Schur-Finiteness (and Bass-Finiteness) Conjecture for Quadric Fibrations and Families of Sextic du Val del Pezzo Surfaces

Gonçalo Tabuada

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Abstract. Let \( Q \to B \) be a quadric fibration and \( T \to B \) a family of sextic du Val del Pezzo surfaces. Making use of the theory of noncommutative mixed motives, we establish a precise relation between the Schur-finiteness conjecture for \( Q \), resp. for \( T \), and the Schur-finiteness conjecture for \( B \). As an application, we prove the Schur-finiteness conjecture for \( Q \), resp. for \( T \), when \( B \) is low-dimensional. Along the way, we obtain a proof of the Schur-finiteness conjecture for smooth complete intersections of two or three quadric hypersurfaces. Finally, we prove similar results for the Bass-finiteness conjecture.

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1 Introduction

Schur-finiteness conjecture

Let \( \mathcal{C} \) be a \( \mathbb{Q} \)-linear, idempotent complete, symmetric monoidal category. Given a partition \( \lambda \) of an integer \( n \geq 1 \), consider the corresponding \( \mathbb{Q} \)-linear representation \( V_\lambda \) of the symmetric group \( S_n \) and the associated idempotent \( e_\lambda \in \mathbb{Q}[S_n] \). Under these notations, the Schur-functor \( S_\lambda: \mathcal{C} \to \mathcal{C} \) sends an object \( a \in \mathcal{C} \) to the direct summand of \( a^\otimes n \) determined by \( e_\lambda \). Following Deligne [11, §1], an object \( a \in \mathcal{C} \) is called Schur-finite if it is annihilated by some Schur-functor. Voevodsky introduced in [39] a triangulated category of geometric mixed motives \( \text{DM}_{gm}(k)_{\mathbb{Q}} \) (over a perfect base field \( k \)). By construction, this
category is $\mathbb{Q}$-linear, idempotent complete, symmetric monoidal, and comes equipped with a $\otimes$-functor $M(-)_Q: \operatorname{Sm}(k) \to \operatorname{DM}_{\operatorname{sm}}(k)_Q$ defined on smooth $k$-schemes of finite type. Given $X \in \operatorname{Sm}(k)$, an important conjecture in the theory of motives is the following:

**Conjecture S(X):** The geometric mixed motive $M(X)_Q$ is Schur-finite.

Thanks to the (independent) work of Guletskii [12] and Mazza [28], the conjecture $S(X)$ holds in the case where $\dim(X) \leq 1$. Thanks to the work of Kimura [21] and Shermenev [31], the conjecture $S(X)$ also holds in the case where $X$ is an abelian variety. Besides these cases (and some other cases scattered in the literature), the Schur-finiteness conjecture remains wide open. The main goal of this note is to prove the Schur-finiteness conjecture in the new cases of quadric fibrations and families of sextic du Val del Pezzo surfaces.

**Quadric fibrations**

Our first main result is the following:

**Theorem 1.** Let $q: Q \to B$ a flat quadric fibration of relative dimension $d - 2$. Assume that $B$ and $Q$ are $k$-smooth, that all the fibers of $q$ have corank $\leq 1$, and that the locus $D \subset B$ of the critical values of the fibration $q$ is $k$-smooth. Under these assumptions, the following holds:

(i) When $d$ is even, we have $S(Q) \iff S(B) + S(\tilde{B})$, where $\tilde{B}$ stands for the discriminant 2-fold cover of $B$ (ramified over $D$).

(ii) When $d$ is odd and $\operatorname{char}(k) \neq 2$, we have $\{S(V_i)\} + \{S(\tilde{D}_i)\} \Rightarrow S(Q)$, where $V_i$ is any affine open of $B$ and $\tilde{D}_i$ is any Galois 2-fold cover of $D_i := D \cap V_i$.

To the best of the author’s knowledge, Theorem 1 is new in the literature. Intuitively speaking, it relates the Schur-finiteness conjecture for the total space $Q$ with the Schur-finiteness conjecture for certain coverings/subschemes of the base $B$. Among other ingredients, its proof makes use of Kontsevich’s noncommutative mixed motives of twisted root stacks; consult §3-§4 below for details.

Making use of Theorem 1, we are now able to prove the Schur-finiteness conjecture in new cases. Here are two low-dimensional examples:

**Corollary 2 (Quadric fibrations over curves).** Let $q: Q \to B$ be a quadric fibration as in Theorem 1 with $B$ a curve. In this case, $S(Q)$ holds.

**Corollary 3 (Quadric fibrations over surfaces).** Let $q: Q \to B$ be a quadric fibration as in Theorem 1 with $B$ a surface and $d$ odd. In this case, the implication $S(B) \Rightarrow S(Q)$ holds.

**Proof.** Given a smooth $k$-surface $X$, we have $S(X) \iff S(U)$ for any open $U$ of $X$. Therefore, thanks to Theorem 1(ii), the proof follows from the fact that when $B$ is a surface, the conjectures $\{S(V_i)\}$ can be replaced by the conjecture $S(B)$.

\[\text{\footnote{Since $B$ is a curve, the locus $D \subset B$ of the critical values of $q$ is necessarily $k$-smooth.}}\]
Corollary 3 can be applied to the case where $B$ is (an open subscheme of) an abelian surface or a smooth projective surface with $p_g = 0$ which satisfies Bloch’s conjecture (see Guletskii-Pedrini [13, §4 Thm. 7]). Recall that Bloch’s conjecture holds for surfaces not of general type (see Bloch-Kas-Leiberman [6]), for surfaces which are rationally dominated by a product of curves (see Kimura [21]), for Godeaux, Catanese and Barlow surfaces (see Voisin [40, 41]), etc.

**Remark 4 (Related work).** Let $q: Q \to B$ be a quadric fibration as in Theorem 1. In the particular case where $Q$ and $B$ are smooth projective, Bouali [9] and Vial [38, §4] “computed” the Chow motive $h(Q)_Q$ of $Q$ using smooth projective $k$-schemes of dimension $\leq \dim(B)$. Since the category of Chow motives (with $\mathbb{Q}$-coefficients) embeds fully-faithfully into $\text{DM}_{gm}(k)$ (see [39, §4]), these computations lead to an alternative “geometric” proof of Corollaries 2-3. Note that in Theorem 1 and in Corollaries 2-3 we do not assume that $Q$ and $B$ are projective; we are (mainly) interested in geometric mixed motives and not in pure motives.

**Intersections of quadrics**

Let $Y \subset \mathbb{P}^{d-1}$ be a smooth complete intersection of $m$ quadric hypersurfaces. The linear span of these quadric hypersurfaces gives rise to a flat quadric fibration $q: Q \to \mathbb{P}^{m-1}$ of relative dimension $d - 2$, with $Q$ $k$-smooth. Under these notations, our second main result is the following:

**Theorem 5.** We have $S(Q) \Rightarrow S(Y)$. When $2m \leq d$, the converse also holds.

By combining Theorem 5 with the above Corollaries 2-3, we hence obtain a proof of the Schur-finiteness conjecture in the following cases:

**Corollary 6 (Intersections of two or three quadrics).** Assume that the quadric fibration $q: Q \to \mathbb{P}^{m-1}$ is as in Theorem 1. In this case, the conjecture $S(Y)$ holds when $Y$ is a smooth complete intersection of two, or of three odd-dimensional, quadric hypersurfaces.

**Families of sextic du Val del Pezzo surfaces**

Recall that a sextic du Val del Pezzo surface $X$ is a projective $k$-scheme with at worst du Val singularities and ample anticanonical class such that $K_X^2 = 6$. Consider a family of sextic du Val del Pezzo surfaces $f: T \to B$, i.e. a flat morphism $f$ such that for every geometric point $x \in B$ the associated fiber $T_x$ is a sextic du Val del Pezzo surface. Following Kuznetsov [26, §5], given $d \in \{2, 3\}$, let us write $\mathcal{M}_d$ for the relative moduli stack of semistable sheaves on fibers of $T$ over $B$ with Hilbert polynomial $h_d(t) := (3t + d)(t + 1)$, and $Z_d$ for the coarse moduli space of $\mathcal{M}_d$. By construction, we have finite flat morphisms $Z_2 \to B$ and $Z_3 \to B$ of degrees 3 and 2, respectively. Under these notations, our third main result is the following:
Theorem 7. Let $f: T \to B$ be a family of sextic du Val del Pezzo surfaces. Assume that $\text{char}(k) \notin \{2, 3\}$ and that $T$ is $k$-smooth. Under these assumptions, we have the equivalence of conjectures $S(T) \Leftrightarrow S(B) + S(Z_2) + S(Z_3)$.

To the best of the author’s knowledge, Theorem 7 is new in the literature. It leads to a proof of the Schur-finiteness conjecture in new cases. Here is an illustrative example:

Corollary 8 (Families of sextic du Val del Pezzo surfaces over curves). Let $f: T \to B$ be a family of sextic du Val del Pezzo surfaces as in Theorem 7 with $B$ a curve. In this case, the conjecture $S(T)$ holds.

Remark 9. Let $f: T \to B$ be a family of sextic du Val del Pezzo surfaces as in Theorem 7. To the best of the author’s knowledge, the associated geometric mixed motive $M(T)_Q$ has not been “computed” (in any non-trivial particular case). Nevertheless, consult Helmsauer [16] for the “computation” of the Chow motive $h(X)_Q$ of certain smooth (projective) del Pezzo surfaces $X$.

Remark 10 (Conservativity conjecture). Given a field $k$ equipped with a complex embedding $\sigma: k \to \mathbb{C}$, recall from Ayoub [3, Conj. 2.1] that the conservativity conjecture asserts that the Betti realization functor $B_{\sigma}: \text{DM}_{gm}(k)_Q \to \mathcal{D}(\mathbb{Q})$ is conservative. As explained in [3, Prop. 2.26], if the conservativity conjecture holds, then every object of the category $\text{DM}_{gm}(k)_Q$ is Schur-finite. In particular, the conjecture $S(X)$ holds for every smooth $k$-scheme of finite type $X$ (when $k$ is equipped with a complex embedding). However, despite the (monumental) work of Ayoub [4], the conservativity conjecture remains wide open.

Bass-finiteness conjecture

Let $k$ be a finite base field and $X$ a smooth $k$-scheme of finite type. The Bass-finiteness conjecture $B(X)$ (see [5, §9]) is one of the oldest and most important conjectures in algebraic $K$-theory. It asserts that the algebraic $K$-theory groups $K_n(X), n \geq 0,$ are finitely generated. In the same vein, given an integer $r \geq 2$, we can consider the conjecture $B(X)_{1/r}$, where $K_n(X)$ is replaced by $K_n(X)_{1/r} := K_n(X) \otimes \mathbb{Z}[1/r]$. Our fourth main result is the following:

Theorem 11. The following holds:

(i) Theorem 1 and Corollaries 2-3 hold similarly for the conjecture $B(-)_{1/2}$. In Corollary 2, the groups $K_n(Q)_{1/2}, n \geq 2,$ are moreover finite.

(ii) Theorem 5 holds similarly for the conjecture $B(-)$.

(iii) Corollary 6 holds similarly for the conjecture $B(-)_{1/2}$. In the case where $Y$ is a smooth complete intersection of two quadric hypersurfaces, the groups $K_n(Y)_{1/2}, n \geq 2,$ are moreover finite.

I hope that Ayoub manages to correct his work [4] in the (near) future.

Corollary 3 (for the conjecture $B(-)_{1/2}$) can also be applied to the case where $B$ is (an open subscheme of) an abelian surface; see [19, Cor. 70 and Thm. 82].
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(iv) **Theorem 7 and Corollary 8 hold similarly for the conjecture B(−)1/6. In Corollary 8, the groups K_n(T)1/6, n ≥ 2, are moreover finite.**

2 Preliminaries

In what follows, all schemes/stacks are of finite type over the perfect base field $k$.

**Dg categories**

For a survey on dg categories we invite the reader to consult [20]. In what follows, we will write dgcat($k$) for the category of (essentially small) dg categories and dg functors. Every (dg) $k$-algebra $A$ gives naturally rise to a dg category with a single object. Another source of examples is provided by schemes/stacks. Given a $k$-scheme $X$ (or stack $X$), the category of perfect complexes of $O_X$-modules perf($X$) admits a canonical dg enhancement perf$_{dg}$(X); consult [20, §4.6] [27] for details. More generally, given a sheaf of $O_X$-algebras $F$, we can consider the dg category of perfect complexes of $F$-modules perf$_{dg}(X;F)$.

**Noncommutative mixed motives**

For a book, resp. survey, on noncommutative motives we invite the reader to consult [33], resp. [32]. Recall from [33, §8.5.1] (see also [22, 23, 24]) the definition of Kontsevich’s triangulated category of noncommutative mixed motives NMot($k$). By construction, this category is idempotent complete, symmetric monoidal, and comes equipped with a $\otimes$-functor $U:dgcat(k) \to NMot(k)$. In what follows, given a $k$-scheme $X$ (or stack $X$) equipped with a sheaf of $O_X$-algebras $F$, we will write $U(X;F) := U(perf_{dg}(X;F))$.

3 Noncommutative mixed motives of twisted root stacks

Let $X$ be a $k$-scheme, $L$ a line bundle on $X$, $\sigma \in \Gamma(X,L)$ a global section, and $r > 0$ an integer. In what follows, we will write $D \subset X$ for the zero locus of $\sigma$. Recall from [10, Def. 2.2.1] (see also [1, Appendix B]) that the associated root stack $\mathcal{X}$ is defined as the following fiber-product of algebraic stacks

$$
\mathcal{X} := \sqrt{(L,\sigma)}/\mathbb{X} \longrightarrow \mathbb{A}^1/\mathbb{G}_m
$$

where $\theta_r$ stands for the morphism induced by the $r^{th}$ power maps on $\mathbb{A}^1$ and $\mathbb{G}_m$. A twisted root stack $(\mathcal{X};F)$ consists of a root stack $\mathcal{X}$ equipped with a sheaf of Azumaya algebras $F$. In what follows, we will write $s$ for the product of
the ranks of $\mathcal{F}$ (at each one of the connected components of $X$). The following result, of independent interest, will play a key role in the proof of Theorem 1.

**Theorem 12.** Assume that $X$ and $D$ are $k$-smooth.

(i) We have an isomorphism $U(X) \simeq U(X) \oplus U(D)^{\oplus(r-1)}$.

(ii) Assume moreover that $\text{char}(k) \neq r$ and that $k$ contains the $r$th roots of unity. Under these extra assumptions, $U(X; \mathcal{F})_{1/r}$ belongs to the smallest thick triangulated subcategory of $\text{NMot}(k)_{1/r}$ containing the non-commutative mixed motives $\{U(V_i)_{1/r}\}$ and $\{U(\mathcal{D}_i)_{1/r}\}$, where $V_i$ is any affine open subscheme of $X$ and $\mathcal{D}_i$ is any Galois $l$-fold cover of $D_i := D \cap V_i$ with $l \nmid r$ and $l \neq 1$.

**Proof.** We start by proving item (i). Following [18, Thm. 1.6], the pull-back functor $p^*$ is fully-faithful and we have the following semi-orthogonal decomposition\(^4\) $\text{perf}(X) = \langle \text{perf}(D_{r-1}), \ldots, \text{perf}(D_1), p^*(\text{perf}(X)) \rangle$. All the categories $\text{perf}(D_i)$ are equivalent (via a Fourier-Mukai type functor) to $\text{perf}(D)$. Therefore, since the functor $U: \text{dgc}(k) \to \text{NMot}(k)$ sends semi-orthogonal decompositions to direct sums, we obtain the searched direct sum decomposition $U(X) \simeq U(X) \oplus U(D)^{\oplus(r-1)}$.

Let us now prove item (ii). We consider first the particular case where $X = \text{Spec}(A)$ is affine and the line bundle $\mathcal{L} = \mathcal{O}_X$ is trivial. Let $\mu_r$ be the group of $r$th roots of unity and $\chi: \mu_r \to k^\times$ a (fixed) primitive character. Under these notations, consider the global quotient $\lfloor \text{Spec}(B)/\mu_r \rfloor$, where $B := A[t]/(t^r - \sigma)$ and the $\mu_r$-action on $B$ is given by $g \cdot t := \chi(g)^{-1}t$ for every $g \in \mu_r$ and by $g \cdot a := a$ for every $a \in A$. As explained in [10, Example 2.4.1], the root stack $X$ agrees, in this particular case, with the global quotient $\lfloor \text{Spec}(B)/\mu_r \rfloor$. By construction, the induced map $\text{Spec}(B) \to X$ is a $r$-fold cover ramified over $D \subset X$. Moreover, for every $l$ such that $l \mid r$ and $l \neq 1$, the associated closed subscheme $\text{Spec}(B)^{\mu_l}$ agrees with the ramification divisor $D \subset \text{Spec}(B)$. Therefore, since the functor $U(\cdot)_{1/r}: \text{dgc}(k) \to \text{NMot}(k)_{1/r}$ is an additive invariant of dg categories in the sense of [33, Def. 2.1] (see [33, §8.4.5]), we conclude from [36, Cor. 1.28(ii)] that, in this particular case, $U(X; \mathcal{F})_{1/r}$ belongs to the smallest thick additive subcategory of $\text{NMot}(k)_{1/r}$ containing the noncommutative mixed motives $U(\text{Spec}(B))^{\mu_l}_{1/r}$ and $\{U(\mathcal{D}_i)_{1/r}\}$, where $\mathcal{D}_i$ is any Galois $l$-fold cover of $D$ with $l \nmid r$ and $l \neq 1$. Furthermore, since the geometric quotient $\text{Spec}(B)/\mu_r$ agrees with $X$ and the latter scheme is $k$-smooth, [36, Thm. 1.22] implies that $U(\text{Spec}(B))^{\mu_l}_{1/r}$ is isomorphic to $U(X)_{1/r}$. This finishes the proof of item (ii) in the particular case where $X$ is affine and the line bundle $\mathcal{L}$ is trivial.

Let us now prove item (ii) in the general case. As explained above, given any affine open subscheme $V_i$ of $X$ which trivializes the line bundle $\mathcal{L}$, the noncommutative mixed motive $U(V_i; \mathcal{F}_i)_{1/r}$, with $V_i := p^{-1}(V_i)$ and $\mathcal{F}_i := \mathcal{F}|_{V_i}$.
belongs to the smallest thick additive subcategory of NMot\((k)_{1/r^s}\) containing 
\(U(V_{1})_{1/r^s}\) and \(\{U(\tilde{D}_i)_{1/r^s}\}\), where \(\tilde{D}\) is any Galois \(l\)-fold cover of \(D_i := D \cap V_i\) with \(l \mid r\) and \(l \neq 1\). Let us then choose an affine open cover \(\{W_i\}\) of \(X\) which

trivializes the line bundle \(L\). Since \(X\) is quasi-compact (recall that \(X\) is of finite

type over \(k\)), this affine open cover admits a finite subcover. Consequently, the

proof follows by induction from the \(\mathbb{Z}[1/r^s]\)-linearization of the distinguished

triangles of Lemma 13 below.

\[\text{Lemma 13. Given an open cover } \{W_1, W_2\} \text{ of } X, \text{ we have an induced Mayer-Vietoris distinguished triangle of noncommutative mixed motives}\]

\[U(X; \mathcal{F}) \rightarrow U(W_1; \mathcal{F}_1) \oplus U(W_2; \mathcal{F}_2) \rightarrow U(W_{12}; \mathcal{F}_{12}) \rightarrow \Sigma U(X; \mathcal{F}), \quad (14)\]

where \(W_{12} := W_1 \cap W_2\) and \(\mathcal{F}_{12} := \mathcal{F}_{W_{12}}\).

\[\text{Proof. Consider the following commutative diagram of dg categories}\]

\[
\begin{array}{c}
\text{perf}_{dg}(X; \mathcal{F})_Z \rightarrow \text{perf}_{dg}(X; \mathcal{F}) \rightarrow \text{perf}_{dg}(W_1; \mathcal{F}_1) \\
| \, \, \, | \\
\text{perf}_{dg}(W_2; \mathcal{F}_2) \rightarrow \text{perf}_{dg}(W_2; \mathcal{F}_2) \rightarrow \text{perf}_{dg}(W_{12}; \mathcal{F}_{12}),
\end{array}
\]

where \(Z\) stands for the closed complement \(X - W_1 = W_2 - W_{12}\) and \(\text{perf}_{dg}(X; \mathcal{F})_Z\), resp. \(\text{perf}_{dg}(W_2; \mathcal{F}_2)_Z\), stands for the full dg subcategory of \(\text{perf}_{dg}(X; \mathcal{F})\), resp. \(\text{perf}_{dg}(W_2; \mathcal{F}_2)\), consisting of those perfect complexes of \(\mathcal{F}\)-modules, resp. \(\mathcal{F}_2\)-modules, that are supported on \(Z\). Both rows are short exact sequences of dg categories in the sense of Drinfeld/Keller (see [20, §4.6]) and the left vertical dg functor is a Morita equivalence. Therefore, since the functor \(U: \text{dgc}(k) \rightarrow \text{NMot}(k)\) is a localizing invariant of dg categories in the sense of [33, §8.1], we obtain the following morphism of distinguished triangles:

\[
\begin{array}{c}
U(\text{perf}_{dg}(X; \mathcal{F})_Z) \rightarrow U(X; \mathcal{F}) \rightarrow U(W_1; \mathcal{F}_1) \rightarrow \Sigma U(\text{perf}_{dg}(X; \mathcal{F})_Z) \\
| \, \, \, | \\
U(\text{perf}_{dg}(W_2; \mathcal{F}_2)_Z) \rightarrow U(W_2; \mathcal{F}_2) \rightarrow U(W_{12}; \mathcal{F}_{12}) \rightarrow \Sigma U(\text{perf}_{dg}(W_2; \mathcal{F}_2)_Z).
\end{array}
\]

Finally, since the middle square is homotopy (co)cartesian, we hence obtain the claimed Mayer-Vietoris distinguished triangle (14).

\[\text{4 Proof of Theorem 1}\]

Following [25, §3] (see also [2, §1.2]), let \(E\) be a vector bundle of rank \(d\) on \(B\), \(q': \mathbb{P}(E) \rightarrow B\) the projectivization of \(E\) on \(B\), \(\mathcal{O}_{\mathbb{P}(E)}(1)\) the Grothendieck line bundle on \(\mathbb{P}(E)\), \(\mathcal{L}\) a line bundle on \(B\), and finally

\[\rho \in \Gamma(B, S^2(E^\vee) \otimes \mathcal{L}^\vee) = \Gamma(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(2) \otimes \mathcal{L}^\vee)\]
a global section. Given this data, recall that \( Q \subset \mathbb{P}(E) \) is defined as the zero
locus of \( \rho \) on \( \mathbb{P}(E) \) and that \( q: Q \to B \) is the restriction of \( q' \) to \( Q \); note that
the relative dimension of \( q \) is equal to \( d - 2 \). Consider also the discriminant
of rank 2

Recall from \([25, \S 3.5]\) (see also \([2, \S 1.6]\)) that when \( d \) is even, we can consider the
discriminant cover \( \tilde{B} := \text{Spec}_B(Z(\mathcal{Cl}_0(q))) \) of \( B \), where \( Z(\mathcal{Cl}_0(q)) \) stands for
the center of the sheaf \( \mathcal{Cl}_0(q) \) of even parts of the Clifford algebra associated to \( q \);
see \([25, \S 3]\) (and also \([2, \S 1.5]\)). By construction, \( \tilde{B} \) is a 2-fold cover ramified
over \( D \). Moreover, since \( D \) is \( k \)-smooth, \( \tilde{B} \) is also \( k \)-smooth.

Recall from \([25, \S 3.6]\) (see also \([2, \S 1.7]\)) that when \( d \) is odd and \( \text{char}(k) \neq 2 \), we
can consider the discriminant stack \( X := \sqrt[2]{(\det(E')^\otimes 2 \otimes (\mathcal{L}')^\otimes d, \text{disc}(q))}/\tilde{B} \). Since \( \text{char}(k) \neq 2 \), \( X \) is a Deligne-Mumford stack with coarse moduli space \( B \).

**Proposition 15.** Under the above assumptions, the following holds:

(i) When \( d \) is even, we have \( U(Q)_{1/2} \simeq U(\tilde{B})_{1/2} \oplus U(B)^{\otimes (d-2)}_{1/2} \).

(ii) When \( d \) is odd and \( \text{char}(k) \neq 2 \), \( U(Q)_{1/2} \) belongs to the smallest thick
triangulated subcategory of \( \text{NMot}(k)_{1/2} \) containing the noncommutative
mixed motives \( \{U(V_i)_{1/2}\} \) and \( \{U(D_i)_{1/2}\} \), where \( V_i \) is any affine open
subscheme of \( B \) and \( D_i \) is any Galois 2-fold cover of \( D := D \cap V_i \).

**Proof.** As proved in \([25, \text{Thm. 4.2}]\) (see also \([2, \text{Thm. 2.2.1}]\)), we have the
following semi-orthogonal decomposition

\[
\text{perf}(Q) = (\text{perf}(B; \mathcal{Cl}_0(q)), \text{perf}(B)_{1}, \ldots, \text{perf}(B)_{d-2})
\]

where \( \text{perf}(B)_j := q'(\text{perf}(B)) \otimes \mathcal{O}_{Q/B}(j) \). All the categories \( \text{perf}(B)_j \) are equivalent
(via a Fourier-Mukai type functor) to \( \text{perf}(B) \). Therefore, since the functor
\( U: \text{dgcat}(\mathcal{Cl}_0(q)) \to \text{perf}(B) \) sends semi-orthogonal decompositions to direct sums,
we obtain the decomposition \( U(Q) \simeq U(B; \mathcal{Cl}_0(q)) \oplus U(B)^{\otimes (d-2)}_{1/2} \).

We start by proving item (i). As explained in \([25, \S 3.5]\) (see also \([2, \S 1.6]\)),
when \( d \) is even, the category \( \text{perf}(B; \mathcal{Cl}_0(q)) \) is equivalent (via a Fourier-Mukai
type functor) to \( \text{perf}(\tilde{B}; \mathcal{F}) \) where \( \mathcal{F} \) is a certain sheaf of Azumaya algebras on \( \tilde{B} \)
of rank \( 2^{d-1} \). This leads to an isomorphism \( U(B; \mathcal{Cl}_0(q)) \simeq U(\tilde{B}; \mathcal{F}) \).

Making use of \([37, \text{Thm. 2.1}]\), we hence conclude that \( U(B; \mathcal{Cl}_0(q))_{1/2} \) is isomorphic to
\( U(\tilde{B}; \mathcal{F})_{1/2} \simeq U(\tilde{B})_{1/2} \). Consequently, we obtain the isomorphism of item (i).

Let us now prove item (ii). As explained in \([25, \S 3.6]\) (see also \([2, \S 1.7]\)), when \( d \) is odd,
the category \( \text{perf}(B; \mathcal{Cl}_0(q)) \) is equivalent (via a Fourier-Mukai type
functor) to \( \text{perf}(X; \mathcal{F}) \) where \( \mathcal{F} \) is a certain sheaf of Azumaya algebras on \( X \)
of rank \( 2^{d-1} \). This leads to an isomorphism \( U(B; \mathcal{Cl}_0(q)) \simeq U(X; \mathcal{F}) \).

By combining Theorem 12(ii) with the isomorphism \( U(Q) \simeq U(X; \mathcal{F}) \oplus U(B)^{\otimes (d-2)}_{1/2} \),
we hence conclude that \( U(Q)_{1/2} \) belongs to the smallest thick triangulated
subcategory of \( \text{NMot}(k)_{1/2} \) containing \( U(B)_{1/2} \), \{\( U(V_i)_{1/2} \)\}, and \{\( U(D_i)_{1/2} \)\}.
where \( V_i \) is any affine open subscheme of \( B \) and \( \tilde{D}_i \) is any Galois 2-fold cover of \( D_i \). We now claim that \( U(B)_{1/2} \) belongs to the smallest thick triangulated subcategory of \( \text{NMot}(k)_{1/2} \) containing \( \{ U(V_i)_{1/2} \} \); note that this would conclude the proof. Choose an affine open cover \( \{ W_i \} \) of \( B \). Since \( B \) is quasi-compact (recall that \( B \) is of finite type over \( k \)), this affine open cover admits a finite subcover. Therefore, similarly to the proof of Theorem 12, our claim follows from an inductive argument using the \( \mathbb{Z}[1/2] \)-linearization of the Mayer-Vietoris distinguished triangles \( U(B) \to U(W_1) \oplus U(W_2) \xrightarrow{\delta} U(W_{12}) \xrightarrow{\partial} \Sigma U(B) \).

As proved in [34, Thm. 2.8], there exists a \( \mathbb{Q} \)-linear, fully-faithful, \( \otimes \)-functor \( \Phi \) making the following diagram commute

\[
\begin{array}{cccc}
\text{Sm}(k) & \xrightarrow{X \mapsto \text{perf}_{dg}(X)} & \text{dgcat}(k) \\
M(-)_{\mathbb{Q}} & & U(-)_{\mathbb{Q}} \\
\text{DM}_{gm}(k)_{\mathbb{Q}} & \pi & \text{NMot}(k)_{\mathbb{Q}} \\
\text{DM}_{gm}(k)_{\mathbb{Q}} \setminus \mathbb{Q}(1)[2] & \Phi & \text{NMot}(k)_{\mathbb{Q}},
\end{array}
\]

where \( \text{DM}_{gm}(k)_{\mathbb{Q}} \setminus \mathbb{Q}(1)[2] \) stands for the orbit category with respect to the Tate motive \( \mathbb{Q}(1)[2] \) and \( \text{Hom}(-,U(k)_{\mathbb{Q}}) \) for the internal Hom of the monoidal structure; note that the functors \( X \mapsto \text{perf}_{dg}(X) \) and \( \text{Hom}(-,U(k)_{\mathbb{Q}}) \) are contravariant. By construction, \( \pi \) is a faithful \( \otimes \)-functor. Therefore, it follows from [28, Lem. 1.11] that we have the following equivalence:

\[ S(X) \Leftrightarrow \text{noncommutative mixed motive } (\Phi \circ \pi)(M(X)_{\mathbb{Q}}) \text{ is Schur-finite.} \]

We now have all the ingredients necessary to conclude the proof of Theorem 1.

**Item (i)**

The above functors \( \pi \) and \( \text{Hom}(-,U(k)_{\mathbb{Q}}) \) are \( \mathbb{Q} \)-linear. Therefore, by combining Proposition 15(i) with the commutative diagram (16), we conclude that

\[ (\Phi \circ \pi)(M(Q)_{\mathbb{Q}}) \simeq (\Phi \circ \pi)(M(\tilde{B})_{\mathbb{Q}}) \oplus (\Phi \circ \pi)(M(B)_{\mathbb{Q}})^{\oplus (d-2)}. \]

Since Schur-finiteness is stable under direct sums and direct summands, the proof of the equivalence \( S(Q) \Leftrightarrow S(\tilde{B}) + S(B) \) follows then from (17)-(18).

**Item (ii)**

Recall from [33, §8.5.1-8.5.2] that, by construction, \( \text{NMot}(k)_{\mathbb{Q}} \) is a \( \mathbb{Q} \)-linear closed symmetric monoidal triangulated category in the sense of Hovey [17, §6-7]. As proved in [12, Thm. 1], this implies that Schur-finiteness has the 2-out-of-3 property with respect to distinguished triangles. The functor \( \text{Hom}(-,U(k)_{\mathbb{Q}}) \)
is triangulated. Hence, by combining Proposition 15(ii) with the commutative
diagram (16), we conclude that \((\Phi \circ \pi)(M(Q)_Q)\) belongs to the smallest thick
triangulated subcategory of \(\text{NMod}(k)_Q\) containing the noncommutative mixed
motives \(\{\Phi \circ \pi(M(V)_Q)\}\) and \(\{\Phi \circ \pi(M(\tilde{D})_Q)\}\), where \(V\) is any affine open
subscheme of \(B\) and \(\tilde{D}\) is any Galois 2-fold cover of \(D\). Since by assumption
the conjectures \(\{S(V_i)\}\) and \(\{S(\tilde{D}_i)\}\) hold, (17) implies that the noncommuta-
tive mixed motives \(\{\Phi \circ \pi(M(V)_Q)\}\) and \(\{\Phi \circ \pi(M(\tilde{D})_Q)\}\) are Schur-finite.
Therefore, making use of the 2-out-of-3 property of Schur-finiteness with re-
spect to distinguished triangles (and of the stability of Schur-finiteness under
direct summands), we conclude that \((\Phi \circ \pi)(M(Q)_Q)\) is also Schur-finite. The
proof follows now from the above equivalence (17).

5 Proof of Theorem 5

Recall from the proof of Proposition 15 that we have the semi-orthogonal de-
composition \(\text{perf}(Q) = \langle \text{perf}(\mathbb{P}^{m-1}; C\mathfrak{L}_0(q)), \text{perf}(\mathbb{P}^{m-1}), \ldots, \text{perf}(\mathbb{P}^{m-1}; d-2) \rangle\),
and consequently the following direct sum decomposition:

\[
U(Q) \simeq U(\mathbb{P}^{m-1}; C\mathfrak{L}_0(q)) \oplus U(\mathbb{P}^{m-1})^{\oplus (d-2)}.
\] (19)

As proved in [25, Thm. 5.5] (see also [2, Thm. 2.3.7]), the following also holds:

(a) When \(2m < d\), we have the following semi-orthogonal decomposition
\(\text{perf}(Y) = \langle \text{perf}(\mathbb{P}^{m-1}; C\mathfrak{L}_0(q)), \mathcal{O}(1), \ldots, \mathcal{O}(d - 2m) \rangle\). Consequently,
since the functor \(U: \text{dgcat}(k) \to \text{NMod}(k)\) sends semi-orthogonal de-
compositions to direct sums, we obtain the following direct sum decomposi-
tion \(U(Y) \simeq U(\mathbb{P}^{m-1}; C\mathfrak{L}_0(q)) \oplus U(k)^{\oplus (d-2m)}\).

(b) When \(2m = d\), the category \(\text{perf}(Y)\) is equivalence (via a Fourier-Mukai
type functor) to \(\text{perf}(\mathbb{P}^{m-1}; C\mathfrak{L}_0(q))\). Consequently, we obtain an isomor-
phism of noncommutative mixed motives \(U(Y) \simeq U(\mathbb{P}^{m-1}; C\mathfrak{L}_0(q))\).

(c) When \(2m > d\), \(\text{perf}(Y)\) is an admissible subcategory of
\(\text{perf}(\mathbb{P}^{m-1}; C\mathfrak{L}_0(q))\). Hence, \(U(Y)\) is a direct summand of \(U(\mathbb{P}^{m-1}; C\mathfrak{L}_0(q))\).

Let us now prove the implication \(S(Q) \Rightarrow S(Y)\). If the conjecture \(S(Q)\)
holds, then it follows from the decomposition (19), from the commutative
diagram (16), from the equivalence (17), and from the stability of Schur-
finiteness under direct summands, that the noncommutative mixed motive
\(\text{Hom}(U(\mathbb{P}^{m-1}; C\mathfrak{L}_0(q))_Q; U(k)_Q)\) is Schur-finite. Making use of the above de-
scriptions (a)-(c) of \(U(Y)\) and of the commutative diagram (16), we hence
conclude that the noncommutative mixed motive \((\Phi \circ \pi)(M(Y)_Q)\) is also Schur-
finite. Consequently, the conjecture \(S(Y)\) follows now from the above equiva-
ience (17). Finally, note that when \(2m \leq d\), a similar argument proves the
converse implication \(S(Y) \Rightarrow S(Q)\).
6 Proof of Theorem 7

Recall first from [26, Prop. 5.12] that since char(k) \not\in \{2, 3\} and T is k-smooth, the k-schemes B, Z_2 and Z_3 are also k-smooth.

**Proposition 20.** We have \( U(T)_{1/6} \simeq U(B)_{1/6} \oplus U(Z_2)_{1/6} \oplus U(Z_3)_{1/6} \).

**Proof.** As proved in [26, Thm. 5.2 and Prop. 5.10], we have the semi-orthogonal decomposition \( \text{perf}(T) = (\text{perf}(B), \text{perf}(Z_2; F_2), \text{perf}(Z_3; F_3)) \), where \( F_2 \) (resp. \( F_3 \)) is a certain sheaf of Azumaya algebras over \( Z_2 \) (resp. \( Z_3 \)) of order 2 (resp. 3). Recall that the functor \( U: \text{dgcat}(k) \to \text{NMot}(k) \) sends semi-orthogonal decompositions to direct sums. Hence, we obtain the direct sum decomposition:

\[
U(T) \simeq U(B) \oplus U(Z_2; F_2) \oplus U(Z_3; F_3).
\]

Since \( F_2 \) (resp. \( F_3 \)) is of order 2 (resp. 3), the rank of \( F_2 \) (resp. \( F_3 \)) is necessarily a power of 2 (resp. 3). Making use of [37, Thm. 2.1], we hence conclude that the noncommutative mixed motive \( U(Z_2; F_2)_{1/2} \) (resp. \( U(Z_3; F_3)_{1/3} \)) is isomorphic to \( U(Z_2)_{1/2} \) (resp. \( U(Z_3)_{1/3} \)). Consequently, the proof follows now from the \( \mathbb{Z}[1/6] \)-linearization of (21).

The functors \( \pi \) and \( \text{Hom}(-, U(k)_\mathbb{Q}) \) in (16) are \( \mathbb{Q} \)-linear. Therefore, similarly to the proof of item (i) of Theorem 1, by combining Proposition 20 with the commutative diagram (16), we conclude that

\[
(\Phi \circ \pi)(M(T)_\mathbb{Q}) \simeq (\Phi \circ \pi)(M(B)_\mathbb{Q}) \oplus (\Phi \circ \pi)(M(Z_2)_\mathbb{Q}) \oplus (\Phi \circ \pi)(M(Z_3)_\mathbb{Q}).
\]

Since Schur-finiteness is stable under direct sums and direct summands, the proof follows then from the combination of (22) with the equivalence (17).

7 Proof of Theorem 11

**Item (i)**

We start by proving the first claim. As explained in [33, §8.6] (see also [35, Thm. 15.10]), given \( X \in \text{Sm}(k) \), we have the isomorphisms of abelian groups:

\[
\text{Hom}_{\text{NMot}(k)}(U(k), \Sigma^{-n}U(X)) \simeq K_n(X) \quad n \in \mathbb{Z}.
\]

Assume that \( d \) is even. By combining Proposition 15(i) with the \( \mathbb{Z}[1/2] \)-linearization of (23), we conclude that \( K_n(Q)_{1/2} \simeq K_n(B)_{1/2} \oplus K_n(B)_{1/2}^{(d-2)} \). Therefore, since finite generation is stable under direct sums and direct summands, we obtain the equivalence \( B(Q)_{1/2} \iff B(B)_{1/2} + B(B)_{1/2} \). Assume now that \( d \) is odd and that \( \text{char}(k) \neq 2 \). Finite generation has the 2-out-of-3 property with respect to (short or long) exact sequences and is stable under direct summands. Therefore, the proof of the following implication

\[
\{B(V_i)_{1/2}\} + \{B(D_i)_{1/2}\} \Rightarrow B(Q)_{1/2}
\]
follows from the combination of Proposition 15(ii) with the $\mathbb{Z}[1/2]$-linearization of (23). Finally, recall from [14, 29, 30] that the conjecture $B(X)$ holds in the case where $\dim(X) \leq 1$. Therefore, the Corollaries 2-3 also hold similarly for the conjecture $B(-)_{1/2}$.

We now prove the second claim. Let $q: Q \to B$ be a quadric fibration as in Theorem 1 with $B$ a curve. Thanks to Corollary 2 (for the conjecture $B(-)_{1/2}$), it suffices to show that the groups $K_n(Q), n \geq 2,$ are torsion. Assume first that $d$ is even. By combining Proposition 15(i) with the $\mathbb{Q}$-linearization of (23), we obtain an isomorphism $K_n(Q) \cong K_n(B)_{\mathbb{Q}} \oplus K_n(B)_{\mathbb{Q}}^{(d-2)}$. Thanks to Proposition 24 below, we have $K_n(B)_{\mathbb{Q}} = K_n(B)_{\mathbb{Q}} = 0$ for every $n \geq 2$.

Therefore, we conclude that the groups $K_n(Q), n \geq 2,$ are torsion. Assume now that $d$ is even and that $\text{char}(k) \neq 2$. Thanks to Proposition 15(ii), $U(Q)_{\mathbb{Q}}$ belongs to the smallest thick triangulated subcategory of $\text{NMot}(k)_{\mathbb{Q}}$ containing the noncommutative mixed motives $\{U(V_i)\}_{\mathbb{Q}}$ and $\{U(D_i)\}_{\mathbb{Q}},$ where $V_i$ is any affine open subscheme of $B$ and $D_i$ is any Galois $2$-fold cover of $D_i$. Moreover, $U(Q)_{\mathbb{Q}}$ may be explicitly obtained from $\{U(V_i)\}_{\mathbb{Q}}$ and $\{U(D_i)\}_{\mathbb{Q}}$ using solely the $\mathbb{Q}$-linearization of the Mayer-Vietoris distinguished triangles. Therefore, since $K_n(V_i)_{\mathbb{Q}} = 0$ for every $n \geq 2$ (see Proposition 24 below) and $K_n(D_i)_{\mathbb{Q}} = 0$ for every $n \geq 1$ (see Quillen’s computation [30] of the algebraic $K$-theory of a finite field), an inductive argument using the $\mathbb{Q}$-linearization of (23) and the $\mathbb{Q}$-linearization of the Mayer-Vietoris distinguished triangles implies that the groups $K_n(Q), n \geq 2,$ are torsion.

**Proposition 24.** We have $K_n(X)_{\mathbb{Q}} = 0, n \geq 2$, for every smooth $k$-curve $X$.

**Proof.** In the particular case where $X$ is affine, this result was proved in [15, Cor. 3.2.3] (see also [14, Thm. 0.5]). In the general case, choose an affine open cover $\{W_i\}$ of $X$. Since $X$ is quasi-compact, this affine open cover admits a finite subcover. Therefore, the proof follows from an inductive argument (similar to the one in the proof of Theorem 12(ii)) using the $\mathbb{Q}$-linearization of (23) and the $\mathbb{Q}$-linearization of the Mayer-Vietoris distinguished triangles. 

**Item (ii)**

If the conjecture $B(Q)$ holds, then it follows from the decomposition (19) and from the isomorphisms (23) that the algebraic $K$-theory groups $K_n(\text{perf}_{dg}(\mathbb{P}^{n-1}; Cl_0(q))), n \geq 0,$ are finitely generated. Therefore, by combining the descriptions (a)-(c) of the noncommutative mixed motive $U(Y)$ (see the proof of Theorem 5) with (23), we conclude that the conjecture $B(Y)$ also holds. Note that when $2m \leq d$, a similar argument proves the converse implication $B(Y) \Rightarrow B(Q)$.

**Item (iii)**

Items (i)-(ii) of Theorem 11 imply that Corollary 6 holds similarly for the conjecture $B(-)_{1/2}$. We now address the second claim. Let $q: Q \to \mathbb{P}^1$ be
the quadric fibration associated to the smooth complete intersection \( Y \) of two quadric hypersurfaces. Thanks to item (i), the groups \( K_n(Q)_{1/2}, n \geq 2, \) are finite. Hence, making use of the decomposition (19), of the \( \mathbb{Z}[1/2] \)-linearization of (23), and of the above descriptions (a)-(c) of \( U(Y) \) (see the proof of Theorem 8), we conclude that the groups \( K_n(Y)_{1/2}, n \geq 2, \) are also finite.

**Item (iv)**

We start by proving the first claim. By combining Proposition 20 with the \( \mathbb{Z}[1/6] \)-linearization of (23), we conclude that

\[
K_n(T)_{1/6} \simeq K_n(B)_{1/6} \oplus K_n(Z_2)_{1/6} \oplus K_n(Z_3)_{1/6}.
\]

Therefore, since finite generation is stable under sums and direct summands, we obtain the equivalence \( B(T)_{1/6} \iff B(B)_{1/6} + B(Z_2)_{1/6} + B(Z_3)_{1/6} \). As mentioned in the proof of item (i), the conjecture \( B(X) \) holds in the case where \( \dim(X) \leq 1 \). Hence, Corollary 8 also holds similarly for the conjecture \( B(-1)_{1/6} \).

We now prove the second claim. Let \( f: T \to B \) be a family of sextic du Val del Pezzo surfaces as in Theorem 7 with \( B \) a curve. Similarly to the proof of item (i) of Theorem 11, it suffices to show that the groups \( K_n(T), n \geq 2, \) are torsion. By combining Proposition 20 with the \( \mathbb{Q} \)-linearization of (23), we obtain an isomorphism \( K_n(T)_{\mathbb{Q}} \simeq K_n(B)_{\mathbb{Q}} \oplus K_n(Z_2)_{\mathbb{Q}} \oplus K_n(Z_3)_{\mathbb{Q}} \). Thanks to Proposition 24, we have moreover \( K_n(B)_{\mathbb{Q}} = K_n(Z_2)_{\mathbb{Q}} = K_n(Z_3)_{\mathbb{Q}} = 0 \) for every \( n \geq 2 \). Therefore, we conclude that the groups \( K_n(T), n \geq 2, \) are torsion.

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**References**


Gonçalo Tabuada
Mathematics Institute
Zeeman Building
University of Warwick
Coventry CV4 7AL
UK
and
Departamento de Matemática and
Centro de Matemática e Aplicações (CMA)
FCT, UNL
Quinta da Torre
2829-516 Caparica
Portugal