epsilon[$ has length l_ϵ and, consequently, there exists a number $\tau_k \in\,]kz, kz + l_\epsilon[\, \cap E_{\epsilon,f}$. In these conditions we can assure that

$$\begin{aligned}
\int_{kz}^{(k+1)z} f(x)\,dx &= \int_{kz-\tau_k}^{(k+1)z-\tau_k} f(y+\tau_k)\,dy \\
&= \int_{kz-\tau_k}^{(k+1)z-\tau_k} f(x+\tau_k)\,dx \\
&= \int_0^z f(x)\,dx + \int_0^z [f(x+\tau_k) - f(x)]\,dx \\
&\quad + \int_{kz-\tau_k}^{0} f(x+\tau_k)\,dx + \int_z^{(k+1)z-\tau_k} f(x+\tau_k)\,dx \\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Since f is a u.a.p. function, it follows that

$$\sup_{x\in\mathbb{R}} |f(x+\tau_k) - f(x)| \le \epsilon,$$

and thus

$$|I_2| = \left|\int_0^z [f(x+\tau_k) - f(x)]\,dx\right| \le \int_0^z |f(x+\tau_k) - f(x)|\,dx \le \int_0^z \epsilon\,dx = \epsilon z.$$

Let

$$A := \sup_{x\in\mathbb{R}} |f(x)|.$$

We can see that the length of the range of integration in both I_3 and I_4 is less than l_ϵ, therefore we have that

$$|I_3| < Al_\epsilon, \quad |I_4| < Al_\epsilon.$$

Hence for every $k \in \{0,\dots,n-1\}$ there exists $\lambda_k \in \mathbb{C}$ such that $|\lambda_k| \leq 1$, and λ_k verifies

$$\int_{kz}^{(k+1)z} f(x)\,dx = \int_0^z f(x)\,dx + \lambda_k(\epsilon z + 2Al_\epsilon).$$

Consequently

$$\frac{1}{z}\sum_{k=0}^{n-1}\int_{kz}^{(k+1)z} f(x)\,dx = \frac{1}{z}\sum_{k=0}^{n-1}\left[\int_0^z f(x)\,dx + \lambda_k(\epsilon z + 2Al_\epsilon)\right],$$

that is,

$$\frac{1}{nz}\int_0^{nz} f(x)\,dx = \frac{1}{z}\int_0^z f(x)\,dx + \frac{1}{n}\sum_{k=0}^{n-1}\lambda_k\left(\epsilon + \frac{2Al_\epsilon}{z}\right), \tag{3.3}$$

where $\lambda_n^* := \frac{1}{n}\sum\limits_{k=0}^{n-1}\lambda_k$ satisfies $|\lambda_n^*| \leq 1$. Consider $\eta > 0$ as small as we please. If

$$\left(\eta > 8\epsilon \Leftrightarrow \epsilon < \frac{\eta}{8}\right), \quad \left(\eta > \frac{16Al_\epsilon}{z} \Leftrightarrow \frac{1}{z} < \frac{\eta}{16Al_\epsilon}\right),$$

then we can see that there exists $\theta_n \in \mathbb{C}$ that verifies $|\theta_n| \leq 1$ and also

$$\frac{1}{nz}\int_0^{nz} f(x)\,dx = \frac{1}{z}\int_0^z f(x)\,dx + \theta_n\frac{\eta}{4}. \tag{3.4}$$

Given $y > z$, there exists $n_y \in \mathbb{N}$ such that $y \in [n_y z, (n_y+1)z[$ and thus there is $\lambda_y \in [0,1]$ such that

$$y = (1-\lambda_y)n_y z + \lambda_y(n_y+1)z = (n_y+\lambda_y)z.$$

Since f is a u.a.p. function, we can assure by Theorem 2.2.1 that f is bounded and hence there exists $M \in \mathbb{R}^+$ such that for every $x \in \mathbb{R}$ we have $|f(x)| \leq M$. Consequently we get that

$$\begin{aligned}\left|\frac{1}{y}\int_0^y f(x)\,dx - \frac{1}{n_y z}\int_0^{n_y z} f(x)\,dx\right| &= \left|\frac{1}{(n_y+\lambda_y)z}\int_0^{(n_y+\lambda_y)z} f(x)\,dx - \frac{1}{n_y z}\int_0^{n_y z} f(x)\,dx\right| \\ &\leq \left|\frac{-\lambda_y}{n_y(n_y+\lambda_y)z}\int_0^{n_y z} f(x)\,dx\right| \\ &\quad + \left|\frac{1}{(n_y+\lambda_y)z}\int_{n_y z}^{(n_y+\lambda_y)z} f(x)\,dx\right| \\ &\leq \frac{\lambda_y}{n_y(n_y+\lambda_y)z}\int_0^{n_y z} M\,dx + \frac{1}{(n_y+\lambda_y)z}\int_{n_y z}^{(n_y+\lambda_y)z} M\,dx\end{aligned}$$

$$=\frac{M\lambda_y}{n_y+\lambda_y}+\frac{M\lambda_y}{n_y+\lambda_y}=\frac{2M\lambda_y}{n_y+\lambda_y}\leq\frac{2M}{n_y},$$

and thus we can assure that

$$\lim_{y\to+\infty}\left|\frac{1}{y}\int_0^y f(x)\,dx-\frac{1}{n_y z}\int_0^{n_y z} f(x)\,dx\right|\leq\lim_{y\to+\infty}\frac{2M}{n_y}=0.$$

Therefore there exists $y_0>0$ such that for every $y>y_0$,

$$\left|\frac{1}{y}\int_0^y f(x)\,dx-\frac{1}{n_y z}\int_0^{n_y z} f(x)\,dx\right|<\frac{\eta}{4}. \tag{3.5}$$

Let $y_1,y_2>y_0$. Then we can assure that there exist $n_1,n_2\in\mathbb{N}$ such that $y_1\in[n_1 z,(n_1+1)z[$ and $y_2\in[n_2 z,(n_2+1)z[$. Applying equation (3.4), we can guarantee that

$$\left|\frac{1}{n_1 z}\int_0^{n_1 z} f(x)\,dx-\frac{1}{n_2 z}\int_0^{n_2 z} f(x)\,dx\right|=\left|\theta_{n_1}\frac{\eta}{4}-\theta_{n_2}\frac{\eta}{4}\right|\leq\frac{\eta}{2}. \tag{3.6}$$

Taking into account inequalities (3.5) and (3.6), it follows that

$$\begin{aligned}\left|\frac{1}{y_1}\int_0^{y_1} f(x)\,dx-\frac{1}{y_2}\int_0^{y_2} f(x)\,dx\right|\leq&\left|\frac{1}{n_1 z}\int_0^{n_1 z} f(x)\,dx-\frac{1}{n_2 z}\int_0^{n_2 z} f(x)\,dx\right|\\&+\left|\frac{1}{y_1}\int_0^{y_1} f(x)\,dx-\frac{1}{n_1 z}\int_0^{n_1 z} f(x)\,dx\right|\\&+\left|\frac{1}{y_2}\int_0^{y_2} f(x)\,dx-\frac{1}{n_2 z}\int_0^{n_2 z} f(x)\,dx\right|\\&<\frac{\eta}{2}+\frac{\eta}{4}+\frac{\eta}{4}=\eta,\end{aligned}$$

and thus we have

$$\left|\frac{1}{y_1}\int_0^{y_1} f(x)\,dx-\frac{1}{y_2}\int_0^{y_2} f(x)\,dx\right|<\eta,$$

for each $y_1,y_2>y_0$. Taking into account the previous inequality and Lemma 3.2.3, we conclude that the limit

$$M_f:=\lim_{y\to+\infty}\left(\frac{1}{y}\int_0^y f(x)\,dx\right)$$

indeed exists and is a finite value as we wanted to prove. □

In the following result we will see an alternative way to compute the mean value of a u.a.p. function.

Theorem 3.2.5 ([4, Chapter 1, Section 3, Theorem 2])**.** *If f is a u.a.p. function, then*

$$M_f=\lim_{z\to\infty}\frac{1}{z}\int_{-\frac{z}{2}}^{\frac{z}{2}} f(x)\,dx.$$

Proof. Let $\epsilon,z>0$. Taking into account equation (3.3) from the previous Theorem, we get that

$$\lambda^*:=\lim_{n\to\infty}\lambda_n^*=\frac{M_f-\frac{1}{z}\int_0^z f(x)\,dx}{\epsilon+\frac{2Al_\epsilon}{z}},$$

where $\lambda_n^* := \frac{1}{n}\sum_{k=0}^{n-1}\lambda_k$ and λ^* satisfies $|\lambda^*| \leq 1$, consequently, the limit of λ_n^* when $n \to \infty$ exists. Organizing both sides of the previous equation, we have that

$$M_f = \frac{1}{z}\int_0^z f(x)\,dx + \lambda^*\left(\epsilon + \frac{2Al_\epsilon}{z}\right). \tag{3.7}$$

The term $\lambda^*\left(\epsilon + \frac{2Al_\epsilon}{z}\right)$ is the error of the representation of M_f by the integral

$$\frac{1}{z}\int_0^z f(x)\,dx,$$

and the term $\lambda^*\left(\epsilon + \frac{2Al_\epsilon}{z}\right)$ depends on ϵ, A, l_ϵ and z. The elements ϵ and z are independent of the function f, but A and l_ϵ depend on the function f. Given $a \in \mathbb{R}$, we can guarantee that the translation function $T_a f$ has the same values for l_ϵ and A as the function f, consequently it follows from equation (3.7) that

$$M_{T_a f} = \frac{1}{z}\int_0^z (T_a f)(x)\,dx + \lambda^*\left(\epsilon + \frac{2Al_\epsilon}{z}\right).$$

Applying Lemma 3.2.2 we can assure that $M_f = M_{T_a f}$, therefore we have, by the previous equation, that

$$M_f = \frac{1}{z}\int_a^{a+z} f(x)\,dx + \lambda^*\left(\epsilon + \frac{2Al_\epsilon}{z}\right),$$

which implies, in particular, that

$$M_f = \frac{1}{z}\int_{-\frac{z}{2}}^{\frac{z}{2}} f(x)\,dx + \lambda^*\left(\epsilon + \frac{2Al_\epsilon}{z}\right).$$

Let $\delta > 0$ and consider ϵ so small that $\epsilon < \frac{\delta}{2}$. Due to the fact that

$$\lim_{z\to\infty}\frac{2Al_\epsilon}{z} = 0,$$

there exists $p \in \mathbb{N}$ such that if $z > p$, then $\frac{2Al_\epsilon}{z} < \frac{\delta}{2}$. In these conditions if $z > p$, then

$$\left|M_f - \frac{1}{z}\int_a^{a+z} f(x)\,dx\right| = \left|\lambda^*\left(\epsilon + \frac{2Al_\epsilon}{z}\right)\right| \leq \epsilon + \frac{2Al_\epsilon}{z} < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

That is, the integral

$$\frac{1}{z}\int_a^{a+z} f(x)\,dx \tag{3.8}$$

converges uniformly in $a \in \mathbb{R}$ to M_f when $z \to \infty$, hence we conclude that

$$M_f = \lim_{z\to\infty}\frac{1}{z}\int_{-\frac{z}{2}}^{\frac{z}{2}} f(x)\,dx.$$

□

Given $x \in \mathbb{R}$ and a function $f : \mathbb{R} \to \mathbb{K}$, we are going to define the function $\widetilde{f_x}$ by

$$\widetilde{f_x}(t) := \widetilde{f}(x,t) := f(t+x)\overline{f}(t),$$

for every $t \in \mathbb{R}$, where the function $\overline{f}$, as usual, is defined by

$$\overline{f}(x) := \overline{f(x)}, \quad x \in \mathbb{R}.$$

In the next result we check that, in fact, the mean value of the function $\widetilde{f_x}$ and the function g_z, with $z \in \mathbb{R}^+$, defined by

$$g_z(x) := \frac{1}{z}\int_0^z \widetilde{f_x}(t)\,dt, \quad x \in \mathbb{R}$$

are u.a.p. functions.

Lemma 3.2.6. *Let $\epsilon > 0$ and let f be a u.a.p. function. Then there exists $l_\epsilon > 0$ such that for all $z > 0$,*

$$\left| M_f - \frac{1}{z}\int_0^z f(x)\,dx \right| \le \epsilon + \frac{2l_\epsilon}{z}\sup_{x\in\mathbb{R}}|f(x)|.$$

Proof. Given $\epsilon > 0$, taking into account equality (3.7), there exists $l_\epsilon > 0$ such that

$$M_f = \frac{1}{z}\int_0^z f(x)\,dx + \lambda^*\left(\epsilon + \frac{2l_\epsilon}{z}\sup_{x\in\mathbb{R}}|f(x)|\right),$$

where $\lambda^* \in \mathbb{C}$ satisfies $|\lambda^*| \le 1$ and $z \in \mathbb{R}^+$. Therefore, organizing the previous equality and applying the absolute value, we have

$$\left| M_f - \frac{1}{z}\int_0^z f(x)\,dx \right| = |\lambda^*|\left|\epsilon + \frac{2l_\epsilon}{z}\sup_{x\in\mathbb{R}}|f(x)|\right| \le \epsilon + \frac{2l_\epsilon}{z}\sup_{x\in\mathbb{R}}|f(x)|$$

as we wanted to prove. □

Theorem 3.2.7 ([4, Chapter 1, Section 3, Theorem 3])**.** *If f is a u.a.p. function, then for every $z > 0$, the functions f_1 and $f_{2,z}$ defined by*

$$f_1(x) := M_{t,\widetilde{f}(x,t)}, \quad f_{2,z}(x) := \frac{1}{z}\int_0^z \widetilde{f}(x,t)\,dt$$

are u.a.p. functions. Moreover, the function $f_{2,z}$ tends to the function f_1 uniformly in $\mathbb{R}$ when $z \to +\infty$.

Proof. Since f is a u.a.p. function, applying Theorem 2.3.1 and 2.3.6 it follows that for each $x \in \mathbb{R}$, $\widetilde{f}(x,t)$ is also a uniformly almost periodic function of variable t. Let $\epsilon_1 > 0$. Due to the fact that for each $x \in \mathbb{R}$ the function $\widetilde{f_x}$ is uniformly almost periodic, there exists $\widetilde{l}_{\epsilon_1} > 0$ such that every interval with length $\widetilde{l}_{\epsilon_1} > 0$ intersects $E_{\epsilon_1,\widetilde{f}}$ and using Theorem 2.2.1 we have that $\widetilde{f}$ is bounded, therefore we have

$$B := \sup_{t\in\mathbb{R}} |\widetilde{f}(x,t)| < \infty.$$

Using equation (3.7) and replacing ϵ, f, A and l_ϵ with $\epsilon_1, \widetilde{f}, B$ and $\widetilde{l_{\epsilon_1}}$ respectively, we can write

$$M_{t,\widetilde{f}(x,t)} = \frac{1}{z}\int_0^z \widetilde{f}(x,t)\,dt + \lambda^*\left(\epsilon_1 + \frac{2\widetilde{l_{\epsilon_1}}B}{z}\right).$$

In these conditions we can see that $B \leq A^2$. Let $\epsilon_1 := 2A\epsilon$ and let $\tau \in E_{\epsilon,f}$. Then we have

$$\begin{aligned}
|\widetilde{f_x}(t+\tau) - \widetilde{f_x}(t)| &= |f(t+x+\tau)\overline{f}(t+\tau) - f(t+x)\overline{f}(t)| \\
&= |f(t+x+\tau)\overline{f}(t+\tau) - f(t+x)\overline{f}(t+\tau) + f(t+x)\overline{f}(t+\tau) - f(t+x)\overline{f}(t)| \\
&\leq |f(t+x+\tau)\overline{f}(t+\tau) - f(t+x)\overline{f}(t+\tau)| + |f(t+x)\overline{f}(t+\tau) - f(t+x)\overline{f}(t)| \\
&= |(f(t+x+\tau) - f(t+x))\overline{f}(t+\tau)| + |(\overline{f}(t+\tau) - \overline{f}(t))f(t+x)| \\
&\leq \epsilon A + \epsilon A = 2\epsilon A = \epsilon_1.
\end{aligned}$$

Therefore the set $E_{\epsilon,f} \subseteq E_{\epsilon_1,\widetilde{f}}$ and, consequently, any interval with length l_ϵ intersects $E_{\epsilon_1,\widetilde{f}}$. Hence we can consider, without any loss of generality, that $l_\epsilon = \widetilde{l_{\epsilon_1}}$. Consequently there exists $\lambda^{**} \in \mathbb{C}$ such that $|\lambda^{**}| \leq 1$, and satisfies

$$M_{t,\widetilde{f}(x,t)} = \frac{1}{z}\int_0^z \widetilde{f}(x,t)\,dt + \lambda^{**}\left(2A\epsilon + \frac{2A^2 l_\epsilon}{z}\right),$$

that is,

$$f_1(x) = f_{2,z}(x) + \lambda^{**}\left(2A\epsilon + \frac{2A^2 l_\epsilon}{z}\right).$$

Let $\delta > 0$. Since we can take ϵ to be as small as we please, we can consider that

$$|2A\epsilon\lambda^{**}| < \frac{\delta}{2}.$$

Taking into account that

$$\lim_{z\to\infty} \frac{2A^2 l_\epsilon \lambda^{**}}{z} = 0,$$

there exists $p \in \mathbb{N}$ such that if $z > p$, then

$$\left|\frac{2A^2 l_\epsilon \lambda^{**}}{z}\right| < \frac{\delta}{2}.$$

Consequently if $z > p$, we shall have for every $x \in \mathbb{R}$,

$$|f_{2,z}(x) - f_1(x)| = \left|\lambda^{**}\left(2A\epsilon + \frac{2A^2 l_\epsilon}{z}\right)\right| \leq |2A\epsilon\lambda^{**}| + \left|\frac{2A^2 l_\epsilon \lambda^{**}}{z}\right| < \frac{\delta}{2} + \frac{\delta}{2} = \delta,$$

therefore $f_{2,z}$ tends to f_1 uniformly in $\mathbb{R}$ when $z \to \infty$.

Let $\tau \in E_{\epsilon,f}$. In these circumstances, analysing the definition of the function $f_{2,z}$, it follows that

$$\begin{aligned}
|f_{2,z}(x+\tau) - f_{2,z}(x)| &= \left|\ \frac{1}{z}\int_0^z [\widetilde{f}(x+\tau,t) - \widetilde{f}(x,t)]\,dt\ \right| \\
&\leq \frac{1}{z}\int_0^z |\ [f(t+x+\tau) - f(t+x)]\overline{f}(t)\ |\,dt
\end{aligned}$$

$$\leq \frac{1}{z}\int_0^z \epsilon A \, dt = A\epsilon,$$

that is, $E_{\epsilon,f} \subseteq E_{A\epsilon, f_{2,z}}$ and consequently $f_{2,z}$ is a u.a.p. function for each $z > 0$. Due to the fact that $f_{2,z}$ tends uniformly to f_1, applying Theorem 2.4.2 to the sequence $(f_n)_{n\in\mathbb{N}}$ defined by

$$f_n(x) := \frac{1}{n}\int_0^n \widetilde{f}(x,t)\, dt,$$

we get that f_1 is also a u.a.p. function as we wanted to prove. □

3.3 Fourier Series of u.a.p. Functions

We are going to define the function e_λ by

$$e_\lambda(x) := e^{i\lambda x}, \quad x \in \mathbb{R}$$

for each $\lambda \in \mathbb{R}$. In these conditions we can assure that for any $\lambda \in \mathbb{R}$, e_λ is a periodic function, therefore it is a u.a.p. function, and we have that

$$M_{e_\lambda} = \begin{cases} \lim\limits_{z\to\infty} \frac{1}{z}\int_0^z e^{i\lambda x}\, dx, & \text{if } \lambda \neq 0 \\ \\ \lim\limits_{z\to\infty} \frac{1}{z}\int_0^z 1\, dx, & \text{if } \lambda = 0 \end{cases} = \begin{cases} 0, & \text{if } \lambda \neq 0 \\ \\ 1, & \text{if } \lambda = 0. \end{cases} \tag{3.9}$$

Let f be a u.a.p. function and $\lambda \in \mathbb{R}$. Since $e_{-\lambda}$ is a u.a.p. function, applying Theorem 2.3.6, we get that $fe_{-\lambda}$ is also a uniformly almost periodic function. Consequently, by Theorem 3.2.4, it follows that $M_{fe_{-\lambda}}$ exists. In this work we will denote for every $\lambda \in \mathbb{R}$,

$$a_f(\lambda) := M_{fe_{-\lambda}}.$$

Theorem 3.3.1 ([4, Chapter 1, Section 3, Theorem 4])**.** *Let f be a uniformly almost periodic function and $N \in \mathbb{N}$. Consider $\{b_1, \dots, b_N\} \subseteq \mathbb{C}$ and $\{\lambda_1, \dots, \lambda_N\} \subseteq \mathbb{R}$ such that for every $i, j \in \{1, \dots, N\}$ if $i \neq j$, then $\lambda_i \neq \lambda_j$. If*

$$h(x) := \sum_{n=1}^{N} b_n e_{\lambda_n}(x),$$

then

$$M_{|f-h|^2} = M_{|f|^2} - \sum_{n=1}^{N} |a_f(\lambda_n)|^2 + \sum_{n=1}^{N} |b_n - a_f(\lambda_n)|^2.$$

Proof. Firstly let us observe that for every $x \in \mathbb{R}$,

$$|f(x) - h(x)|^2 = (f(x) - h(x)) \cdot (\overline{f(x) - h(x)}) = (f(x) - h(x)) \cdot (\overline{f(x)} - \overline{h(x)})$$
$$= \left(f(x) - \sum_{n=1}^{N} b_n e^{i\lambda_n x} \right) \cdot \left(\overline{f}(x) - \sum_{n=1}^{N} \overline{b}_n e^{-i\lambda_n x} \right)$$

$$= f(x)\overline{f}(x) - \sum_{n=1}^{N} f(x)\overline{b}_n e^{-i\lambda_n x} - \sum_{n=1}^{N} \overline{f}(x) b_n e^{i\lambda_n x} + \sum_{n=1}^{N}\sum_{m=1}^{N} b_n \overline{b}_m e^{i(\lambda_n - \lambda_m)x}.$$

Using the previous statement, we can assure that

$$M_{|f-h|^2} = M_{|f|^2} - \sum_{n=1}^{N} \overline{b}_n M_{f e_{-\lambda_n}} - \sum_{n=1}^{N} b_n M_{\overline{f} e_{\lambda_n}} + \sum_{n=1}^{N}\sum_{m=1}^{N} b_n \overline{b}_m M_{e_{\lambda_n - \lambda_m}},$$

and observing equation (3.9), it follows that $M_{e_0} = 1$ and $M_{e_\delta} = 0$ for any $\delta \neq 0$, consequently

$$\sum_{n=1}^{N}\sum_{m=1}^{N} b_n \overline{b}_m M_{e_{\lambda_n - \lambda_m}} = \sum_{n=1}^{N} |b_n|^2.$$

Hence we can conclude that

$$\begin{aligned} M_{|f-h|^2} &= M_{|f|^2} - \sum_{n=1}^{N} \overline{b}_n a_f(\lambda_n) - \sum_{n=1}^{N} b_n \overline{a_f(\lambda_n)} + \sum_{n=1}^{N} b_n \overline{b}_n \\ &= M_{|f|^2} - \sum_{n=1}^{N} a_f(\lambda_n)\overline{a_f(\lambda_n)} + \sum_{n=1}^{N} (b_n - a_f(\lambda_n)) \cdot (\overline{b}_n - \overline{a_f(\lambda_n)}) \\ &= M_{|f|^2} - \sum_{n=1}^{N} |a_f(\lambda_n)|^2 + \sum_{n=1}^{N} |b_n - a_f(\lambda_n)|^2. \end{aligned}$$

□

The following result shows us that, given a u.a.p. function f, the set of values $\lambda \in \mathbb{R}$ for which $a_f(\lambda) \neq 0$ is at most a countable set.

Theorem 3.3.2 ([4, Chapter 1, Section 3, Theorem 5])**.** *If f is a u.a.p. function, then there exists at most a countable set of values of $\lambda \in \mathbb{R}$ for which $a_f(\lambda) \neq 0$.*

Proof. Applying Theorem 3.3.1, we can see that $h(x) := \sum_{n=1}^{N} b_n e^{i\lambda_n x}$ gives us the best approximation in mean to f. If we consider $b_n = a_f(\lambda_n)$ for every $n \in \{1, \ldots N\}$, then we have

$$M_{|f-h|^2} = M_{|f|^2} - \sum_{n=1}^{N} |a_f(\lambda_n)|^2 \Leftrightarrow M_{|f|^2} = M_{|f-h|^2} + \sum_{n=1}^{N} |a_f(\lambda_n)|^2. \tag{3.10}$$

Since $M_{|f|^2}, M_{|f-h|^2}$ and $\sum_{n=1}^{N} |a_f(\lambda_n)|^2$ have non-negative values, it follows, by equation (3.10), that

$$\sum_{n=1}^{N} |a_f(\lambda_n)|^2 \leq M_{|f|^2}. \tag{3.11}$$

Due to the fact that the previous inequality is true for any $N \in \mathbb{N}$ of real numbers $\lambda_1, \ldots, \lambda_N$, we get that for every $\epsilon > 0$ there corresponds at most a finite number of values of λ for which $|a_f(\lambda)| > \epsilon$. If $a_f(\lambda) \neq 0$, then there exists $m \in \mathbb{N}$ such that

$$|a_f(\lambda)| > 1 \ \vee \ \frac{1}{m+1} < |a_f(\lambda)| \leq \frac{1}{m},$$

that is, each of these inequalities is satisfied by a finite number of values of λ. Consequently there exists at most a countable set of values of λ that verify $a_f(\lambda) \neq 0$ as we wanted to prove. □

Taking into account the previous theorem, we can assure that there exists $B \subseteq \mathbb{N}$ such that $|\{\lambda_n : a_f(\lambda_n) \neq 0 \wedge n \in \mathbb{N}\}| = |B|$. In these conditions we say that $\lambda_1, \lambda_2, \ldots$ are the Fourier exponents and $a_f(\lambda_1), a_f(\lambda_2), \ldots$ are the Fourier coefficients of the function f. The series

$$\sum_{n=1}^{\infty} a_f(\lambda_n) e^{i\lambda_n x}$$

is called the Fourier series of the function f, and we write it as being

$$f(x) \sim \sum_{n=1}^{\infty} a_f(\lambda_n) e^{i\lambda_n x}.$$

Since inequality (3.11) is true for any $N \in \mathbb{N}$, we can assure that

$$\sum_{n=1}^{\infty} |a_f(\lambda_n)|^2 \leq M_{|f|^2}.$$

The above inequality is called the Bessel inequality for uniformly almost periodic functions.

Example 3.3.3. Let f be a purely periodic function with period 2π. In these conditions we know that its Fourier series is defined by

$$f(x) \sim \sum_{n=-\infty}^{\infty} A_n e^{inx}$$

where the constants A_n for every $n \in \mathbb{Z}$, satisfy the Parseval Identity [2, Theorem 8.63]

$$A_n := \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx}\, dx, \qquad \sum_{n=-\infty}^{\infty} |A_n|^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2\, dx.$$

However, since f is a periodic function, it follows that f is a u.a.p. function and taking into account that f and e_{-n} are periodic functions with period 2π, we have that for each $n \in \mathbb{Z}$

$$\begin{aligned} M_{fe_{-n}} &= \lim_{y \to \infty} \frac{1}{y} \int_0^y f(x) e^{-inx}\, dx \\ &= \lim_{m \to \infty} \frac{1}{m2\pi} \int_0^{m2\pi} f(x) e^{-inx}\, dx \\ &= \lim_{m \to \infty} \frac{m}{m2\pi} \int_0^{2\pi} f(x) e^{-inx}\, dx \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx}\, dx \end{aligned}$$

and also

$$\begin{aligned} M_{|f|^2} &= \lim_{y\to\infty} \frac{1}{y} \int_0^y |f(x)|^2 \, dx \\ &= \lim_{m\to\infty} \frac{1}{m2\pi} \int_0^{m2\pi} |f(x)|^2 \, dx \\ &= \lim_{m\to\infty} \frac{m}{m2\pi} \int_0^{2\pi} |f(x)|^2 \, dx \\ &= \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2 \, dx. \end{aligned}$$

The statements made in this example assure that the coefficients A_n are also Fourier coefficients of f in the new sense, and there cannot be any other Fourier coefficients in the new definition. Hence if we consider a periodic function, the definition of the Fourier series we were used to coincides with the new one.

Theorem 3.3.4 ([4, Chapter 1, Section 3, Theorem 8]). *If f is a u.a.p. function represented by the sum of a uniformly convergent trigonometric series $f(x) = \sum_{n=1}^{\infty} a_n e^{i\lambda_n x}$, then the Fourier series of f coincides with this series.*

Proof. Firstly let us observe that for every $\lambda \in \mathbb{R}$, the series $\sum_{n=1}^{\infty} a_n e^{i(\lambda_n - \lambda)x}$ is a uniformly convergent series because $\sum_{n=1}^{\infty} a_n e^{i\lambda_n x}$ is a uniformly convergent series by our hypothesis. In these conditions it follows that

$$M_{fe_{-\lambda}} = \lim_{y\to\infty} \frac{1}{y} \int_0^y e^{-i\lambda x} \sum_{n=1}^{\infty} a_n e^{i\lambda_n x} \, dx = \sum_{n=1}^{\infty} a_n M_{e_{\lambda_n - \lambda}}.$$

Taking into account equation (3.9), we can see that

$$M_{e_{\lambda_n - \lambda}} = \begin{cases} 0, & \text{if } \lambda \neq \lambda_n \\ 1, & \text{if } \lambda = \lambda_n. \end{cases}$$

Consequently if there exists $n \in \mathbb{N}$ such that $\lambda = \lambda_n$, then $M_{fe_{-\lambda}} = a_n$, otherwise we have $M_{fe_{-\lambda}} = 0$. Therefore for every $n \in \mathbb{N}$, we conclude that $a_f(\lambda_n) = M_{fe_{-\lambda_n}} = a_n$. □

As our intuition would tell us, any non-negative u.a.p. real function with mean value equal to 0 must be the null function.

Theorem 3.3.5 ([4, Chapter 1, Section 3, Theorem 10]). *If f is a u.a.p. real function, $f(x) \geq 0$ for every $x \in \mathbb{R}$ and $M_f = 0$, then $f(x) = 0$ for any $x \in \mathbb{R}$.*

Proof. Let us suppose, by contradiction, that there exists $x_0 \in \mathbb{R}$ such that

$$f(x_0) = m > 0.$$

Since f is a u.a.p. function, it follows that f is continuous and therefore there exists $\delta > 0$ such that for every $x \in]x_0 - \delta, x_0 + \delta[$,

$$f(x) > \frac{2}{3} \cdot m. \tag{3.12}$$

Since $\frac{1}{3}m > 0$ and f is a u.a.p. function, there is $l_{\frac{1}{3}m} > 0$ such that any interval with length $l_{\frac{1}{3}m}$ intersects $E_{l_{\frac{1}{3}m},f}$. Consider, without loss of generality, that $l_{\frac{1}{3}m} > 2\delta$. In these conditions for any interval I with length $l_{\frac{1}{3}m}$, there exists $\tau \in E_{\frac{1}{3}m,f}$ such that $x_0 + \tau \in I$ and thus I contains at least one of the intervals $]x_0 + \tau - \delta, x_0 + \tau[$ or $]x_0 + \tau, x_0 + \tau + \delta[$, because I has length equal to $l_{\frac{1}{3}m} > 2\delta$. Taking into account that

$$\sup_{x\in\mathbb{R}} |f(x+\tau) - f(x)| \leq \frac{1}{3}m \Rightarrow \sup_{x\in\mathbb{R}} |f(x-\tau) - f(x)| \leq \frac{1}{3}m,$$

we can guarantee, using inequality (3.12), that for every $x \in]x_0 + \tau - \delta, x_0 + \tau[$ and $x \in]x_0 + \tau, x_0 + \tau + \delta[$,

$$|f(x) - f(x-\tau)| \leq \frac{1}{3}m \Rightarrow -\frac{1}{3}m \leq f(x) - f(x-\tau) \leq \frac{1}{3}m \Rightarrow f(x) \geq f(x-\tau) - \frac{1}{3}m > \frac{1}{3}m.$$

Consequently in each interval of length $l_{\frac{1}{3}m}$ there exists a sub-interval of length δ, such that f verifies $f(x) > \frac{1}{3}m$, for each x in that sub-interval. Taking into account the above inequality and the fact that f is a non negative function, we have for every $\lambda \in \mathbb{R}$,

$$\int_{\lambda}^{\lambda + l_{\frac{1}{3}m}} f(x)\, dx \geq \frac{m\delta}{3}.$$

Therefore we get

$$0 = M_f = \lim_{n\to\infty} \frac{1}{n l_{\frac{1}{3}m}} \int_0^{n l_{\frac{1}{3}m}} f(x)\, dx \geq \frac{m\delta}{3} > 0,$$

which is a contradiction, thus $f(x) = 0$ for any $x \in \mathbb{R}$ as we wanted. □

Let $(f_k)_{k\in\mathbb{N}}$ be a sequence of u.a.p. functions. We know that the set of the Fourier exponents of each function f_k is a countable set for every $k \in \mathbb{N}$. Let us denote all of those exponents by $\{\lambda_n^{(k)} : n \in \mathbb{N}\}$. In these conditions the Fourier series of each function f_k can be represented by

$$f_k(x) \sim \sum_{n=1}^{\infty} a_{f_k}(\lambda_n^{(k)}) e^{i\lambda_n^{(k)} x},$$

where $\{a_{f_k}(\lambda_n^{(k)}) : n \in \mathbb{N}\}$ represents the Fourier coefficients of the function f_k for any $k \in \mathbb{N}$. It is important to observe that a countable union of countable sets is a countable set. Thus if we add at most a countable number of terms in each Fourier series of each function f_k for which $a_{f_k}(\lambda_n^{(k)}) = 0$, then we can consider, without loss of generality, that every function f_k has the same Fourier exponents, that is, if $i, j \in \mathbb{N}$ and $i \neq j$, then we can represent the Fourier series of f_i and f_j by

$$f_i(x) \sim \sum_{n=1}^{\infty} a_{f_i}(\lambda_n) e^{i\lambda_n x}, \quad f_j(x) \sim \sum_{n=1}^{\infty} a_{f_j}(\lambda_n) e^{i\lambda_n x}.$$

Theorem 3.3.6 ([4, Chapter 1, Section 3, Theorem 11]). *Let $(f_k)_{k\in\mathbb{N}}$ a sequence of u.a.p. functions such that for every $k \in \mathbb{N}$,*

$$f_k(x) \sim \sum_{n=1}^{\infty} a_{f_k}(\lambda_n) e^{i\lambda_n x}.$$

If $(f_k)_{k\in\mathbb{N}}$ converges uniformly to a function f, then the Fourier series of f is given by

$$f(x) \sim \sum_{n=1}^{\infty} a_f(\lambda_n) e^{i\lambda_n x},$$

where $a_f(\lambda_n) = \lim_{k\to\infty} a_{f_k}(\lambda_n)$, for any $n \in \mathbb{N}$.

Proof. Firstly let us observe that, by definition, $a_f(\lambda_n) = M_{f e_{-\lambda_n}}$ and $a_{f_k}(\lambda_n) = M_{f_k e_{-\lambda_n}}$ for every $n, k \in \mathbb{N}$. Since $(f_k)_{k\in\mathbb{N}}$ converges uniformly to f, it follows that

$$\lim_{k\to\infty} \sup_{x\in\mathbb{R}} |f_k(x) - f(x)| = 0.$$

In these conditions we can assure that for every $\lambda \in \mathbb{R}$,

$$\begin{aligned}
\lim_{k\to\infty} \left| M_{f_k e_{-\lambda}} - M_{f e_{-\lambda}} \right| &= \lim_{k\to\infty} \left| \lim_{y\to\infty} \frac{1}{y} \int_0^y [f_k(x) - f(x)] e^{-i\lambda x}\, dx \right| \\
&\le \lim_{k\to\infty} \left(\lim_{y\to\infty} \frac{1}{y} \int_0^y |f_k(x) - f(x)|\, dx \right) \\
&\le \lim_{k\to\infty} \left(\lim_{y\to\infty} \frac{1}{y} \int_0^y \sup_{x\in\mathbb{R}} |f_k(x) - f(x)|\, dx \right) \\
&\le \lim_{k\to\infty} \left(\sup_{x\in\mathbb{R}} |f_k(x) - f(x)| \right) = 0.
\end{aligned}$$

Therefore $\lim_{k\to\infty} a_{f_k}(\lambda_n) = a_f(\lambda_n)$ for any $n \in \mathbb{N}$, and we conclude that

$$f(x) \sim \sum_{n=1}^{\infty} a_f(\lambda_n) e^{i\lambda_n x}.$$

□

3.4 Uniqueness Theorem for Fourier Series

If f is a u.a.p. function, then we shall denote

$$\phi_f(\lambda, z) := \frac{1}{z} \int_0^z f(x) e^{-i\lambda x}\, dx,$$

for every $\lambda \in \mathbb{R}$ and $z > 0$. In the next result we will see that the function $\phi_f(\lambda, z)$ tends to the null function uniformly in $z \in [1, +\infty[$ when $|\lambda| \to \infty$.

Lemma 3.4.1 ([4, Chapter 1, Section 4, Lemma 2]). *If f is a u.a.p. function, then $\phi_f(\lambda, z)$ tends to 0 uniformly in $z \in [1, +\infty[$ when $|\lambda| \to \infty$.*

Proof. Let $\lambda \in \mathbb{R} \setminus \{0\}$ and $z \geq 1$. In these conditions it follows that

$$\begin{aligned}
\phi_f(\lambda, z) + \phi_f(\lambda, z) =& \frac{1}{z}\int_0^z f(x)e^{-i\lambda x}\,dx + \frac{1}{z}\int_0^z f(x)e^{-i\lambda x}\,dx \\
=& \frac{1}{z}\int_0^z f(x)e^{-i\lambda x}\,dx + \frac{1}{z}\int_{\frac{\pi}{\lambda}}^{\frac{\pi}{\lambda}+z} f\left(y - \frac{\pi}{\lambda}\right)e^{-i\lambda(y-\frac{\pi}{\lambda})}\,dy \\
=& \frac{1}{z}\int_0^z f(x)e^{-i\lambda x}\,dx - \frac{1}{z}\int_{\frac{\pi}{\lambda}}^{\frac{\pi}{\lambda}+z} f\left(x - \frac{\pi}{\lambda}\right)e^{-i\lambda x}\,dx \\
=& \frac{1}{z}\int_0^z f(x)e^{-i\lambda x}\,dx - \frac{1}{z}\int_{\frac{\pi}{\lambda}}^{0} f\left(x - \frac{\pi}{\lambda}\right)e^{-i\lambda x}\,dx \\
&- \frac{1}{z}\int_0^z f\left(x - \frac{\pi}{\lambda}\right)e^{-i\lambda x}\,dx - \frac{1}{z}\int_z^{\frac{\pi}{\lambda}+z} f\left(x - \frac{\pi}{\lambda}\right)e^{-i\lambda x}\,dx \\
=& \frac{1}{z}\int_0^z \left(f(x) - f\left(x - \frac{\pi}{\lambda}\right)\right)e^{-i\lambda x}\,dx - \frac{1}{z}\int_{\frac{\pi}{\lambda}}^{0} f\left(x - \frac{\pi}{\lambda}\right)e^{-i\lambda x}\,dx \\
&- \frac{1}{z}\int_z^{\frac{\pi}{\lambda}+z} f\left(x - \frac{\pi}{\lambda}\right)e^{-i\lambda x}\,dx.
\end{aligned}$$

Consequently we have

$$\begin{aligned}
\phi_f(\lambda, z) =& \frac{1}{2z}\int_0^z \left(f(x) - f\left(x - \frac{\pi}{\lambda}\right)\right)e^{-i\lambda x}\,dx - \frac{1}{2z}\int_{\frac{\pi}{\lambda}}^{0} f\left(x - \frac{\pi}{\lambda}\right)e^{-i\lambda x}\,dx \\
&- \frac{1}{2z}\int_z^{\frac{\pi}{\lambda}+z} f\left(x - \frac{\pi}{\lambda}\right)e^{-i\lambda x}\,dx =: I_1 + I_2 + I_3.
\end{aligned}$$

Let

$$A = \sup_{x\in\mathbb{R}} |f(x)|, \quad \omega(\tau) = \sup_{x\in\mathbb{R}} |f(x+\tau) - f(x)|.$$

We can see that

$$|I_1| \leq \frac{1}{2}\omega\left(\frac{\pi}{|\lambda|}\right), \quad |I_2| \leq \frac{A\pi}{2z|\lambda|}, \quad |I_3| \leq \frac{A\pi}{2z|\lambda|},$$

therefore

$$|\phi_f(\lambda, z)| = |I_1 + I_2 + I_3| \leq |I_1| + |I_2| + |I_3| \leq \frac{1}{2}\omega\left(\frac{\pi}{|\lambda|}\right) + \frac{A\pi}{z|\lambda|} \leq \frac{1}{2}\omega\left(\frac{\pi}{|\lambda|}\right) + \frac{A\pi}{|\lambda|}.$$

Hence we can conclude that

$$\lim_{|\lambda|\to\infty} |\phi_f(\lambda, z)| \leq \lim_{|\lambda|\to\infty} \left(\frac{1}{2}\omega\left(\frac{\pi}{|\lambda|}\right) + \frac{A\pi}{|\lambda|}\right) = 0,$$

that is, $\phi_f(\lambda, z)$ tends to 0 uniformly in $z \in [1, +\infty[$ when $|\lambda| \to \infty$. □

The following three preparatory results will help us to establish Theorem 3.4.6.

Lemma 3.4.2 ([4, Chapter 1, Section 4, Lemma 3])**.** *Let f be a u.a.p. function and $\epsilon > 0$. If $M_f = 0$, then there exist $\delta_1, \delta_2 > 0$ such that*

$$\left|\frac{1}{z}\int_0^z f(x)e^{-i\lambda x}\,dx\right| < \epsilon$$

for every $z > \delta_2$ and for any $\lambda \in]-\delta_1, \delta_1[$.

Proof. Let $a \in \mathbb{R}$ and $\epsilon > 0$. Applying Lemma 3.2.2, we have $M_f = M_{T_a f}$ and since $M_f = 0$, it follows that $M_{T_a f} = 0$, consequently there exists $\delta_2 > 0$ such that

$$\left| \frac{1}{z_1} \int_0^{z_1} f(x+a)\, dx \right| \leq \frac{\epsilon}{2} \tag{3.13}$$

for every $z_1 > \frac{\delta_2}{2}$. In these conditions it is not hard to see that any $z_2 > \delta_2$ can be written as $z_2 = nz_1$, where $n \in \mathbb{N}$ and $z_1 \in \left]\frac{\delta_2}{2}, \delta_2\right[$. Let us consider

$$A = \sup_{x \in \mathbb{R}} |f(x)|.$$

Due to the fact that $e^{-i0x} = 1$, there exists $\delta_1 > 0$ such that

$$|e^{-i\lambda x} - 1| < \frac{\epsilon}{2A} \tag{3.14}$$

for every $\lambda \in]-\delta_1, \delta_1[$ and for any $x \in [0, \delta_2]$. Due to the fact that $z_2 = nz_1$, we get

$$\begin{aligned} \frac{1}{z_2} \int_0^{z_2} f(x) e^{-i\lambda x}\, dx &= \frac{1}{nz_1} \sum_{k=0}^{n-1} \int_{kz_1}^{(k+1)z_1} f(y) e^{-i\lambda y}\, dy \\ &= \frac{1}{nz_1} \sum_{k=0}^{n-1} e^{-i\lambda k z_1} \int_0^{z_1} f(x + kz_1) e^{-i\lambda x}\, dx. \end{aligned}$$

Using both inequalities (3.13) and (3.14) we have that for every $k \in \{0, \ldots, n-1\}$,

$$\begin{aligned} \left| \frac{1}{z_1} \int_0^{z_1} f(x+kz_1) e^{-i\lambda x}\, dx \right| &= \left| \frac{1}{z_1} \int_0^{z_1} f(x+kz_1)(e^{-i\lambda x} - 1 + 1)\, dx \right| \\ &\leq \left| \frac{1}{z_1} \int_0^{z_1} f(x+kz_1)(e^{-i\lambda x} - 1)\, dx \right| + \left| \frac{1}{z_1} \int_0^{z_1} f(x+kz_1)\, dx \right| \\ &< \frac{\epsilon}{2Az_1} \int_0^{z_1} A\, dx + \left| \frac{1}{z_1} \int_0^{z_1} f(x+kz_1)\, dx \right| \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Since

$$\frac{1}{z_2} \int_0^{z_2} f(x) e^{-i\lambda x}\, dx = \frac{1}{nz_1} \sum_{k=0}^{n-1} e^{-i\lambda k z_1} \int_0^{z_1} f(x+kz_1) e^{-i\lambda x}\, dx,$$

it follows that

$$\begin{aligned} \left| \frac{1}{z_2} \int_0^{z_2} f(x) e^{-i\lambda x}\, dx \right| &= \left| \frac{1}{nz_1} \sum_{k=0}^{n-1} e^{-i\lambda k z_1} \int_0^{z_1} f(x+kz_1) e^{-i\lambda x}\, dx \right| \\ &\leq \frac{1}{nz_1} \sum_{k=0}^{n-1} \left| e^{-i\lambda k z_1} \int_0^{z_1} f(x+kz_1) e^{-i\lambda x}\, dx \right| \\ &= \frac{1}{n} \sum_{k=0}^{n-1} \left| \frac{1}{z_1} \int_0^{z_1} f(x+kz_1) e^{-i\lambda x}\, dx \right| \\ &< \frac{1}{n} \sum_{k=0}^{n-1} \epsilon = \frac{\epsilon n}{n} = \epsilon. \end{aligned}$$

Consequently

$$\left|\frac{1}{z_2}\int_0^{z_2} f(x)e^{-i\lambda x}\,dx\right| < \epsilon$$

for any $z_2 > \delta_2$ and for every $\lambda \in]-\delta_1, \delta_1[$ as we wanted to prove. □

Corollary 3.4.3 ([4, Chapter 1, Section 4, Lemma 4]). *Let f be a u.a.p. function, $\epsilon > 0$ and $\mu \in \mathbb{R}$. If $M_{fe_\mu} = 0$, then there exist $\delta_1^{(\mu)}, \delta_2^{(\mu)} > 0$ such that*

$$\left|\frac{1}{z}\int_0^{z} f(x)e^{-i\lambda x}\,dx\right| < \epsilon$$

for every $z > \delta_2^{(\mu)}$ and for any $\lambda \in]\mu - \delta_1^{(\mu)}, \mu + \delta_1^{(\mu)}[$.

Proof. We can assure that fe_μ is a u.a.p. function and consequently, applying Lemma 3.4.2, there exist $\delta_1, \delta_2 > 0$ such that

$$\left|\frac{1}{z}\int_0^{z} f(x)e^{i\mu x}e^{-i\lambda x}\,dx\right| = \left|\frac{1}{z}\int_0^{z} f(x)e^{-ix(\lambda-\mu)}\,dx\right| < \epsilon$$

for every $z > \delta_2$ and for any $(\lambda - \mu) \in]-\delta_1, \delta_1[$, that is, for each $\lambda \in]\mu - \delta_1, \mu + \delta_1[$ as we wanted to prove. □

Lemma 3.4.4 ([4, Chapter 1, Section 4, Lemma 5]). *Let $\epsilon > 0$ and f be a u.a.p. function that verifies $a_f(\lambda) = 0$ for every $\lambda \in \mathbb{R}$. Then there exists $z_0 > 0$, such that*

$$\left|\frac{1}{z}\int_0^{z} f(x)e^{-i\lambda x}\,dx\right| < \epsilon$$

for every $z > z_0$ and for each $\lambda \in \mathbb{R}$.

Proof. Let $\epsilon > 0$. Applying Lemma 3.4.1, we can assure that there exists $\lambda_0 > 0$, such that

$$\left|\frac{1}{z}\int_0^{z} f(x)e^{-i\lambda x}\,dx\right| < \epsilon$$

for every $z \geq 1$ and for any λ that satisfies $|\lambda| > \lambda_0$. Given $\mu \in [-\lambda_0, \lambda_0]$ and using Corollary 3.4.3, it follows that there exist $z_\mu, \delta_\mu > 0$, such that

$$\left|\frac{1}{z}\int_0^{z} f(x)e^{-i\lambda x}\,dx\right| < \epsilon$$

for every $z > z_\mu$ and for any $\lambda \in]\mu - \delta_\mu, \mu + \delta_\mu[$. In these conditions it is obvious that

$$\bigcup_{\mu \in [-\lambda_0, \lambda_0]}]\mu - \delta_\mu, \mu + \delta_\mu[$$

is an open cover of the interval $[-\lambda_0, \lambda_0]$, therefore taking into account the Heine-Borel Theorem we get that $[-\lambda_0, \lambda_0]$ is compact and thus there exist $\mu_1, \ldots, \mu_n \in \mathbb{R}$, where $n \in \mathbb{N}$, such that

$$[-\lambda_0, \lambda_0] \subseteq \bigcup_{k=1}^{n} [\mu_k - \delta_{\mu_k}, \mu_k + \delta_{\mu_k}].$$

In these circumstances we can assure that

$$\left|\frac{1}{z}\int_0^z f(x)e^{-i\lambda x}\,dx\right| < \epsilon$$

for every $z > z_{\mu_k}$, for any $\lambda \in]\mu_k - \delta_{\mu_k}, \mu_k + \delta_{\mu_k}[$ and for each $k \in \{1,\dots,n\}$. Consequently, if we consider $z_0 > \max\{1, z_{\mu_1}, \dots, z_{\mu_n}\}$, then we can conclude that for every $z > z_0$, for any $\lambda \in]\mu_k - \delta_{\mu_k}, \mu_k + \delta_{\mu_k}[$, for each $k \in \{1,\dots,n\}$ and for every λ that verifies $|\lambda| > \lambda_0$,

$$\left|\frac{1}{z}\int_0^z f(x)e^{-i\lambda x}\,dx\right| < \epsilon.$$

That is, the inequality stated previously is indeed true for any $z > z_0$ and for every $\lambda \in \mathbb{R}$ as we wanted to prove. □

Lemma 3.4.5. *Let $z > 0$ and let $f_{1,z} : \mathbb{R} \to \mathbb{C}$ be a continuous periodic function of period z. Suppose that $(A_k(z))_{k\in\mathbb{Z}}$, is the sequence of its Fourier coefficients. Then the functions*

$$f_{2,z}(x) := \frac{1}{z}\int_0^z f_{1,z}(x+t)\overline{f}_{1,z}(t)\,dt, \quad x \in \mathbb{R},$$

$$f_{3,z}(x) := \frac{1}{z}\int_0^z f_{2,z}(x+t)\overline{f}_{2,z}(t)\,dt, \quad x \in \mathbb{R},$$

are continuous periodic functions of period z, and $(|A_k(z)|^2)_{k\in\mathbb{Z}}$ and $(|A_k(z)|^4)_{k\in\mathbb{Z}}$ are the sequences of their Fourier coefficients, respectively.

Proof. Since $f_{1,z}$ is a continuous and periodic function, applying Weierstrass theorem, it follows that

$$A := \sup_{x\in[0,z]} |f_{1,z}(x)| < \infty.$$

Taking into account Lemma 2.1.5 and Theorem 2.2.2, we have that the function $f_{1,z}$ is uniformly continuous on $\mathbb{R}$.

Let $\epsilon > 0$. Then there exists $\delta > 0$ such that for each $x, y, t \in \mathbb{R}$, the inequality $|x - y| < \delta$ implies that $|f_{1,z}(x+t) - f_{1,z}(y+t)| < \epsilon$. For such x, y, we have

$$|f_{2,z}(x) - f_{2,z}(y)| \le \frac{A}{z}\int_0^z |f(x+t) - f(y+t)|\,dt \le \epsilon,$$

that is, $f_{2,z}$ is continuous on $\mathbb{R}$. The periodicity of $f_{2,z}$ follows immediately from the periodicity of $f_{1,z}$. Repeating the same argument with $f_{2,z}$ in place of $f_{1,z}$, we see that $f_{3,z}$ is a continuous periodic function of period z.

Since $f_{1,z}$ is a periodic function, applying Fubini's Theorem [2, Theorem 5.32], it follows that

$$\begin{aligned}\frac{1}{z}\int_0^z f_{2,z}(x)e^{\frac{-2\pi i k x}{z}}\,dx &= \frac{1}{z}\int_0^z \left(\frac{1}{z}\int_0^z \widetilde{f_1}(x,t)\,dt\right)e^{\frac{-2\pi i k x}{z}}\,dx \\ &= \frac{1}{z}\int_0^z \left(\frac{1}{z}\int_0^z f_1(x+t)\overline{f_1}(t)\,dt\right)e^{\frac{-2\pi i k x}{z}}\,dx\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{z}\int_0^z \left(\frac{1}{z}\int_0^z f_1(x+t)\overline{f_1}(t)e^{\frac{-2\pi ik(x-t+t)}{z}}\,dt\right)dx \\
&= \frac{1}{z^2}\int_0^z \left(\overline{f_1}(t)e^{\frac{2\pi ikt}{z}}\int_0^z f_1(x+t)e^{\frac{-2\pi ik(x+t)}{z}}\,dx\right)dt \\
&= \frac{1}{z^2}\int_0^z \left(\overline{f_1}(t)e^{\frac{2\pi ikt}{z}}\int_t^{z+t} f_1(y)e^{\frac{-2\pi iky}{z}}\,dx\right)dt \\
&= \frac{1}{z^2}\int_0^z \left(\overline{f_1}(t)e^{\frac{2\pi ikt}{z}}\int_0^z f_1(x)e^{\frac{-2\pi ikx}{z}}\,dx\right)dt \\
&= \left(\frac{1}{z}\int_0^z \overline{f_1}(t)e^{\frac{2\pi ikt}{z}}\,dt\right)\left(\frac{1}{z}\int_0^z f_1(x)e^{\frac{-2\pi ikx}{z}}\,dx\right) \\
&= \overline{\left(\frac{1}{z}\int_0^z f_1(t)e^{\frac{-2\pi ikt}{z}}\,dt\right)}\left(\frac{1}{z}\int_0^z f_1(x)e^{\frac{-2\pi ikx}{z}}\,dx\right) \\
&= \overline{A}_k(z)A_k(z) = |A_k(z)|^2,
\end{aligned}$$

that is,

$$\frac{1}{z}\int_0^z f_{2,z}(x)e^{\frac{-2\pi ikx}{z}}\,dx = |A_k(z)|^2 \tag{3.15}$$

for every $k \in \mathbb{Z}$. In these conditions we can assure that the Fourier series of $f_{2,z}$ is

$$f_{2,z}(x) \sim \sum_{k=-\infty}^{\infty} |A_k(z)|^2 e^{\frac{2\pi ikx}{z}},$$

and repeating the same reasoning for the function $f_{3,z}$, we guarantee that its Fourier series is defined by

$$f_{3,z}(x) \sim \sum_{k=-\infty}^{\infty} |A_k(z)|^4 e^{\frac{2\pi ikx}{z}}$$

as we wanted to prove. □

Now we have everything that we need in order to prove that the mean value of the square of the absolute value of a u.a.p. function is always equal to 0, supposing that each Fourier coefficient of that function is 0.

Theorem 3.4.6 ([4, Chapter 1, Section 4, Lemma 6])**.** *If f is a u.a.p. function and $a_f(\lambda) = 0$ for every $\lambda \in \mathbb{R}$, then $M_{|f|^2} = 0$.*

Proof. Let $\epsilon, z > 0$ and let $f_{1,z}$ be a periodic function with period z, such that $f_{1,z}(x) = f(x)$ for each $x \in [0, z[$. Applying Lemma 3.4.4, we can assure that there exists $z_0 > 0$ such that for each $z > z_0$ and for every $k \in \mathbb{Z}$,

$$|A_k(z)| := \left|\frac{1}{z}\int_0^z f_{1,z}(x)e^{\frac{-2\pi kxi}{z}}\,dx\right| = \left|\frac{1}{z}\int_0^z f(x)e^{\frac{-2\pi kxi}{z}}\,dx\right| < \epsilon.$$

Taking into account the Parseval Identity [2, Theorem 8.63], it follows that

$$\sum_{k=-\infty}^{\infty} |A_k(z)|^2 = \frac{1}{z}\int_0^z |f_{1,z}(x)|^2\,dx = \frac{1}{z}\int_0^z |f(x)|^2\,dx,$$

which implies that

$$\sum_{k=-\infty}^{\infty} |A_k(z)|^2 \leq \frac{1}{z}\int_0^z A^2\, dx = A^2,$$

where

$$A := \sup_{x\in\mathbb{R}} |f(x)|.$$

Consequently, we get that

$$\sum_{k=-\infty}^{\infty} |A_k(z)|^4 < \epsilon^2 \sum_{k=-\infty}^{\infty} |A_k(z)|^2 \leq \epsilon^2 A^2. \tag{3.16}$$

Let us recall that, given a function $h : \mathbb{R} \to \mathbb{C}$, we define the function $\widetilde{h}$ by

$$\widetilde{h}(x,t) := h(t+x)\overline{h}(t), \quad x,t \in \mathbb{R}.$$

Consider the functions

$$f_{2,z}(x) := \frac{1}{z}\int_0^z \widetilde{f}_{1,z}(x,t)\, dt, \quad f_{3,z}(x) := \frac{1}{z}\int_0^z \widetilde{f}_{2,z}(x,t)\, dt.$$

Applying Lemma 3.4.5, we conclude that $f_{2,z}$ and $f_{3,z}$ are continuous z-periodic functions and $(|A_k(z)|^2)_{k\in\mathbb{Z}}$ and $(|A_k(z)|^4)_{k\in\mathbb{Z}}$ are the sequence of their Fourier coefficients, respectively. Due to the fact that $f_{3,z}$ is a continuous function and using inequality (3.16), we have that

$$\sum_{k=-\infty}^{\infty} |A_k(z)|^4 e^{\frac{2\pi ikx}{z}}$$

is a uniformly convergent series. Hence applying the corollary of Fejér's Theorem[12, p. 19], the sum of the previous series coincides with the function $f_{3,z}$, that is,

$$f_{3,z}(x) = \sum_{k=-\infty}^{\infty} |A_k(z)|^4 e^{\frac{2\pi ikx}{z}}, \quad x \in \mathbb{R}.$$

In these circumstances we have that

$$f_{3,z}(0) = \sum_{k=-\infty}^{\infty} |A_k(z)|^4.$$

Taking into account inequality (3.16) and the fact that the inequality $|A_k(z)| < \epsilon$ holds if $z > z_0$, we get that

$$\lim_{z\to\infty} f_{3,z}(0) = 0,$$

hence

$$\lim_{z\to\infty} \frac{1}{z}\int_0^z |f_{2,z}(x)|^2\, dx = \lim_{z\to\infty} f_{3,z}(0) = 0.$$

Since f is a u.a.p. function, it follows that for every $\epsilon > 0$, the set $E_{\epsilon,f}$ is relatively dense and thus we can consider an increasing sequence $(z_n)_{n\in\mathbb{N}}$ of positive numbers such that $z_n \in E_{\frac{1}{n},f}$ for any $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} z_n = +\infty.$$

Consequently, we have

$$\lim_{n\to\infty} \frac{1}{z_n}\int_0^{z_n} |f_{2,z_n}(x)|^2\, dx = 0. \tag{3.17}$$

Taking into account that $z_n \in E_{\frac{1}{n},f}$, we can see that for each $x, t \in [0, z_n]$,

$$|f(x+t-z_n) - f(x+t)| \le \frac{1}{n}. \tag{3.18}$$

Given $n \in \mathbb{N}$, put $z = z_n$. Analysing the definition of the function $f_{2,z}$, due to the fact that $f_{1,z}$ is a periodic function and $z_n \in E_{\frac{1}{n},f}$, we can assure that for every $x \in [0, z_n]$ we have

$$\begin{aligned}
f_{2,z_n}(x) &= \frac{1}{z_n}\int_0^{z_n} \widetilde{f}_{1,z}(x,t)\, dt = \frac{1}{z_n}\left(\int_0^{z_n-x} \widetilde{f}_{1,z}(x,t)\, dt + \int_{z_n-x}^{z_n} \widetilde{f}_{1,z}(x,t)\, dt\right)\\
&= \frac{1}{z_n}\left(\int_0^{z_n-x} \widetilde{f}_{1,z}(x,t)\, dt + \int_{z_n-x}^{z_n} \widetilde{f}_{1,z}(x-z_n,t)\, dt\right)\\
&= \frac{1}{z_n}\left(\int_0^{z_n-x} \widetilde{f}(x,t)\, dt + \int_{z_n-x}^{z_n} \widetilde{f}(x-z_n,t)\, dt\right)\\
&= \frac{1}{z_n}\left(\int_0^{z_n} \widetilde{f}(x,t)\, dt - \int_{z_n-x}^{z_n} \widetilde{f}(x,t)\, dt + \int_{z_n-x}^{z_n} \widetilde{f}(x-z_n,t)\, dt\right)\\
&= \frac{1}{z_n}\left(\int_0^{z_n} \widetilde{f}(x,t)\, dt + \int_{z_n-x}^{z_n} [f(x+t-z_n) - f(x+t)]\overline{f}(t)\, dt\right).
\end{aligned} \tag{3.19}$$

For each $x \in [0, z_n]$, using inequality (3.18), one has

$$\left|\frac{1}{z_n}\int_{z_n-x}^{z_n} [f(x+t-z_n) - f(x+t)]\overline{f}(t)\, dt\right| \le \frac{1}{z_n}\int_{z_n-x}^{z_n} \frac{A}{n}\, dt = \frac{Ax}{z_n n} \le \frac{A}{n}.$$

Consequently, there exists $\lambda_{n,x} \in \mathbb{C}$ such that $|\lambda_{n,x}| \le 1$ and verifies

$$\frac{1}{z_n}\int_{z_n-x}^{z_n} [f(x+t-z_n) - f(x+t)]\overline{f}(t)\, dt = \frac{\lambda_{n,x} A}{n}.$$

Taking into account the previous equality and equation (3.19), it follows that

$$\frac{1}{z_n}\int_0^{z_n} \widetilde{f}(x,t)\, dt = f_{2,z_n}(x) - \frac{\lambda_{n,x} A}{n}. \tag{3.20}$$

Let $g(x) := M_{t,\widetilde{f}(x,t)}$ for each $x \in \mathbb{R}$. Applying Theorem 3.2.7, we can see that the function g is a u.a.p. function and

$$h_z(x) := \frac{1}{z}\int_0^z \widetilde{f}(x,t)\, dt, \quad z > 0,\ x \in \mathbb{R}$$

tends to g uniformly in $\mathbb{R}$ when $z \to \infty$, hence we get that

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left|\frac{1}{z_n}\int_0^{z_n} \widetilde{f}(x,t)\, dt - g(x)\right| = \lim_{n\to\infty} \epsilon_n = 0,$$

where

$$\epsilon_n := \sup_{x\in\mathbb{R}} \left|\frac{1}{z_n}\int_0^{z_n} \widetilde{f}(x,t)\, dt - g(x)\right|, \quad n \in \mathbb{N}. \tag{3.21}$$

It follows from equations (3.20) and (3.21) that for each $x \in [0, z_n]$,

$$|g(x)| \leq \epsilon_n + \left| \frac{1}{z_n} \int_0^{z_n} \widetilde{f}(x,t)\, dt \right| \leq \epsilon_n + |f_{2,z_n}(x)| + \frac{A}{n}.$$

Moreover, we can see, by definition of the function $f_{2,z}$, that

$$|f_{2,z_n}(x)| \leq \frac{1}{z_n} \int_0^{z_n} |f_{1,z_n}(x+t)\overline{f}_{1,z_n}(t)|\, dt \leq A^2.$$

Hence, for each $x \in [0, z_n]$, one has

$$|g(x)|^2 \leq |f_{2,z_n}(x)|^2 + 2A^2\left(\epsilon_n + \frac{A}{n}\right) + \left(\epsilon_n + \frac{A}{n}\right)^2.$$

Therefore

$$\frac{1}{z_n} \int_0^{z_n} |g(x)|^2\, dx \leq \frac{1}{z_n} \int_0^{z_n} |f_{2,z_n}(x)|^2\, dx + 2A^2\left(\epsilon_n + \frac{A}{n}\right) + \left(\epsilon_n + \frac{A}{n}\right)^2.$$

Consequently, using the previous inequality and equation (3.17), we get that

$$\lim_{n\to\infty} \frac{1}{z_n} \int_0^{z_n} |g(x)|^2\, dx = 0.$$

Fix $\gamma > 0$. By Lemma 3.2.6, there exists $l_\gamma > 0$ such that for all $n \in \mathbb{N}$,

$$0 \leq M_{|g|^2} \leq \frac{1}{z_n} \int_0^{z_n} |g(x)|^2\, dx + \gamma + \frac{2l_\gamma}{z_n} A^2.$$

Since $z_n \to +\infty$ as $n \to \infty$, passing to the limit when $n \to +\infty$, we get that

$$0 \leq M_{|g|^2} \leq \gamma.$$

Passing to the limit as $\gamma \to 0$, we finally obtain $M_{|g|^2} = 0$. Applying Theorem 3.3.5, we get that $|g(x)|^2 = 0$ for every $x \in \mathbb{R}$, and we conclude that

$$g(0) = M_{|f|^2} = 0$$

as we wanted to prove. □

Fortunately, as in classical Fourier theory (see, e.g., [12, Chapter 1, Theorem 2.7]), we can also guarantee that u.a.p. functions with the same Fourier series must be equal to each other.

Theorem 3.4.7 ([4, Chapter 1, Section 4, Theorem 7])**.** *If f and g are u.a.p. functions with the same Fourier series, then $f = g$.*

Proof. Let $h = f - g$. Since h is a u.a.p. function and f and g have the same Fourier series, it follows that the function h satisfies $a_h(\lambda) = 0$ for any $\lambda \in \mathbb{R}$. Applying Theorem 3.4.6 we can assure that $M_{|h|^2} = 0$, consequently using Theorem 3.3.5 we have that $h(x) = 0$ for every $x \in \mathbb{R}$, and we conclude that $f = g$. □

We finish this section by establishing the Parseval Identity for u.a.p. functions.

Theorem 3.4.8 ([4, Chapter 1, Section 4, Theorem 8]). *Let f be a u.a.p. function and consider $\sum_{n=1}^{\infty} a_f(\lambda_n)e^{i\lambda_n x}$ its Fourier series. Then*

$$M_{|f|^2} = \sum_{n=1}^{\infty} |a_f(\lambda_n)|^2,$$

where the above equation is called the Parseval Identity for u.a.p. functions.

Proof. Let f be a u.a.p. function and $\sum_{n=1}^{\infty} a_f(\lambda_n)e^{i\lambda_n x}$ its Fourier Series. For each $x \in \mathbb{R}$, consider the function $g(x) = M_{t,\widetilde{f}(x,t)}$. Taking into account the proof of Theorem 3.4.6 and using a similar reasoning for the function g that we used on equation (3.15), we get that

$$M_{ge_{-\lambda_n}} = |a_f(\lambda_n)|^2$$

for each $n \in \mathbb{N}$, and thus the Fourier series of g is

$$g(x) \sim \sum_{n=1}^{\infty} |a_f(\lambda_n)|^2 e^{i\lambda_n x}.$$

Applying Theorem 3.3.1, we can assure that

$$\sum_{n=1}^{\infty} |a_f(\lambda_n)|^2 \leq M_{|f|^2},$$

that is, the series $\sum_{n=1}^{\infty} |a_f(\lambda_n)|^2$ is convergent, consequently the series

$$\varphi(x) := \sum_{n=1}^{\infty} |a_f(\lambda_n)|^2 e^{i\lambda_n x}$$

is absolutely convergent and also uniformly convergent. Since φ is a uniformly convergent series, using Theorem 2.3.4 we have that φ is a u.a.p. function and by Theorem 3.3.4, the Fourier series of φ coincides with the series for which φ is represented. Hence the functions g and φ have the same Fourier series, consequently applying Theorem 3.4.7 we get that $g = \varphi$, that is,

$$g(x) = \sum_{n=1}^{\infty} |a_f(\lambda_n)|^2 e^{i\lambda_n x}.$$

In these conditions we can see that $g(0) = \sum_{n=1}^{\infty} |a_f(\lambda_n)|^2$ and due to the fact that $g(x) = M_{t,\widetilde{f}(x,t)}$, it follows that $g(0) = M_{|f|^2}$ and we conclude that

$$\sum_{n=1}^{\infty} |a_f(\lambda_n)|^2 = M_{|f|^2}$$

as we wanted to prove. □

3.5 Approximation of u.a.p. Functions by Trigonometric Polynomials

We start this section by defining two new classes of functions, the class of finite sums of trigonometric polynomial functions and its closure in $L^\infty(\mathbb{R})$.

Definition 3.5.1. Let $APP(\mathbb{R})$ denote the vector space over $\mathbb{C}$ of all finite sums of trigonometric polynomial functions, that is, the set of all finite sums of the form

$$\sum_{k=1}^{n} c_k e^{i\lambda_k x},$$

where $c_k \in \mathbb{C}$ and $\lambda_k \in \mathbb{R}$ for every $k \in \{1,\ldots,n\}$. We define $AP(\mathbb{R})$ as the smallest closed subset of $L^\infty(\mathbb{R})$ that contains $APP(\mathbb{R})$, that is,

$$AP(\mathbb{R}) = \text{clos}_{L^\infty(\mathbb{R})}(APP(\mathbb{R})), \tag{3.22}$$

where $\text{clos}_{L^\infty(\mathbb{R})}(APP(\mathbb{R}))$ denotes the closure of $APP(\mathbb{R})$ in $L^\infty(\mathbb{R})$.

The following figures show us the behaviour of three trigonometric polynomial functions.

Figure 3.1: $f(x) = 10e^{ix} + (3+5i)e^{2ix} - 3ie^{\pi ix} + 2e^{5ix}$, $\quad x \in [-6,6]$, $\quad x \in [-300,300]$.

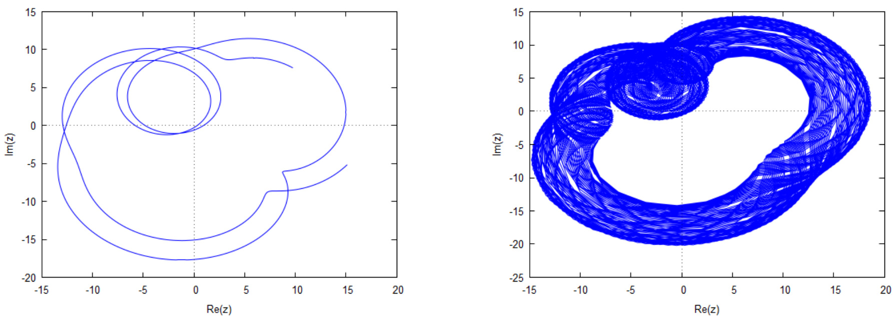


Figure 3.2: $f(x) = 4e^{-7ix} + 10e^{ix} + 6ie^{\frac{\pi ix}{3}}$, $\quad x \in [-10,10]$, $\quad x \in [-400,400]$.

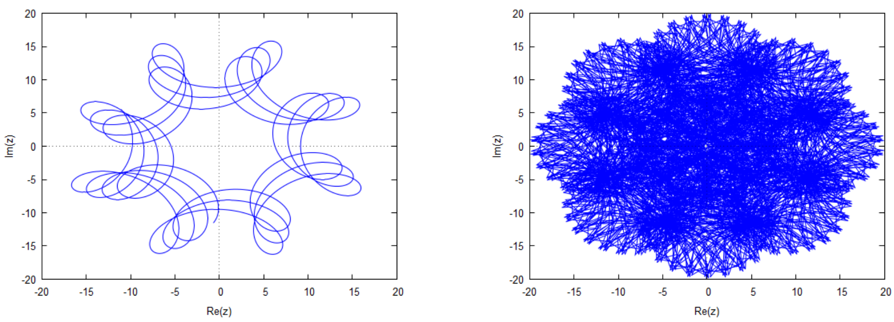
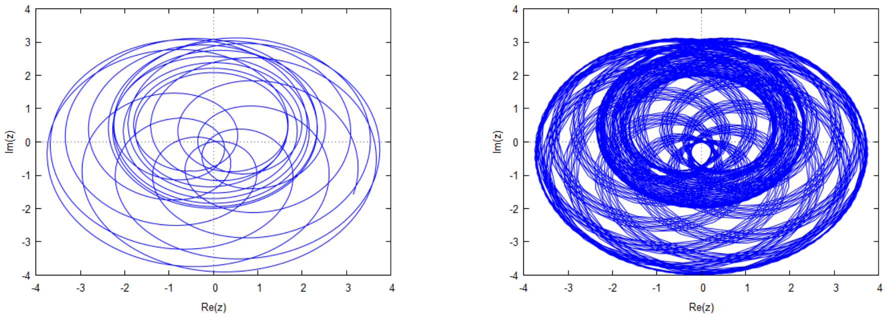


Figure 3.3: $f(x) = e^{ix} + 2e^{\pi ix} + ie^{2ix}$, $x \in [-20, 20]$, $x \in [-200, 200]$.



In the following result we check that $AP(\mathbb{R})$ is, indeed, contained in the set of u.a.p. functions.

Theorem 3.5.2. *As we defined before, let $U(\mathbb{R})$ be the set of all uniformly almost periodic functions and let $AP(\mathbb{R})$ be the closure of $APP(\mathbb{R})$ in $L^\infty(\mathbb{R})$. Then $AP(\mathbb{R}) \subseteq U(\mathbb{R})$.*

Proof. Let us consider a function $\varphi \in AP(\mathbb{R})$. Since $AP(\mathbb{R}) = \text{clos}_{L^\infty(\mathbb{R})}(APP(\mathbb{R}))$, we can assure that there exists a sequence $(p_n)_{n\in\mathbb{N}}$ of terms in $APP(\mathbb{R})$ such that

$$\lim_{n\to\infty} \|\varphi - p_n\|_{L^\infty(\mathbb{R})} = 0.$$

Firstly let us note that for every $n \in \mathbb{N}$, p_n is a uniformly almost periodic function because it is a sum of functions of the form $c \cdot e^{i\lambda x}$, with $c \in \mathbb{C}$ and $\lambda \in \mathbb{R}$, and we know that those monomials are purely periodic functions and thus u.a.p. functions. Observing that the sequence $(p_n)_{n\in\mathbb{N}}$ converges uniformly to φ, we just need to apply the Theorem 2.4.2 and we can conclude that φ is indeed uniformly almost periodic, that is, $AP(\mathbb{R}) \subseteq U(\mathbb{R})$. □

The next result show us that the opposite is also true, that is, the set of u.a.p. functions is contained in $AP(\mathbb{R})$.

Theorem 3.5.3 ([4, Chapter 1, Section 5, Theorem 2])**.** *Let f be a u.a.p. function, $\epsilon > 0$ and*

$$f(x) \sim \sum_{n=1}^{\infty} a_f(\lambda_n) e^{i\lambda_n x}.$$

In these conditions there exists a trigonometric polynomial function P, whose exponents are Fourier exponents of f and

$$\sup_{x\in\mathbb{R}} |f(x) - P(x)| \leq \epsilon.$$

Proof. Given $k \in \mathbb{N}$, consider

$$f_k(x) = f(x) - \sum_{n=1}^{k} a_f(\lambda_n) e^{i\lambda_n x},$$

for any $x \in \mathbb{R}$. Since f_k is a u.a.p. function, using equation (3.10) and Theorem 3.4.8, it follows that

$$M_{|f|^2} = M_{|f_k|^2} + \sum_{n=1}^{k} |a_f(\lambda_n)|^2,$$

which implies that

$$M_{|f_k|^2} = \sum_{n=k+1}^{\infty} |a_f(\lambda_n)|^2.$$

Let $\eta > 0$. Using the previous equality and due to the fact that $\sum_{n=1}^{\infty} |a_f(\lambda_n)|^2$ is a convergent series, there is $p \in \mathbb{N}$ that verifies

$$M_{|f_p|^2} < \eta.$$

Taking into account the statement (3.8) of Theorem 3.2.5, we can assure that

$$\frac{1}{z} \int_0^z |f_p(x+s)|^2 \, dx$$

tends to $M_{|f_p|^2}$ uniformly in $s \in \mathbb{R}$ when $z \to \infty$, consequently there exists $z_0 > 0$, such that

$$\left| \frac{1}{z} \int_0^z |f_p(x+s)|^2 \, dx - M_{|f_p|^2} \right| < \eta,$$

for every $z > z_0$ and for any $s \in \mathbb{R}$. Let $s \in \mathbb{R}$. Using the inequality stated previously and the fact that $M_{|f_p|^2} < \eta$, we get

$$\frac{1}{z} \int_0^z |f_p(x+s)|^2 \, dx < 2\eta,$$

that is,

$$\int_0^z |f_p(x+s)|^2 \, dx < 2\eta z. \tag{3.23}$$

Since f is a u.a.p. function, there is $l_{\frac{\epsilon}{3}} > 0$ such that any interval with length $l_{\frac{\epsilon}{3}}$ intersects $E_{\frac{\epsilon}{3},f}$. Consider $z = N(l_{\frac{\epsilon}{3}} + 1) > z_0$, where $N \in \mathbb{N}$, and for every $k \in \{0, \dots, N-1\}$, let

$$\tau_k \in [k(l_{\frac{\epsilon}{3}} + 1), k(l_{\frac{\epsilon}{3}} + 1) + l_{\frac{\epsilon}{3}}] \cap E_{\frac{\epsilon}{3},f}.$$

Using Theorem 2.2.2 we have that f is uniformly continuous, consequently there exists $\delta \in]0,1[$ such that for any $x_1, x_2 \in \mathbb{R}$,

$$|x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \frac{\epsilon}{3}.$$

Let

$$B = \bigcup_{k=0}^{N-1}]\tau_k, \tau_k + \delta[$$

and consider the function

$$\chi_B(x) = \begin{cases} 1, & \text{if } x \in B \\ 0, & \text{if } x \in]0,z[\setminus B. \end{cases}$$

Applying Hölder's inequality [2, Theorem 7.9], we have

$$\left|\int_0^z f_p(x+s)\chi_B(x)\,dx\right|^2 \leq \int_0^z |f_p(x+s)|^2\,dx \int_0^z (\chi_B(x))^2\,dx. \tag{3.24}$$

Observing the definition of the function χ_B and the fact that if $i,j \in \{0,\ldots,N-1\}$ and $i \neq j$, then $]\tau_i, \tau_i+\delta[\, \cap\,]\tau_j, \tau_j+\delta[= \emptyset$, we can see that

$$\int_0^z (\chi_B(x))^2\,dx = N\delta,$$

and also

$$\begin{aligned}\int_0^z f_p(x+s)\chi_B(x)\,dx &= \sum_{k=0}^{N-1} \int_{\tau_k}^{\tau_k+\delta} f_p(x+s)\,dx \\ &= \sum_{k=0}^{N-1} \int_0^{\delta} f_p(y+\tau_k+s)\,dy \\ &= \sum_{k=0}^{N-1} \int_0^{\delta} f_p(x+\tau_k+s)\,dx.\end{aligned}$$

Taking into account the previous equation and inequalities (3.23) and (3.24), it follows that

$$\left|\sum_{k=0}^{N-1} \int_0^{\delta} f_p(x+\tau_k+s)\,dx\right| < \sqrt{2\eta z N\delta},$$

which implies, using the fact that $z = N(l_{\frac{\epsilon}{3}}+1)$, that

$$\left|\frac{1}{N\delta}\sum_{k=0}^{N-1} \int_0^{\delta} f_p(x+\tau_k+s)\,dx\right| < \sqrt{\frac{2\eta(l_{\frac{\epsilon}{3}}+1)}{\delta}}.$$

If $\eta < \frac{\epsilon^2\delta}{18(l_{\frac{\epsilon}{3}}+1)}$, then we obtain

$$\left|\frac{1}{N\delta}\sum_{k=0}^{N-1} \int_0^{\delta} f_p(x+\tau_k+s)\,dx\right| < \frac{\epsilon}{3}. \tag{3.25}$$

For $k \in \{0,\ldots,N-1\}$, let

$$P_k(s) := \frac{1}{N\delta}\int_0^{\delta} \sum_{n=1}^{p} a_f(\lambda_n) e^{i\lambda_n(x+\tau_k+s)}\,dx = \frac{1}{N\delta}\sum_{n=1}^{p} e^{i\lambda_n s} \int_0^{\delta} a_f(\lambda_n) e^{i\lambda_n(x+\tau_k)}\,dx.$$

In these circumstances we can assure that P_k is a trigonometric polynomial function whose exponents belong to the set of Fourier exponents of f and analyzing the definition of the function f_p and P_k, we get that

$$\frac{1}{N\delta}\int_0^{\delta} f_p(x+\tau_k+s)\,dx = \frac{1}{N\delta}\int_0^{\delta} f(x+\tau_k+s)\,dx - P_k(s). \tag{3.26}$$

Let $P(s) := \sum_{k=0}^{N-1} P_k(s)$. Since inequality (3.25) is satisfied and due to the fact that equation (3.26) holds, we can assure that

$$\left| \frac{1}{N\delta} \sum_{k=0}^{N-1} \int_0^\delta f(x+\tau_k+s)\, dx - P(s) \right| < \frac{\epsilon}{3}. \tag{3.27}$$

Noting that f is uniformly continuous and that

$$|(x+\tau_k+s)-(\tau_k+s)| = |x| < \delta$$

for every $k \in \{0,\ldots,N-1\}$ and $x \in]0,\delta[$, we can see that

$$|f(x+\tau_k+s)-f(\tau_k+s)| < \frac{\epsilon}{3},$$

consequently, we get

$$\begin{aligned} \left| \frac{1}{\delta} \int_0^\delta f(x+\tau_k+s)\, dx - f(\tau_k+s) \right| &= \left| \frac{1}{\delta} \int_0^\delta (f(x+\tau_k+s)-f(\tau_k+s))\, dx \right| \\ &\le \frac{1}{\delta} \int_0^\delta |f(x+\tau_k+s)-f(\tau_k+s)|\, dx \\ &< \frac{1}{\delta} \int_0^\delta \frac{\epsilon}{3}\, dx = \frac{\epsilon}{3}. \end{aligned}$$

Taking into account that

$$|f(\tau_k+s)-f(s)| \le \frac{\epsilon}{3}$$

due to the fact that $\tau_k \in E_{\frac{\epsilon}{3},f}$ for each $k \in \{0,\ldots,N-1\}$, it follows that

$$\begin{aligned} \left| \frac{1}{\delta} \int_0^\delta f(x+\tau_k+s)\, dx - f(s) \right| &= \left| \frac{1}{\delta} \int_0^\delta f(x+\tau_k+s)\, dx - f(s) + f(\tau_k+s) - f(\tau_k+s) \right| \\ &\le \left| \frac{1}{\delta} \int_0^\delta f(x+\tau_k+s)\, dx - f(\tau_k+s) \right| + |f(\tau_k+s)-f(s)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{2\epsilon}{3}. \end{aligned}$$

Therefore one has

$$\begin{aligned} \left| \frac{1}{N\delta} \sum_{k=0}^{N-1} \int_0^\delta f(x+\tau_k+s)\, dx - f(s) \right| &= \left| \frac{1}{N} \sum_{k=0}^{N-1} \left(\frac{1}{\delta} \int_0^\delta f(x+\tau_k+s)\, dx - f(s) \right) \right| \\ &\le \frac{1}{N} \sum_{k=0}^{N-1} \left| \frac{1}{\delta} \int_0^\delta f(x+\tau_k+s)\, dx - f(s) \right| \\ &< \frac{1}{N} \sum_{k=0}^{N-1} \frac{2\epsilon}{3} = \frac{2\epsilon}{3}. \end{aligned}$$

Hence, using the previous inequality and inequality (3.27), we conclude that

$$|f(s)-P(s)| = \left| f(s) - P(s) + \frac{1}{N\delta} \sum_{k=0}^{N-1} \int_0^\delta f(x+\tau_k+s)\, dx - \frac{1}{N\delta} \sum_{k=0}^{N-1} \int_0^\delta f(x+\tau_k+s)\, dx \right|$$

$$\leq \left| \frac{1}{N\delta} \sum_{k=0}^{N-1} \int_0^{\delta} f(x+\tau_k+s)\,dx - f(s) \right| + \left| \frac{1}{N\delta} \sum_{k=0}^{N-1} \int_0^{\delta} f(x+\tau_k+s)\,dx - P(s) \right|$$
$$< \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,$$

that is,

$$\sup_{x\in\mathbb{R}} |f(x) - P(x)| \leq \epsilon$$

as we wanted to prove. □

We finish this chapter by observing the most important results that we obtained so far, that is, any u.a.p. function has three equivalent definitions.

Theorem 3.5.4. *Let $f : \mathbb{R} \to \mathbb{K}$ be a function. Then the following statements are equivalent.*

1. $f \in U(\mathbb{R})$;
2. $f \in N(\mathbb{R})$;
3. $f \in AP(\mathbb{R})$.

Proof. Applying Theorems 3.1.4 and 3.1.5, we can assure that $f \in U(\mathbb{R})$ if and only if $f \in N(\mathbb{R})$. Taking account Theorems 3.5.2 and 3.5.3, we have $f \in U(\mathbb{R})$ if and only if $f \in AP(\mathbb{R})$, consequently, we have the equivalence between the statements as we wanted to prove. □

4

Fourier Transform on the Space L^2

In this chapter we are going to start by studying the $L^p(\mathbb{R})$ spaces with $p \in [1,+\infty]$. Following that we will define the Fourier Transform in $L^1(\mathbb{R})$, and after analysing some important properties of it we will be able to extend this definition to the space $L^2(\mathbb{R})$.

4.1 L^p Spaces and Step Functions

Definition 4.1.1. Let $f : \mathbb{R} \to \mathbb{C}$ be a measurable function and $p \in [1,+\infty[$. We say that $f \in L^p(\mathbb{R})$ if, and only if,

$$\int_{-\infty}^{+\infty} |f(x)|^p \, dx < \infty,$$

and we define its norm by

$$\|f\|_{L^p(\mathbb{R})} := \left(\int_{-\infty}^{+\infty} |f(x)|^p \, dx \right)^{\frac{1}{p}}.$$

If $p = \infty$, then we say that $f \in L^\infty(\mathbb{R})$ if, and only if,

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}} |f(x)| := \inf\{t > 0 : \mu(\{x \in \mathbb{R} : |f(x)| > t\}) = 0\} < \infty,$$

and we define its norm as

$$\|f\|_{L^\infty(\mathbb{R})} := \operatorname*{ess\,sup}_{x\in\mathbb{R}} |f(x)|.$$

In this work, given $p \in [1,+\infty]$, we will denote $\|f\|_{L^p(\mathbb{R})}$ by $\|f\|_p$. It is important to observe that, in these spaces, we consider that two functions are the same if they are identical almost everywhere.

Definition 4.1.2. Let $f : \mathbb{R} \to \mathbb{C}$ be a function. We say that f is a simple function if and only if f takes on only finitely many values, that is, if there exist $\{\lambda_1, \dots, \lambda_n\} \subseteq \mathbb{C} \setminus \{0\}$ and measurable sets $E_1, \dots, E_n \subseteq \mathbb{R}$ such that

$$f = \lambda_1 \chi_{E_1} + \cdots + \lambda_n \chi_{E_n},$$

where

$$\chi_E(x) := \begin{cases} 1, & \text{if } x \in E, \\ 0, & \text{if } x \in \mathbb{R} \setminus E. \end{cases}$$

is the characteristic function of a set $E \subseteq \mathbb{R}$.

Definition 4.1.3. Let $f : \mathbb{R} \to \mathbb{C}$ be a function, $\{\alpha_1, \ldots, \alpha_n\} \subseteq \mathbb{C} \setminus \{0\}$ and $I_1, \ldots, I_n$ intervals of $\mathbb{R}$, with $n \in \mathbb{N}$. We say that f is a step function if, and only if,

$$f = \alpha_1 \chi_{I_1} + \cdots + \alpha_n \chi_{I_n},$$

where χ_E is the characteristic function of a set E.

Analyzing both of the previous definitions we can see immediately that every step function is a simple function. For $p = 1$, the following result is proved in [2, Theorem 3.47]. For $1 < p < \infty$, the proof is analogous.

Theorem 4.1.4. *Let $p \in [1, +\infty[$. If $f \in L^p(\mathbb{R})$, then for every $\epsilon > 0$ there exists a step function $g \in L^p(\mathbb{R})$ such that $\|f - g\|_p < \epsilon$.*

Proof. Let $\epsilon > 0$, consider without loss of generality, $f : \mathbb{R} \to [0, +\infty[$ and suppose that $f \in L^p(\mathbb{R})$, where $p \geq 1$. Applying Theorem [2, Theorem 2.89], there exists a sequence of simple functions $(\varphi_n)_{n \in \mathbb{N}}$, such that for each $x \in \mathbb{R}$ and for every $k \in \mathbb{N}$, one has

$$\lim_{n \to +\infty} \varphi_n(x) = f(x)$$

and also

$$|\varphi_k(x)| \leq |\varphi_{k+1}(x)| \leq |f(x)|. \tag{4.1}$$

Due to the fact that $f \in L^p(\mathbb{R})$ and (φ_n) is a sequence of functions that satisfies inequality (4.1), we get that $\varphi_n \in L^p(\mathbb{R})$ for every $n \in \mathbb{N}$. Taking into account that these simple functions form a sequence that verifies inequality (4.1), we can assure that for any $x \in \mathbb{R}$,

$$|f(x) - \varphi_n(x)|^p \leq 2^p(|f(x)|^p + |\varphi_n(x)|^p) \leq 2^{p+1}|f(x)|^p.$$

Moreover, $|f|^p \in L^1(\mathbb{R})$ because $f \in L^p(\mathbb{R})$. Using the Dominated Convergence Theorem [2, Theorem 3.31], we get that

$$\lim_{n \to +\infty} \int_{-\infty}^{+\infty} |f(x) - \varphi_n(x)|^p \, dx = 0,$$

and we conclude that simple functions are dense on $L^p(\mathbb{R})$. Consequently given $\epsilon > 0$, we can assure that there exist measurable subsets $A_1, \ldots, A_n$ of $\mathbb{R}$ and nonzero complex numbers $a_1, \ldots, a_n$ such that $\mu(A_k) < \infty$ for each $k \in \{1, \ldots, n\}$, where $\mu(A)$ denotes the Lebesgue measure of any set A, and also

$$\left\| f - \sum_{k=1}^{n} a_k \chi_{A_k} \right\|_p < \frac{\epsilon}{2}.$$

For each $k \in \{1,\dots,n\}$, there is an open subset G_k of $\mathbb{R}$ that contains A_k and whose Lebesgue measure is as close as we want to $\mu(A_k)$. Each open subset of $\mathbb{R}$, including each G_k, is a countable union of disjoint open intervals (see, e.g, [1, Theorem 3.11]). Thus for each k, there exists a set E_k that is a finite union of bounded open intervals contained in G_k whose Lebesgue measure is as close as we want to $\mu(G_k)$. Hence for each k, there is a set E_k that is a finite union of bounded intervals such that

$$\mu(E_k \setminus A_k) + \mu(A_k \setminus E_k) \leq \mu(G_k \setminus A_k) + \mu(G_k \setminus E_k) < \left(\frac{\epsilon}{2|a_k|n}\right)^p,$$

that is,

$$\|\chi_{A_k} - \chi_{E_k}\|_p = \left(\int_{-\infty}^{+\infty} |\chi_{A_k}(x) - \chi_{E_k}(x)|^p \, dx\right)^{\frac{1}{p}} = (\mu(E_k \setminus A_k) + \mu(A_k \setminus E_k))^{\frac{1}{p}} < \frac{\epsilon}{2|a_k|n}.$$

Therefore applying Minkowski's Theorem [2, Theorem 7.14], we have

$$\begin{aligned}
\left\|f - \sum_{k=1}^{n} a_k \chi_{E_k}\right\|_p &\leq \left\|f - \sum_{k=1}^{n} a_k \chi_{A_k}\right\|_p + \left\|\sum_{k=1}^{n} a_k \chi_{A_k} - \sum_{k=1}^{n} a_k \chi_{E_k}\right\|_p \\
&< \frac{\epsilon}{2} + \sum_{k=1}^{n} |a_k| \, \|\chi_{A_k} - \chi_{E_k}\|_p \\
&< \epsilon,
\end{aligned}$$

and we conclude that for every $\epsilon > 0$, there exists a step function $g = \sum_{k=1}^{n} a_k \chi_{E_k} \in L^p(\mathbb{R})$ such that $\|f - g\|_p < \epsilon$. □

As a consequence of the previous theorem, in the next result we will be able to prove that, in fact, the space $L^p(\mathbb{R}) \cap L^q(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ for each $p, q \in [1, +\infty[$.

Corollary 4.1.5. *Let $p, q \in [1, +\infty[$. Then the space $L^q(\mathbb{R}) \cap L^p(\mathbb{R})$ is dense on $L^p(\mathbb{R})$.*

Proof. Let $f \in L^p(\mathbb{R})$ and let $\epsilon > 0$. Then, applying the previous theorem, there exists a step function $g \in L^p(\mathbb{R})$ such that

$$\|f - g\|_p < \epsilon.$$

Taking into account that $g \in L^q(\mathbb{R})$ for each $q \in [1, +\infty[$ because g is a step function, we get that $g \in L^q(\mathbb{R}) \cap L^p(\mathbb{R})$ for every $q \in [1, +\infty[$, and we conclude that the space $L^q(\mathbb{R}) \cap L^p(\mathbb{R})$ is dense on $L^p(\mathbb{R})$ for each $q \in [1, +\infty[$ as we wanted to prove. □

We finish this section with two preparatory results that will help us to establish Theorem 4.4.3.

Theorem 4.1.6. *Let $p \in [1, \infty[$ and consider $f \in L^p(\mathbb{R})$. Then the function*

$$\Phi(t) := \|f - T_{-t} f\|_p$$

is bounded and uniformly continuous on $\mathbb{R}$.

Proof. Let $f \in L^p(\mathbb{R})$. Then there exists $M_1 \in \mathbb{R}$ such that

$$\left(\int_{-\infty}^{+\infty} |f(x)|^p \, dx\right)^{\frac{1}{p}} \leq M_1,$$

and since

$$\int_{-\infty}^{+\infty} |f(x)|^p \, dx = \int_{-\infty}^{+\infty} |f(x+\lambda)|^p \, dx$$

for every $\lambda \in \mathbb{R}$, it follows that the translation function $T_\lambda f \in L^p(\mathbb{R})$ for each $\lambda \in \mathbb{R}$. Applying Minkowski's Theorem [2, Theorem 7.14], we have that

$$\begin{aligned}
\Phi(t) &= \left(\int_{-\infty}^{+\infty} |f(x) - f(x-t)|^p \, dx\right)^{\frac{1}{p}} \\
&\leq \left(\int_{-\infty}^{+\infty} |f(x)|^p \, dx\right)^{\frac{1}{p}} + \left(\int_{-\infty}^{+\infty} |f(x-t)|^p \, dx\right)^{\frac{1}{p}} \\
&= \left(\int_{-\infty}^{+\infty} |f(x)|^p \, dx\right)^{\frac{1}{p}} + \left(\int_{-\infty}^{+\infty} |f(y)|^p \, dy\right)^{\frac{1}{p}} \leq (M_1 + M_1) =: M_2
\end{aligned}$$

for every $t \in \mathbb{R}$, that is, Φ is a bounded function.

Let $a, b \in \mathbb{R}$ be such that $a < b$ and consider the function

$$\phi(t) := \|\chi_{[a,b]} - T_{-t}\chi_{[a,b]}\|_p,$$

for each $t \in \mathbb{R}$. Let us suppose that $s, t \in \mathbb{R}$ with $s < t$. Then

$$\begin{aligned}
|\phi(t) - \phi(s)| &= |\ \|\chi_{[a,b]} - T_{-t}\chi_{[a,b]}\|_p - \|\chi_{[a,b]} - T_{-s}\chi_{[a,b]}\|_p\ | \\
&\leq \|T_{-t}\chi_{[a,b]} - T_{-s}\chi_{[a,b]}\|_p \\
&= \left(\int_{-\infty}^{+\infty} |\chi_{[a,b]}(x-t) - \chi_{[a,b]}(x-s)|^p \, dx\right)^{\frac{1}{p}} \\
&= \left(\int_{a+s}^{b+t} |\chi_{[a+t,b+t]}(x) - \chi_{[a+s,b+s]}(x)|^p \, dx\right)^{\frac{1}{p}} \\
&= \begin{cases} (2(b-a))^{\frac{1}{p}}, & \text{if } b+s \leq a+t, \\ \\ (2(t-s))^{\frac{1}{p}}, & \text{if } a+t < b+s, \end{cases} \\
&\leq 2^{\frac{1}{p}}(t-s)^{\frac{1}{p}}.
\end{aligned} \tag{4.2}$$

For $\epsilon_1 > 0$, put $\delta_1 := \frac{\epsilon_1^p}{2}$. If $0 \leq t - s < \delta_1$, then it follows from (4.2) that

$$|\phi(t) - \phi(s)| \leq 2^{\frac{1}{p}}(t-s)^{\frac{1}{p}} < 2^{\frac{1}{p}}\delta_1^{\frac{1}{p}} \leq \epsilon_1,$$

therefore we conclude that ϕ is uniformly continuous on $\mathbb{R}$.

Now let $a_i < b_i$ for each $i \in \{1, \ldots, n\}$ with $n \in \mathbb{N}$, let $\{\lambda_1, \ldots, \lambda_n\} \subseteq \mathbb{C}$, and consider the function

$$\varphi(t) := \left\| \sum_{i=1}^{n} \lambda_i \chi_{[a_i,b_i]} - T_{-t}\left(\sum_{i=1}^{n} \lambda_i \chi_{[a_i,b_i]}\right) \right\|_p,$$

for each $t \in \mathbb{R}$. If $s < t$, then as before one has

$$\begin{aligned}
|\varphi(t) - \varphi(s)| &\leq \left\| T_{-t}\left(\sum_{i=1}^{n} \lambda_i \chi_{[a_i,b_i]}\right) - T_{-s}\left(\sum_{i=1}^{n} \lambda_i \chi_{[a_i,b_i]}\right)\right\|_p \\
&= \left(\int_{-\infty}^{+\infty} \left|\sum_{i=1}^{n} \lambda_i \chi_{[a_i,b_i]}(x-t) - \sum_{i=1}^{n} \lambda_i \chi_{[a_i,b_i]}(x-s)\right|^p dx\right)^{\frac{1}{p}} \\
&\leq \sum_{i=1}^{n} |\lambda_i| \, \|T_{-t}\chi_{[a_i,b_i]} - T_{-s}\chi_{[a_i,b_i]}\|_p. \qquad (4.3)
\end{aligned}$$

Let $\epsilon_2 > 0$ and let

$$\delta_2 := \left(1 + \sum_{i=1}^{n} |\lambda_i|\right)^{-p} \frac{\epsilon_2^p}{2}.$$

If $0 \leq t - s < \delta_2$, then it follows from (4.2) and (4.3) that

$$\begin{aligned}
|\varphi(t) - \varphi(s)| &\leq \sum_{i=1}^{n} |\lambda_i| 2^{\frac{1}{p}} (t-s)^{\frac{1}{p}} \\
&< \left(\sum_{i=1}^{n} |\lambda_i|\right) 2^{\frac{1}{p}} \delta_2^{\frac{1}{p}} \\
&\leq \left(\sum_{i=1}^{n} |\lambda_i|\right)\left(1 + \sum_{i=1}^{n} |\lambda_i|\right)^{-1} 2^{\frac{1}{p}} \frac{\epsilon_2}{2^{\frac{1}{p}}} < \epsilon_2,
\end{aligned}$$

consequently, φ is uniformly continuous on $\mathbb{R}$. Therefore the function

$$\psi_g : \mathbb{R} \to \mathbb{R}, \quad t \mapsto \|g - T_{-t}g\|_p$$

is uniformly continuous on $\mathbb{R}$, for every step function g.

Let $\epsilon_3 > 0$. Applying Theorem 4.1.4, there exists a step function $g \in L^p(\mathbb{R})$ such that

$$\|f - g\|_p < \frac{\epsilon_3}{3}.$$

Taking into account inequality (4.3), there is $\delta_3 > 0$ such that for all $x, y \in \mathbb{R}$, one has

$$|x - y| < \delta_3 \Rightarrow \|T_{-x}g - T_{-y}g\|_p < \frac{\epsilon_3}{3}.$$

Hence if $x, y \in \mathbb{R}$ satisfy $|x - y| < \delta_3$, then we get that

$$\begin{aligned}
|\Phi(x) - \Phi(y)| &= \left|\|f - T_{-x}f\|_p - \|f - T_{-y}f\|_p\right| \\
&\leq \|T_{-y}f - T_{-x}f\|_p \\
&= \|T_{-y}f - T_{-y}g + T_{-y}g - T_{-x}g + T_{-x}g - T_{-x}f\|_p \\
&\leq \|T_{-y}f - T_{-y}g\|_p + \|T_{-y}g - T_{-x}g\|_p + \|T_{-x}g - T_{-x}f\|_p \\
&= \|f - g\|_p + \|T_{-y}g - T_{-x}g\|_p + \|f - g\|_p \\
&< \frac{\epsilon_3}{3} + \frac{\epsilon_3}{3} + \frac{\epsilon_3}{3}
\end{aligned}$$

$$= \epsilon_3,$$

and we can conclude that the function Φ is uniformly continuous on $\mathbb{R}$ as we wanted to prove. □

Lemma 4.1.7. *Let us suppose that $\Phi : \mathbb{R} \to \mathbb{R}^+$ is a bounded uniformly continuous function and let $p \in [1, +\infty[$. Then the function $\Psi := \Phi^p$ is bounded and uniformly continuous on $\mathbb{R}$.*

Proof. Since Φ is a bounded function, it follows immediately that Ψ is also a bounded function. Let $x, y \in \mathbb{R}^+$. Then, applying the Lagrange theorem (see, e.g., [1, Theorem 5.16]), there exists $\xi \in [\min\{x,y\}, \max\{x,y\}]$, such that

$$|x^p - y^p| = p\xi^{p-1}|x-y|.$$

Consequently, one has

$$|x^p - y^p| \leq p(\max\{x,y\})^{p-1}|x-y|. \tag{4.4}$$

Let $\epsilon > 0$. Since Φ is uniformly continuous, there exists $\delta > 0$ such that for every $s, t \in \mathbb{R}$, one has

$$|s-t| < \delta \Rightarrow |\Phi(s) - \Phi(t)| < \frac{\epsilon}{p\left(\sup\limits_{\xi \in \mathbb{R}} \Phi(\xi)\right)^{p-1}}. \tag{4.5}$$

It is important to observe that $\sup\limits_{\xi \in \mathbb{R}} \Phi(\xi)$ is a finite value because Φ is a bounded function. Taking into account inequalities (4.4) and (4.5), we see that if $|s-t| < \delta$, then

$$\begin{aligned}
|\Psi(s) - \Psi(t)| &= |(\Phi(s))^p - (\Phi(t))^p| \\
&\leq p(\max\{\Phi(s), \Phi(t)\})^{p-1}|\Phi(s) - \Phi(t)| \\
&\leq p\left(\sup_{\xi \in \mathbb{R}} \Phi(\xi)\right)^{p-1} |\Phi(s) - \Phi(t)| < \epsilon,
\end{aligned}$$

that is, Ψ is uniformly continuous on $\mathbb{R}$. □

4.2 Proprieties of the Fourier Transform on L^1

We start this section by presenting the definition of the Fourier transform in $L^1(\mathbb{R})$.

Definition 4.2.1. Let $f \in L^1(\mathbb{R})$. We define the Fourier transform of f by the function $\widehat{f} : \mathbb{R} \to \mathbb{C}$, such that

$$\widehat{f}(t) := \int_{-\infty}^{\infty} f(x) e^{-2\pi i t x}\, dx$$

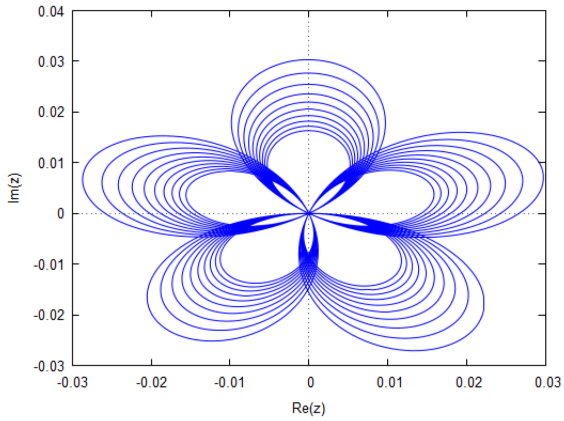
for each $t \in \mathbb{R}$.

In the following example, we will analyse the behaviour of the Fourier transform of the characteristic function.

Example 4.2.2. Let $a,b \in \mathbb{R}$, let $[a,b] \subset \mathbb{R}$ and consider the characteristic function $\chi_{[a,b]}$. We can guarantee that $\chi_{[a,b]} \in L^1(\mathbb{R})$ and

$$\widehat{\chi_{[a,b]}}(t) = \int_{-\infty}^{+\infty} \chi_{[a,b]}(x) e^{-2\pi i t x}\, dx = \int_a^b e^{-2\pi i t x}\, dx$$
$$= \begin{cases} b-a, & \text{if } t = 0, \\ \frac{e^{-2\pi i b t} - e^{-2\pi i a t}}{-2\pi i t}, & \text{if } t \neq 0. \end{cases}$$

Figure 4.1: $\widehat{\chi_{[-1,4]}}(t), \quad t \in [10,20]$.



Now we will see that the Fourier transform of a $L^1(\mathbb{R})$ function is uniformly continuous in $\mathbb{R}$.

Theorem 4.2.3 ([2, Theorem 11.49])**.** *If $f \in L^1(\mathbb{R})$, then $\widehat{f}$ is uniformly continuous on $\mathbb{R}$ and*

$$\|\widehat{f}\|_\infty \leq \|f\|_1, \quad \lim_{|t| \to \infty} \widehat{f}(t) = 0.$$

Proof. Let $x, y \in \mathbb{R}$. Since $|e^{-2\pi i y x}| = 1$, it follows that

$$|\widehat{f}(t)| = \left| \int_{-\infty}^{+\infty} f(x) e^{-2\pi i t x}\, dx \right| \leq \int_{-\infty}^{+\infty} |f(x)|\, dx = \|f\|_1$$

for any $t \in \mathbb{R}$, consequently we get that $\|\widehat{f}\|_\infty \leq \|f\|_1$. Given $t, h \in \mathbb{R}$, we have that

$$|\widehat{f}(t+h) - \widehat{f}(t)| = \left| \int_{-\infty}^{+\infty} f(x) e^{-2\pi i (t+h) x} - f(x) e^{-2\pi i t x}\, dx \right|$$
$$\leq \int_{-\infty}^{+\infty} |f(x)|\, |e^{-2\pi i h x} - 1|\, dx.$$

Let $(h_n)_{n \in \mathbb{N}}$ be a sequence of real numbers such that $\lim_{n \to +\infty} h_n = 0$ and consider the function

$$\varphi_n(x) = |f(x)|\, |e^{-2\pi i h_n x} - 1|,$$

for each $x \in \mathbb{R}$ and $n \in \mathbb{N}$. In these conditions we can assure that $\varphi_n \in L^1(\mathbb{R})$ for any $n \in \mathbb{N}$ because $f \in L^1(\mathbb{R})$ and, using the previous equality, we have for every $n \in \mathbb{N}$ and $x \in \mathbb{R}$,

$$\lim_{n \to +\infty} \varphi_n(x) = 0, \quad |\varphi_n(x)| \leq 2\, |f(x)|.$$

Consequently applying the Dominated Convergence Theorem [2, Theorem 3.31], it follows that

$$\lim_{n\to+\infty}\int_{-\infty}^{+\infty}|f(x)|\,|e^{-2\pi i h_n x}-1|\,dx=0.$$

Due to the fact that

$$0\le|\widehat{f}(t+h)-\widehat{f}(t)|\le\int_{-\infty}^{+\infty}|f(x)|\,|e^{-2\pi i h x}-1|\,dx$$

for every $t,h\in\mathbb{R}$, we can assure that

$$\lim_{n\to+\infty}|\widehat{f}(t+h_n)-\widehat{f}(t)|=0,$$

that is, $\widehat{f}$ is a uniformly continuous function. Taking into account Example 4.2.2 if $[a,b]\subset\mathbb{R}$, then we can guarantee that

$$\lim_{|t|\to\infty}\widehat{\chi}_{[a,b]}(t)=0. \tag{4.6}$$

Taking into account Theorem 4.1.4, there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of step functions in $L^1(\mathbb{R})$, which imply by the previous arguments that each of the functions $\widehat{f}$ is uniformly continuous, such that

$$\lim_{k\to\infty}\|f-f_k\|_1=0,$$

hence, due to the fact $\|\widehat{f}\|_\infty\le\|f\|_1$ we have

$$\lim_{k\to\infty}\|\widehat{f}-\widehat{f_k}\|_\infty=0.$$

In these conditions we can guarantee that the sequence $(\widehat{f_k})_{k\in\mathbb{N}}$ is a sequence of uniformly continuous functions that converges uniformly in $\mathbb{R}$ to $\widehat{f}$. Taking into account equation (4.6) and the fact that the uniform limit of uniformly continuous functions in $\mathbb{R}$ each of which has limit 0 when $|t|\to+\infty$ also has limit 0 when $|t|\to+\infty$, we can conclude that

$$\lim_{|t|\to\infty}\widehat{f}(t)=0,$$

as we wanted to prove. □

In the following theorem, we establish some algebraic properties of the Fourier transform.

Theorem 4.2.4 ([2, Theorem 11.55]). *Let $f,h\in L^1(\mathbb{R})$ and $\lambda,t\in\mathbb{R}$. Then the following proprieties hold.*

1. *If $g(x)=f(x-\lambda)$ for every $x\in\mathbb{R}$, then $\widehat{g}(t)=\widehat{f}(t)e^{-2\pi i\lambda t}$.*
2. *If $g(x)=e^{2\pi i\lambda x}f(x)$ for any $x\in\mathbb{R}$, then $\widehat{g}(t)=\widehat{f}(t-\lambda)$.*
3. *If $\lambda\neq0$ and $g(x)=f(\lambda x)$ for each $x\in\mathbb{R}$, then $\widehat{g}(t)=\frac{1}{|\lambda|}\widehat{f}\left(\frac{t}{\lambda}\right)$.*

4. *If $\alpha, \beta \in \mathbb{C}$ and $g(x) = \alpha f(x) + \beta h(x)$ for every $x \in \mathbb{R}$, then $\widehat{g}(t) = \alpha \widehat{f}(t) + \beta \widehat{h}(t)$.*

5. *If $g(x) = \overline{f}(x)$ for any $x \in \mathbb{R}$, then $\widehat{g}(t) = \overline{\widehat{f}(-t)}$.*

Proof. If $g(x) = f(x-\lambda)$ for every $x \in \mathbb{R}$, then

$$\begin{aligned}\widehat{g}(t) &= \int_{-\infty}^{+\infty} g(x)e^{-2\pi i t x}\, dx = \int_{-\infty}^{+\infty} f(x-\lambda)e^{-2\pi i t x}\, dx \\ &= \int_{-\infty}^{+\infty} f(y)e^{-2\pi i t(y+\lambda)}\, dy = e^{-2\pi i \lambda t}\int_{-\infty}^{+\infty} f(x)e^{-2\pi i t x}\, dx \\ &= \widehat{f}(t)e^{-2\pi i \lambda t}.\end{aligned}$$

If $g(x) = e^{2\pi i \lambda x} f(x)$ for any $x \in \mathbb{R}$, then

$$\begin{aligned}\widehat{g}(t) &= \int_{-\infty}^{+\infty} g(x)e^{-2\pi i t x}\, dx = \int_{-\infty}^{+\infty} f(x)e^{2\pi i \lambda x}e^{-2\pi i t x}\, dx \\ &= \int_{-\infty}^{+\infty} f(x)e^{-2\pi i x(t-\lambda)}\, dx = \widehat{f}(t-\lambda).\end{aligned}$$

If $\lambda \neq 0$ and $g(x) = f(\lambda x)$ for each $x \in \mathbb{R}$, then

$$\begin{aligned}\widehat{g}(t) &= \int_{-\infty}^{+\infty} g(x)e^{-2\pi i t x}\, dx = \int_{-\infty}^{+\infty} f(\lambda x)e^{-2\pi i t x}\, dx \\ &= \frac{1}{|\lambda|}\int_{-\infty}^{+\infty} f(y)e^{\frac{-2\pi i t y}{\lambda}}\, dy = \frac{1}{|\lambda|}\widehat{f}\left(\frac{t}{\lambda}\right).\end{aligned}$$

If $\alpha, \beta \in \mathbb{C}$ and $g(x) = \alpha f(x) + \beta h(x)$ for every $x \in \mathbb{R}$, then

$$\begin{aligned}\widehat{g}(t) &= \int_{-\infty}^{+\infty} g(x)e^{-2\pi i t x}\, dx = \int_{-\infty}^{+\infty} (\alpha f(x) + \beta h(x))e^{-2\pi i t x}\, dx \\ &= \alpha\int_{-\infty}^{+\infty} f(x)e^{-2\pi i t x}\, dx + \beta\int_{-\infty}^{+\infty} h(x)e^{-2\pi i t x}\, dx = \alpha\widehat{f}(t) + \beta\widehat{h}(t).\end{aligned}$$

If $g(x) = \overline{f}(x)$ for any $x \in \mathbb{R}$, then

$$\begin{aligned}\widehat{g}(t) &= \int_{-\infty}^{+\infty} g(x)e^{-2\pi i t x}\, dx = \int_{-\infty}^{+\infty} \overline{f}(x)e^{-2\pi i t x}\, dx \\ &= \overline{\int_{-\infty}^{+\infty} f(x)e^{2\pi i t x}\, dx} = \overline{\widehat{f}(-t)}.\end{aligned}$$

□

We finish this section by verifying that the integral, in $\mathbb{R}$, of the product between the Fourier transform of a function and other function, is equal to the integral, in $\mathbb{R}$, of the product of the former function and the Fourier transform of the latter function.

Theorem 4.2.5 ([2, Theorem 11.59])**.** *If $f, g \in L^1(\mathbb{R})$, then*

$$\int_{-\infty}^{+\infty} \widehat{f}(t)g(t)\, dt = \int_{-\infty}^{+\infty} f(t)\widehat{g}(t)\, dt.$$

Proof. Since $f, g \in L^1(\mathbb{R})$, applying Theorem 4.2.3, it follows that $\widehat{f}, \widehat{g} \in L^\infty(\mathbb{R})$, therefore both integrals are well defined. Taking into account the definition of the Fourier Transform, Tonelli's Theorem [2, Theorem 5.28] and Fubini's Theorem [2, Theorem 5.32], we can assure that

$$\begin{aligned}\int_{-\infty}^{+\infty} \widehat{f}(t)g(t)\,dt &= \int_{-\infty}^{+\infty} g(t)\int_{-\infty}^{+\infty} f(x)e^{-2\pi i t x}\,dx\,dt\\ &= \int_{-\infty}^{+\infty} f(x)\int_{-\infty}^{+\infty} g(t)e^{-2\pi i t x}\,dt\,dx\\ &= \int_{-\infty}^{+\infty} f(x)\widehat{g}(x)\,dx = \int_{-\infty}^{+\infty} f(t)\widehat{g}(t)\,dt.\end{aligned}$$

□

4.3 Convolution and Fourier Transform

We start this section by recalling the definition of convolution between two functions.

Definition 4.3.1. Let $f, g : \mathbb{R} \to \mathbb{C}$ be measurable functions. We define the convolution of f and g by

$$(f * g)(t) := \int_{-\infty}^{+\infty} f(x)g(t-x)\,dx,$$

for every $t \in \mathbb{R}$ for which the integral is defined. Analysing the definition of convolution we can see that $f * g = g * f$.

In the next result, we will prove that the norm, in $L^p(\mathbb{R})$, of the convolution between a $L^1(\mathbb{R})$ function and a $L^p(\mathbb{R})$ function is always less or equal to the product between their corresponding norms.

Theorem 4.3.2 ([2, Theorem 11.64])**.** *Let $f \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R})$, with $p \in [1, +\infty]$. In these conditions we can guarantee that $f * g$ is defined for almost every $x \in \mathbb{R}$ and*

$$\|f * g\|_p \le \|f\|_1 \|g\|_p.$$

Proof. For every $p \ge 1$ and $y \in \mathbb{R}$, we have

$$\int_{-\infty}^{+\infty} |f(y)|\,|g(x-y)|^p\,dx = |f(y)|\int_{-\infty}^{+\infty} |g(x-y)|^p\,dx = |f(y)|\,\|g\|_p^p < \infty,$$

and, moreover,

$$\int_{-\infty}^{+\infty} |f(y)|\,dy \int_{-\infty}^{+\infty} |g(x-y)|^p\,dx = \|f\|_1 \|g\|_p^p < \infty.$$

Let $p = 1$. Applying Fubini's Theorem [2, Theorem 5.32] we can guarantee that

$$\begin{aligned}\|f * g\|_1 &= \int_{-\infty}^{+\infty} |(f * g)(x)|\,dx\\ &= \int_{-\infty}^{+\infty} \left|\int_{-\infty}^{+\infty} f(t)g(x-t)\,dt\right|\,dx\end{aligned}$$

$$\begin{aligned}
&\leq \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |f(t)g(x-t)| \, dt \, dx \\
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |f(t)| \, |g(x-t)| \, dx \, dt \\
&= \int_{-\infty}^{+\infty} |f(t)| \int_{-\infty}^{+\infty} |g(x-t)| \, dx \, dt \\
&= \int_{-\infty}^{+\infty} |f(t)| \int_{-\infty}^{+\infty} |g(y)| \, dy \, dt \\
&= \left(\int_{-\infty}^{+\infty} |f(t)| \, dt \right) \left(\int_{-\infty}^{+\infty} |g(y)| \, dy \right) \\
&= \|f\|_1 \|g\|_1.
\end{aligned}$$

Put $p = \infty$. In these conditions we can assure that

$$\begin{aligned}
|(f * g)(x)| &= \left| \int_{-\infty}^{+\infty} f(t)g(x-t) \, dt \right| \\
&\leq \int_{-\infty}^{+\infty} |f(t)| \, |g(x-t)| \, dt \\
&\leq \operatorname*{ess\,sup}_{y \in \mathbb{R}} |g(y)| \int_{-\infty}^{+\infty} |f(t)| \, dt \\
&= \|g\|_\infty \|f\|_1,
\end{aligned}$$

consequently we have

$$\|f * g\|_\infty \leq \|f\|_1 \|g\|_\infty.$$

Let $p, q \in]1, +\infty[$ be such that $p^{-1} + q^{-1} = 1$. Using Hölder's inequality [2, Theorem 7.9] it follows that

$$\begin{aligned}
|(f * g)(x)| &= \left| \int_{-\infty}^{+\infty} f(t)g(x-t) \, dt \right| \\
&\leq \int_{-\infty}^{+\infty} |f(t)| \, |g(x-t)| \, dt \\
&= \int_{-\infty}^{+\infty} |f(t)|^{\frac{1}{q}} \, |f(t)|^{\frac{1}{p}} \, |g(x-t)| \, dt \\
&\leq \left(\int_{-\infty}^{+\infty} |f(t)| \, dt \right)^{\frac{1}{q}} \left(\int_{-\infty}^{+\infty} |f(t)| \, |g(x-t)|^p \, dt \right)^{\frac{1}{p}} \\
&= \|f\|_1^{\frac{1}{q}} \left(\int_{-\infty}^{+\infty} |f(t)| \, |g(x-t)|^p \, dt \right)^{\frac{1}{p}}.
\end{aligned}$$

Applying the previous inequality and Fubini's Theorem [2, Theorem 5.32] we have

$$\begin{aligned}
\int_{-\infty}^{+\infty} |(f * g)(x)|^p \, dx &\leq \|f\|_1^{\frac{p}{q}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |f(t)| \, |g(x-t)|^p \, dt \, dx \\
&= \|f\|_1^{\frac{p}{q}} \int_{-\infty}^{+\infty} |f(t)| \int_{-\infty}^{+\infty} |g(x-t)|^p \, dx \, dt \\
&= \|f\|_1^{\frac{p}{q}} \int_{-\infty}^{+\infty} |f(t)| \int_{-\infty}^{+\infty} |g(y)|^p \, dy \, dt
\end{aligned}$$

$$
\begin{aligned}
&= \|f\|_1^{\frac{p}{q}} \left(\int_{-\infty}^{+\infty} |f(t)|\, dt\right)\left(\int_{-\infty}^{+\infty} |g(y)|^p\, dy\right)\\
&= \|f\|_1^{\frac{p}{q}} \|f\|_1 \|g\|_p^p\\
&= \|f\|_1^{\frac{p}{q}+1} \|g\|_p^p\\
&= \|f\|_1^p \|g\|_p^p
\end{aligned}
$$

that is, $\|f * g\|_p \leq \|f\|_1 \|g\|_p$, as we wanted to prove. □

Taking into account the definition of the Fourier transform and convolution, in the following theorem we can see that the Fourier transform of the convolution of two $L^1(\mathbb{R})$ functions, is equal to the product of their corresponding Fourier transforms.

Theorem 4.3.3 ([2, Theorem 11.66]). *If $f, g \in L^1(\mathbb{R})$, then*

$$\widehat{(f * g)}(t) = \widehat{f}(t)\widehat{g}(t),$$

for each $t \in \mathbb{R}$.

Proof. Let $t \in \mathbb{R}$. Applying Fubini's Theorem [2, Theorem 5.32] we can assure that

$$
\begin{aligned}
\widehat{(f * g)}(t) &= \int_{-\infty}^{+\infty} (f * g)(x) e^{-2\pi i t x}\, dx\\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(y) g(x-y) e^{-2\pi i t x}\, dy\, dx\\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(y) g(x-y) e^{-2\pi i t(x-y+y)}\, dx\, dy\\
&= \int_{-\infty}^{+\infty} f(y) e^{-2\pi i t y} \int_{-\infty}^{+\infty} g(x-y) e^{-2\pi i t(x-y)}\, dx\, dy\\
&= \int_{-\infty}^{+\infty} f(y) e^{-2\pi i t y} \int_{-\infty}^{+\infty} g(z) e^{-2\pi i t z}\, dz\, dy\\
&= \left(\int_{-\infty}^{+\infty} f(y) e^{-2\pi i t y}\, dy\right)\left(\int_{-\infty}^{+\infty} g(z) e^{-2\pi i t z}\, dz\right)\\
&= \widehat{f}(t)\widehat{g}(t).
\end{aligned}
$$

□

4.4 Plancherel's Theorem

We present now the definition of the Poisson kernel and the Poisson integral, which will play an important role in order to establish that the Fourier transform preserves $L^2(\mathbb{R})$ norms.

Definition 4.4.1. Let $y > 0$. We define the Poisson kernel $P_y : \mathbb{R} \to]0, +\infty[$ by

$$P_y(x) := \frac{y}{\pi(x^2 + y^2)}.$$

In these conditions we have

$$\int_{-\infty}^{+\infty} |P_y(x)|\, dx = \left[\pi^{-1} \arctan\left(\frac{x}{y}\right)\right]_{-\infty}^{+\infty} = 1,$$

therefore we can assure that $P_y \in L^1(\mathbb{R})$.

Definition 4.4.2. Let $f \in L^p(\mathbb{R})$ and $y > 0$, where $p \in [1, +\infty]$. We define the Poisson integral $\mathcal{P}_y f : \mathbb{R} \to \mathbb{C}$ by

$$(\mathcal{P}_y f)(t) := \int_{-\infty}^{+\infty} f(x) P_y(t-x)\, dx,$$

for any $t \in \mathbb{R}$. Analysing the definition of $\mathcal{P}_y f$ we can see that $\mathcal{P}_y f = f * P_y$.

In the following theorem, we are going to check that the Poisson integral of a function in $L^p(\mathbb{R})$ gives us a good approximation to that function.

Theorem 4.4.3 ([2, Theorem 11.74])**.** *If $p \in [1, +\infty[$ and $f \in L^p(\mathbb{R})$, then*

$$\lim_{y \downarrow 0} \|f - \mathcal{P}_y f\|_p = 0.$$

Proof. Let $x \in \mathbb{R}$ and $y > 0$. Applying Hölder's inequality [2, Theorem 7.9], using the definition of the Poisson integral and using the fact that

$$\int_{-\infty}^{+\infty} P_y(x)\, dx = 1,$$

we have that

$$\begin{aligned}
|f(x) - (\mathcal{P}_y f)(x)| &= \left| \int_{-\infty}^{+\infty} f(x) P_y(t)\, dt - \int_{-\infty}^{+\infty} f(x-t) P_y(t)\, dt \right| \\
&= \left| \int_{-\infty}^{+\infty} [f(x) - f(x-t)] P_y(t)\, dt \right| \\
&\leq \int_{-\infty}^{+\infty} |f(x) - f(x-t)| P_y(t)\, dt \\
&\leq \left(\int_{-\infty}^{+\infty} |f(x) - f(x-t)|^p P_y(t)\, dt \right)^{\frac{1}{p}} \left(\int_{-\infty}^{+\infty} 1^q P_y(t)\, dt \right)^{\frac{1}{q}} \\
&= \left(\int_{-\infty}^{+\infty} |f(x) - f(x-t)|^p P_y(t)\, dt \right)^{\frac{1}{p}},
\end{aligned}$$

where $p^{-1} + q^{-1} = 1$ and the second inequality comes from applying Hölder's inequality [2, Theorem 7.9] to the measure $d\mu = P_y(t)\, dt$ defined by

$$\mu(E) := \int_E P_y(t)\, dt$$

for any measurable set $E \subseteq \mathbb{R}$. Consider the function $g : \mathbb{R} \to [0, +\infty[$, defined by

$$g(t) = \int_{-\infty}^{+\infty} |f(x) - f(x-t)|^p\, dx$$

for each $t \in \mathbb{R}$. In these conditions we have

$$g(0) = \int_{-\infty}^{+\infty} |f(x) - f(x)|^p \, dx = 0.$$

Taking into account Theorem 4.1.6 and Lemma 4.1.7, we can guarantee that the function g is bounded and uniformly continuous on $\mathbb{R}$. Since P_y is an even function, due to the fact that

$$|f(x) - (\mathscr{P}_y f)(x)| \leq \left(\int_{-\infty}^{+\infty} |f(x) - f(x-t)|^p P_y(t) \, dt \right)^{\frac{1}{p}},$$

and using Fubini's Theorem [2, Theorem 5.32], it follows that

$$\begin{aligned} \|f - \mathscr{P}_y f\|_p^p &\leq \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |f(x) - f(x-t)|^p P_y(t) \, dt \, dx \\ &= \int_{-\infty}^{+\infty} P_y(t) \int_{-\infty}^{+\infty} |f(x) - f(x-t)|^p \, dx \, dt \\ &= \int_{-\infty}^{+\infty} P_y(-t) g(t) \, dt = (\mathscr{P}_y g)(0). \end{aligned} \tag{4.7}$$

Let $\epsilon > 0$. Since g is a uniformly continuous function on $\mathbb{R}$, there exists $\delta > 0$ such that for every $x_1, x_2 \in \mathbb{R}$,

$$|x_1 - x_2| < \delta \;\Rightarrow\; |g(x_1) - g(x_2)| < \epsilon.$$

Consider $z \in \mathbb{R}$. Analyzing the definition of the Poisson kernel and using the fact that

$$\int_{-\infty}^{+\infty} P_y(x) \, dx = 1,$$

we get that

$$\begin{aligned} |g(z) - (\mathscr{P}_y g)(z)| &= \left| g(z) - \int_{-\infty}^{+\infty} g(t) P_y(z-t) \, dt \right| \\ &= \left| \int_{-\infty}^{+\infty} g(z) P_y(z-t) \, dt - \int_{-\infty}^{+\infty} g(t) P_y(z-t) \, dt \right| \\ &\leq \int_{-\infty}^{+\infty} |g(z) - g(t)| P_y(z-t) \, dt \\ &= \int_{\{x \in \mathbb{R} : |z-x| < \delta\}} |g(z) - g(t)| P_y(z-t) \, dt \\ &\quad + \int_{\{x \in \mathbb{R} : |z-x| \geq \delta\}} |g(z) - g(t)| P_y(z-t) \, dt \\ &< \epsilon \int_{\{x \in \mathbb{R} : |z-x| < \delta\}} P_y(z-t) \, dt \\ &\quad + 2 \, \|g\|_\infty \int_{\{x \in \mathbb{R} : |z-x| \geq \delta\}} P_y(z-t) \, dt. \end{aligned}$$

In these conditions we have

$$\lim_{y \downarrow 0} \int_{\{x \in \mathbb{R} : |z-x| < \delta\}} P_y(z-t) \, dt = \lim_{y \downarrow 0} \int_{z-\delta}^{z+\delta} P_y(z-t) \, dt$$

$$
\begin{aligned}
&= \lim_{y \downarrow 0} \int_{-\delta}^{\delta} P_y(x)\, dx \\
&= \lim_{y \downarrow 0} \left[\frac{1}{\pi} \arctan\left(\frac{x}{y}\right) \right]_{-\delta}^{\delta} = 1
\end{aligned}
$$

and

$$
\begin{aligned}
\lim_{y \downarrow 0} \int_{\{x \in \mathbb{R} : |z-x| \geq \delta\}} P_y(z-t)\, dt &= \lim_{y \downarrow 0} \int_{-\infty}^{z-\delta} P_y(z-t)\, dt + \lim_{y \downarrow 0} \int_{z+\delta}^{+\infty} P_y(z-t)\, dt \\
&= \lim_{y \downarrow 0} \int_{\delta}^{+\infty} P_y(x)\, dx + \lim_{y \downarrow 0} \int_{-\infty}^{-\delta} P_y(x)\, dx \\
&= \lim_{y \downarrow 0} \left[\frac{1}{\pi} \arctan\left(\frac{x}{y}\right) \right]_{\delta}^{+\infty} + \lim_{y \downarrow 0} \left[\frac{1}{\pi} \arctan\left(\frac{x}{y}\right) \right]_{-\infty}^{-\delta} = 0.
\end{aligned}
$$

Using the previous statements, we have

$$
\lim_{y \downarrow 0} |g(z) - (\mathscr{P}_y g)(z)| < \epsilon,
$$

consequently, $\mathscr{P}_y g$ converges pointwise on $\mathbb{R}$ to the function g as $y \downarrow 0$. Hence we get

$$
\lim_{y \downarrow 0} (\mathscr{P}_y g)(0) = g(0),
$$

and due to the fact that $g(0) = 0$, we have $\lim_{y \downarrow 0} (\mathscr{P}_y g)(0) = 0$. Taking into account inequality (5.6), we get

$$
\lim_{y \downarrow 0} \|f - \mathscr{P}_y f\|_p^p \leq \lim_{y \downarrow 0} (\mathscr{P}_y g)(0) = 0,
$$

therefore

$$
\lim_{y \downarrow 0} \|f - \mathscr{P}_y f\|_p = 0
$$

as we wanted to prove. □

The following example is an immediate consequence of the Fourier transform and its algebraic properties, and will be used in the proof of Theorems 4.4.5 and 4.5.4.

Example 4.4.4. Let $f(x) = e^{-2\pi|x|}$ for every $x \in \mathbb{R}$. Given $t \in \mathbb{R}$, we have

$$
\begin{aligned}
\widehat{f}(t) &= \int_{-\infty}^{+\infty} e^{-2\pi|x|} e^{-2\pi i t x}\, dx \\
&= \int_{-\infty}^{0} e^{2\pi x - 2\pi i t x}\, dx + \int_{0}^{+\infty} e^{-2\pi x - 2\pi i t x}\, dx \\
&= \frac{1}{2\pi(1 - it)} + \frac{1}{2\pi(1 + it)} \\
&= \frac{1}{\pi(1 + t^2)}.
\end{aligned}
$$

Consider now $g_y(x) = e^{-2\pi y|x|}$ and $h_{y,z}(x) = e^{2\pi izx-2\pi y|x|}$ for each $x, z \in \mathbb{R}$ and $y > 0$. Applying the previous equality and Theorem 4.2.4, we get

$$\widehat{g_y}(t) = \frac{1}{y\pi\left(1+\left(\frac{t}{y}\right)^2\right)} = \frac{y}{\pi(t^2+y^2)},$$

consequently using Theorem 4.2.4 it follows that

$$\widehat{h_{y,z}}(t) = \frac{\pi^{-1}y}{y^2+(t-z)^2}.$$

The next result shows us that if we apply the Fourier transform to a function four times, then we go back to the original function.

Theorem 4.4.5 ([2, Theorem 11.76])**.** *Let $f \in L^1(\mathbb{R})$ be such that $\widehat{f} \in L^1(\mathbb{R})$. Then*

$$f(x) = \int_{-\infty}^{+\infty} \widehat{f}(t)e^{2\pi ixt}\,dt = (\widehat{\widehat{f}})(-x)$$

for almost every $x \in \mathbb{R}$.

Proof. Let $x \in \mathbb{R}$, $y > 0$ and $g_{x,y}(t) := e^{2\pi ixt-2\pi y|t|}$. Taking into account Example 4.4.4 and applying Theorem 4.2.5, we get

$$\begin{aligned}
(\mathscr{P}_y f)(x) &= \int_{-\infty}^{+\infty} f(t)P_y(x-t)\,dt \\
&= \int_{-\infty}^{+\infty} f(t)\frac{y\pi^{-1}}{(x-t)^2+y^2}\,dt \\
&= \int_{-\infty}^{+\infty} f(t)\widehat{g_{x,y}}(t)\,dt \\
&= \int_{-\infty}^{+\infty} \widehat{f}(t)g_{x,y}(t)\,dt \\
&= \int_{-\infty}^{+\infty} \widehat{f}(t)e^{2\pi ixt-2\pi y|t|}\,dt.
\end{aligned}$$

Due to the fact that $\widehat{f} \in L^1(\mathbb{R})$ and using the Dominated Convergence Theorem [2, Theorem 3.31], we have for every $x \in \mathbb{R}$,

$$\lim_{y\downarrow 0}\int_{-\infty}^{+\infty} \widehat{f}(t)e^{2\pi ixt-2\pi y|t|}\,dt = (\widehat{\widehat{f}})(-x).$$

Since $f \in L^1(\mathbb{R})$ and using Theorem 4.4.3, it follows that

$$\lim_{y\downarrow 0}\|f - \mathscr{P}_y f\|_1 = 0.$$

Taking into account the previous equality, we know from [2, Theorem 7.23] that there exists a sequence of positive numbers $(y_n)_{n\in\mathbb{N}}$, such that

$$\lim_{n\to\infty} y_n = 0, \quad \lim_{n\to\infty}(\mathscr{P}_{y_n} f)(x) = f(x),$$

for almost every $x \in \mathbb{R}$, consequently by the previous statements we get that

$$f(x) = \lim_{n\to\infty} (\mathscr{P}_{y_n} f)(x) = \lim_{y\downarrow 0} (\mathscr{P}_y f)(x) = \lim_{y\downarrow 0} \int_{-\infty}^{+\infty} \widehat{f}(t) e^{2\pi i x t - 2\pi y|t|}\, dt = \widehat{(\widehat{f})}(-x),$$

for almost any $x \in \mathbb{R}$ as we wanted to prove. □

The following example will be used in the proof of Theorem 4.4.7

Example 4.4.6. Let $g_y(x) = e^{-2\pi y|x|}$, for each $x \in \mathbb{R}$ and $y > 0$. Applying Theorem 4.4.5 and analysing Example 4.4.4, we have

$$\widehat{P_y}(x) = \widehat{(\widehat{g_y})}(x) = g_y(-x) = e^{-2\pi y|x|}.$$

We finish this section by establishing Plancherel's Theorem, which states that the Fourier transform preserves $L^2(\mathbb{R})$ norms.

Theorem 4.4.7 ([2, Theorem 11.82]). *If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then $\|\widehat{f}\|_2 = \|f\|_2$.*

Proof. Let us suppose that $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $\widehat{f} \in L^1(\mathbb{R})$. Consider the function $g : \mathbb{R} \to \mathbb{C}$ such that $g(x) = \overline{f(-x)}$. Applying Theorem 4.2.4, we can assure that

$$\widehat{g}(t) = \overline{\widehat{f}(t)}$$

for each $t \in \mathbb{R}$. Using Theorems 4.2.5 and 4.4.5, it follows that

$$\begin{aligned} \|f\|_2^2 &= \int_{-\infty}^{+\infty} f(x)\overline{f(x)}\, dx = \int_{-\infty}^{+\infty} f(-x)\overline{f(-x)}\, dx \\ &= \int_{-\infty}^{+\infty} \widehat{(\widehat{f})}(x) g(x)\, dx = \int_{-\infty}^{+\infty} \widehat{f}(x)\widehat{g}(x)\, dx \\ &= \int_{-\infty}^{+\infty} \widehat{f}(x)\overline{\widehat{f}(x)}\, dx = \|\widehat{f}\|_2^2, \end{aligned}$$

therefore the theorem is proved when $\widehat{f} \in L^1(\mathbb{R})$. Consider now an arbitrary function $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, $y > 0$ and $x \in \mathbb{R}$. Taking account Theorems 4.3.2 and 4.3.3 and Example 4.4.6, we can guarantee that $f * P_y \in L^1(\mathbb{R})$ and

$$\widehat{(f * P_y)}(x) = \widehat{f}(x)\widehat{(P_y)}(x) = \widehat{f}(x) e^{-2\pi y|x|}. \tag{4.8}$$

Applying Theorem 4.2.3, it follows that $\widehat{f} \in L^\infty(\mathbb{R})$ and also

$$\int_{-\infty}^{+\infty} |\widehat{f * P_y}(x)|\, dx = \int_{-\infty}^{+\infty} |\widehat{f}(x) e^{-2\pi y|x|}|\, dx \le \|\widehat{f}\|_\infty \int_{-\infty}^{+\infty} e^{-2\pi y|x|}\, dx < \infty,$$

which implies that $\widehat{f * P_y} \in L^1(\mathbb{R})$. Applying Theorem 4.3.2 we get that $P_y * f = f * P_y \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and since $\widehat{f * P_y} \in L^1(\mathbb{R})$, using the first case it follows that

$$\|f * P_y\|_2 = \|\widehat{f * P_y}\|_2.$$

Applying Theorem 4.4.3, we can assure that

$$\lim_{y\downarrow 0}\|f - f * P_y\|_2 = 0,$$

which implies that

$$\lim_{y\downarrow 0}\|f * P_y\|_2 = \|f\|_2.$$

Taking into account equation (4.8) and the Monotone Convergence Theorem [2, Theorem 3.11], we can conclude that

$$\lim_{y\downarrow 0}\|\widehat{f * P_y}\|_2 = \|\widehat{f}\|_2,$$

that is, $\|f\|_2 = \|\widehat{f}\|_2$, as we wanted to prove. □

4.5 Fourier Transform on L^2

Applying Corollary 4.1.5, we get that the space $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is dense on $L^2(\mathbb{R})$. Taking into account the previous theorem, we can extend by continuity the map $f \mapsto \widehat{f}$ uniquely to a bounded linear map from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$, which we will define as shown below.

Definition 4.5.1. If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then we define the Fourier transform of f by the bounded operator $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ such that

$$\mathcal{F}f := \widehat{f}.$$

If $f \in L^2(\mathbb{R}) \setminus L^1(\mathbb{R})$, then we define the Fourier transform of f by

$$\mathcal{F}f := \lim_{n\to+\infty} \widehat{f_n},$$

where $(f_n)_{n\in\mathbb{N}}$ is a sequence in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ such that $\lim_{n\to+\infty}\|f - f_n\|_2 = 0$.

Definition 4.5.2. Let X be a Hilbert space and $T : X \to X$ be a linear bounded transformation. We say that T is a unitary operator if and only if

$$TT^* = T^*T = I,$$

where $T^* : X \to X$ is the only operator that satisfies $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for each $x, y \in X$.

Now we will recall a well-known result of Functional Analysis, which states that in a Hilbert space, a linear bounded operator is unitary if and only if it is a surjective isometry.

Theorem 4.5.3 ([2, Theorem 10.61]). *Let X be a Hilbert space and $T : X \to X$ be a linear bounded operator. Then T is unitary if and only if T is a surjective isometry.*

Proof. Suppose that T is unitary. Given $x \in X$ we have

$$\|Tx\|^2 = \langle Tx, Tx\rangle = \langle x, T^*Tx\rangle = \langle x, x\rangle = \|x\|^2,$$

consequently we can assure that T is an isometry. Let $y \in X$. In these conditions there exists $x = T^*y \in X$ such that

$$Tx = TT^*y = y,$$

that is, T is surjective.

Let us suppose now that T is a surjective isometry. Taking into account that T is an isometry, it follows that T is injective and thus T is a bounded bijective linear map. Consequently, applying the Bounded Inverse Theorem [2, Theorem 6.83], it follows that T is an invertible bounded linear operator. For every $x \in X$, we have

$$\|Tx\|^2 - \|x\|^2 = \langle Tx, Tx\rangle - \langle x, x\rangle = \langle (T^*T - I)x, x\rangle$$

and since T is an isometry, it follows that

$$\langle (T^*T - I)x, x\rangle = 0$$

for each $x \in X$. We know from [2, Theorem 10.46] that $\langle (T^*T - I)x, x\rangle = 0$ for any $x \in X$ if and only if $T^*T - I = 0$, that is, $T^*T = I$. Taking into account the uniqueness of T^{-1}, $T^{-1}T = TT^{-1} = I$ and $T^*T = I$, we get that $T^{-1} = T^*$ and hence T is a unitary operator as we wanted to prove. □

With the help of the previous theorem, we finish this chapter by presenting and proving that the Fourier transform on $L^2(\mathbb{R})$ is an isometry. Moreover, we also check that the Fourier transform on $L^2(\mathbb{R})$ is a unitary operator and that, applying the Fourier transform four times, we get the identity operator on $L^2(\mathbb{R})$.

Theorem 4.5.4 ([2, Theorem 11.87]). *If $\mathcal{F}$ is the Fourier transform on $L^2(\mathbb{R})$, then the following properties hold.*

1. *$\mathcal{F}$ is an isometry on $L^2(\mathbb{R})$.*

2. *$\mathcal{F}^4 = I$.*

3. *$\mathcal{F}$ is a unitary operator on $L^2(\mathbb{R})$.*

Proof. Let $f \in L^2(\mathbb{R})$. Since $\mathcal{F}$ is obtained by continuously extending, in the norm of $L^2(\mathbb{R})$, the Fourier transform from $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ to $L^2(\mathbb{R})$, it follows by Theorem 4.4.7 that

$$\|\mathcal{F}f\|_2 = \|f\|_2,$$

that is, $\mathcal{F}$ is an isometry on $L^2(\mathbb{R})$. Consider now an arbitrary function $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and let $y > 0$. In these conditions we know, by definition, that the function $P_y \in L^1(\mathbb{R})$, hence applying Theorem 4.3.2 it follows that

$$P_y * f = f * P_y \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}).$$

Due to the fact that $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, applying Theorems 4.2.3 and 4.4.7 we have that $\widehat{f} \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$, and observing the definition of the Poisson kernel we know that

$P_y \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Consequently using Theorem 4.2.3 and analyzing Example 4.4.4, we guarantee that $\widehat{P_y} \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ and thus, using Theorem 4.3.3, we can assure that

$$\widehat{f * P_y} = \widehat{f}\,\widehat{P_y},$$

therefore applying Hölder's inequality [2, Theorem 7.9], we have

$$\widehat{f * P_y} = \widehat{f}\widehat{P_y} \in L^1(\mathbb{R}).$$

Due to the fact that $f * P_y \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, it follows, by Theorem 4.4.7, that $\widehat{f * P_y} \in L^2(\mathbb{R})$ and thus we get

$$\widehat{f * P_y} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}).$$

Since $f * P_y, \widehat{f * P_y} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, using Theorem 4.4.5 we have that

$$\mathcal{F}^4(f * P_y) = f * P_y,$$

thus if we take the limit in $L^2(\mathbb{R})$ when $y \downarrow 0$ in both sides of the previous equation and observe that $\mathcal{P}_y f = f * P_y$, then applying Theorem 4.4.3 we get $\mathcal{F}^4 f = f$, that is, $\mathcal{F}^4 = I$. Let $f_2 \in L^2(\mathbb{R})$. Then there exists $f_1 = \mathcal{F}^3 f_2 \in L^2(\mathbb{R})$ such that

$$f_2 = \mathcal{F} f_1,$$

that is, $\mathcal{F}$ is a surjective operator. Taking account that $\mathcal{F}$ is an isometry, bounded and surjective, we can conclude, by the previous theorem that $\mathcal{F}$ is a unitary operator on $L^2(\mathbb{R})$ as we wanted to prove. □

5

Banach Algebras of Almost Periodic Fourier Multipliers

In the final chapter, we start by presenting several definitions and theorems regarding Banach algebras, maximal ideals and multiplicative linear functionals. After that, we will define the algebra $AP_p(\mathbb{R})$ as the closure of $APP(\mathbb{R})$ in the norm of the set of Fourier multipliers, which are functions that belong to $L^\infty(\mathbb{R})$ and satisfy certain properties. Moreover, we also verify that $AP_p(\mathbb{R})$ is embedded densely in $AP(\mathbb{R})$. Following that, we define the algebra $APW(\mathbb{R})$ as the set of all trigonometric convergent series and we prove that $APW(\mathbb{R})$ is embedded densely not only in $AP_p(\mathbb{R})$, but also in $AP(\mathbb{R})$. In fact, we will also see that the Banach algebra $l^1(\mathbb{R})$ is isometrically isomorphic to $APW(\mathbb{R})$ and, with that being done, we prove that the Gelfand space of $APW(\mathbb{R})$ is homeomorphic to the Gelfand space of $AP(\mathbb{R})$. We finish this work by establishing that the algebra $AP_p(\mathbb{R})$ is inverse-closed in $AP(\mathbb{R})$.

5.1 Basic Definitions, Banach Algebras and C*-Algebras

In this chapter, we will always consider non-null algebras over $\mathbb{C}$.

Definition 5.1.1. Let D be a non empty set. We say that D is a directed set if and only if D is a set with a partial order relation, $\leq$, such that for each $x, y \in D$ there exists $z \in D$ that verifies

$$x \leq z, \quad y \leq z.$$

Definition 5.1.2. Let (X, τ) be a topological space and D a directed set. A net on X is a function defined by

$$x : D \to X, \quad \alpha \mapsto x(\alpha) := x_\alpha.$$

Definition 5.1.3. Let (X, τ) be a topological space and let (x_α) be a net on X, defined on a directed set D. We say that (x_α) converges to $x \in X$, and we denote it by $x_\alpha \underset{\alpha}{\to} x$, if and only if for each neighborhood V of the element x there exists $p \in D$ such that

$$\alpha \geq p \Rightarrow x_\alpha \in V.$$

Definition 5.1.4. Let $\mathcal{A}$ be a vector space over $\mathbb{C}$. We say that $\mathcal{A}$ is an algebra if and only if there exists a binary operation $\bullet\ \mathcal{A}\times\mathcal{A}\to\mathcal{A}$ such that for every $x,y,z\in\mathcal{A}$ and for each $\lambda\in\mathbb{C}$,

1. $x\bullet(y\bullet z)=(x\bullet y)\bullet z$;
2. $(\lambda x)\bullet y = x\bullet(\lambda y)=\lambda(x\bullet y)$;
3. $x\bullet(y+z)=x\bullet y+x\bullet z$;
4. $(x+y)\bullet z = x\bullet z+y\bullet z$.

In this work we will denote $x\bullet y = xy$ for any $x,y\in\mathcal{A}$ and we will always consider that $\mathcal{A}\neq\{0\}$. We say that an algebra is commutative if it satisfies

$$xy = yx$$

for each $x,y\in\mathcal{A}$. Moreover, we say that $\mathcal{A}$ is a unital algebra (or an algebra with unit) if there exists an element $e\in\mathcal{A}$ such that

$$ea = ae = a$$

for every $a\in\mathcal{A}$, and an element $u\in\mathcal{A}$ is invertible in $\mathcal{A}$ if there is $v\in\mathcal{A}$ such that

$$uv = vu = e.$$

Definition 5.1.5. Let $\mathcal{A}$ be an algebra with unit and let $\mathcal{B}$ be a subalgebra of $\mathcal{A}$. We say that $\mathcal{B}$ is a unital subalgebra of $\mathcal{A}$ if the unit of $\mathcal{A}$ belongs to $\mathcal{B}$.

Definition 5.1.6. Let $\mathcal{A}$ be an algebra with unit and $\mathcal{B}$ a unital subalgebra of $\mathcal{A}$. We say that $\mathcal{B}$ is inverse-closed in $\mathcal{A}$ if and only if every element of $\mathcal{B}$ which is invertible in $\mathcal{A}$, is also invertible in $\mathcal{B}$.

Definition 5.1.7. Let $\mathcal{A}$ be an algebra over $\mathbb{C}$. We say that $\mathcal{A}$ is a normed algebra if and only if there exists a norm $\|\cdot\|:\mathcal{A}\to\mathbb{R}_0^+$ that verifies for every $x,y\in\mathcal{A}$,

$$\|xy\|\le\|x\|\,\|y\|.$$

Definition 5.1.8. Let $\mathcal{A}$ be a normed algebra over $\mathbb{C}$. We say that $\mathcal{A}$ is a Banach algebra if and only if $\mathcal{A}$ is a complete space, that is, a space where all Cauchy sequences are convergent.

The following theorem is a well known fact from Functional Analysis, and it will be useful later in this work.

Theorem 5.1.9 ([3, Theorem 1.2.1])**.** *Let $\mathcal{A}$ be a unital Banach algebra with unit e and let $x\in\mathcal{A}$. If $\|x\|<1$, then $e-x$ is an invertible element and we have*

$$(e-x)^{-1}=\sum_{n=0}^{\infty}x^n.$$

Proof. Let $x \in \mathscr{A}$ be such that $\|x\| < 1$. Since $\mathscr{A}$ is a Banach algebra, it follows that $\|x^n\| \leq \|x\|^n$ for each $n \in \mathbb{N}$. Due to the fact that $\|x\| < 1$ we get that the series

$$\sum_{n=0}^{\infty} \|x\|^n$$

is convergent in $\mathbb{R}$. Hence, using the fact that $\|x^n\| \leq \|x\|^n$, we get that the series

$$\sum_{n=0}^{\infty} \|x^n\|$$

is also convergent in $\mathbb{R}$, which implies that

$$\sum_{n=0}^{\infty} x^n$$

is absolutely convergent in $\mathscr{A}$. Taking into account that $\mathscr{A}$ is a Banach space and the fact that the series

$$\sum_{n=0}^{\infty} x^n$$

is absolutely convergent, we can guarantee that this series is convergent to some element $s \in \mathscr{A}$. Consider the sequence of partial sums $(s_n)_{n\in\mathbb{N}}$ defined by

$$s_n := \sum_{k=0}^{n} x^k.$$

Let $n \in \mathbb{N}$. Then

$$\|(e-x)s_n - e\| = \|s_n - xs_n - e\| = \|-x^{n+1}\| \leq \|x\|^{n+1} \underset{n\to\infty}{\longrightarrow} 0$$

and also

$$\|s_n(e-x) - e\| = \|s_n - s_n x - e\| = \|-x^{n+1}\| \leq \|x\|^{n+1} \underset{n\to\infty}{\longrightarrow} 0.$$

Therefore the equality

$$s(e-x) = (e-x)s = e$$

holds, consequently $e - x$ is invertible and its inverse is given by $(e-x)^{-1} = s$. □

Definition 5.1.10. Let $\mathscr{A}$ be a Banach algebra. We say that $\mathscr{A}$ is a $*$-algebra if and only if there exists a function $^* : \mathscr{A} \to \mathscr{A}$, which we call involution, such that for any $x, y \in \mathscr{A}$ and for each $\lambda \in \mathbb{C}$,

1. $(x+y)^* = x^* + y^*$;
2. $(\lambda x)^* = \overline{\lambda} x^*$;
3. $(x^*)^* = x$;
4. $(xy)^* = y^* x^*$.

Moreover, if $\mathscr{A}$ satisfies the C^*-property, that is,

$$\|a^* a\| = \|a\|^2$$

for each $a \in \mathscr{A}$, we say that $\mathscr{A}$ is a C^*-algebra.

5.2 Ideals and Invertibility

In this section we will recall some important properties of maximal ideals and their relations with invertible elements of a Banach algebra.

Definition 5.2.1. Let $\mathscr{A}$ be an algebra and $\mathcal{I}$ a subalgebra of $\mathscr{A}$. We say that $\mathcal{I}$ is an ideal of $\mathscr{A}$ if and only if

$$ia \in \mathcal{I}, \quad ai \in \mathcal{I},$$

for every $i \in \mathcal{I}$ and for each $a \in \mathscr{A}$.

Definition 5.2.2. Let $\mathscr{A}$ be an algebra and $\mathcal{I}$ an ideal of $\mathscr{A}$. We say that $\mathcal{I}$ is a proper ideal of $\mathscr{A}$ if and only if $\mathcal{I} \neq \mathscr{A}$.

Definition 5.2.3. Let $\mathscr{A}$ be an algebra and $\mathcal{I}$ an ideal of $\mathscr{A}$. We say that $\mathcal{I}$ is a maximal ideal of $\mathscr{A}$ if there is no proper ideal $\mathcal{K}$ of $\mathscr{A}$ such that $\mathcal{I} \subsetneq \mathcal{K}$.

In the following theorem we will see that invertible elements do not belong to proper ideals.

Theorem 5.2.4 ([3, Theorem 1.3.2]). *Let $\mathscr{A}$ be a unital commutative Banach algebra and $\mathcal{I}$ a proper ideal of $\mathscr{A}$. If an element is invertible in $\mathscr{A}$ then that element does not belong to $\mathcal{I}$.*

Proof. Suppose that $x \in \mathscr{A}$ is invertible and suppose, by contradiction, that $x \in \mathcal{I}$. Since $\mathcal{I}$ is an ideal, we can assure that $x^{-1}x = e \in \mathcal{I}$ and consequently $\mathcal{I} = \mathscr{A}$ which is impossible because $\mathcal{I}$ is a proper ideal. □

In the next result we will prove that if an element does not belong to a maximal ideal, then it must be invertible.

Theorem 5.2.5 ([14, Theorem 1.3.2]). *Let $\mathscr{A}$ be a unital commutative Banach algebra and $x \in \mathscr{A}$. If there is no maximal ideal $\mathcal{I}$ of the algebra $\mathscr{A}$ such that $x \in \mathcal{I}$, then x is invertible.*

Proof. Let us suppose that x is not invertible and consider the subalgebra $\mathcal{I}$ of $\mathscr{A}$ defined by

$$\mathcal{I} := \mathscr{A}x = \{ax : a \in \mathscr{A}\}.$$

In these conditions $\mathcal{I}$ is an ideal of the algebra $\mathscr{A}$ because if $i \in \mathcal{I}$ and $y \in \mathscr{A}$, then there exists $a \in \mathscr{A}$ such that $i = ax$ and thus

$$iy = (ax)y = (ay)x \in \mathcal{I}, \quad yi = y(ax) = (ya)x \in \mathcal{I}.$$

Let e be the unit of $\mathscr{A}$. Since $x = ex$, it follows that $x \in \mathcal{I}$. The ideal $\mathcal{I}$ must be a proper ideal because otherwise we would have $\mathcal{I} = \mathscr{A}$, which is equivalent to say that $e \in \mathcal{I}$, therefore we would have $e = bx$ for some $b \in \mathscr{A}$ which is impossible because we are assuming that the element x is not invertible. Hence $\mathcal{I}$ is a proper ideal that contains x. Taking into account Krull's Lemma [3, Proposition 1.3.1], we can guarantee that every proper ideal is contained in some maximal ideal and we conclude that x belongs to some maximal ideal $\mathcal{K}$ as we wanted to prove. □

We finish this section by proving that any maximal ideal of a unital commutative Banach algebra is closed.

Theorem 5.2.6 ([3, Theorem 1.3.2]). *Let $\mathscr{A}$ be a unital commutative Banach algebra and let $\mathcal{J}$ be a maximal ideal of the algebra $\mathscr{A}$. Then* $\text{clos}_{\mathscr{A}}(\mathcal{J})$ *is an ideal of $\mathscr{A}$ and $\mathcal{J}$ is closed in $\mathscr{A}$.*

Proof. Let $\mathscr{A}$ be a unital commutative Banach algebra and let $\mathcal{J}$ be a maximal ideal of the algebra $\mathscr{A}$. Consider $a \in \mathscr{A}$ and $x \in \text{clos}_{\mathscr{A}}(\mathcal{J})$. In these conditions there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of terms in $\mathcal{J}$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$, consequently, we have

$$\lim_{n\to\infty} \|ax - ax_n\| \leq \lim_{n\to\infty} \|a\| \, \|x_n - x\| = 0.$$

Therefore $(ax_n)_{n\in\mathbb{N}}$ is a sequence of elements in $\mathcal{J}$, because $\mathcal{J}$ is an ideal of the algebra $\mathscr{A}$, and we conclude that $ax = xa \in \text{clos}_{\mathscr{A}}(\mathcal{J})$, that is, $\text{clos}_{\mathscr{A}}(\mathcal{J})$ is an ideal of $\mathscr{A}$ that contains $\mathcal{J}$.

Since $\mathcal{J}$ is a maximal ideal of $\mathscr{A}$ and $\mathcal{J} \subseteq \text{clos}_{\mathscr{A}}(\mathcal{J})$, it follows that $\text{clos}_{\mathscr{A}}(\mathcal{J}) = \mathcal{J}$ or $\text{clos}_{\mathscr{A}}(\mathcal{J}) = \mathscr{A}$. Suppose, by contradiction, that $\text{clos}_{\mathscr{A}}(\mathcal{J}) = \mathscr{A}$. It is known from [3, Theorem 1.2.3] that, in a unital Banach algebra, the set of invertible elements is an open set. Taking into account that $e \in \mathscr{A} = \text{clos}_{\mathscr{A}}(\mathcal{J})$, we can assure that there exists a sequence $(y_n)_{n\in\mathbb{N}}$ of elements in $\mathcal{J}$ such that

$$\lim_{n\to\infty} \|y_n - e\| = 0.$$

Using the previous equation and the fact that the set of invertible elements is open, we can guarantee that there exists $p \in \mathbb{N}$ such that if $n > p$, then $y_n \in G(\mathscr{A})$ where $G(\mathscr{A})$ is the set of invertible elements of the algebra $\mathscr{A}$. But we know that if an element is invertible, then it cannot belong to a maximal ideal, thus $y_{p+1} \in \mathcal{J} \cap G(\mathscr{A})$ is a contradiction and we conclude that $\text{clos}_{\mathscr{A}}(\mathcal{J}) = \mathcal{J}$. □

5.3 Multiplicative Linear Functionals

In this section we will start by proving important properties of multiplicative linear functionals. Moreover, we will also study Gelfand's theory regarding the relation between these functionals and maximal ideals.

Definition 5.3.1. Let $\mathscr{A}$ be an algebra and $\varphi : \mathscr{A} \to \mathbb{C}$ a function defined on $\mathscr{A}$. We say that φ is a multiplicative linear functional if and only if

- $\varphi(a+b) = \varphi(a) + \varphi(b)$,
- $\varphi(\lambda a) = \lambda\varphi(a)$,
- $\varphi(ab) = \varphi(a)\varphi(b)$,

for each $a, b \in \mathscr{A}$ and $\lambda \in \mathbb{C}$. We will denote by $\mathfrak{M}(\mathscr{A})$ the set of all non-null multiplicative linear functionals of the algebra $\mathscr{A}$.

In the following theorem we will analyse some important properties of multiplicative linear functionals.

Theorem 5.3.2 ([3, Theorem 1.4.4]). *Let $\mathcal{A}$ be a unital Banach algebra with unit e and let $\varphi \in \mathfrak{M}(\mathcal{A})$. Then the following properties hold:*

1. $\varphi(e) = 1$;

2. φ *is bounded and* $\|\varphi\| = 1$.

Proof. Since φ is a non-null multiplicative linear functional, there is $x \in \mathcal{A}$ such that $\varphi(x) \neq 0$. Then

$$\varphi(x) = \varphi(xe) = \varphi(x)\varphi(e) \Rightarrow \varphi(e) = 1.$$

Suppose, by contradiction, that there exists $a \in \mathcal{A}$ such that $\|a\| = 1$ and $|\varphi(a)| > 1$. In these conditions we can assure, applying Theorem 5.1.9, that the element $e - \frac{1}{\varphi(a)}a$ is invertible in $\mathcal{A}$ because $\mathcal{A}$ is a Banach algebra and $\left\|\frac{1}{\varphi(a)}a\right\| < 1$, consequently the element

$$\varphi(a)\left(e - \frac{1}{\varphi(a)}a\right) = \varphi(a)e - a$$

is also an invertible element. Taking into account that $\varphi(\varphi(a)e - a) = 0$, it follows that

$$1 = \varphi(e) = \varphi((\varphi(a)e - a)(\varphi(a)e - a)^{-1}) = \varphi(\varphi(a)e - a)\varphi((\varphi(a)e - a)^{-1}) = 0,$$

which is impossible. Hence we have that

$$\|\varphi\| = \sup_{\|a\|=1} |\varphi(a)| \leq 1,$$

and due to the fact that $\varphi(e) = 1$, we conclude that $\|\varphi\| = 1$ as we wanted to prove. □

Theorem 5.3.3 ([3, Theorem 1.4.8]). *Let $\mathcal{A}$ be a unital commutative Banach algebra and let $a \in \mathcal{A}$. Then a is invertible in $\mathcal{A}$ if and only if $\varphi(a) \neq 0$, for each $\varphi \in \mathfrak{M}(\mathcal{A})$.*

Proof. Let a be an invertible element in $\mathcal{A}$ and let $\varphi \in \mathfrak{M}(\mathcal{A})$. Then

$$e = aa^{-1} \Rightarrow 1 = \varphi(a)\varphi(a^{-1}),$$

which implies that $\varphi(a) \neq 0$. For the proof of the sufficiency part, we refer to [3, Theorem 1.4.8] or any other book on the theory of Banach algebras. □

Definition 5.3.4. Let $\mathcal{A}$ be a unital commutative Banach algebra and $a \in \mathcal{A}$. We define the function

$$\Gamma_a : \mathfrak{M}(\mathcal{A}) \to \mathbb{C}, \quad \varphi \mapsto \varphi(a)$$

as the Gelfand transform of the element a.

The following theorem is an important theorem regarding Gelfand's theory, which we will not prove in this work.

Theorem 5.3.5 ([3, Theorem 1.4.6])**.** *Let $\mathscr{A}$ be a unital commutative Banach algebra. Then $\mathcal{J}$ is a maximal ideal of $\mathscr{A}$ if and only if there exists a non-null multiplicative linear functional φ, defined on $\mathscr{A}$, that verifies*

$$\ker(\varphi) = \mathcal{J}.$$

Taking into account the previous theorem, there is a unique correspondence between the non-null multiplicative linear functionals and the maximal ideals of a unital commutative Banach algebra. Consequently, we will also denote by $\mathfrak{M}(\mathscr{A})$ the set of all maximal ideals in $\mathscr{A}$.

Definition 5.3.6. We will equip the space $\mathfrak{M}(\mathscr{A})$ with the Gelfand topology, that is, the topology given by arbitrary unions of finite intersections of sets of the form

$$\{\Gamma_a^{-1}(U) : U \text{ is open in } \mathbb{C}\}.$$

In these conditions $\mathfrak{M}(\mathscr{A})$ is a subset of $\mathscr{A}'$, where $\mathscr{A}'$ denotes de dual space of $\mathscr{A}$, and the Gelfand topology coincides with the weak-* topology of $\mathscr{A}'$, that is, the smallest topology that makes continuous every function of the form

$$f_a : \mathscr{A}' \to \mathbb{C}, \quad \varphi \mapsto \varphi(a),$$

where $a \in \mathscr{A}$. It is known from [13, Proposition 5.2.1] that if $\mathscr{A}$ is a Banach space then $\mathscr{A}'$ is a Hausdorff space under the weak-* topology. As basis of neighborhoods of a functional $\varphi_0 \in \mathscr{A}'$ we have the family of open sets

$$U(\varphi_0, \epsilon, a_1, \dots, a_n) := \{\varphi \in \mathscr{A}' : |\varphi(a_i) - \varphi_0(a_i)| < \epsilon,\ i \in \{1, \dots, n\}\},$$

where $\epsilon > 0$, $n \in \mathbb{N}$ and $a_1, \dots, a_n \in \mathscr{A}$.

Definition 5.3.7. Given a net (φ_α) in $\mathscr{A}'$, we say that (φ_α) converges weakly-* to $\varphi \in \mathscr{A}'$, and we write $\varphi_\alpha \underset{\alpha(w*)}{\to} \varphi$, if and only if $\varphi_\alpha(a) \underset{\alpha}{\to} \varphi(a)$ for each $a \in \mathscr{A}$.

In the following result we will prove that the Gelfand space $\mathfrak{M}(\mathscr{A})$ is a compact Hausdorff space under the weak-* topology.

Theorem 5.3.8 ([3, Theorem 2.1.3])**.** *Let $\mathscr{A}$ be a unital commutative Banach algebra. Then $\mathfrak{M}(\mathscr{A})$ is a compact Hausdorff space under the weak-* topology.*

Proof. We know from Alaoglu's Theorem [15, Theorem 1.4] that the closed ball

$$B(\mathscr{A}) = \{\varphi \in \mathscr{A}' : \|\varphi\| \leq 1\}$$

is a compact space under the weak-* topology. Taking in account the definition of $\mathfrak{M}(\mathscr{A})$ and the fact that any non-null multiplicative linear functional has norm equal to 1, we can assure that $\mathfrak{M}(\mathscr{A}) \subseteq B(\mathscr{A})$. Let $(\varphi_\alpha)_{\alpha \in D}$, where D is a directed set, be a net of elements in $\mathfrak{M}(\mathscr{A})$ such that that $\varphi_\alpha \underset{\alpha(w*)}{\to} \varphi \in B(\mathscr{A})$ and let $x, y \in \mathscr{A}$. Then

$$\varphi(xy) = \lim_\alpha \varphi_\alpha(xy) = \lim_\alpha \varphi_\alpha(x) \lim_\alpha \varphi_\alpha(y) = \varphi(x)\varphi(y),$$

consequently φ is a multiplicative linear functional in $\mathscr{A}$. Since $(\varphi_\alpha)_{\alpha\in D}$ is a net of elements in $\mathfrak{M}(\mathscr{A})$ such that $\varphi_\alpha(e) = 1$ for each $\alpha \in \mathscr{A}$, it follows that $\varphi(e) = 1$ and thus φ is a non-null multiplicative linear functional, that is, $\varphi \in \mathfrak{M}(\mathscr{A})$. Hence $\mathfrak{M}(\mathscr{A})$ is a closed subspace of $B(\mathscr{A})$ and due to the fact that $\mathfrak{M}(\mathscr{A})$ is a closed subspace of a compact Hausdorff space, we can guarantee that $\mathfrak{M}(\mathscr{A})$ is a compact Hausdorff space as we wanted to prove. □

Definition 5.3.9. Let $\mathscr{A}$ be a Banach algebra and $E \subseteq \mathscr{A}$. We denote by $\mathrm{alg}_{\mathscr{A}}(E)$ the smallest closed subalgebra of $\mathscr{A}$ that contains E, that is,

$$\mathrm{alg}_{\mathscr{A}}(E) := \mathrm{clos}_{\mathscr{A}} \left\{ \sum_{j=1}^{m} \lambda_j \prod_{k=1}^{n_j} x_{j,k} : m, n_1, \dots, n_m \in \mathbb{N},\ \lambda_j \in \mathbb{C},\ x_{j,k} \in E \right\}.$$

Moreover, we will call polynomials in elements $x_{1,1}, \dots, x_{1,n_1}, \dots, x_{m,1}, \dots, x_{m,n_m} \in E$ to the elements of the form

$$P(x_{1,1}, \dots, x_{1,n_1}, \dots, x_{m,1}, \dots, x_{m,n_m}) := \sum_{j=1}^{m} \lambda_j \prod_{k=1}^{n_j} x_{j,k}$$

where $m, n_1, \dots, n_m \in \mathbb{N}$ and $\lambda_1, \dots, \lambda_m \in \mathbb{C}$.

Definition 5.3.10. Let $\mathscr{A}$ be a unital Banach algebra with unit e. We say that $K \subseteq \mathscr{A}$ is a system of generators of $\mathscr{A}$ if and only if $e \notin K$ and

$$\mathscr{A} = \mathrm{alg}_{\mathscr{A}}(K \cup \{e\}).$$

We finish this section by proving that, in fact, the Gelfand topology can be defined in terms of a system of generators.

Theorem 5.3.11 ([7, Chapter 1, Section 5, Theorem 3])**.** *Let $\mathscr{A}$ be a unital Banach algebra with unit e, let K be a system of generators of $\mathscr{A}$ and let $\varphi_0 \in \mathfrak{M}(\mathscr{A})$. Then the sets of the form*

$$U(\varphi_0, \epsilon, x_1, \dots, x_m) := \{\varphi \in \mathfrak{M}(\mathscr{A}) : |\varphi(x_i) - \varphi_0(x_i)| < \epsilon,\ i \in \{1, \dots, m\}\} \tag{5.1}$$

where $\epsilon > 0$, $m \in \mathbb{N}$ and $x_1, \dots, x_m \in K$, form a basis of neighbourhoods of the element φ_0.

Proof. Let $\epsilon > 0$, let $n \in \mathbb{N}$ and let $y_1, \dots, y_n \in \mathscr{A}$. We just need to prove that there exists a neighbourhood of the form (5.1) contained in the neighbourhood

$$U(\varphi_0, \epsilon, y_1, \dots, y_n) := \{\varphi \in \mathfrak{M}(\mathscr{A}) : |\varphi(y_i) - \varphi_0(y_i)| < \epsilon,\ i \in \{1, \dots, n\}\}.$$

Since K is a system of generators of $\mathscr{A}$, it follows that there exist polynomials in elements in K, $P_i(x_{1,1,i}, \dots, x_{1,n_{1,i},i}, \dots, x_{m_i,1,i}, \dots, x_{m_i,n_{m_i,i},i})$, with $i \in \{1, \dots, n\}$, that verify

$$\|P_i - y_i\| < \frac{\epsilon}{3}, \quad P_i := P_i(x_{1,1,i}, \dots, x_{1,n_{1,i},i}, \dots, x_{m_i,1,i}, \dots, x_{m_i,n_{m_i,i},i}) := \sum_{j=1}^{m_i} \lambda_{j,i} \prod_{k=1}^{n_{j,i}} x_{j,k,i}$$

for each $i \in \{1,\dots,n\}$. Given $\varphi \in \mathfrak{M}(\mathscr{A})$ and $x,y,z \in \mathscr{A}$, using the fact that $|\varphi(a)| \le \|a\|$ for each $a \in \mathscr{A}$, we have

$$\begin{aligned}|\varphi(xyz)-\varphi_0(xyz)| &\le |\varphi(x)\varphi(y)\varphi(z)-\varphi(x)\varphi(y)\varphi_0(z)| + |\varphi(x)\varphi(y)\varphi_0(z)-\varphi(x)\varphi_0(y)\varphi_0(z)| \\ &\quad + |\varphi(x)\varphi_0(y)\varphi_0(z)-\varphi_0(x)\varphi_0(y)\varphi_0(z)| \\ &= |\varphi_0(yz)|\,|\varphi(x)-\varphi_0(x)| + |\varphi(x)\varphi_0(z)|\,|\varphi(y)-\varphi_0(y)| \\ &\quad + |\varphi(xy)|\,|\varphi(z)-\varphi_0(z)| \\ &\le \|y\|\,\|z\|\,|\varphi(x)-\varphi_0(x)| + \|x\|\,\|z\|\,|\varphi(y)-\varphi_0(y)| + \|x\|\,\|y\|\,|\varphi(z)-\varphi_0(z)|.\end{aligned}$$

Therefore applying a similar reasoning as we did in the previous inequality, we get that

$$\begin{aligned}|\varphi(z_1\cdots z_m)-\varphi_0(z_1\cdots z_m)| &= \left|\prod_{k=1}^{m}\varphi(z_k)-\prod_{k=1}^{m}\varphi_0(z_k)\right| \\ &\le \sum_{k=1}^{m}\|z_1\|\cdots\|z_{k-1}\|\,\|z_{k+1}\|\cdots\|z_m\|\,|\varphi(z_k)-\varphi_0(z_k)|, \qquad (5.2)\end{aligned}$$

for every finite product of elements $z_1,\dots,z_m \in \mathscr{A}$ with $m \in \mathbb{N}$. Let

$$\delta := \min_{i\in\{1,\dots,n\}} \epsilon\left(3\sum_{j=1}^{m_i}|\lambda_{j,i}|\sum_{k=1}^{n_{j,i}}\|x_{j,1,i}\|\cdots\|x_{j,k-1,i}\|\,\|x_{j,k+1,i}\|\cdots\|x_{j,n_{j,i},i}\|\right)^{-1}$$

and let $\varphi \in U(\varphi_0,\delta,x_{1,1,1},\dots,x_{1,n_{1,1},1},\dots,x_{m_1,1,1},\dots,x_{m_1,n_{m_1,1},1},\dots,x_{1,1,n},\dots,x_{m_n,n_{m_n,n},n})$. Then for each $i \in \{1,\dots,n\}$, one has

$$\begin{aligned}|\varphi(P_i)-\varphi_0(P_i)| &\le \sum_{j=1}^{m_i}|\lambda_{j,i}|\,|\varphi(x_{j,1,i}\cdots x_{j,n_{j,i},i})-\varphi_0(x_{j,1,i}\cdots x_{j,n_{j,i},i})| \\ &\le \sum_{j=1}^{m_i}|\lambda_{j,i}|\sum_{k=1}^{n_{j,i}}\|x_{j,1,i}\|\cdots\|x_{j,k-1,i}\|\,\|x_{j,k+1,i}\|\cdots\|x_{j,n_{j,i},i}\|\,|\varphi(x_{j,k,i})-\varphi_0(x_{j,k,i})| \\ &< \sum_{j=1}^{m_i}|\lambda_{j,i}|\sum_{k=1}^{n_{j,i}}\|x_{j,1,i}\|\cdots\|x_{j,k-1,i}\|\,\|x_{j,k+1,i}\|\cdots\|x_{j,n_{j,i},i}\|\,\delta \\ &\le \frac{\epsilon}{3}.\end{aligned}$$

Therefore $U(\varphi_0,\delta,x_{1,1,1},\dots,x_{1,n_{1,1},1},\dots,x_{m_1,1,1},\dots,x_{m_1,n_{m_1,1},1},\dots,x_{1,1,n},\dots,x_{m_n,n_{m_n,n},n})$ is contained in the set $U(\varphi_0,\frac{\epsilon}{3},P_1,\dots,P_n)$. Let $\varphi \in U(\varphi_0,\frac{\epsilon}{3},P_1,\dots,P_n)$. Then for each $i \in \{1,\dots,n\}$, we have

$$\begin{aligned}|\varphi(y_i)-\varphi_0(y_i)| &\le |\varphi(y_i)-\varphi(P_i)| + |\varphi(P_i)-\varphi_0(P_i)| + |\varphi_0(P_i)-\varphi_0(y_i)| \\ &\le \|\varphi\|\,\|P_i-y_i\| + \|\varphi_0\|\,\|P_i-y_i\| + |\varphi(P_i)-\varphi_0(P_i)| \\ &= 2\|P_i-y_i\| + |\varphi(P_i)-\varphi_0(P_i)| < 2\frac{\epsilon}{3}+\frac{\epsilon}{3} = \epsilon,\end{aligned}$$

for every $i \in \{1\dots,n\}$.

Consequently, $U(\varphi_0,\frac{\epsilon}{3},P_1,\dots,P_n)$ is contained in the neighbourhood $U(\varphi_0,\epsilon,y_1,\dots,y_n)$ and thus the set $U(\varphi_0,\delta,x_{1,1,1},\dots,x_{1,n_{1,1},1},\dots,x_{m_1,1,1},\dots,x_{m_1,n_{m_1,1},1},\dots,x_{1,1,n},\dots,x_{m_n,n_{m_n,n},n})$ is contained in the neighbourhood $U(\varphi_0,\epsilon,y_1,\dots,y_n)$ as we wanted to prove. □

5.4 Extensions of Multiplicative Linear Functionals

In this section we will prove that the Gelfand space of a unital commutative Banach algebra is homeomorphic to a closed subset of the Gelfand space of its embedded densely subalgebras. Moreover, we will also see an important theorem regarding extensions of multiplicative linear functionals.

Definition 5.4.1. Let $\mathscr{A}$ and $\mathscr{B}$ be normed algebras. We say that the algebra $\mathscr{A}$ is embedded densely into the algebra $\mathscr{B}$ if and only if $\mathscr{A} \subseteq \mathscr{B}$, $\text{clos}_{\mathscr{B}}(\mathscr{A}) = \mathscr{B}$ and if there exists $c > 0$ such that for each $x \in \mathscr{A}$,

$$\|x\|_{\mathscr{B}} \leq c\|x\|_{\mathscr{A}}.$$

The following two results are going to be crucial in order to prove Theorem 5.7.5 and Theorem 5.8.1.

Theorem 5.4.2 ([8, Chapter 7, Section 3, Proposition 1])**.** *Let $\mathscr{A}$ and $\mathscr{B}$ be unital commutative Banach algebras such that $\mathscr{A}$ is embedded densely in $\mathscr{B}$. Then the space $\mathfrak{M}(\mathscr{B})$ is homeomorphic to a closed subset of $\mathfrak{M}(\mathscr{A})$.*

Proof. Let $\mathscr{R}$ be the set of the multiplicative linear functionals of $\mathfrak{M}(\mathscr{A})$ which admits a unique extension to the multiplicative linear functionals of $\mathfrak{M}(\mathscr{B})$, that is,

$$\mathscr{R} := \{f \in \mathfrak{M}(\mathscr{A}) | \ \exists^1 g \in \mathfrak{M}(\mathscr{B}) : f(x) = g(x), \text{ for each } x \in \mathscr{A}\}$$

and consider

$$\Phi : \mathfrak{M}(\mathscr{B}) \to \mathscr{R}, \ g \mapsto g_{|\mathscr{A}}$$

a function defined on the space $\mathfrak{M}(\mathscr{B})$. We know from Theorem [3, Theorem 1.4.7] that any commutative unital Banach algebra always contains a maximal ideal, therefore both the sets $\mathfrak{M}(\mathscr{B})$ and $\mathfrak{M}(\mathscr{A})$ are not the empty set. It is important to observe that the set $\mathscr{R}$ is also different from the empty set. Indeed, we know that $\mathfrak{M}(\mathscr{B}) \neq \emptyset$ therefore there exists an element $\varphi \in \mathfrak{M}(\mathscr{B})$, consequently, we have that $\varphi_{|\mathscr{A}} \in \mathfrak{M}(\mathscr{A})$ and φ is the only multiplicative linear functional that is an extension of $\varphi_{|\mathscr{A}}$ in $\mathscr{B}$. For if $x \in \mathscr{B}$ and if $\varphi_1, \varphi_2 \in \mathfrak{M}(\mathscr{B})$ are extensions of $\varphi_{|\mathscr{A}}$, then due to the fact that $\mathscr{A}$ is embedded densely into $\mathscr{B}$ it follows that there exists a sequence $(x_n)_{n\in\mathbb{N}}$, with elements in $\mathscr{A}$, such that

$$\lim_{n\to\infty} \|x - x_n\|_{\mathscr{B}} = 0.$$

Hence we get that

$$\varphi_1(x) = \lim_{n\to\infty} \varphi_1(x_n) = \lim_{n\to\infty} \varphi_{|\mathscr{A}}(x_n) = \lim_{n\to\infty} \varphi_2(x_n) = \varphi_2(x)$$

which implies that $\varphi_1 = \varphi_2$ and thus $\mathscr{R} \neq \emptyset$. Taking into account the definition of the set $\mathscr{R}$ and the definition of the function Φ, we can see that Φ is surjective. Given $f, g \in \mathfrak{M}(\mathscr{B})$, it follows by the uniqueness of these extensions that

$$\Phi(f) = \Phi(g) \Leftrightarrow f_{|\mathscr{A}} = g_{|\mathscr{A}} \Leftrightarrow f = g$$

therefore the function Φ is injective and thus bijective. Let $\epsilon > 0$, $\widetilde{f_0} \in \mathfrak{M}(\mathcal{B})$, $n \in \mathbb{N}$, $x_1, \ldots, x_n \in \mathcal{A}$, $f_0 := \widetilde{f_0}_{|\mathcal{A}}$, and let

$$V(f_0, \epsilon, x_1, \ldots, x_n) := \{f \in \mathcal{R} : |f(x_j) - f_0(x_j)| < \epsilon,\ j \in \{1, \ldots, n\}\}$$

be a neighborhood of $\Phi(\widetilde{f_0}) = f_0$. Then

$$U(\widetilde{f_0}, \epsilon, x_1, \ldots, x_n) := \{g \in \mathfrak{M}(\mathcal{B}) : |g(x_j) - \widetilde{f_0}(x_j)| < \epsilon,\ j \in \{1, \ldots, n\}\}$$

is a neighborhood of $\widetilde{f_0}$ that verifies $\Phi(U(\widetilde{f_0}, \epsilon, x_1, \ldots, x_n)) \subseteq V(f_0, \epsilon, x_1, \ldots, x_n)$, consequently Φ is continuous. We know that, under the weak-* topology, $\mathfrak{M}(\mathcal{A})$ and $\mathfrak{M}(\mathcal{B})$ are Hausdorff and compact spaces on the dual space of the algebras $\mathcal{A}$ and $\mathcal{B}$ respectively (see Theorem 5.3.8). Since Φ is a bijective continuous function and $\mathfrak{M}(\mathcal{B})$ is compact, it follows that $\Phi(\mathfrak{M}(\mathcal{B})) = \mathcal{R}$ is compact. But we also know that $\mathcal{R}$ is a subset of $\mathfrak{M}(\mathcal{A})$ which is Hausdorff, consequently $\mathcal{R}$ is closed. Taking into account that Φ is a continuous bijective correspondence between a compact and a Hausdorff space, we can assure that Φ is a homeomorphism as we wanted to prove. □

Theorem 5.4.3 ([8, Chapter 7, Section 3, Proposition 2])**.** *Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be unital commutative Banach algebras such that $\mathcal{A}$ is embedded densely in $\mathcal{B}$ and $\mathcal{B}$ is embedded densely in $\mathcal{C}$. If every non-null multiplicative linear functional of the algebra $\mathcal{A}$ is extensible to a unique multiplicative linear functional of the algebra $\mathcal{C}$, then the same happens for the pair $\mathcal{B}$ and $\mathcal{C}$.*

Proof. Let $g \in \mathfrak{M}(\mathcal{B})$ and $f = g_{|\mathcal{A}}$. In these conditions we get that $f \in \mathfrak{M}(\mathcal{A})$, consequently by our hypothesis there exists a unique $h \in \mathfrak{M}(\mathcal{C})$ such that $f = h_{|\mathcal{A}}$ and therefore $g_{|\mathcal{A}} = h_{|\mathcal{A}}$. Then $h_{|\mathcal{B}}$ is an extension of $h_{|\mathcal{A}} = g_{|\mathcal{A}}$, and this extension is unique. For if $x \in \mathcal{B}$ and if $h_1, h_2 \in \mathfrak{M}(\mathcal{B})$ are two extensions of $h_{|\mathcal{A}}$, then due to the fact that the algebra $\mathcal{A}$ is embedded densely into $\mathcal{B}$, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of elements in $\mathcal{A}$ such that

$$\lim_{n\to\infty} \|x - x_n\|_{\mathcal{B}} = 0.$$

Hence we have

$$h_1(x) = \lim_{n\to\infty} h_1(x_n) = \lim_{n\to\infty} h_{|\mathcal{A}}(x_n) = \lim_{n\to\infty} h_2(x_n) = h_2(x),$$

therefore $h_1 = h_2$. Thus we must have $h_{|\mathcal{B}} = g$ because $g \in \mathfrak{M}(\mathcal{B})$ is also an extension of $g_{|\mathcal{A}}$, which implies that h is the unique extension of $h_{|\mathcal{B}} = g$ because the algebra $\mathcal{B}$ is embedded densely into the algebra $\mathcal{C}$. For if $y \in \mathcal{C}$ and if $h_3, h_4 \in \mathfrak{M}(\mathcal{C})$ are two extensions of g, then due to the fact that the algebra $\mathcal{B}$ is embedded densely into $\mathcal{C}$, there exists a sequence $(y_n)_{n\in\mathbb{N}}$ of elements in $\mathcal{B}$ such that

$$\lim_{n\to\infty} \|y - y_n\|_{\mathcal{C}} = 0.$$

Consequently, it follows that

$$h_3(y) = \lim_{n\to\infty} h_3(y_n) = \lim_{n\to\infty} g(y_n) = \lim_{n\to\infty} h_4(y_n) = h_4(y)$$

and we conclude that h is the unique extension of g as we wanted to prove. □

5.5 Banach Algebra $l^1(\mathbb{R})$

In this section we will give the definition of a character in the complex unit circle and we will recall the Banach algebra $l^1(\mathbb{R})$. After that, we are going to prove that the Gelfand space of $l^1(\mathbb{R})$ is homeomorphic to the space of all characters. Moreover, we will prove that the set of multiplicative linear functionals of $l^1(\mathbb{R})$, corresponding to continuous characters, is dense in the Gelfand space of $l^1(\mathbb{R})$.

Definition 5.5.1. Let $(\mathbb{T},\cdot)$ denote the multiplicative group of the complex unit circle and $(\mathbb{R},+)$ the additive group of real numbers. We say that $\chi : (\mathbb{R},+) \to (\mathbb{T},\cdot)$ is a character if and only if χ is a homomorphism, that is,

$$\chi(x+y) = \chi(x)\chi(y)$$

for each $x,y \in \mathbb{R}$. The set of characters forms a group under the usual multiplication, which we will denote by $\mathbb{X}$.

Taking into account the previous definition, we can see that if χ is a character, then $\chi(0) = 1$ and also that

$$\chi(-\lambda) = \frac{1}{\chi(\lambda)} = \frac{\overline{\chi(\lambda)}}{|\chi(\lambda)|^2} = \overline{\chi(\lambda)},$$

for each $\lambda \in \mathbb{R}$. It is immediate that the function $e_\lambda(x) := e^{i\lambda x}$ is a character for every $\lambda \in \mathbb{R}$. In fact, it is known from [11, Chapter 14, Section A, Example 1] that if f is a continuous character, then $f = e_\mu$ for some $\mu \in \mathbb{R}$.

Definition 5.5.2. Let $x : \mathbb{R} \to \mathbb{C}$ be a complex function. We say that x is absolutely summable if and only if it is different from zero on an at most countable set and if

$$\|x\| := \sum_{\lambda\in\mathbb{R}} |x(\lambda)| < \infty.$$

In fact, the set of all absolutely summable functions on $\mathbb{R}$ forms a vector space under the usual sum and multiplication by complex numbers. With the norm stated above, it is well known that this set is a Banach space and we will denote it by $l^1(\mathbb{R})$.

We can introduce in $l^1(\mathbb{R})$ an operation of multiplication of elements, where the product of two elements $x,y \in l^1(\mathbb{R})$ is defined by

$$(x*y)(t) := \sum_{\lambda\in\mathbb{R}} x(t-\lambda)y(\lambda), \quad t \in \mathbb{R}.$$

This product is indeed well defined due to the fact that

$$\sum_{\lambda\in\mathbb{R}} |x(t-\lambda)|\,|y(\lambda)| \le \|x\| \sum_{\lambda\in\mathbb{R}} |y(\lambda)| = \|x\|\,\|y\| < \infty,$$

and, with this new operation, the space $l^1(\mathbb{R})$ becomes a unital commutative Banach algebra.

Definition 5.5.3. A basis of neighbourhoods of an element $\chi_0 \in \mathbb{X}$ is given by

$$\{\chi \in \mathbb{X} : |\chi(x_j) - \chi_0(x_j)| < \epsilon,\ j \in \{1, \dots, n\}\}$$

where $\epsilon > 0$ and $x_1, \dots, x_n \in \mathbb{R}$.

Theorem 5.5.4 ([7, Chapter 5, Section 29, Theorem 1])**.** *Let $H := l^1(\mathbb{R})$. Then the space $\mathfrak{M}(H)$ is homeomorphic to the space $\mathbb{X}$, where the first set is equipped with the weak-* topology and the latter set equipped with the topology described in Definition 5.5.3.*

Proof. Consider the function

$$\delta(t) := \begin{cases} 1, & \text{if } t = 0, \\ 0, & \text{if } t \neq 0, \end{cases} \tag{5.3}$$

for each $t \in \mathbb{R}$. Taking into account the definition of the function δ, we have that $\delta \in H$. Given $t, \lambda, \mu \in \mathbb{R}$, we can guarantee that

$$(T_{-\lambda,\delta} * T_{-\mu,\delta})(t) = \sum_{s \in \mathbb{R}} T_{-\lambda,\delta}(t-s) T_{-\mu,\delta}(s) = \sum_{s \in \mathbb{R}} \delta(t - s - \lambda)\delta(s - \mu) = T_{-\lambda-\mu,\delta}(t), \tag{5.4}$$

where $T_{\lambda,\delta} := T_\lambda \delta$ denotes the translation function of δ with respect to $\lambda \in \mathbb{R}$ given in Definition 2.1.7. Let $\varphi \in \mathfrak{M}(H)$ and let

$$\chi_\varphi(\lambda) := \varphi(T_{-\lambda}\delta),$$

for each $\lambda \in \mathbb{R}$. Since $\varphi \in \mathfrak{M}(H)$ and equation (5.4) holds, it follows that

$$\chi_\varphi(\lambda + \mu) = \varphi(T_{-\lambda-\mu}\delta) = \varphi(T_{-\lambda}\delta * T_{-\mu}\delta) = \varphi(T_{-\lambda}\delta)\varphi(T_{-\mu}\delta) = \chi_\varphi(\lambda)\chi_\varphi(\mu), \tag{5.5}$$

for any $\lambda, \mu \in \mathbb{R}$. Due to the fact that $|\varphi(h)| \le \|h\|_H$ for each $h \in H$, we can assure that

$$|\chi_\varphi(\lambda)| = |\varphi(T_{-\lambda}\delta)| \le \|T_{-\lambda}\delta\|_H = 1$$

for every $\lambda \in \mathbb{R}$, and since $\delta = T_0\delta$ is the unit of the algebra H, we have $\chi_\varphi(0) = \varphi(T_0\delta) = 1$. Hence, applying equality (5.5) with $\mu = -\lambda$, we get that $|\chi_\varphi(\lambda)| = 1$ for each $\lambda \in \mathbb{R}$ and we conclude that χ_φ is indeed a character of $(\mathbb{R}, +)$. Due to the fact that every element $x \in H$ can be represented in the form

$$x = \sum_{\lambda \in \mathbb{R}} x(\lambda) T_{-\lambda}\delta,$$

where the previous series converges in the norm of H, it follows from the definition of the character χ_φ that

$$\varphi(x) = \sum_{\lambda \in \mathbb{R}} x(\lambda)\chi_\varphi(\lambda). \tag{5.6}$$

Given $\chi \in \mathbb{X}$, consider the function

$$h_\chi : H \to \mathbb{C}, \quad x \mapsto \sum_{\lambda \in \mathbb{R}} x(\lambda)\chi(\lambda).$$

In these conditions the function h_χ is a multiplicative linear functional defined on H, because it is a linear functional by definition, and satisfies

$$\begin{aligned} h_\chi(x*y) &= \sum_{\lambda\in\mathbb{R}}(x*y)(\lambda)\chi(\lambda) = \sum_{\lambda\in\mathbb{R}}\sum_{\mu\in\mathbb{R}}x(\lambda-\mu)y(\mu)\chi(\lambda) = \sum_{\mu\in\mathbb{R}}y(\mu)\sum_{\lambda\in\mathbb{R}}x(\lambda-\mu)\chi(\lambda) \\ &= \sum_{\mu\in\mathbb{R}}y(\mu)\sum_{\lambda\in\mathbb{R}}x(\lambda-\mu)\chi(\lambda-\mu+\mu) = \sum_{\mu\in\mathbb{R}}y(\mu)\chi(\mu)\sum_{\lambda\in\mathbb{R}}x(\lambda-\mu)\chi(\lambda-\mu) \\ &= \left(\sum_{\mu\in\mathbb{R}}y(\mu)\chi(\mu)\right)\left(\sum_{k\in\mathbb{R}}x(k)\chi(k)\right) = h_\chi(y)h_\chi(x) = h_\chi(x)h_\chi(y). \end{aligned}$$

Since

$$h_\chi(\delta) = \sum_{\lambda\in\mathbb{R}}\delta(\lambda)\chi(\lambda) = 1,$$

it follows that h_χ is different from the null function and therefore $h_\chi \in \mathfrak{M}(H)$. Let

$$\Phi : \mathfrak{M}(H) \to \mathbb{X}, \quad \varphi \mapsto \chi_\varphi, \tag{5.7}$$

and let $\chi \in \mathbb{X}$. Then

$$\chi_{h_\chi}(\lambda) = h_\chi(T_{-\lambda}\delta) = \sum_{t\in\mathbb{R}}\delta(t-\lambda)\chi(t) = \chi(\lambda),$$

for each $\lambda \in \mathbb{R}$, and, consequently, $\chi_{h_\chi} = \chi$. Hence there exists $f = h_\chi \in \mathfrak{M}(H)$ such that $\Phi(f) = \chi$, that is, Φ is surjective. Given $\varphi_1, \varphi_2 \in \mathfrak{M}(H)$, for every $\lambda \in \mathbb{R}$ and $x \in H$ we guarantee, using equation (5.6), that

$$\Phi(\varphi_1) = \Phi(\varphi_2) \Rightarrow \chi_{\varphi_1}(\lambda) = \chi_{\varphi_2}(\lambda) \Rightarrow \varphi_1(x) = \varphi_2(x) \Rightarrow \varphi_1 = \varphi_2,$$

thus Φ is injective, and taking into account that Φ is surjective, we can conclude that Φ is bijective. The functions $T_{-\lambda}\delta$, with $\lambda \in \mathbb{R}\setminus\{0\}$, form a system of generators of the algebra H and therefore, applying Theorem 5.3.11, the sets of type

$$\{\varphi \in \mathfrak{M}(H) : |\varphi(T_{-\lambda_k}\delta) - \varphi_0(T_{-\lambda_k}\delta)| = |\chi_\varphi(\lambda_k) - \chi_{\varphi_0}(\lambda_k)| < \epsilon,\ k \in \{1,\ldots,n\}\} \tag{5.8}$$

form a basis of neighborhoods of the element $\varphi_0 \in \mathfrak{M}(H)$, where $\epsilon > 0, n \in \mathbb{N}$ and $\lambda_1,\ldots,\lambda_n \in \mathbb{R}\setminus\{0\}$. Let $\varphi_0 \in \mathfrak{M}(H)$ and let

$$V := V(\chi_{\varphi_0},\epsilon,\lambda_1,\ldots,\lambda_n) := \{\chi_\varphi \in \mathbb{X} : |\chi_\varphi(\lambda_k) - \chi_{\varphi_0}(\lambda_k)| < \epsilon,\ k \in \{1,\ldots,n\}\}$$

be a neighborhood of $\Phi(\varphi_0) = \chi_{\varphi_0}$. Then there is a neighborhood

$$U := U(\varphi_0,\epsilon,\lambda_1,\ldots,\lambda_n) := \{\varphi \in \mathfrak{M}(H) : |\varphi(T_{-\lambda_k}\delta) - \varphi_0(T_{-\lambda_k}\delta)| < \epsilon,\ k \in \{1,\ldots,n\}\}$$

of φ_0 such that $\Phi(U) \subseteq V$ and we conclude that Φ is continuous. Using a similar reasoning to the function Φ^{-1}, we can assure that Φ^{-1} is continuous and we conclude that Φ is a homeomorphism between $\mathfrak{M}(H)$ and $\mathbb{X}$ as we wanted to prove. □

In this work, since $\mathfrak{M}(H)$ is homeomorphic to $\mathbb{X}$, we will denote by $\mathfrak{M}_C(H)$ the set of non-null multiplicative linear functionals of H, corresponding to continuous characters

$$\chi_\lambda(t) = e^{i\lambda t}$$

for each $\lambda \in \mathbb{R}$ and $t \in \mathbb{R}$, that is,

$$\mathfrak{M}_C(H) := \{\varphi \in \mathfrak{M}(H) | \, \exists \lambda \in \mathbb{R} : \, \Phi(\varphi) = e_\lambda\},$$

where Φ is the function defined in statement (5.7).

Theorem 5.5.5 ([7, Chapter 5, Section 29, Theorem 2])**.** *Let $H := l^1(\mathbb{R})$. Then the set $\mathfrak{M}_C(H)$ is dense in $\mathfrak{M}(H)$.*

Proof. Let $\varphi_0 \in \mathfrak{M}(H)$ and let

$$U(\varphi_0, \epsilon, x_1, \ldots, x_t) := \{\varphi \in \mathfrak{M}(H) : |\varphi(x_j) - \varphi_0(x_j)| < \epsilon, j \in \{1, \ldots, t\}\}$$

be an arbitrary neighbourhood of φ_0, where $\epsilon > 0$, $t \in \mathbb{N}$ and $x_1, \ldots, x_t \in H$. Let δ be given by (5.3). Taking into account the proof of the previous theorem and statement (5.8), we can assure that $U(\varphi_0, \epsilon, x_1, \ldots, x_t)$ contains a neighbourhood $U'(\varphi_0, \epsilon_1, \lambda_1, \ldots, \lambda_n)$ of the form

$$U' := U'(\varphi_0, \epsilon_1, \lambda_1, \ldots, \lambda_n) := \{\varphi \in \mathfrak{M}(H) : |\varphi(T_{-\lambda_k}\delta) - \varphi_0(T_{-\lambda_k}\delta)| < \epsilon_1, \; k \in \{1, \ldots, n\}\},$$

where $\lambda_1, \ldots, \lambda_n \in \mathbb{R} \setminus \{0\}$ and $\epsilon_1 > 0$. We can select numbers $\{\lambda_{k_1}, \ldots, \lambda_{k_m}\} \subseteq \{\lambda_1, \ldots, \lambda_n\}$, where $m \in \mathbb{N}$, such that the numbers $\{\lambda_{k_1}, \ldots, \lambda_{k_m}\}$ are linearly independent over the field of rational numbers, that is, given $\alpha_{k_1}, \ldots, \alpha_{k_m} \in \mathbb{Q}$,

$$\alpha_{k_1}\lambda_{k_1} + \cdots + \alpha_{k_m}\lambda_{k_m} = 0 \Rightarrow \alpha_{k_1} = \cdots = \alpha_{k_m} = 0,$$

and also that every λ_k, with $k \in \{1, \ldots, n\}$, can be expressed as a linear combination of the values $\{\lambda_{k_1}, \ldots, \lambda_{k_m}\}$ with rational coefficients, where $j \in \{1, \ldots, m\}$. Consequently, for each $i \in \{1, \ldots, n\}$, there exist $\alpha_{k_1,i}, \ldots, \alpha_{k_m,i} \in \mathbb{Q}$ such that

$$-\lambda_i = \alpha_{k_1,i}\lambda_{k_1} + \cdots + \alpha_{k_m,i}\lambda_{k_m}.$$

Let L be the least common multiple of the denominators of the coefficients of all these terms, that is, let

$$L := \text{lcm}\,(\alpha_{k_1,1}, \ldots, \alpha_{k_m,1}, \ldots, \alpha_{k_1,n}, \ldots, \alpha_{k_m,n}).$$

Then the numbers λ_k, with $k \in \{1, \ldots, n\}$, can be expressed in terms of the numbers

$$\mu_j := \frac{\lambda_{k_j}}{L},$$

where $j \in \{1, \ldots, m\}$, in the form of linear combinations with integer coefficients, that is, for each $i \in \{1, \ldots, n\}$ there exist $\beta_{k_1,i}, \ldots, \beta_{k_m,i} \in \mathbb{Z}$ such that

$$-\lambda_i = \beta_{k_1,i}\mu_1 + \cdots + \beta_{k_m,i}\mu_m$$

where, in fact,

$$\beta_{k_1,i} = L \cdot \alpha_{k_1,i}, \ldots, \beta_{k_m,i} = L \cdot \alpha_{k_m,i}$$

for every $i \in \{1,\ldots,n\}$. In these conditions the elements μ_j are also linearly independent over the rational numbers, and the functions $T_{-\lambda_k}\delta$, with $k \in \{1,\ldots,n\}$, are products of the functions $T_{-\mu_j}\delta$ and $T_{\mu_j}\delta$, with $j \in \{1,\ldots,m\}$, where this product refers to the product defined in $l^1(H)$.

Let us establish an auxiliary fact. Let $\beta \in \mathbb{Z} \setminus \{0\}$ and let $\mu \in \mathbb{R}$. If $\beta > 0$, then it is immediate that for every $\varphi \in \mathfrak{M}(H)$, one has

$$|\varphi(T_{\beta\mu}\delta) - \varphi_0(T_{\beta\mu}\delta)| = |(\varphi(T_\mu\delta))^{|\beta|} - (\varphi_0(T_\mu\delta))^{|\beta|}|.$$

On the other hand, if $\beta < 0$, then observing that

$$\chi_\varphi(-\lambda) = \overline{\chi_\varphi(\lambda)}$$

for each $\lambda \in \mathbb{R}$, we can guarantee that

$$\begin{aligned} |\varphi(T_{\beta\mu}\delta) - \varphi_0(T_{\beta\mu}\delta)| &= |(\varphi(T_{-\mu}\delta))^{|\beta|} - (\varphi_0(T_{-\mu}\delta))^{|\beta|}| \\ &= |(\chi_\varphi(\mu))^{|\beta|} - (\chi_{\varphi_0}(\mu))^{|\beta|}| \\ &= |(\overline{\chi_\varphi(-\mu)})^{|\beta|} - (\overline{\chi_{\varphi_0}(-\mu)})^{|\beta|}| \\ &= |(\chi_\varphi(-\mu))^{|\beta|} - (\chi_{\varphi_0}(-\mu))^{|\beta|}| \\ &= |(\varphi(T_\mu\delta))^{|\beta|} - (\varphi_0(T_\mu\delta))^{|\beta|}|. \end{aligned}$$

Thus, for all $\beta \in \mathbb{Z} \setminus \{0\}$ and $\mu \in \mathbb{R}$,

$$|\varphi(T_{\beta\mu}\delta) - \varphi_0(T_{\beta\mu}\delta)| = |(\varphi(T_\mu\delta))^{|\beta|} - (\varphi_0(T_\mu\delta))^{|\beta|}|. \tag{5.9}$$

If $j \in \{1,\ldots,n\}$, then using inequality (5.2), equality (5.9) and the fact that $\|T_\lambda\delta\| = 1$ for every $\lambda \in \mathbb{R}$, one has

$$\begin{aligned} |\varphi(T_{-\lambda_j}\delta) - \varphi_0(T_{-\lambda_j}\delta)| &= |\varphi(T_{\beta_{k_1,j}\mu_1+\cdots+\beta_{k_m,j}\mu_m}\delta) - \varphi_0(T_{\beta_{k_1,j}\mu_1+\cdots+\beta_{k_m,j}\mu_m}\delta)| \\ &\leq \sum_{s=1}^{m} |\varphi(T_{\beta_{k_s,j}\mu_s}\delta) - \varphi_0(T_{\beta_{k_s,j}\mu_s}\delta)| \\ &= \sum_{s=1}^{m} \left|(\varphi(T_{\mu_s}\delta))^{|\beta_{k_s,j}|} - (\varphi_0(T_{\mu_s}\delta))^{|\beta_{k_s,j}|}\right| \\ &\leq \sum_{s=1}^{m} |\beta_{k_s,j}|\, |\varphi(T_{\mu_s}\delta) - \varphi_0(T_{\mu_s}\delta)|. \end{aligned} \tag{5.10}$$

Let

$$\epsilon_2 := \epsilon_1 \min_{j\in\{1,\ldots,n\}} \left(1 + \sum_{s=1}^{m} |\beta_{k_s,j}|\right)^{-1}$$

and let

$$U'' := U''(\varphi_0, \epsilon_2, \mu_1, \ldots, \mu_m) := \{\varphi \in \mathfrak{M}(H) : |\varphi(T_{-\mu_j}\delta) - \varphi_0(T_{-\mu_j}\delta)| < \epsilon_2,\ j \in \{1,\ldots,m\}\}.$$

If $\varphi \in U'$, then it follows from inequality (5.10) that for all $j \in \{1,\dots,n\}$,

$$|\varphi(T_{-\lambda_j}\delta) - \varphi_0(T_{-\lambda_j}\delta)| \le \sum_{s=1}^{m} |\beta_{k_s,j}|\, |\varphi(T_{\mu_s}\delta) - \varphi_0(T_{\mu_s}\delta)| < \epsilon_2 \sum_{s=1}^{m} |\beta_{k_s,j}| < \epsilon_1,$$

that is, $\varphi \in U'$ and thus we proved that $U'' \subseteq U'$.

We can consider that the character corresponding to φ_0 verifies $\chi_{\varphi_0}(\mu_j) = e^{2\pi i a_j}$, for some $a_j \in \mathbb{R}$ and $j \in \{1,\dots,m\}$ because the function e_λ is surjective in $\mathbb{T}$ for any $\lambda \in \mathbb{R} \setminus \{0\}$. Due to the fact that the function e_λ is uniformly continuous in $\mathbb{R}$ for each $\lambda \in \mathbb{R}$, there exists $\eta_{\epsilon_2} > 0$ such that for every $t_1, t_2 \in \mathbb{R}$ we have

$$|t_1 - t_2| < \eta_{\epsilon_2} \Rightarrow |e^{it_1} - e^{it_2}| < \epsilon_2. \tag{5.11}$$

Taking into account Kronecker's Theorem [10, Theorem 444], there is $t_0 \in \mathbb{R}$ and $p_1,\dots,p_m \in \mathbb{Z}$ satisfying

$$|a_j - t_0\mu_j - p_j| < \frac{\eta_{\epsilon_2}}{2\pi},$$

that is,

$$|2\pi a_j - 2\pi p_j - 2\pi t_0 \mu_j| < \eta_{\epsilon_2}$$

for each $j \in \{1,\dots,m\}$. Then, using inequality (5.11), we get that

$$|e^{2\pi i a_j - 2\pi i p_j} - e^{2\pi i t_0 \mu_j}| = |e^{2\pi i a_j} e^{-2\pi i p_j} - e^{2\pi i t_0 \mu_j}| = |e^{2\pi i t_0 \mu_j} - e^{2\pi i a_j}| < \epsilon_2,$$

that is

$$|e^{2\pi i t_0 \mu_j} - \chi_{\varphi_0}(\mu_j)| < \epsilon_2$$

for each $j \in \{1,\dots,m\}$. Consequently the neighbourhood $U''(\varphi_0, \epsilon_2, \mu_1, \dots, \mu_m)$ contains the element $\varphi^* \in \mathfrak{M}_C(H)$ corresponding to the continuous character $e_{2\pi t_0}(x) := e^{2\pi t_0 i x}$, and since $U''(\varphi_0, \epsilon_2, \mu_1, \dots, \mu_m) \subseteq U(\varphi_0, \epsilon, x_1, \dots, x_t)$, it follows that $\varphi^* \in U(\varphi_0, \epsilon, x_1, \dots, x_t)$ as we wanted to prove. □

5.6 C^*-Algebra $AP(\mathbb{R})$ and Banach Algebra $AP_p(\mathbb{R})$

Let us recall that we defined the space $AP(\mathbb{R})$ as the closure, in $L^\infty(\mathbb{R})$, of the space $APP(\mathbb{R})$ of all finite sums of trigonometric polynomial functions. The following theorem show us that $AP(\mathbb{R})$ is, in fact, a commutative C^*-algebra.

Theorem 5.6.1. *The set $AP(\mathbb{R})$ is a commutative C^*-subalgebra of $L^\infty(\mathbb{R})$.*

Proof. Since $L^\infty(\mathbb{R})$ is a Banach space and since $AP(\mathbb{R})$ is a closed subspace of $L^\infty(\mathbb{R})$, because it is the closure of $APP(\mathbb{R})$, it follows that $AP(\mathbb{R})$ is indeed a Banach space. Let $f, g, h \in AP(\mathbb{R})$ and $\lambda \in \mathbb{C}$. Then there exist sequences $(f_n)_{n\in\mathbb{N}}$, $(g_n)_{n\in\mathbb{N}}$ and $(h_n)_{n\in\mathbb{N}}$ of terms in $APP(\mathbb{R})$ such that

$$\lim_{n\to\infty} \|f - f_n\|_{L^\infty(\mathbb{R})} = 0,$$
$$\lim_{n\to\infty} \|g - g_n\|_{L^\infty(\mathbb{R})} = 0,$$

$$\lim_{n\to\infty} \|h - h_n\|_{L^\infty(\mathbb{R})} = 0.$$

Then, for each $x \in \mathbb{R}$, we have

1. $(fg)(x) = f(x)g(x) = \lim_{n\to\infty} f_n(x) \lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} f_n(x)g_n(x) = \lim_{n\to\infty} g_n(x)f_n(x)$

   $= \lim_{n\to\infty} g_n(x) \lim_{n\to\infty} f_n(x) = g(x)f(x) = (gf)(x);$
2. $(f(gh))(x) = \lim_{n\to\infty} f_n(x)(\lim_{n\to\infty} g_n(x) \lim_{n\to\infty} h_n(x)) = (\lim_{n\to\infty} f_n(x) \lim_{n\to\infty} g_n(x)) \lim_{n\to\infty} h_n(x)$

   $= ((fg)h)(x);$
3. $(\lambda f(x))g(x) = (\lambda \lim_{n\to\infty} f_n(x)) \lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} f_n(x)(\lambda \lim_{n\to\infty} g_n(x)) = f(x)(\lambda g(x));$
4. $f(x)(g(x) + h(x)) = \lim_{n\to\infty} f_n(x)(\lim_{n\to\infty} g_n(x) + \lim_{n\to\infty} h_n(x))$

   $= \lim_{n\to\infty} f_n(x) \lim_{n\to\infty} g_n(x) + \lim_{n\to\infty} f_n(x) \lim_{n\to\infty} h_n(x) = f(x)g(x) + f(x)h(x);$
5. $(f(x) + g(x))h(x) = (\lim_{n\to\infty} f_n(x) + \lim_{n\to\infty} g_n(x)) \lim_{n\to\infty} h_n(x)$

   $= \lim_{n\to\infty} f_n(x) \lim_{n\to\infty} h_n(x) + \lim_{n\to\infty} g_n(x) \lim_{n\to\infty} h_n(x) = f(x)h(x) + g(x)h(x);$
6. $\|fg\|_{L^\infty(\mathbb{R})} = \sup_{x\in\mathbb{R}} |f(x)g(x)| \le \sup_{x\in\mathbb{R}} |f(x)| \sup_{x\in\mathbb{R}} |g(x)| = \|f\|_{L^\infty(\mathbb{R})} \|g\|_{L^\infty(\mathbb{R})}.$

Consequently $AP(\mathbb{R})$ is a commutative Banach algebra. Consider the operation $*$ to be the conjugate operation that we know it is well defined in $\mathbb{C}$. It follows that for every $f, g \in AP(\mathbb{R})$, for any $\lambda \in \mathbb{C}$ and for each $x \in \mathbb{R}$,

1. $(f(x) + g(x))^* = \overline{f(x) + g(x)} = \overline{\lim_{n\to\infty} f_n(x)} + \overline{\lim_{n\to\infty} g_n(x)} = f^*(x) + g^*(x);$
2. $(\lambda f(x))^* = \overline{\lambda f(x)} = \overline{\lambda}\ \overline{\lim_{n\to\infty} f_n(x)} = \overline{\lambda} f^*(x);$
3. $((f(x))^*)^* = \overline{\overline{\lim_{n\to\infty} f_n(x)}} = f(x);$
4. $(f(x)g(x))^* = \overline{f(x)g(x)} = \overline{\lim_{n\to\infty} g_n(x)}\ \overline{\lim_{n\to\infty} f_n(x)} = g^*(x)f^*(x).$

That is, $*$ verifies all the proprieties of an involution and thus $AP(\mathbb{R})$ is a commutative *-algebra. Let $\varphi \in AP(\mathbb{R})$. Then

$$\|\varphi^*\varphi\|_{L^\infty(\mathbb{R})} = \|\overline{\varphi}\varphi\|_{L^\infty(\mathbb{R})} = \sup_{x\in\mathbb{R}} |\overline{\varphi(x)}\varphi(x)| = \sup_{x\in\mathbb{R}} |(\varphi(x))^2| = \left(\sup_{x\in\mathbb{R}} |\varphi(x)|\right)^2 = \|\varphi\|^2_{L^\infty(\mathbb{R})},$$

consequently for every $f \in AP(\mathbb{R})$ we have

$$\|f^*f\|_{L^\infty(\mathbb{R})} = \|f\|^2_{L^\infty(\mathbb{R})}.$$

Hence $AP(\mathbb{R})$ verifies the C^*-property and we conclude that $AP(\mathbb{R})$ is indeed a commutative C^*-subalgebra of $L^\infty(\mathbb{R})$. □

Definition 5.6.2. Let $\phi \in L^\infty(\mathbb{R})$ and $1 < p < \infty$. We say that ϕ is a Fourier multiplier on $L^p(\mathbb{R})$ if and only if the map $f \mapsto (\mathcal{F}^{-1}\phi\mathcal{F})(f)$ maps $L^2(\mathbb{R}) \cap L^p(\mathbb{R})$ into itself and extends to a unique bounded operator on $L^p(\mathbb{R})$, where the latter operator is denoted by W^0_ϕ.

In this work we will denote by $M_p(\mathbb{R})$ the set of all Fourier multipliers on $L^p(\mathbb{R})$ and it is known, from [9, Proposition 2.5.13], that $M_p(\mathbb{R})$ is a Banach algebra under the norm

$$\|\phi\|_{M_p(\mathbb{R})} := \|W^0_\phi\|_{\mathcal{B}(L^p(\mathbb{R}))}, \quad \phi \in M_p(\mathbb{R}). \tag{5.12}$$

Example 5.6.3. Given $\lambda \in \mathbb{R}$, $p \in]1, +\infty[$ and $f \in L^p(\mathbb{R})$, consider the translation operator $U_\lambda \in \mathcal{B}(L^p(\mathbb{R}))$ defined by

$$f \mapsto U_\lambda f, \quad (U_\lambda f)(t) := f(t-\lambda),$$

for every $t \in \mathbb{R}$. Let $f \in L^2(\mathbb{R}) \cap L^p(\mathbb{R})$ and $x \in \mathbb{R}$. Then there exists a sequence $(f_n)_{n\in\mathbb{N}} \subseteq L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ such that

$$\lim_{n\to\infty} \|f - f_n\|_2 = 0.$$

Consequently we have

$$\widehat{(U_\lambda f_n)}(x) = \int_{-\infty}^{+\infty} f_n(t-\lambda)e^{-2\pi i t x}\, dt = \int_{-\infty}^{+\infty} f_n(s)e^{-2\pi i (s+\lambda)x}\, ds = e^{-2\pi i \lambda x}\widehat{f_n}(x) = (\widehat{\mathcal{F}^{-1}\phi\mathcal{F}} f_n)(x),$$

where $\phi(x) := e^{-2\pi i \lambda x}$, which implies, applying the limit when $n \to \infty$, that

$$\mathcal{F}(U_\lambda f)(x) = \mathcal{F}(\mathcal{F}^{-1}\phi\mathcal{F} f)(x).$$

Taking into account the previous equality and the fact that the space $L^2(\mathbb{R}) \cap L^p(\mathbb{R})$ is dense in $L^p(\mathbb{R})$, it follows that $U_\lambda = W^0_\phi$ and therefore, using the fact that $M_p(\mathbb{R})$ is an algebra, we get that the set $APP(\mathbb{R})$ is contained in the set $M_p(\mathbb{R})$.

Definition 5.6.4. Consider $1 < p < \infty$ and let $M_p(\mathbb{R})$ be the set of all Fourier multipliers on $L^p(\mathbb{R})$. Then we define $AP_p(\mathbb{R})$ as the closure of $APP(\mathbb{R})$ in $M_p(\mathbb{R})$, that is,

$$AP_p(\mathbb{R}) := \text{clos}_{M_p(\mathbb{R})}(APP(\mathbb{R})).$$

Theorem 5.6.5. *Let $p \in]1, +\infty[$. Then the algebra $AP_p(\mathbb{R})$ is embedded densely in $AP(\mathbb{R})$.*

Proof. Taking into account the definition of the set $M_p(\mathbb{R})$ we have $M_p(\mathbb{R}) \subseteq L^\infty(\mathbb{R})$, and it is know from [6, Proposition 2.4] that

$$\|\phi\|_{L^\infty(\mathbb{R})} \le \|W^0_\phi\|_{\mathcal{B}(L^p(\mathbb{R}))} = \|\phi\|_{M_p(\mathbb{R})}, \quad \phi \in M_p(\mathbb{R}).$$

Consequently, $M_p(\mathbb{R})$ is continuously embedded into $L^\infty(\mathbb{R})$ and thus, using the fact that

$$\|\phi\|_{L^\infty(\mathbb{R})} = \|\phi\|_{AP(\mathbb{R})}, \quad \|\phi\|_{M_p(\mathbb{R})} = \|\phi\|_{AP_p(\mathbb{R})},$$

it follows that the space $AP_p(\mathbb{R})$ is continuously embedded into the space $AP(\mathbb{R})$. Let $\Phi \in AP(\mathbb{R})$. Then there exists a sequence $(\Phi_n)_{n\in\mathbb{N}} \subseteq APP(\mathbb{R})$ such that

$$\lim_{n\to\infty} \|\Phi - \Phi_n\|_{L^\infty(\mathbb{R})} = 0.$$

Since $APP(\mathbb{R}) \subseteq AP_p(\mathbb{R})$, we have that the sequence $(\Phi_n)_{n\in\mathbb{N}}$ satisfies $\Phi_n \in AP_p(\mathbb{R})$ for each $n \in \mathbb{N}$, and also

$$\lim_{n\to\infty} \|\Phi - \Phi_n\|_{AP(\mathbb{R})} = \lim_{n\to\infty} \|\Phi - \Phi_n\|_{L^\infty(\mathbb{R})} = 0.$$

Therefore

$$\text{clos}_{AP(\mathbb{R})}(AP_p(\mathbb{R})) = AP(\mathbb{R}),$$

that is, $AP_p(\mathbb{R})$ is embedded densely into $AP(\mathbb{R})$. □

5.7 Banach Algebra $APW(\mathbb{R})$

In this section we will define the Banach algebra $APW(\mathbb{R})$ and we will prove that $APW(\mathbb{R})$ is embedded densely in $AP_p(\mathbb{R})$ and in $AP(\mathbb{R})$. Moreover, we will prove that $l^1(\mathbb{R})$ is isometrically isomorphic to $APW(\mathbb{R})$ and that the Gelfand space of $APW(\mathbb{R})$ is homeomorphic to the Gelfand space of $AP(\mathbb{R})$.

Definition 5.7.1. Let $APW(\mathbb{R})$ denote the set of all functions $f : \mathbb{R} \to \mathbb{C}$ which can be written in the form

$$f(x) := \sum_{j=1}^{\infty} a_j e^{i\lambda_j x}$$

and satisfy

$$\|f\|_{APW(\mathbb{R})} := \sum_{j=1}^{\infty} |a_j| < \infty,$$

where λ_j are arbitrary distinct real numbers and a_j are arbitrary complex numbers. It is known, from a similar result from [12, Chapter 1, Section 6.1, Lemma 1], that the space $APW(\mathbb{R})$, under the usual operations of multiplication by a scalar, sum of two functions and multiplication of two functions, is a commutative unital Banach algebra.

Theorem 5.7.2. *For each $p \in]1, +\infty[$, the algebra $APW(\mathbb{R})$ is embedded densely in the algebra $AP_p(\mathbb{R})$.*

Proof. Taking into account the definition of the spaces $APW(\mathbb{R})$ and $AP_p(\mathbb{R})$, it follows that $APW(\mathbb{R}) \subseteq AP_p(\mathbb{R})$. Let $\phi \in AP_p(\mathbb{R})$. Then there exists a sequence $(\phi_n)_{n\in\mathbb{N}}$ such that $\phi_n \in APP(\mathbb{R}) \subseteq M_p(\mathbb{R})$ for every $n \in \mathbb{N}$, and verify

$$\lim_{n\to\infty} \|\phi - \phi_n\|_{M_p(\mathbb{R})} = 0.$$

Since $\phi_n \in APP(\mathbb{R})$, there are $\alpha_{n,1}, \ldots, \alpha_{n,k_n} \in \mathbb{C}$ and distinct numbers $\lambda_{n,1}, \ldots, \lambda_{n,k_n} \in \mathbb{R}$ such that

$$\phi_n(x) = \alpha_{n,1} e^{i\lambda_{n,1}x} + \cdots + \alpha_{n,k_n} e^{i\lambda_{n,k_n}x}.$$

Analyzing Example 5.6.3 and using the fact that the Fourier transform is a linear operator, we can assure that for each $n \in \mathbb{N}$, one has

$$\alpha_{n,1} U_{\frac{1}{-2\pi}\lambda_{n,1}} + \cdots + \alpha_{n,k_n} U_{\frac{1}{-2\pi}\lambda_{n,k_n}} = W^0_{\phi_n}$$

where $U_{\frac{1}{-2\pi}\lambda_{n,1}},\ldots,U_{\frac{1}{-2\pi}\lambda_{n,k_n}} \in \mathcal{B}(L^p(\mathbb{R}))$ are translation operators. As a consequence of the definition, any translation operator has norm equal to 1, hence

$$\|\phi_n\|_{AP_p(\mathbb{R})} = \|\phi_n\|_{M_p(\mathbb{R})} = \|W^0_{\phi_n}\|_{\mathcal{B}(L^p(\mathbb{R}))} \leq |\alpha_{n,1}| + \cdots + |\alpha_{n,k_n}| = \|\phi_n\|_{APW(\mathbb{R})}$$

for all $n \in \mathbb{N}$, therefore

$$\|\phi\|_{AP_p(\mathbb{R})} \leq \|\phi\|_{APW(\mathbb{R})},$$

that is, $APW(\mathbb{R})$ is continuously embedded into $AP_p(\mathbb{R})$. Consequently we get that

$$\text{clos}_{AP_p(\mathbb{R})}(APW(\mathbb{R})) \subseteq AP_p(\mathbb{R}).$$

Let $f \in AP_p(\mathbb{R}) = \text{clos}_{M_p(\mathbb{R})}(APP(\mathbb{R}))$. Then there exists a sequence $(f_n)_{n\in\mathbb{N}}$ such that $f_n \in APP(\mathbb{R})$ for each $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} \|f - f_n\|_{M_p(\mathbb{R})} = 0.$$

Due to the fact that $APP(\mathbb{R}) \subseteq APW(\mathbb{R})$, we have that $f_n \in APW(\mathbb{R})$ for every $n \in \mathbb{N}$. In these conditions

$$\lim_{n\to\infty} \|f - f_n\|_{AP_p(\mathbb{R})} = \lim_{n\to\infty} \|f - f_n\|_{M_p(\mathbb{R})} = 0,$$

therefore $(f_n)_{n\in\mathbb{N}}$ is a sequence of elements in $APW(\mathbb{R})$ that converge uniformly in the norm $AP_p(\mathbb{R})$ to f, that is, $f \in \text{clos}_{AP_p(\mathbb{R})}(APW(\mathbb{R}))$ and we conclude that

$$\text{clos}_{AP_p(\mathbb{R})}(APW(\mathbb{R})) = AP_p(\mathbb{R}).$$

□

Theorem 5.7.3. *The algebra $APW(\mathbb{R})$ is embedded densely into the algebra $AP(\mathbb{R})$.*

Proof. Let $p \in]1,+\infty[$. Taking into account the previous theorem and Theorem 5.6.5, it follows that $APW(\mathbb{R})$ is continuously embedded into $AP_p(\mathbb{R})$ and $AP_p(\mathbb{R})$ is continuously embedded into $AP(\mathbb{R})$, that is, every element $f \in APW(\mathbb{R})$ satisfies

$$\|f\|_{AP(\mathbb{R})} \leq \|f\|_{AP_p(\mathbb{R})} \leq \|f\|_{APW(\mathbb{R})}.$$

Hence we have that

$$\text{clos}_{AP(\mathbb{R})}(APW(\mathbb{R})) \subseteq AP(\mathbb{R}).$$

Let $f \in AP(\mathbb{R}) = \text{clos}_{L^\infty(\mathbb{R})}(APP(\mathbb{R}))$. Then there exists a sequence $(f_n)_{n\in\mathbb{N}}$ such that $f_n \in APP(\mathbb{R})$ for each $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} \|f - f_n\|_{L^\infty(\mathbb{R})} = 0.$$

Due to the fact that $APP(\mathbb{R}) \subseteq APW(\mathbb{R})$, we have that $f_n \in APW(\mathbb{R})$ for every $n \in \mathbb{N}$. In these conditions

$$\lim_{n\to\infty} \|f - f_n\|_{AP(\mathbb{R})} = \lim_{n\to\infty} \|f - f_n\|_{L^\infty(\mathbb{R})} = 0,$$

therefore $(f_n)_{n\in\mathbb{N}}$ is a sequence of elements in $APW(\mathbb{R})$ that converge uniformly in the norm $AP(\mathbb{R})$ to f, that is, $f \in \text{clos}_{AP(\mathbb{R})}(APW(\mathbb{R}))$ and we conclude that

$$\text{clos}_{AP(\mathbb{R})}(APW(\mathbb{R})) = AP(\mathbb{R}).$$

□

In the next result we will see that, in fact, the Banach algebras $l^1(\mathbb{R})$ and $APW(\mathbb{R})$ are isometrically isomorphic.

Theorem 5.7.4. *The Banach algebra $l^1(\mathbb{R})$ is isometrically isomorphic to the Banach algebra $APW(\mathbb{R})$.*

Proof. Let us consider the operator

$$T : l^1(\mathbb{R}) \to APW(\mathbb{R}), \quad x \mapsto f(t) := \sum_{\lambda \in \mathbb{R}} x(\lambda) e^{i\lambda t}.$$

In these conditions for every $x, y \in l^1(\mathbb{R})$ and for each $\alpha \in \mathbb{C}$, we have

$$T(x+y) = \sum_{\lambda\in\mathbb{R}} (x+y)(\lambda)e^{i\lambda t} = \sum_{\lambda\in\mathbb{R}} x(\lambda)e^{i\lambda t} + \sum_{\lambda\in\mathbb{R}} y(\lambda)e^{i\lambda t} = T(x) + T(y),$$

$$T(\alpha x) = \sum_{\lambda\in\mathbb{R}} (\alpha x)(\lambda)e^{i\lambda t} = \alpha \sum_{\lambda\in\mathbb{R}} x(\lambda)e^{i\lambda t} = \alpha T(x),$$

$$\begin{aligned} T(x*y) &= \sum_{\lambda\in\mathbb{R}} (x*y)(\lambda)e^{i\lambda t} = \sum_{\lambda\in\mathbb{R}}\sum_{s\in\mathbb{R}} x(\lambda-s)y(s)e^{i(\lambda-s)t}e^{ist} \\ &= \left(\sum_{\mu\in\mathbb{R}} x(\mu)e^{i\mu t}\right)\left(\sum_{s\in\mathbb{R}} y(s)e^{ist}\right) = T(x)T(y). \end{aligned}$$

Consequently, T is a linear operator that preserves the multiplication between the two algebras. Due to the fact that

$$\|T\| = \sup_{x\in l^1(\mathbb{R})\setminus\{0\}} \frac{\|T(x)\|_{APW(\mathbb{R})}}{\|x\|_{l^1(\mathbb{R})}} = \sup_{x\in l^1(\mathbb{R})\setminus\{0\}} \frac{\sum_{\lambda\in\mathbb{R}} |x(\lambda)|}{\sum_{\lambda\in\mathbb{R}} |x(\lambda)|} = 1 < \infty,$$

it follows that T is a bounded operator. Taking into account that $l^1(\mathbb{R})$ and $APW(\mathbb{R})$ are Banach spaces, we just need to prove that T is bijective and isometric and the proof is done. Let $f(t) := \sum_{j=1}^{\infty} a_j e^{i\lambda_j t} \in APW(\mathbb{R})$ and consider the function

$$x(\lambda) := \begin{cases} a_j, & \text{if} \quad \exists j \in \mathbb{N} : \ \lambda = \lambda_j, \\ 0, & \qquad \text{otherwise,} \end{cases}$$

for each $\lambda \in \mathbb{R}$. Then, by construction, we have that

$$\sum_{\lambda\in\mathbb{R}} |x(\lambda)| = \sum_{j=1}^{\infty} |a_j| < \infty$$

and also that $T(x) = f$, which implies that T is surjective. Let $x, y \in l^1(\mathbb{R})$. Taking into account the Parseval Identity (see Theorem 3.4.8), one has

$$T(x) = T(y) \Rightarrow \sum_{\lambda\in\mathbb{R}} x(\lambda)e^{i\lambda t} = \sum_{\lambda\in\mathbb{R}} y(\lambda)e^{i\lambda t}$$

$$\begin{aligned}
&\Rightarrow \sum_{\lambda\in\mathbb{R}}(x-y)(\lambda)e^{i\lambda t}=0\\
&\Rightarrow 0=M_0=\sum_{\lambda\in\mathbb{R}}|(x-y)(\lambda)|^2\\
&\Rightarrow (x-y)(\lambda)=0, \quad \text{for each } \lambda\in\mathbb{R}\\
&\Rightarrow x=y.
\end{aligned}$$

Therefore T is a bounded bijective linear operator, and thus the Banach algebra $l^1(\mathbb{R})$ is isomorphic to the Banach algebra $APW(\mathbb{R})$. Due to the fact that

$$\|T(x)\|_{APW(\mathbb{R})}=\sum_{\lambda\in\mathbb{R}}|x(\lambda)|=\|x\|_{l^1(\mathbb{R})}$$

for each $x\in l^1(\mathbb{R})$, it follows that T is isometric and we conclude that $l^1(\mathbb{R})$ is isometrically isomorphic to $APW(\mathbb{R})$ as we wanted to prove. □

As we analysed before, the Banach algebra $APW(\mathbb{R})$ can be identified with the Banach algebra $l^1(\mathbb{R})$, consequently, there exits a homeomorphism, Ψ, between $\mathfrak{M}(APW(\mathbb{R}))$ and $\mathfrak{M}(l^1(\mathbb{R}))$ defined by

$$\Psi:\mathfrak{M}(APW(\mathbb{R}))\to\mathfrak{M}(l^1(\mathbb{R})),\quad \varphi\mapsto\varphi\circ T,$$

where T is the isometric isomorphism defined in the previous theorem. Therefore, applying Theorem 5.5.5, we have that

$$\mathfrak{M}_C(APW(\mathbb{R})):=\{\varphi\in\mathfrak{M}(APW(\mathbb{R}))|\,\exists\lambda\in\mathbb{R}:\ \Phi(\Psi(\varphi))=e_\lambda\},$$

where Φ is the function defined in statement (5.7), is dense in $\mathfrak{M}(APW(\mathbb{R}))$.

The proof of the following theorem is analogous to that one of [8, Chapter 7, Section 3, Theorem 3.3], where the authors considered the similar problem for some algebras $\mathcal{S}$ and $\mathcal{S}_2$ containing the subalgebras $APW(\mathbb{R})$ and $AP(\mathbb{R})$, respectively.

Theorem 5.7.5. *The spaces $\mathfrak{M}(APW(\mathbb{R}))$ and $\mathfrak{M}(AP(\mathbb{R}))$ are homeomorphic.*

Proof. Taking into account Theorem 5.4.2 and the fact that the Banach algebra $APW(\mathbb{R})$ is embedded densely into $AP(\mathbb{R})$, we get that the space $\mathfrak{M}(AP(\mathbb{R}))$ is homeomorphic to the closed set $\mathcal{R}\subseteq\mathfrak{M}(APW(\mathbb{R}))$ defined by

$$\mathcal{R}:=\{f\in\mathfrak{M}(APW(\mathbb{R}))|\,\exists^1 g\in\mathfrak{M}(AP(\mathbb{R})):f(x)=g(x),\ \text{for each } x\in APW(\mathbb{R})\}.$$

Let $\varphi\in\mathfrak{M}_C(APW(\mathbb{R}))$. Then $\varphi\in\mathfrak{M}(APW(\mathbb{R}))$ and there exists $\lambda\in\mathbb{R}$ such that $\Phi(\Psi(\varphi))=e_\lambda$, where Φ is the function defined in statement (5.7) and Ψ is the homeomorphism between $\mathfrak{M}(APW(\mathbb{R}))$ and $\mathfrak{M}(l^1(\mathbb{R}))$ defined above. Taking into account definition (5.6), we know that the function $\varphi\circ T$ satisfies

$$(\varphi\circ T)(x)=\sum_{s\in\mathbb{R}}x(s)e^{i\lambda s},$$

for each $x \in l^1(\mathbb{R})$. Let us consider the function $\widetilde{\varphi}$ defined by

$$\widetilde{\varphi}(y) := \begin{cases} \varphi(y) = \varphi(T(x)) = \sum_{s\in\mathbb{R}} x(s)e^{i\lambda s}, & \text{if} \quad y \in APW(\mathbb{R}), \\ \lim_{n\to\infty} \varphi(y_n) = \lim_{n\to\infty} \varphi(T(x_n)) = \lim_{n\to\infty} \sum_{s\in\mathbb{R}} x_n(s)e^{i\lambda s}, & \text{if} \quad y \in AP(\mathbb{R}) \setminus APW(\mathbb{R}), \end{cases}$$

where $(y_n)_{n\in\mathbb{N}}$ is any sequence of elements in $APW(\mathbb{R})$ that converges to y, which exists because $APW(\mathbb{R})$ is embedded densely in $AP(\mathbb{R})$, and $x, (x_n)_{n\in\mathbb{N}} \in l^1(\mathbb{R})$ are the only elements that verify $y = T(x)$ and $y_n = T(x_n)$ for each $n \in \mathbb{N}$. In these conditions $\widetilde{\varphi}$ is the only extension of φ to $AP(\mathbb{R})$, that is, $\varphi \in \mathcal{R}$ and therefore it follows that $\mathfrak{M}_C(APW(\mathbb{R})) \subseteq \mathcal{R}$.

Due to the fact $\mathfrak{M}_C(APW(\mathbb{R}))$ is dense in $\mathfrak{M}(APW(\mathbb{R}))$ and $\mathfrak{M}_C(APW(\mathbb{R})) \subseteq \mathcal{R} \subseteq \mathfrak{M}(APW(\mathbb{R}))$, we get that $\mathcal{R}$ is dense in $\mathfrak{M}(APW(\mathbb{R}))$, and due to the fact that $\mathcal{R}$ is a closed set we have that $\mathcal{R} = \mathfrak{M}(APW(\mathbb{R}))$ as we wanted to prove. ☐

5.8 Inverse Closedness of $AP_p(\mathbb{R})$ in $AP(\mathbb{R})$ and in $L^\infty(\mathbb{R})$

In order to establish the inverse closedness of $AP_p(\mathbb{R})$ in $AP(\mathbb{R})$, we are going to prove that $\mathfrak{M}(AP(\mathbb{R}))$ is homeomorphic to $\mathfrak{M}(AP_p(\mathbb{R}))$, and we will also characterize the invertible elements in $AP_p(\mathbb{R})$.

The proof of the following theorem is analogous to that one of [8, Chapter 7, Section 3, Theorem 3.4], where the authors considered the similar problem for some algebras $\mathcal{S}$ and $\mathcal{S}_2$ containing the subalgebras $APW(\mathbb{R})$ and $AP(\mathbb{R})$, respectively.

Theorem 5.8.1. *Let $\varphi \in \mathfrak{M}(AP(\mathbb{R}))$. Then $\psi := \varphi_{|AP_p(\mathbb{R})}$ belongs to $\mathfrak{M}(AP_p(\mathbb{R}))$, and all of the non-null multiplicative linear functionals of $AP_p(\mathbb{R})$ are exhausted by the functionals of this kind, that is, $\mathfrak{M}(AP(\mathbb{R}))$ is homeomorphic to $\mathfrak{M}(AP_p(\mathbb{R}))$.*

Proof. Applying Theorem 5.7.5, it follows that $\mathfrak{M}(APW(\mathbb{R}))$ and $\mathfrak{M}(AP(\mathbb{R}))$ are homeomorphic. Taking into account Theorem 5.4.3, that $APW(\mathbb{R})$ is embedded densely in $AP_p(\mathbb{R})$ and $AP_p(\mathbb{R})$ is embedded densely in $AP(\mathbb{R})$, it follows that $\mathfrak{M}(AP(\mathbb{R}))$ is homeomorphic to $\mathfrak{M}(AP_p(\mathbb{R}))$ as we wanted to prove. ☐

We are now in position to prove the main result of this section.

Theorem 5.8.2 ([5, Proposition 19.4])**.** *Let $p \in]1, +\infty[$. An element $f \in AP_p(\mathbb{R})$ is invertible if and only if*

$$\inf_{x\in\mathbb{R}} |f(x)| > 0.$$

Proof. Let $f \in AP_p(\mathbb{R})$ be an invertible element. Taking into account Theorem 5.6.5, it follows that

$$AP_p(\mathbb{R}) \subseteq AP(\mathbb{R}),$$

consequently, f is also invertible in $AP(\mathbb{R})$ and, applying Theorem 3.5.4, we get that $f \in U(\mathbb{R})$ and also that

$$\inf_{x\in\mathbb{R}} |f(x)| > 0.$$

Suppose now that $f \in AP_p(\mathbb{R})$ verifies

$$\inf_{x\in\mathbb{R}} |f(x)| > 0.$$

In these conditions, using Theorem 2.2.4, we get that f is invertible in $AP(\mathbb{R})$ and, consequently, applying Theorem 5.3.3, we get that

$$\varphi(f) \neq 0$$

for each $\varphi \in \mathfrak{M}(AP(\mathbb{R}))$. Suppose, by contradiction, that there exists $\varphi_0 \in \mathfrak{M}(AP_p(\mathbb{R}))$ such that $\varphi_0(f) = 0$. Then, taking into account Theorem 5.8.1, it follows that $\mathfrak{M}(AP(\mathbb{R}))$ is homeomorphic to $\mathfrak{M}(AP_p(\mathbb{R}))$ and thus there is $\varphi_0^* \in \mathfrak{M}(AP(\mathbb{R}))$ that verifies

$$\varphi_0(a) = \varphi_0^*(a)$$

for every $a \in AP_p(\mathbb{R})$. Therefore we have

$$\varphi_0^*(f) = \varphi_0(f) = 0,$$

which is impossible because f is invertible in $AP(\mathbb{R})$. Hence $\varphi(f) \neq 0$ for any $\varphi \in \mathfrak{M}(AP_p(\mathbb{R}))$ which implies, by Theorem 5.3.3, that f is invertible in $AP_p(\mathbb{R})$ as we wanted to prove. □

Applying the previous theorem, our desired result is immediate.

Theorem 5.8.3 ([5, Proposition 19.4])**.** *The algebra $AP_p(\mathbb{R})$ is inverse-closed in $AP(\mathbb{R})$, and therefore is inverse-closed in $L^\infty(\mathbb{R})$.*

Proof. Let $\psi \in AP_p(\mathbb{R})$ such that ψ is invertible in $AP(\mathbb{R})$. In these conditions, using Theorems 2.2.4 and 3.5.4, we have that $\psi \in U(\mathbb{R})$ and

$$\inf_{x\in\mathbb{R}} |\psi(x)| > 0.$$

Applying Theorem 5.8.2, we conclude that ψ is also invertible in $AP_p(\mathbb{R})$ as we wanted to prove. □

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