Identification results for inverse source problems in unsteady Stokes flows

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Abstract. We establish identification results in an inverse source problem for the 2D unsteady Stokes equations. This identification problem is cast in the context of a non destructive evaluation problem that consists in retrieving a pair of body and divergence forces from the corresponding Cauchy data. Results are established for data obtained from a single and several measurements.

Keywords: Inverse source problems, unsteady Stokes equations, Brinkman flows.

1. Introduction

In this paper we consider an inverse source problem for the Stokes/Brinkman system of equations

$$\left\{\begin{array}{ll} (\Delta-\lambda)\mathbf{u}_\lambda-\nabla p_\lambda=\mathbf{f} & \text{in } \Omega \\ \nabla\cdot\mathbf{u}_\lambda=g & \text{in } \Omega \\ \mathbf{u}_\lambda=0 & \text{on } \Gamma \end{array}\right. ,\ \lambda\geq 0. \tag{1}$$

We study identification results for the pair of body forces $\mathbf{f}$ and divergence source g, from boundary traction data $T(\mathbf{u}_\lambda,p_\lambda)\mathbf{n}|_\Gamma$. Brinkman's equations are a model proposed by H. C. Brinkman for viscous fluid flows in porus materials (cf. [12]) and has been considered in many engineering problems, for instance in petroleum engineering (eg. [11]) and microfluidics (eg.[17]). The constant λ plays the role of the medium's resistance to the flow (eg. [6]). For $\lambda=0$ we obtain Stokes flow whilst for $\lambda=\infty$ we obtain a Darcy's fluid flow.

Brinkman's equations can also be obtained by taking Laplace transform with respect to time in the unsteady Stokes equations $-\frac{\partial\mathbf{u}}{\partial t}+\Delta\mathbf{u}-\nabla p=\mathbf{f}$. In this case, $\lambda=s$ is the transform argument. There are some papers on inverse source problems for unsteady Stokes problems. For, instance in [3] the authors deal with the determination of the body force $\mathbf{f}(t,x)=r(t)\mathbf{g}(x)$ (see also [10]) from data in an arbitrary subdomain $\omega\subset\Omega$. However, these are intrusive methods.

Our aim is to study a non intrusive inverse problem. For Stokes flows there are several literature dealing with non intrusive problems. For instance, in [19] the authors studied the identification of obstacles from boundary data. Several methods were also considered for this inverse obstacle problem. For instance, an iterative method was applied in [20] whilst in [7] a method based on integral equations was considered. For inverse obstacle problems in Brinkman's systems, we refer the paper [16] where the authors studied the factorization method for reconstructing the shape of the obstacle. We refer also the inverse source problem studied in [14] that consists in the identification and reconstruction of point forces in steady state Stokes problems.

Inverse problems for the reconstruction of body sources from boundary data have been studied by several authors. We refer the paper [18] where the problem was studied in the context of Laplace equation, considering only one single boundary measurement. For the Helmholtz equation, the problem was studied in eg. [8] (see also [21]) in a multifrequency setting. The same type of problem was studied more recently for unsteady Lamé equations in [9]. In these problems, the identification of the source was obtained by showing that the corresponding Fourier coefficients are retrievable from the boundary data (albeit requiring several measurements). Although we apply some of the techniques from in [8] and [9] it will be clear that they cannot be fully adapted to the unsteady Stokes problem. In fact, it is easy to see that even in the multifrequency setting there is always a part of the body force invisible from the boundary data. For the divergence source term the same type of phenomenon occurs. The question is then, which Fourier coefficients can be retrieved form the boundary data.

The paper is organized as follows. In section two we introduce the direct and inverse problems and give the functional setting. In section three we study the identification from one single measurement. We show in particular that, in absence of body forces, the only retrievable part of the divergence term g is the harmonic part. This is somehow related to the fact that for incompressible fluids ($g = 0$) and moreover for $g = constant$, the pressure is an harmonic function. In section four, we address the multifrequency problem and show that the coefficients of the divergence free part of the body force and the harmonic part of g can be retrieved from boundary data. In order to deal with the resolvent equations, we have considered some adapted functions whose properties are included as an Appendix to the paper.

Although we focus on the 2D problem, it seems that many of the following results can be adapted for the 3D case.

2. Direct and inverse problems

Let $\Omega \subseteq \mathbb{R}^2$ be an open, bounded, and simply connected set with C^2 boundary $\Gamma := \partial\Omega$. Let λ be a non negative constant, which we call the frequency. The body force $\mathbf{f}$ is an element in the product space $\mathbf{L}^2(\Omega) := (L^2(\Omega))^2$ whilst for the divergence term g, notice that the following compatibility condition

$$\int_\Gamma \nabla \cdot \mathbf{u} dx = \int_\Gamma \mathbf{u} \cdot \mathbf{n} d\sigma$$

implies

$$\int_\Omega g dx = 0.$$

The appropriate setting for g is then

$$L_0^2(\Omega) := \left\{ g \in L^2(\Omega) : \int_\Omega g dx = 0 \right\} \cong L^2(\Omega)/\mathbb{R}.$$

For $(\mathbf{f}, g) \in \mathbb{L}^2(\Omega) := \mathbf{L}^2(\Omega) \times L_0^2(\Omega)$, the Brinkman/Stokes problem (1) is well posed with $(\mathbf{u}_\lambda, p_\lambda) \in \mathbf{H}^1(\Omega) \times L_0^2(\Omega)$ (eg. [4]).

The direct problem consists in the determination of the traction vector $T(\mathbf{u}_\lambda, p_\lambda)\mathbf{n}|_\Gamma$,

$$T(\mathbf{u}_\lambda, p_\lambda)\mathbf{n}|_\Gamma := (-p_\lambda I + 2\epsilon(\mathbf{u}_\lambda))\mathbf{n}|_\Gamma$$

where $\epsilon(\mathbf{u}_\lambda) = \frac{1}{2}\left(\nabla \mathbf{u}_\lambda + \nabla \mathbf{u}_\lambda^\top\right)$ is the stress-strain tensor of $\mathbf{u}_\lambda$ and $\mathbf{n}$ is the outward pointing normal vector at Γ. In the above functional setting, the Neumann data $T(\mathbf{u}_\lambda, p_\lambda)\mathbf{n}|_\Gamma$ belongs to $\mathbf{H}^{-1/2}(\Gamma)$.

The *inverse problem* consists in retrieving the pair of body forces and divergence source, $(\mathbf{f}, g)$, from the generated traction data (measured data) $T(\mathbf{u}_\lambda, p_\lambda)\mathbf{n}|_\Gamma$, with $(\mathbf{u}_\lambda, p_\lambda)$ satisfying (1). Here we are assuming the knowledge of the non slip boundary condition $\mathbf{u}_\lambda|_\Gamma = 0$, the geometry of the domain and the frequency $\lambda \geq 0$.

It is clear that, independently of λ, some pairs $(\mathbf{f}, g)$ generate null boundary data and cannot be identified. For instance, consider pairs $(\nabla q, 0)$, with $q \in H_0^1(\Omega)$. Then, $q\mathbf{n}|_\Gamma = 0$ and is obvious that this source cannot be identified from boundary data because

$$\begin{cases} (\Delta - \lambda)0 - \nabla(-q) = \nabla q & \text{in } \Omega \\ \nabla \cdot 0 = 0 & \text{in } \Omega \\ T(0, -q)\mathbf{n} = 0 & \text{on } \Gamma \end{cases}.$$

We start with identification results from one measurement (fixed frequency).

3. Identification of body forces and divergence from one measurement

In order to address the identification question, it will be more convenient to formulate the inverse problem as a (linear) equation. Recall that, for a given a pair $(\mathbf{f}, g) \in \mathbb{L}^2(\Omega)$, there corresponds an unique $(\mathbf{u}_\lambda, p_\lambda) \in \mathbf{H}^1(\Omega) \times L_0^2(\Omega)$ which solves (1). This defines a linear map $\Lambda_\lambda : \mathbb{L}^2(\Omega) \to \mathbf{H}^{-1/2}(\Gamma)$,

$$\Lambda_\lambda(\mathbf{f}, g) := T(\mathbf{u}_\lambda, p_\lambda)\mathbf{n}|_\Gamma.$$

The inverse problem is then to, given $\mathbf{h}$, solve the equation $\Lambda_\lambda(\mathbf{f}, g) = \mathbf{h}$.

The kernel of the map

$$\ker \Lambda_\lambda = \left\{(\mathbf{f}, g) \in \mathbb{L}^2(\Omega) : \mathbf{f} = (\Delta - \lambda)\mathbf{u} - \nabla p, \nabla \cdot \mathbf{u} = g,\ \mathbf{u}|_\Gamma = T(\mathbf{u}, p)\mathbf{n}|_\Gamma = 0\right\}$$

gives the sources that leave trivial information on the boundary. His orthogonal complement is the space of sources that can be identified from the corresponding traction data. We give a characterization of such space. Let

$$F_\lambda := \left\{(\mathbf{u}, p) \in \mathbf{H}^1(\Omega) \times L^2_0(\Omega) : (\Delta - \lambda)\mathbf{u} = \nabla p,\ \ \nabla \cdot \mathbf{u} = constant\right\}. \tag{2}$$

Theorem 1. *The identity*

$$F_\lambda^\perp = \ker \Lambda_\lambda$$

holds and it follows

$$\mathbb{L}^2(\Omega) = \overline{F_\lambda}^{\mathbb{L}^2(\Omega)} \oplus \ker \Lambda_\lambda. \tag{3}$$

Hence, the $\overline{F_\lambda}^{\mathbb{L}^2(\Omega)}$ *part of a pair of sources* $(\mathbf{f}, g)$ *is fully identified from* $\Lambda_\lambda(\mathbf{f}, g)$.

Proof. Recall Green's second identity (eg. [13])

$$\int_\Omega \left[((\Delta - \lambda)\mathbf{u} - \nabla p) \cdot \mathbf{v} - \mathbf{u} \cdot ((\Delta - \lambda)\mathbf{v} - \nabla q)\right] dx + \int_\Omega (\nabla \cdot \mathbf{u} q - p \nabla \cdot \mathbf{v}) dx$$
$$= \int_{\partial\Omega} (T(\mathbf{u}, p)\mathbf{n} \cdot \mathbf{v} - \mathbf{u} \cdot T(\mathbf{v}, q)\mathbf{n})\, d\sigma. \tag{4}$$

We start by proving the inclusion $F_\lambda^\perp \subseteq \ker \Lambda_\lambda$. Let $(\mathbf{f}, g) \in F_\lambda^\perp$ so that

$$0 = \int_\Omega (\mathbf{f}, g) \cdot \overline{(\mathbf{v}, q)} dx, \quad \forall (\mathbf{v}, q) \in F_\lambda. \tag{5}$$

Notice that exists a pair $(\mathbf{u}, p) \in \mathbf{H}^1_0(\Omega) \times L^2_0(\Omega)$ satisfying (1) for the input source $(\mathbf{f}, g)$.

From Green's second identity (4) we obtain

$$0 = \int_\Omega (\mathbf{f}, g) \cdot \overline{(\mathbf{v}, q)} dx = \int_\Omega p \nabla \cdot \overline{\mathbf{v}} dx + \int_\Gamma T(\mathbf{u}, p)\mathbf{n} \cdot \overline{\mathbf{v}} d\sigma, \tag{6}$$

for all $(\mathbf{v}, q)$ such that $(\Delta - \lambda)\mathbf{v} - \nabla q = 0$.

On the other hand, any $\mathbf{v} \in \mathbf{H}^{1/2}(\Gamma)$ can be lifted to $\widehat{\mathbf{v}} \in \mathbf{H}^1(\Omega)$ such that

$$\begin{cases} (\Delta - \lambda)\widehat{\mathbf{v}} - \nabla q = 0 & \text{in } \Omega \\ \nabla \cdot \widehat{\mathbf{v}} = \frac{1}{|\Omega|} \int_\Gamma \mathbf{v} \cdot \mathbf{n} dx & \text{in } \Omega \end{cases}, \quad q \in L^2_0(\Omega).$$

Clearly $(\widehat{\mathbf{v}}, q) \in F_\lambda$ and

$$\int_\Omega p \nabla \cdot \overline{\widehat{\mathbf{v}}} dx = \frac{1}{|\Omega|} \int_\Gamma \overline{\mathbf{v}} \cdot \mathbf{n} d\Gamma \int_\Omega p dx = 0,$$

because $p \in L^2_0(\Omega)$. Substituting in identity (6) we obtain

$$\int_\Gamma T(\mathbf{u},p)\mathbf{n}\cdot\overline{\mathbf{v}}d\Gamma = 0,\ \ \forall\mathbf{v}\in\mathbf{H}^{1/2}(\Gamma).$$

It follows $||T(\mathbf{u},p)\mathbf{n}||_{\mathbf{H}^{-1/2}(\Gamma)} = 0$ hence $T(\mathbf{u},p)\mathbf{n} = 0$ and $(\mathbf{f},g)\in\ker\Lambda_\lambda$.

Conversely, suppose $(\mathbf{f},g)\in\ker\Lambda_\lambda$. Given any $(\mathbf{v},q)\in F_\lambda$ we obtain

$$\begin{aligned}\int_\Omega(\mathbf{f},g)\cdot\overline{(\mathbf{v},q)}dx &= \int_\Omega \mathbf{u}\cdot((\Delta-\lambda)\overline{\mathbf{v}}-\nabla\overline{q})dx \\ &+ \int_\Omega p\nabla\cdot\overline{\mathbf{v}}dx + \int_\Gamma(T(\mathbf{u},p)\mathbf{n}\cdot\overline{\mathbf{v}} - \mathbf{u}\cdot T(\overline{\mathbf{v}},\overline{q})\mathbf{n})d\Gamma = 0 \qquad (7)\end{aligned}$$

so that $(\mathbf{f},g)\in F_\lambda^\perp$. □

As discussed in the begging of this section, we cannot identify the $\nabla H^1_0(\Omega)$ part of a source $\mathbf{f}$. Taking into account the orthogonal decomposition (A.2) we can only aim to identify the divergence free part of the body force. This implies (eg. [5]) that the divergence free part, $\mathbf{f}_{\nabla\cdot}$ is of the form

$$\mathbf{f}_{\nabla\cdot} = \mathbf{curl}\, q, \qquad (8)$$

for some stream function $q\in H^1(\Omega)$ (unique up to a constant). Here **curl** represents the 2D vector curl (eg. [5]),

$$\mathbf{curl}\, q = \nabla^\perp q = (\partial_2 q, -\partial_1 q).$$

3.1. Identification from steady/unsteady Stokeslets

Let

$$\Phi^0(x) = \frac{1}{4\pi}\left[-\log|x|\delta_{ij} + \frac{x\otimes x}{|x|^2}\right]$$

denote the 2D *steady Stokeslet* tensor and

$$\Phi^\lambda(x) := \frac{1}{2\pi}\left[d_1(\sqrt{\lambda}|x|)\delta_{ij} + \frac{x\otimes x}{|x|^2}d_2(\sqrt{\lambda|x|})\right],\ \ \lambda>0$$

the *unsteady Stokeslet* tensor, with d_1 and d_2 defined by

$$d_1(\zeta) = \frac{\zeta K_0(\zeta)+K_1(\zeta)}{\zeta} - \frac{1}{\zeta^2},\ \ d_2(\zeta) = \frac{2}{\zeta^2} - K_2(\zeta)$$

and K_i is the modified Bessel function of order i.

By Φ^λ_y we denote the Stokeslet centered at y, that is $\Phi^\lambda_y := \Phi^\lambda(\bullet - y)$. Furthermore, we consider the pressure vector

$$\Psi_y(x) := \frac{x-y}{2\pi|x-y|^2}.$$

The pair (Φ_y^λ, Ψ_y) is a fundamental solution for the Brinkman/Stokes system (eg. [6]), that is,

$$\left\{ \begin{array}{l} (\Delta - \lambda)\Phi_y^\lambda - \nabla \Psi_y = -\delta_y I \\ \nabla \cdot \Phi_y^\lambda = 0 \end{array} \right. , \quad \lambda \geq 0.$$

Suppose that the point sources y at located at some artificial boundary $\widehat{\Gamma}$ enclosing the domain Ω. Then, we have the following boundary density result (cf. [2]).

Theorem 2. *The set* $\{\Phi_y^0 \mathrm{e}_i : y \in \widehat{\Gamma}\}$ *spans a dense subset in the boundary space* $\mathbf{H}_{\mathbf{n}}^{1/2}(\Gamma) + \mathbb{R}^2$, *with*

$$\mathbf{H}_{\mathbf{n}}^{1/2}(\Gamma) := \left\{ \mathbf{u} \in \mathbf{H}^{1/2}(\Gamma) : \int_\Gamma \mathbf{u} \cdot \mathbf{n} d\sigma = 0 \right\}.$$

Remark 3. *1) The constants can be dropped if we take, for instance,* $\widehat{\Gamma}$ *enclosing a domain in the exterior of* Ω *(see for instance [15]).*

2) Similar results are valid for the Stokes resolvent ($\lambda > 0$). *In this case however, there is no need to add constants (cf. [1]).*

From well posedness of direct problem the steady/unsteady Stokelests set $\left\{(\Phi_y^\lambda e_i, \Psi_y e_i)|_\Omega : y \in \widehat{\Gamma}\right\}$ spans a dense subspace in

$$\left\{(\mathbf{u}, p) \in \mathbf{H}^1(\Omega) \times L_0^2(\Omega) : (\Delta - \lambda)\mathbf{u} = \nabla p, \ \nabla \cdot \mathbf{u} = 0\right\} \subseteq F_\lambda.$$

This means that the retrievable part of the pair body force and divergence can be approximated using superpositions of fundamental solutions.

It is not clear however the space obtained from superposition of Stokeslets. Still, since Ψ_y is harmonic we can expect to retrieve the harmonic part of the divergence term g. Indeed, we now show that for a pure divergence problem we can identify the harmonic part of g.

3.2. Identification for a pure divergence problem

In the absence of body forces (ie, $\mathbf{f} = 0$), follows from Theorem 1 that

$$L_0^2(\Omega) = \widetilde{F}_\lambda \oplus \ker \Lambda_\lambda^0,$$

where

$$\widetilde{F}_\lambda = \left\{p \in L_0^2(\Omega) : \nabla p = (\Delta - \lambda)\mathbf{u}, \ \nabla \cdot \mathbf{u} = constant\right\}$$

and

$$\Lambda_\lambda^0(g) := \Lambda_\lambda(0, g).$$

In particular,

$$\ker \Lambda_\lambda^0 = \left\{g = \nabla \cdot \mathbf{u} \in L_0^2(\Omega) : (\Delta - \lambda)\mathbf{u} = \nabla p, \ \mathbf{u}|_\Gamma = T(\mathbf{u}, p)\mathbf{n}|_\Gamma = 0\right\}$$

We now show that the spaces $\ker \Lambda_\lambda^0$ are all equal.

Theorem 4. *Given $g = \nabla \cdot \mathbf{u} \in \ker \Lambda^0_\lambda$, we have*

$$\mathbf{u} = \nabla p_0,$$

for some $p_0 \in H^1_0(\Omega) \cap H^2(\Omega)$.

Proof. Let $\mathbf{v} = \phi_{\kappa,\mathbf{d}}$ be a shear wave (S wave), that is

$$\phi_{\kappa,\mathbf{d}}(x) := \mathbf{d}^\perp e^{i\kappa x\cdot\mathbf{d}}, \tag{9}$$

$\kappa > 0$, $\mathbf{d} = (d_1, d_2) \in S^1 := \partial B(0,1)$ and $\mathbf{d}^\perp = (d_2, -d_1)$. Due to remark 15 in Appendix A, it suffices to show that

$$\int_\Omega \mathbf{u} \cdot \overline{\mathbf{v}} dx = 0, \quad \forall \kappa > 0, \quad \mathbf{d} \in S^1.$$

This is a consequence of Green's identity. In fact, considering the test function $(\mathbf{v}, 0)$, integration by parts gives

$$\int_\Omega (((\Delta - \lambda)\mathbf{u} - \nabla p) \cdot \overline{\mathbf{v}} - \mathbf{u} \cdot (\Delta - \lambda)\overline{\mathbf{v}})\, dx + \int_\Omega (\nabla \cdot \mathbf{u} \times 0 - p\nabla \cdot \overline{\mathbf{v}})\, dx = 0$$

where we have omitted the vanishing boundary integrals. This identity reduces to

$$(\kappa^2 + \lambda) \int_\Omega \mathbf{u} \cdot \overline{\mathbf{v}} dx = 0$$

and the conclusion follows. □

Corollary 5. *We have*

$$\ker \Lambda^0_{\lambda_1} = \ker \Lambda^0_{\lambda_2}.$$

Proof. Let $g \in \ker \Lambda^0_{\lambda_1}$. From previous proposition, $g = \nabla \cdot \mathbf{u}$, with

$$\mathbf{u} = \nabla p_0, \quad p_0 \in H^1_0(\Omega).$$

On the other hand, $(\Delta - \lambda_1)\mathbf{u} - \nabla p = 0$ so that we can write

$$(\Delta - \lambda_2)\mathbf{u} - \nabla(p + (\lambda_1 - \lambda_2)p_0) = 0.$$

Since

$$T(\mathbf{u}, p + (\lambda_1 - \lambda_2)p_0)\mathbf{n}|_\Gamma = T(\mathbf{u}, p)\mathbf{n}|_\Gamma + T(0, (\lambda_1 - \lambda_2)p_0)\mathbf{n}|_\Gamma = (\lambda_1 - \lambda_2)p_0\mathbf{n}|_\Gamma = 0,$$

where the last identity holds because $p_0 \in H^1_0(\Omega)$, we conclude that $g \in \ker \Lambda^0_{\lambda_2}$. The other inclusion can be established using the same ideias. □

An immediate consequence is that the class of $L^2_0(\Omega)$ divergence sources that can be retrieved from traction data does not depend on the considered frequency λ. Indeed, the spaces $\widetilde{F}_\lambda$ are all equal and it is sufficient to consider $\widetilde{F}_0$ sources.

Theorem 6. *Given $g \in \ker \Lambda_0^0$ there exists $\mathbf{u} \in \mathbf{H}^1(\Omega)$ such that*

$$\Delta \mathbf{u} - \nabla g = 0, \quad g = \nabla \cdot \mathbf{u}$$

with $\mathbf{u} = T(\mathbf{u}, g)\mathbf{n} = 0$ on Γ.

Proof. Let $g \in \ker \Lambda_0^0$ so that $g = \nabla \cdot \mathbf{u}$ with

$$\Delta \mathbf{u} = \nabla p, \quad \mathbf{u}|_\Gamma = T(\mathbf{u}, p)\mathbf{n}|_\Gamma = 0.$$

We show that $g = p$.

Consider the pair of test functions $(\mathbf{v}, q) = (-i/\kappa \psi_{\kappa,\mathbf{d}}, e^{i\kappa \bullet \cdot \mathbf{d}})$ where

$$\psi_{\kappa,\mathbf{d}} := \mathbf{d} e^{i\kappa x\cdot \mathbf{d}}, \ \ \kappa > 0, \ \ \mathbf{d} \in S^1 \tag{10}$$

is a pressure wave. Green's formula yields the identity

$$\int_\Omega \mathbf{u} \cdot (\Delta \mathbf{v} - \nabla q) dx = \int_\Omega (p \nabla \cdot \mathbf{v} - \nabla \cdot \mathbf{u} q)\, dx.$$

Since $\Delta \mathbf{v} - \nabla q = 0$ and $\nabla \cdot \mathbf{v} = q$ then above equation can be written as

$$\int_\Omega (p - \nabla \cdot \mathbf{u})\, q dx = 0.$$

It follows that

$$\mathcal{F}(p - \nabla \cdot \mathbf{u})(\frac{\kappa}{2\pi}\mathbf{d}) = 0, \ \ \forall \kappa > 0, \ \mathbf{d} \in S^1,$$

where $\mathcal{F}$ is the Fourier transform. Hence, $p = \nabla \cdot \mathbf{u} = g$. □

We can now give the following characterization for $\widetilde{F}_0$.

Theorem 7. *We have*

$$\widetilde{F}_0 = \overline{\{q \in H^1(\Omega) : \Delta \mathbf{v} - \nabla \nabla \cdot \mathbf{v} = \nabla q, \ \ \mathbf{v} \in \mathbf{H}^2(\Omega)\}}^{L_0^2(\Omega)}.$$

Proof. Define

$$G = \left\{ q \in H^1(\Omega) : \Delta \mathbf{v} - \nabla \nabla \cdot \mathbf{v} = \nabla q, \ \ \mathbf{v} \in \mathbf{H}^2(\Omega) \right\}. \tag{11}$$

Due to identity $\widetilde{F}_0^\perp = \ker \Lambda_0^0$, it is sufficient to show that

$$G^\perp = \ker \Lambda_0^0.$$

Take $p_0 \in G^\perp$. Then,

$$\int_\Omega p_0 \overline{q} dx = 0, \ \ \forall \overline{q} \in G.$$

On the other hand, for $p_0 \in L_0^2(\Omega)$ there exists $(\mathbf{u}, p) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$ such that $\Delta \mathbf{u} = \nabla p$ and $\nabla \cdot \mathbf{u} = p_0$, both in Ω.

We establish that $T(\mathbf{u},p)\mathbf{n}|_\Gamma = 0$ from where the inclusion $G^\perp \subset \ker \Lambda_0^0$ follows. An application of Green's formula gives

$$\int_\Omega \mathbf{u}\cdot\nabla\nabla\cdot\overline{\mathbf{v}}dx - \int_\Omega (\nabla\cdot\mathbf{u}\overline{q} - p\nabla\cdot\overline{\mathbf{v}})\,dx = -\int_\Gamma T(\mathbf{u},p)\mathbf{n}\cdot\overline{\mathbf{v}}d\sigma.$$

Now if we consider fundamental solutions $(\Phi_y^0 e_i, \Psi_y^0 e_i) \in \mathbf{H}^2(\Omega)\times H^1(\Omega)$ centered at point sources in the conditions of Theorem 2 we have

$$\Delta\Phi_y^0 e_i - \nabla\nabla\cdot\Phi_y^0 e_i = \nabla\Psi_y^0 e_i$$

hence $\Psi_y^0 e_i \in G$ and substituting in above equation we obtain

$$\int_\Gamma T(\mathbf{u},p)\mathbf{n}\cdot\Phi_y^0 e_i d\sigma = 0.$$

The conclusion $T(\mathbf{u},p)\mathbf{n}|_\Gamma = 0$ follows from the density results in Theorem 2.

It remains to see that $\ker\Lambda_0^0$ is contained in $G^\perp$. Take $g = \nabla\cdot\mathbf{u} \in \ker\Lambda_0^0$ and $\overline{q} \in G$. For $\overline{\mathbf{v}} \in \mathbf{H}^2(\Omega) : \Delta\overline{\mathbf{v}} - \nabla\nabla\cdot\overline{\mathbf{v}} = \nabla\overline{q}$ put $\overline{w} = \nabla\cdot\overline{\mathbf{v}} \in H^1(\Omega)$. Notice that $\overline{w}|_\Gamma \in H^{1/2}(\Gamma)$ and

$$\int_\Omega \mathbf{u}\cdot\nabla\overline{w}dx + \int_\Omega \nabla\cdot\mathbf{u}\overline{w}dx = \int_\Gamma \mathbf{u}\cdot\mathbf{n}\overline{w}d\sigma$$

holds. Since by hypothesis, $\mathbf{u}\cdot\mathbf{n}|_\Gamma = 0$ we can write

$$\int_\Omega \mathbf{u}\cdot\nabla\overline{w}dx = -\int_\Omega g\overline{w}dx. \tag{12}$$

On the other hand, Green's formula gives

$$\int_\Omega \mathbf{u}\cdot\nabla\overline{w}dx - \int_\Omega (g\overline{q} - p\overline{w})\,dx = 0. \tag{13}$$

From Theorem 6 we have $g = p$. Substituting this identity in equation (12) and combining with (13) we obtain

$$\int_\Omega g\overline{q}dx = 0$$

hence, $g \in G^\perp$. □

Remark 8. *The given representation for $\widetilde{F}_0$ relies on solutions to the system of equations*

$$\Delta\mathbf{v} - \nabla\nabla\cdot\mathbf{v} = \nabla q.$$

Clearly, the above left hand side can be seen as particular case of Navier operator

$$\mu\Delta\mathbf{v} + (\lambda+\mu)\nabla\nabla\cdot\mathbf{v},$$

with Lamé coefficients $\lambda = -2$ and $\mu = 1$. Since this operator admits fundamental solutions for $\mu \neq 0$ and $\lambda + 2\mu \neq 0$ then,

$$\mu\Delta\mathbf{v} + (\lambda+\mu)\nabla\nabla\cdot\mathbf{v} = \nabla q$$

admits a solution, for any $\nabla q \in \mathbf{L}^2(\Omega)$. *However, for* $\lambda = -2$ *and* $\mu = 1$ *we have* $\lambda + 2\mu = 0$ *so that above observation is not valid. Indeed, we will show that for this case, the system has a solution when* ∇q *is divergence-free.*

Define the vorticity map (scalar curl) by

$$curl\ \mathbf{u} = \nabla \cdot \mathbf{u}^{\perp},$$

$\mathbf{u}^{\perp} = (u_2, -u_1)$. It is well known (eg. [5]) that for a sufficiently regular field $\mathbf{v}$,

$$\mathbf{curl}\ curl\ \mathbf{v} = -\Delta \mathbf{v} + \nabla \nabla \cdot \mathbf{v}.$$

Therefore, we can also write

$$G = \{q \in H^1(\Omega) : \mathbf{curl}\ curl\ \mathbf{v} = -\nabla q,\ \ \mathbf{v} \in \mathbf{H}^2(\Omega)\}.$$

Theorem 9. *The following identity holds*

$$\widetilde{F}_0 = \overline{\{q \in H^1(\Omega) : \Delta q = 0\}}^{L^2_0(\Omega)}.$$

Proof. Let $\mathbf{v} \in \mathbf{H}^2(\Omega)$ and set $w = curl\ \mathbf{v} \in H^1(\Omega)$. Clearly,

$$\nabla \cdot \mathbf{curl}\ w = 0$$

hence, if $q \in G$ then $\nabla \cdot \nabla q = \Delta q = 0$. Since $\widetilde{F}_0 = \overline{G}^{L^2_0(\Omega)}$ we conclude

$$\widetilde{F}_0 \subseteq \overline{\{q \in H^1(\Omega) : \Delta q = 0\}}^{L^2_0(\Omega)}.$$

Conversely, suppose that $q \in H^1(\Omega)$ is harmonic with zero mean. From the identity

$$\int_\Omega \mathbf{u} \cdot \nabla q dx + \int_\Omega \nabla \cdot \mathbf{u} q dx = \int_\Gamma \mathbf{u} \cdot \mathbf{n} q d\sigma \qquad (14)$$

we see that $q \in \ker {\Lambda^0_0}^{\perp}$. Indeed, if $\nabla \cdot \mathbf{u} \in \ker \Lambda^0_0$ then $\mathbf{u}|_\Gamma = 0$ hence the right hand side of above equation vanishes. On the other hand, from Theorem 4, $\mathbf{u} = \nabla p$ with $p \in H^1_0(\Omega)$. Hence

$$\int_\Omega \mathbf{u} \cdot \nabla q dx = \int_\Omega \nabla p \cdot \nabla q dx = \int_\Gamma p \partial_{\mathbf{n}} q d\sigma - \int_\Omega p \Delta q dx = 0$$

Substituting in (14) follows

$$\int_\Omega \nabla \cdot \mathbf{u} q dx = 0$$

which means that $q \in \ker {\Lambda^0_0}^{\perp}$. □

For the sake of completeness we can now give a characterization for the kernel of Λ^0_0. It is given by (eg. [18])

$$\ker \Lambda^0_0 = \left\{ g \in L^2_0(\Omega) : g = \Delta p,\ \ p \in H^2_0(\Omega) \right\}.$$

From the results presented in this section we can now conclude the following:

Theorem 10. *The space* $\widetilde{F}_0$ *is the* $L^2_0(\Omega)$ *closure of the space formed by harmonic functions* $\Delta q = 0$. *Moreover, independently of the frequency* λ, *we can identify the harmonic part of the divergence source* g *from the boundary data* $\Lambda^0_\lambda(g)$.

4. Identification from several measurements

We now address the identification from measurements associated with several values of the frequency λ.

Let $U = [a, b] \subset \mathbb{R}^+$ denote a set of frequencies. Recall that for a given frequency, $\Lambda_\lambda(\mathbf{f}, g)$ represents the boundary data generated by the pair $(\mathbf{f}, g) \in \mathbf{L}^2(\Omega) \times L^2_0(\Omega)$. We now define

$$\Lambda_U(\mathbf{f}, g) := (\Lambda_\lambda(\mathbf{f}, g))_{\lambda \in U}$$

to represent the set of measurements associated with the pair $(\mathbf{f}, g)$, for $\lambda \in U$. In this multiple frequency setting we can write, from decomposition (3),

$$\mathbb{L}^2(\Omega) = \overline{F_U}^{\mathbb{L}^2(\Omega)} \oplus \ker \Lambda_U \tag{15}$$

with

$$F_U = \sum_{\lambda \in U} F_\lambda$$

and F_λ defined as in (2).

In Appendix A.2 we constructed a set of divergence free functions $\upsilon_{\lambda,\mathbf{d}} \in L^2(\Omega)$ satisfying

$$(\Delta - \lambda^2)\upsilon_{\lambda,\mathbf{d}} = 0$$

and spanning a dense subspace in $\ker \nabla\cdot$. Clearly, the pairs $(\upsilon_{\lambda,\mathbf{d}}, 0)$ belong to $\overline{F}_{\lambda^2}^{\mathbb{L}^2(\Omega)}$ and

$$span\left\{(\upsilon_{\sqrt{\lambda},\mathbf{d}}, 0) : \lambda \in U,\ \mathbf{d} \in S^1\right\} \subseteq \overline{F}_U^{\mathbb{L}^2(\Omega)}.$$

Theorem 11. *The kernel of* $\Lambda_{\mathbb{R}^+}$ *is*

$$\ker \Lambda_{\mathbb{R}^+} = \nabla(H^1_0(\Omega)) \times \ker \Lambda^0_0.$$

As consequence the space of identifiable sources from the whole set of measurements $\Lambda_{\mathbb{R}^+}(\mathbf{f}, g)$ *is*

$$\overline{F_{\mathbb{R}^+}}^{\mathbb{L}^2(\Omega)} = \left\{\mathbf{f} \in \mathbf{L}^2(\Omega) : \nabla \cdot \mathbf{f} = 0\right\} \times \left\{g \in L^2_0(\Omega) : \Delta g = 0\right\}, \tag{16}$$

ie, the divergence-free part of the body force and the harmonic part of the divergence term.

Proof. Let $(\mathbf{f}, g) \in \ker \Lambda_{\mathbb{R}^+}$. Since $(\upsilon_{\sqrt{\lambda},\mathbf{d}}, 0)$ belongs to $F_{\mathbb{R}^+}$ then,

$$0 =< (\mathbf{f}, g), (\upsilon_{\sqrt{\lambda},\mathbf{d}}, 0) >_{\mathbb{L}^2(\Omega)} = \int_\Omega \mathbf{f} \cdot \overline{\upsilon}_{\sqrt{\lambda},\mathbf{d}} dx,\ \ \forall \lambda > 0,\ \mathbf{d} \in S^1.$$

By density (Theorem 18) it now follows

$$\mathbf{f} = \nabla q_0, \quad q_0 \in H_0^1(\Omega).$$

In particular, we have

$$\begin{cases} (\Delta - \lambda)\mathbf{v} - \nabla(q + q_0) = 0 & \text{in } \Omega \\ \nabla \cdot \mathbf{v} = g & \text{in } \Omega \\ \mathbf{v} = 0 & \text{on } \Gamma \\ T(\mathbf{v}, q + q_0)\mathbf{n} = 0 & \text{on } \Gamma \end{cases}$$

hence,

$$g \in \cap_{\lambda \in U} \ker \Lambda_\lambda^0 = \ker \Lambda_0^0,$$

where the equality follows from Corollary 5. This entails that

$$\ker \Lambda_U \subseteq \nabla(H_0^1(\Omega)) \times \ker \Lambda_0^0.$$

The other inclusion is direct. Notice that from (A.2),

$$\nabla(H_0^1(\Omega))^\perp = \left\{ \mathbf{f} \in \mathbf{L}^2(\Omega) : \nabla \cdot \mathbf{f} = 0 \right\}.$$

Theorem 9 gives

$$\ker \Lambda_0^{0\perp} = \left\{ g \in L_0^2(\Omega) : \Delta g = 0 \right\}$$

and these two identities give (16). □

Remark 12. *From the computational point of view, it is not practical to use the functions* $v_{\lambda,\mathbf{d}}$ *to retrieve the divergence free part of* $\mathbf{f}$*. One consequence of the density of fundamental solutions* $(\Phi_y^\lambda e_i, \Psi_y^\lambda e_i)$, $y \in \widehat{\Gamma}$ *in* $\{(\mathbf{u}, p) \in \mathbf{L}^2(\Omega) \times L^2(\Omega) : (\Delta - \lambda)\mathbf{u} = \nabla p,\ \nabla \cdot \mathbf{u} = 0\}$ *is that*

$$\mathit{span} \left\{ \Phi_y^\lambda e_i : \lambda \in \mathbb{R}^+,\ y \in \widehat{\Gamma} \right\}$$

is dense in the whole space $\ker \nabla\cdot$*. This means that the divergence free part of the source can be approximated as a superposition of Stokeslets. For the source g we can use for instance (normalized) fundamental solutions of Laplace's equation, with point sources y placed at* $\widehat{\Gamma}$*.*

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Appendix A. Density results

Let $\Omega \subset \mathbb{R}^2$ be an open and non-empty simply connected set with C^2 boundary $\Gamma := \partial\Omega$. We denote by $\mathbf{H}(\nabla\cdot, \Omega)$ the Hilbert space of $\mathbf{L}^2(\Omega)$ functions with $L^2(\Omega)$ divergence. We are particulary interested in the kernel of the divergence operator $\nabla\cdot : \mathbf{H}(\nabla\cdot, \Omega) \to L^2(\Omega)$, $\mathbf{f} \mapsto \nabla \cdot \mathbf{f}$, which we shall denote by $\ker \nabla\cdot$. Hence,

$$\ker \nabla\cdot = \left\{ \mathbf{f} \in \mathbf{L}^2(\Omega) : \nabla \cdot \mathbf{f} = 0 \right\}. \tag{A.1}$$

It is well known that (eg. [5]),

$$\mathbf{L}^2(\Omega) = \ker \nabla \cdot \oplus \nabla \left(H_0^1(\Omega) \right) \tag{A.2}$$

and

$$\mathbf{L}^2(\Omega) = \{ \mathbf{f} \in \ker \nabla\cdot : \mathbf{f} \cdot \mathbf{n}|_\Gamma = 0 \} \oplus \nabla \left(H^1(\Omega) \right). \tag{A.3}$$

Recall that for $\mathbf{f} \in \mathbf{L}^2(\Omega)$ the normal trace $\mathbf{f} \cdot \mathbf{n}|_\Gamma$ can be defined as an element in $\mathbf{H}^{-1/2}(\Gamma)$.

Appendix A.1. Shear and pressure waves

Related to some of these spaces are sets of the shear and pressure waves. A shear wave, $\phi_{\lambda,\mathbf{d}}$ (see (9)), can be seen as the *curl* of a plane wave $e^{i\lambda x\cdot\mathbf{d}}$ in the sense that

$$\phi_{\lambda,\mathbf{d}}(x) = -\frac{i}{\lambda}\mathbf{curl}\, e^{i\lambda x\cdot\mathbf{d}}. \tag{A.4}$$

The set of shear waves for a range of frequencies $\lambda \in U$ is denoted by

$$SW_U := \left\{ \phi_{\lambda,\mathbf{d}} : \; \lambda \in U, \;\; \mathbf{d} \in S^1 \right\}. \tag{A.5}$$

A pressure wave, $\psi_{\lambda,\mathbf{d}}$ (see (10)) is a function that can be seen as the gradient of a plane wave,

$$\psi_{\lambda,\mathbf{d}}(x) := -\frac{i}{\lambda}\nabla e^{i\lambda x\cdot\mathbf{d}}.$$

where λ and $\mathbf{d}$ have the same meaning as above. The set of pressure waves for a set of frequencies $\lambda \in U$ will be denoted by

$$PW_U := \left\{ \psi_{\lambda,\mathbf{d}} : \; \lambda \in U, \;\; \mathbf{d} \in S^1 \right\}. \tag{A.6}$$

Notice that

$$SW_U \subset \ker \nabla \cdot \quad \text{and} \quad PW_U \subset \nabla(H^1(\Omega)).$$

Furthermore, we have.

Theorem 13. *Let $U =]a, b[\subseteq \mathbb{R}^+$ be an interval. The set SW_U spans a dense subspace in* $\ker \nabla\cdot$ *and PW_U spans a dense subspace in $\nabla(H^1(\Omega))$.*

Proof. Clearly, span SW_U is contained in $\ker \nabla\cdot$. To see the density claim, take $\mathbf{f} \in \ker \nabla\cdot$ such that

$$\int_\Omega \mathbf{f} \cdot \overline{\phi}_{\lambda,\mathbf{d}} dx = \int_\Omega \mathbf{f} \cdot \mathbf{d}^\perp e^{-i\lambda x\cdot\mathbf{d}} dx = 0.$$

Then,

$$\mathcal{F}(\chi_\Omega \mathbf{f})(\frac{\lambda}{2\pi}\mathbf{d}) \cdot \mathbf{d}^\perp = 0,$$

where $\mathcal{F}$ denotes the Fourier transform of $\chi_\Omega \mathbf{f}$. On the other hand, the condition $\nabla\cdot\mathbf{f} = 0$ gives

$$2\pi i \mathcal{F}(\chi_\Omega \mathbf{f})(\xi) \cdot \xi = 0$$

hence, from

$$\mathcal{F}(\chi_\Omega \mathbf{f})(\xi) = (\mathcal{F}(\chi_\Omega \mathbf{f})(\xi) \cdot \mathbf{d})\, \mathbf{d} + \left(\mathcal{F}(\chi_\Omega \mathbf{f})(\xi) \cdot \mathbf{d}^\perp\right) \mathbf{d}^\perp$$

we obtain

$$\mathcal{F}(\chi_\Omega \mathbf{f})(\frac{\lambda}{2\pi}\mathbf{d}) = 0.$$

From $int(U \times S^1) \cong B(0,]a, b[) := B(0, b) \setminus \overline{B(0, a)}$ we conclude that

$$\mathcal{F}(\chi_\Omega \mathbf{f})(\xi) = 0, \ \ \xi \in B(0,]a, b[).$$

This gives that the Fourier transform of $\chi_\Omega \mathbf{f}$ is null in the open set $B(0,]a, b[)$ and by analytic continuation is null in the whole plane and the conclusion $\mathbf{f} = 0$ follows.

Regarding the pressures waves, the density follows from the density of span $\left\{e^{i\lambda x\cdot\mathbf{d}} : \ \lambda \in U, \ \ \mathbf{d} \in S^1\right\}$ in $H^1(\Omega)$ and continuity of the gradient operator $\nabla : H^1(\Omega) \to \mathbf{L}^2(\Omega)$. □

Remark 14. *The density of shear waves in* $\ker \nabla\cdot$ *can also be established from the identity (A.4) and the fact that the* **curl** *of* $H^1(\Omega)$ *functions is (eg. [5])*

$$\mathbf{curl}(H^1(\Omega)) = \ker \nabla \cdot .$$

Remark 15. *From decomposition (A.2) it follows that*

$$\int_\Omega \mathbf{f} \cdot \overline{\phi}_{\kappa,\mathbf{d}} dx = 0, \quad \forall \kappa \in U, \ \ \mathbf{d} \in S^1$$

if and only if $\mathbf{f} = \nabla p$, *with* $p \in H^1_0(\Omega)$.

On the other hand, from (A.3)

$$\int_\Omega \mathbf{f} \cdot \overline{\psi}_{\kappa,\mathbf{d}} dx = 0, \quad \forall \kappa \in U, \ \ \mathbf{d} \in S^1$$

if and only if $\nabla \cdot \mathbf{f} = 0$ *in* Ω *and* $\mathbf{f} \cdot \mathbf{n}|_\Gamma = 0$.

Appendix A.2. Density results for solutions to modified Helmholtz equations

Shear and pressure waves have the good properties established in Theorem 13. However, they do not satisfy the modified Helmholtz (vector) equation

$$(\Delta - \lambda)\mathbf{f} = 0. \tag{A.7}$$

Instead, shear waves satisfy Helmholtz vector equations $(\Delta + \lambda)\phi_{\sqrt{\lambda},\mathbf{d}} = 0$. In this section we construct a family of functions with similar properties to those of shear waves but satisfying the modified equations (A.7).

Let

$$\mathcal{H}_\lambda := \{f \in H^m(\Omega) : (\Delta + \lambda)f = 0 \text{ in } \Omega\}$$

and assume and that the boundary $\Gamma = \partial\Omega$ is C^{m+2}, with $m \geq 0$.

Suppose that $\lambda > 0$ is not an eigenfrequency for the Laplace Dirichlet operator. Then, given $f \in H^m(\Omega)$ there exists an unique $u_\lambda \in H_0^1(\Omega) \cap H^{m+2}(\Omega)$ such that

$$(\Delta + \lambda)u_\lambda = f$$

and the map

$$f \mapsto (\Delta - \lambda)u_\lambda, \Theta : H^m(\Omega) \to H^m(\Omega)$$

is well defined.

Theorem 16. *The linear map* Θ *is a homemorphism and satisfies*

$$\Theta(\mathcal{H}_{-\lambda}) = \mathcal{H}_\lambda. \tag{A.8}$$

Proof. The map is clearly linear. It follows that if $f \in \ker\Theta$, then $(\Delta - \lambda)u_\lambda = 0$. Since $u_\lambda \in H_0^1(\Omega)$ then $u_\lambda = 0$ and, in particular, $f = 0$. This shows that the map is injective. On the other hand, for any $g \in H^m(\Omega)$ there exists $u_\lambda \in H_0^1(\Omega) \cap H^{m+2}(\Omega)$ satisfying $(\Delta - \lambda)u_k = g$. Define $f := (\Delta + \lambda)u_k$ and notice that $\Theta(f) = g$. Continuity follows from the well posedness of the Helmholtz and modified Helmholtz equations.

To see (A.8) notice that, if $f = (\Delta + \lambda)u_\lambda$ with $(\Delta - \lambda)f = 0$ then,

$$(\Delta + \lambda)\Theta(f) = (\Delta^2 - \lambda^2)u_\lambda = (\Delta - \lambda)f = 0$$

hence, $\Theta(\mathcal{H}_{-\lambda}) \subset \mathcal{H}_\lambda$. For the other inclusion the verification can be made in the same manner. □

In particular, given a plane wave $e^{i\lambda x\cdot\mathbf{d}} (\in \mathcal{H}_{\lambda^2})$ there exists an unique $\rho_{\lambda,\mathbf{d}} \in \mathcal{H}_{-\lambda^2}$ such that

$$\Theta(\rho_{\lambda,\mathbf{d}}) = e^{i\lambda x\cdot\mathbf{d}}.$$

Moreover,

Theorem 17. *The space*

$$span\{\rho_{\lambda,\mathbf{d}} : \lambda \in \mathbb{R}^+,\ \mathbf{d} \in S^1\} \subset \sum_{\lambda} \mathcal{H}_{-\lambda^2}$$

is dense in $H^m(\Omega)$.

Proof. This follows from the density of $span\left\{e^{i\lambda x.\mathbf{d}} : \lambda \in \mathbb{R}^+,\ \mathbf{d} \in S^1\right\}$ in $H^m(\Omega)$ and above properties of Θ.

□

Next step is to take functions

$$\upsilon_{\lambda,\mathbf{d}} := \mathbf{curl}\ \rho_{\lambda,\mathbf{d}}. \tag{A.9}$$

These shear waves for negative frequency have the desired properties:

Theorem 18. *The functions* $\upsilon_{\lambda,\mathbf{d}}$ *are divergence free and satisfy (eventually in the sense of distributions) the vector resolvent equation*

$$(\Delta - \lambda^2)\upsilon_{\lambda,\mathbf{d}} = 0.$$

Moreover, the set span $\{\upsilon_{\lambda,\mathbf{d}} : \lambda \in \mathbb{R}^+,\ \ \mathbf{d} \in S^1\}$ *is dense in* $\ker \nabla\cdot = \mathbf{curl} H^1(\Omega)$.