A canonical construction for nonnegative integral matrices with given line sums *

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Abstract

Let p be a positive integer and let $\mathcal{A}^{(p)}(R,S)$ be the class of nonnegative integral matrices with entries less than or equal to p, with row-sum partition R, and column-sum partition S .

In this paper we state a new necessary and sufficient condition for $\mathcal{A}^{(p)}(R,S) \neq \emptyset$. This condition generalizes the well known Gale-Ryser theorem. We also present a canonical construction for matrices in $\mathcal{A}^{(p)}(R,S)$.

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1 Introduction

A *partition* of (weight) $t \geq 0$ is a nonincreasing sequence of nonnegative integers whose sum is t. The number of nonzero elements in λ is called the *length*

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of λ and is denoted by $l(\lambda)$. When λ is a partition of weight t we denote λ as a finite sequence

$$(\lambda_1, \lambda_2, \ldots, \lambda_{l(\lambda)}),$$

where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{l(\lambda)} > 0$.

The *conjugate partition* of $(\lambda_1, \ldots, \lambda_{l(\lambda)})$ is the partition λ^* defined by

$$\lambda_j^* = |\{i : \ l(\lambda) \geq i \geq 1, \ \lambda_i \geq j\}|, \quad \text{for } 1 \leq j \leq \lambda_1.$$

It is easy to see that $(\lambda^*)^* = \lambda$.

Let α and β be partitions of the same weight. We say that α is *dominated* or *majorized* by β, $\alpha \preceq \beta$, when

$$\alpha_1 + \ldots + \alpha_i \leq \beta_1 + \ldots + \beta_i, \qquad \text{for } i \geq 1.$$

It is well-known that $\alpha \preceq \beta$ if and only if $\beta^* \preceq \alpha^*$.

Let λ and μ be partitions of the same weight. The union of λ and μ, $\lambda \cup \mu$, is the partition obtained by ordering the integers

$$\lambda_1, \ldots, \lambda_{l(\lambda)}, \mu_1, \ldots, \mu_{l(\mu)},$$

in nonincreasing order. The union of p copies of λ will be denoted $\bigcup^p \lambda$ (see [2]).

In a wide variety of combinatorial problems, matrices whose entries are only zeros and ones, the $(0,1)$-matrices, play a fundamental role. These matrices have been studied by many researchers. Consequently, remarkable results appeared in the literature. One of these results is the Gale-Ryser theorem which asserts that if R and S are partitions of the same weight, then there exists a matrix of zeros and ones whose row-sum partition is R and column-sum partition is S if and only if $S \preceq R^*$ (see [1], [3] or [6]).

Let p be a positive integer. We denote by $\mathcal{A}^{(p)}(R,S)$ the class of nonnegative integral matrices with entries less than or equal to p, whose row-sum partition is R and column-sum partition is S (R and S are partitions of the same weight). Mirsky [4, 5] solved the more general problem of finding conditions for the existence of matrices in $\mathcal{A}^{(p)}(R,S)$.

Theorem 1.1 (*Mirsky*) [4, 5] *Let $R = (R_1, \ldots, R_m)$ and $S = (S_1, \ldots, S_n)$ be partitions of the same weight, and let p be a positive integer. Then $\mathcal{A}^{(p)}(R,S)$ is nonempty if and only if*

$$pkl + \sum_{i=k+1}^{m} R_i - \sum_{j=1}^{l} S_j \geq 0, \ \ k = 0, 1, \ldots, m; \ l = 0, 1, \ldots, n.$$

In [1], Brualdi showed a necessary and sufficient condition for the existence of these matrices.

Theorem 1.2 (*Brualdi*) [1] *Let $R = (R_1, \ldots, R_m)$ and $S = (S_1, \ldots, S_n)$ be partitions of the same weight, and let p be a positive integer. Then $\mathcal{A}^{(p)}(R, S)$ is nonempty if and only if*

$$\sum_{j=1}^{k} S_j \leq \sum_{i=1}^{m} \min\{R_i, pk\}, \quad k = 1, \ldots, n.$$

More recently, using the domination relation between two partitions, Dias da Silva and Fonseca extended the Gale-Ryser theorem proving a necessary and sufficient condition for $\mathcal{A}^{(p)}(R, S) \neq \emptyset$ [2].

Theorem 1.3 (*Dias da Silva, Fonseca*) [2] *Let R and S be partitions of the same weight, and let p be a positive integer. Then*

$$\mathcal{A}^{(p)}(R, S) \text{ is nonempty if and only if } \left(\bigcup^{p} S\right) \preceq \left(\bigcup^{p} R\right)^{*}.$$

□

In Section 2 we define a new relation between two partitions of the same weight. This relation allow us to obtain a new extension of the Gale-Ryser theorem. In Section 3, when $\mathcal{A}^{(p)}(R, S)$ is nonempty, we give a canonical construction for a matrix in $\mathcal{A}^{(p)}(R, S)$. We also present illustrative examples.

2 An extension of the Gale-Ryser theorem

The purpose of this section is to state another necessary and sufficient condition for the existence of matrices in $\mathcal{A}^{(p)}(R, S)$, in the spirit of the Gale-Ryser theorem (Theorem 2.4). First we generalize the domination relation between two partitions.

Definition 2.1 *Let α and β be partitions of t and let p be a positive integer. We say that α is p-dominated (or p-majorized) by β, writing $\alpha \preceq_p \beta$, when*

$$\sum_{i=1}^{k} \alpha_i \leq \sum_{i=1}^{pk} \beta_i, \quad \text{for } k \geq 1.$$

□

Note that to write $\alpha \preceq_1 \beta$ is the same that to write $\alpha \preceq \beta$.

In the next result we prove the equivalence between the p-domination relation of two partitions and the usual domination relation of two specific partitions.

Theorem 2.2 *Let α and β be partitions of the same weight. Let p be a positive integer. Then*

$$\left(\bigcup^{p} \alpha\right) \preceq \left(\bigcup^{p} \beta\right)^{*} \qquad \textit{if and only if} \qquad \alpha \preceq_p \beta^{*}.$$

Proof: Assume that $(\bigcup^p \alpha) \preceq (\bigcup^p \beta)^*$. Let $l \geq 1$ and $v = pl$. So

$$p\alpha_1 + \ldots + p\alpha_l \leq p\beta_1^* + \ldots + p\beta_v^*,$$

and

$$\alpha_1 + \ldots + \alpha_l \leq \beta_1^* + \ldots + \beta_v^*.$$

Consequently, $\alpha \preceq_p \beta^*$.

Assume that $(\bigcup^p \alpha) \not\preceq (\bigcup^p \beta)^*$. Consequently, there is an integer u, with $u = pk + t \neq 0$, $k \geq 0$, and $0 \leq t < p$, such that

$$p\alpha_1 + \ldots + p\alpha_k + t\alpha_{k+1} > p\beta_1^* + \ldots + p\beta_u^*. \tag{1}$$

If $t = 0$ then $\alpha_1 + \ldots + \alpha_k > \beta_1^* + \ldots + \beta_u^*$. Therefore, $\alpha \not\preceq_p \beta^*$.

Assume that $t \neq 0$ and u is the least integer that satisfies (1). Now,

$$p\alpha_1 + \ldots + p\alpha_k + (t-1)\alpha_{k+1} \leq p\beta_1^* + \ldots + p\beta_{u-1}^*$$

and $\alpha_{k+1} > p\beta_u^*$. Because $\beta_u^* \geq \beta_{u+1}^* \geq \ldots,$ we have

$$p\alpha_1 + \ldots + p\alpha_k + p\alpha_{k+1} > p\beta_1^* + \ldots + p\beta_{u+p-t}^*.$$

So,

$$\alpha_1 + \ldots + \alpha_k + \alpha_{k+1} > \beta_1^* + \ldots + \beta_{u+p-t}^*.$$

Consequently, $\alpha \not\preceq_p \beta^*$. □

Corollary 2.3 *Let α and β be partitions of the same weight. Then*

$$\alpha \preceq_p \beta \qquad \textit{if and only if} \qquad \beta^* \preceq_p \alpha^*.$$

Proof: We have

$$\begin{aligned} \alpha \preceq_p \beta \quad & \text{if and only if} \quad \alpha \preceq_p (\beta^*)^* \\ & \text{if and only if} \quad (\textstyle\bigcup^p \alpha) \preceq (\bigcup^p \beta^*)^* \qquad \text{(Theorem 2.2)} \\ & \text{if and only if} \quad (\textstyle\bigcup^p \beta^*) \preceq (\bigcup^p \alpha)^* \\ & \text{if and only if} \quad \beta^* \preceq_p \alpha^* \qquad \text{(Theorem 2.2).} \end{aligned}$$

□

The next result is an easy consequence of Theorems 1.3 and 2.2.

Theorem 2.4 *Let R and S be partitions of the same weight and let p be a positive integer. Then*

$$\mathcal{A}^{(p)}(R,S) \textit{ is nonempty if and only if } S \preceq_p R^*.$$

□

3 A canonical construction for matrices in $\mathcal{A}^{(p)}(R,S)$

Let t, m, n and p be positive integers. Let R and S be partitions of t such that $l(R)=m$, $l(S)=n$, and $S \preceq_p R^*$ (so, $\mathcal{A}^{(p)}(R,S) \neq \emptyset$). In this section we describe a direct algorithm that constructs a matrix $A \in \mathcal{A}^{(p)}(R,S)$. The proof of the algorithm is a generalization of the proof of the Gale-Ryser algorithm (see Theorem 3.1.1 in [1]).

Definition 3.1 *Let t, m, n and p be positive integers. Let R and S be partitions of t such that $l(R)=m$, $l(S)=n$, and $S \preceq_p R^*$.*

For each $1 \leq l \leq n$, let d_l and b_l be the nonnegative integers such that

$$S_l = d_l p + b_l, \qquad \text{with } 0 \leq b_l < p.$$

We denote by $\overline{A_S}$ the m-by-n matrix, $[a_{ij}]$, such that, for $1 \leq i \leq m$ and $1 \leq j \leq n$,

$$a_{ij} = \begin{cases} p & \text{if} \quad i \leq d_j \\ b_j & \text{if} \quad i = d_j + 1 \\ 0 & \quad \text{otherwise} \end{cases}.$$

□

Example 3.2 Let $R=(7,7,6,5)$, $S=(6,5,5,5,4)$ be partitions of 25 and $p=3$. We have

$$S \preceq_3 R^*.$$

Since

$$S_1 = 6 = 2 \times 3 + 0$$

we have

$$d_1 = 2 \text{ and } b_1 = 0.$$

Therefore, the first column of $\overline{A_S}$ is

$$\begin{bmatrix} 3 \\ 3 \\ 0 \\ 0 \end{bmatrix}.$$

The other columns of $\overline{A_S}$ are constructed in analogue fashion. Hence,

$$\overline{A_S} = \begin{bmatrix} 3 & 3 & 3 & 3 & 3 \\ 3 & 2 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

□

The matrix $\overline{A_S}$ given in Definition 3.1 is the starting point of our algorithm and its column-sum partition is S.

Remark 3.3 Since S is a partition we have

$$S_1 \geq S_2 \geq \ldots \geq S_n.$$

Consequently, the integers d_l of Definition 3.1 satisfy

$$d_1 \geq d_2 \geq \ldots \geq d_n \geq 0.$$

Therefore $d = (d_1, d_2, \ldots, d_n)$ is a nonincreasing sequence of nonnegative integers, that is, d is a partition. □

Lemma 3.4 *Let R and S be partitions of t such that $R = (R_1, \ldots, R_m)$, where $m = l(R)$, and $R \preceq_p S^*$. Let g be the sum of the integers in mth row of $\overline{A_S}$. Then*

$$g \leq R_m.$$

Proof: If $\overline{A_S} = [a_{ij}]$ then, by Definition 3.1, we get

$$\begin{aligned}
\textstyle\sum_{j=1}^{n} a_{mj} &= |\{r: \ d_r = m\}|p + \sum_{l \in \{r: \ d_r = m-1\}} b_l \\
&= |\{r: \ S_r = mp\}|p + \sum_{l=1}^{p-1} |\{r: \ d_r = m-1, \ b_r \geq l\}| \\
&= \sum_{l=1}^{p} |\{r: \ S_r \geq (m-1)p + l\}| \\
&= \sum_{l=1}^{p} S^*_{(m-1)p+l} \\
&= \sum_{l=(m-1)p+1}^{mp} S^*_l .
\end{aligned}$$

On the other hand, because $R \preceq_p S^*$, we get

$$R_1 + \ldots + R_m = \sum_{i=1}^{mp} S^*_i$$

and

$$R_1 + \ldots + R_{m-1} \leq \sum_{i=1}^{(m-1)p} S^*_i .$$

Consequently, $R_m \geq \sum_{i=(m-1)p+1}^{mp} S_i^* = \sum_{j=1}^{n} a_{mj} = g$. □

Definition 3.5 *Let d^* be the conjugate partition of d (the partition $(d_1, \ldots, d_n)$ defined in Definition 3.1) and let*

$$v_i^* = d_i^* p + \sum_{j \in \{l:\ 1 \leq l \leq n,\ d_l = i-1\}} b_j, \quad \text{with} \ \ 1 \leq i \leq m.$$

□

Remark 3.6

1. Since, for each $1 \leq i \leq m$, $d_i^* = d_{i+1}^* + |\{l :\ d_l = i\}|$ and

$$\sum_{j \in \{l:\ 1 \leq l \leq n,\ d_l = i\}} b_j \leq |\{l :\ d_l = i\}| p,$$

(recall that $0 \leq b_j < p$) we conclude that

$$\begin{aligned} v_i^* &= d_i^* p + \sum_{j \in \{l:\ 1 \leq l \leq n,\ d_l = i-1\}} b_j \\ &\geq d_i^* p \\ &= (d_{i+1}^* + |\{l :\ d_l = i\}|) p \\ &\geq d_{i+1}^* p + \sum_{j \in \{l:\ 1 \leq l \leq n,\ d_l = i\}} b_j \\ &= v_{i+1}^*. \end{aligned}$$

Thus,

$$v_1^* \geq v_2^* \geq \ldots \geq v_m^*.$$

2. By definition, $v_i^* = d_i^* p + \sum_{j \in \{l:\ 1 \leq l \leq n,\ d_l = i-1\}} b_j$. Using Definition 3.1, this is the sum of the integers in row i of $\overline{A_S}$. Therefore, $(v_1^*, v_2^*, \ldots, v_m^*)$ is the row-sum partition of $\overline{A_S}$.

3. If k, g are positive integers such that $1 \leq g < p$, then

$$\begin{aligned} S_{(k-1)p+g}^* &= |\{l :\ S_l \geq (k-1)p + g\}| \\ &= |\{l :\ d_l \geq k\}| + |\{l :\ d_l = k-1, \ \text{and} \ b_l \geq g\}| \\ &= d_k^* + |\{l :\ d_l = k-1, \ \text{and} \ b_l \geq g\}|, \end{aligned}$$

and

$$S_{kp}^* = |\{l :\ S_l \geq kp\}| = |\{l :\ d_l \geq k\}| = d_k^*.$$

Thus,

$$\sum_{i=(k-1)p+1}^{kp} S_i^* = d_k^* p + \sum_{j \in \{l:\ d_l = k-1\}} b_j = v_k^*.$$

4. Using last remark, when k is a positive integer, we get

$$\sum_{i=1}^{kp} S_i^* = v_1^* + \ldots + v_k^*.$$

$\square$

Lemma 3.7 *Let R and S be partitions of t such that $R = (R_1, \ldots, R_m)$, where $m = l(R)$, and $R \preceq_p S^*$. Let h be the sum of the integers in the first row of $\overline{A_S}$. Then*

$$R_m \leq h.$$

Proof: Since $S \preceq_p R^*$, we have $R \preceq_p S^*$. Using Remark 3.6, this implies that

$$R_m \leq R_1 \leq \sum_{i=1}^{p} S_i^* = v_1^* = h.$$

$\square$

Algorithm to construct a matrix in $\mathcal{A}^{(p)}(R, S)$

Let $R = (R_1, \ldots, R_m)$ and $S = (S_1, \ldots, S_n)$ be two partitions of the same weight and let p be a positive integer. Assume that $S \preceq_p R^*$.

1. Begin with the m-by-n matrix $\overline{A_S}$, of Definition 3.1.
2. Let $E = [e_{ij}] = \overline{A_S}$.
3. Let z be the number of rows in E.
4. If $\sum_{j=1}^{n} e_{zj} = R_z$, then go to step 9, with $E = C$.
   Otherwise, if $\sum_{j=1}^{n} e_{zj} < R_z$, then go to step 5.
5. Let (r, l) be the greatest pair (by lexicographic order) such that

$$1 \leq r \leq z-1, \quad 1 \leq l \leq n, \quad e_{rl} \neq 0, \quad \text{and} \quad e_{zl} \neq p.$$

6. Let

$$f_{r,l} = \min \left\{ p - e_{zl}, \quad e_{rl}, \quad R_z - \sum_{j=1}^{n} e_{zj} \right\}.$$

7. Let $C = [c_{ij}]$ be the z-by-n matrix such that

$$c_{ij} = \begin{cases} e_{ij} & \text{if } (ij) \notin \{(rl), \ (zl)\} \\ e_{rl} - f_{r,l} & \text{if } (ij) = (rl) \\ e_{zl} + f_{r,l} & \text{if } (ij) = (zl) \end{cases}$$

8. If $\sum_{j=1}^{n} c_{zj} < R_z$, then repeat the process, beginning in step 5, with $E = [e_{ij}] = C$.

   Otherwise, if $\sum_{j=1}^{n} c_{zj} = R_z$, then go to step 9.

9. If $z > 2$, then we construct the $(z-1)$-by-n matrix G obtained from C by removing the last row. Repeat the process, beginning in step 3, with $E = [e_{ij}] = G$.

   Otherwise, if $z = 2$, then put in C the rows removed in each process of step 9 and stop the algorithm

Before showing that the algorithm constructs a matrix $A \in \mathcal{A}^{(p)}(R,S)$ we illustrate it with two examples.

Example 3.8 *1.* As in Example 3.2, let $R = (7,7,6,5)$, $S = (6,5,5,5,4)$ and $p = 3$. So,

$$\overline{A_S} = \begin{bmatrix} 3 & 3 & 3 & 3 & 3 \\ 3 & 2 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus,

$$\sum_{j=1}^{5} a_{4j} = 0, \qquad R_4 = 5.$$

Using the algorithm, the pair $(2,5)$ is the greatest pair (by lexicographic order) such that $a_{25} = 1 \neq 0$ and $a_{45} = 0 \neq p = 3$. Thus, we get

$$f_{2,5} = \min\{3-0, 1, 5-0\} = 1.$$

So, the matrix C is equal to

$$\begin{bmatrix} 3 & 3 & 3 & 3 & 3 \\ 3 & 2 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

By step 8 and because $\sum_{j=1}^{5} c_{4j} = 1 < R_4 = 5$, we repeat the process, beginning in step 5, with the matrix $E = C$.

Now, we obtain $f_{2,4} = 2$ and a new matrix

$$C = \begin{bmatrix} 3 & 3 & 3 & 3 & 3 \\ 3 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \end{bmatrix}.$$

By step 8 and because $\sum_{j=1}^{5} c_{4j} = 3 < R_4 = 5$, we repeat the process, beginning in step 5, with the matrix $E = C$.

Now, we obtain $f_{2,3} = 2$ and a new matrix

$$C = \begin{bmatrix} 3 & 3 & 3 & 3 & 3 \\ 3 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 1 \end{bmatrix}.$$

By step 8 and because $\sum_{j=1}^{5} c_{4j} = 5 = R_4 = 5$, we go to step 9. So, we remove the last row of C and we repeat the process, beginning in step 3, with the matrix

$$E = \begin{bmatrix} 3 & 3 & 3 & 3 & 3 \\ 3 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Repeating three times the algorithm (steps 5 to 8), we obtain the matrix

$$C = \begin{bmatrix} 3 & 3 & 3 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 & 1 \end{bmatrix}.$$

By step 8 and because $\sum_{j=1}^{5} c_{3j} = 6 = R_3 = 6$, we go to step 9. So, we remove the last row of C and we repeat the process, beginning in step 3, with the matrix

$$E = \begin{bmatrix} 3 & 3 & 3 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Repeating three times the algorithm (steps 5 to 8), we get the matrix

$$C = \begin{bmatrix} 3 & 3 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & 2 \end{bmatrix}.$$

By step 8, because $\sum_{j=1}^{5} c_{2j} = 7 = R_2 = 7$ we go to step 9. Because the number of rows in this matrix is two, the algorithm stops. We put in the last matrix the rows removed in each iteration of step 9 and we have the matrix

$$A = \begin{bmatrix} 3 & 3 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & 2 \\ 3 & 2 & 0 & 0 & 1 \\ 0 & 0 & 2 & 2 & 1 \end{bmatrix}.$$

Schematizing

$$\begin{bmatrix} 3 & 3 & 3 & 3 & 3 \\ 3 & 2 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \left[\begin{array}{ccccc} 3 & 3 & 3 & 3 & 3 \\ 3 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 2 & 2 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccccc} 3 & 3 & 3 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 3 & 2 & 0 & 0 & 1 \\ 0 & 0 & 2 & 2 & 1 \end{array}\right] \qquad \left[\begin{array}{ccccc} 3 & 3 & 1 & 0 & 0 \\ \hline 0 & 0 & 2 & 3 & 2 \\ 3 & 2 & 0 & 0 & 1 \\ 0 & 0 & 2 & 2 & 1 \end{array}\right]$$

Thus, the matrix A constructed by this algorithm is the matrix

$$\begin{bmatrix} 3 & 3 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & 2 \\ 3 & 2 & 0 & 0 & 1 \\ 0 & 0 & 2 & 2 & 1 \end{bmatrix}$$

2. Let $R = (8, 7, 6, 6)$, $S = (9, 9, 9)$ and $p = 4$. In this case,

$$\overline{A_S} = \begin{bmatrix} 4 & 4 & 4 \\ 4 & 4 & 4 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Schematizing, by the algorithm we have

$$\begin{bmatrix} 4 & 4 & 4 \\ 4 & 4 & 4 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \left[\begin{array}{ccc} 4 & 4 & 4 \\ 4 & 4 & 1 \\ 0 & 0 & 0 \\ \hline 1 & 1 & 4 \end{array}\right]$$

$$\left[\begin{array}{ccc} 4 & 4 & 4 \\ 3 & 0 & 0 \\ \hline 1 & 4 & 1 \\ 1 & 1 & 4 \end{array}\right] \qquad \left[\begin{array}{ccc} 4 & 4 & 0 \\ \hline 3 & 0 & 4 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{array}\right]$$

Thus, the matrix A constructed by this algorithm is the matrix

$$\begin{bmatrix} 4 & 4 & 0 \\ 3 & 0 & 4 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix}$$

Now, we will prove the algorithm.

Lemma 3.9 *Let R and S be partitions of t such that $R = (R_1, \ldots, R_m)$, where $m = l(R)$, and $R \preceq_p S^*$. Let $C = [c_{i,j}]$ be the first matrix obtained from $\overline{A_S}$, using the algorithm, whose mth row-sum is R_m. Then*

1. *If $1 \leq j < l \leq n$, then*
$$\sum_{h=1}^{m-1} c_{hj} \geq \sum_{h=1}^{m-1} c_{hl}.$$
2. *If $1 \leq j < l < m$, then*
$$\sum_{h=1}^{n} c_{jh} \geq \sum_{h=1}^{n} c_{lh}.$$
3. *Let C' be the matrix obtained from C by removing the last row. Then, there is a partition $S' = (S'_1, \ldots, S'_n)$ of $S_1 + \ldots + S_n - R_m$ such that $C' = \overline{A_{S'}}$.*

Proof: Let $\overline{A_S} = [a_{i,j}]$.

1. Let $1 \leq j < l \leq n$. We will prove that
$$\sum_{h=1}^{m-1} c_{hj} \geq \sum_{h=1}^{m-1} c_{hl}. \tag{2}$$

Let r and s be the largest integers such that $a_{r,j} \neq 0$ and $a_{s,l} \neq 0$. Using the definition of $\overline{A_{S'}}$, we know that $r \geq s$. We have to consider several cases:

Case 1: $a_{r,j} = c_{r,j}$ and $a_{s,l} = c_{s,l}$. Using the construction of C and the definition of $\overline{A_S}$, we have $c_{i,j} = a_{i,j}$ and $c_{i,l} = a_{i,l}$, for $i = 1, \ldots, m$. Thus,
$$S_j = \sum_{h=1}^{m-1} c_{hj} \geq \sum_{h=1}^{m-1} c_{hl} = S_l.$$

Case 2: $a_{r,j} = c_{r,j}$ and $a_{s,l} \neq c_{s,l}$. In this case, using the construction of C and the definition of $\overline{A_{S'}}$, we have $c_{s,j} = a_{s,j} \geq a_{s,l} > c_{s,l}$, $c_{i,j} = a_{i,j} = p \geq c_{i,l}$, for $i = 1, \ldots, s-1$. Then
$$\sum_{h=1}^{m-1} c_{hj} > \sum_{h=1}^{m-1} c_{hl}.$$

Case 3: $a_{r,j} \neq c_{r,j}$ and $a_{s,l} = c_{s,l}$. In this case $r > s$, $c_{t,j} = a_{t,j} = p = c_{t,l}$, for $t = 1, \ldots s-1$ and $c_{s,j} = a_{s,j} = p \geq a_{s,l} = c_{s,l}$. Since $c_{t,l} = 0$, for $t = s+1, \ldots, m-1$, we get (2).

Case 4: $a_{r,j} \neq c_{r,j}$ and $a_{s,l} \neq c_{s,l}$.

Suppose that $r = s = m$. This implies that $a_{r-1,l} = a_{r-1,j} = p$. Since $a_{m,j} \geq a_{m,l} \neq 0$, by the algorithm (steps 5 and 6) we have

$$c_{r-1,j} \geq c_{r-1,l} \quad \text{and} \quad c_{i,j} = c_{i,l} = p, \quad \text{for } i = 1, \ldots, r-2.$$

So, we get (2).

Suppose that $r = s < m$. Since $a_{r,j} + a_{r-1,j} \geq a_{r,l} + a_{r-1,l} > p$ and the fact that if $c_{r-1,j} \neq a_{r-1,j} = p$ then $c_{r-1,l} \neq a_{r-1,l} = p$, we conclude that $c_{r,j} + c_{r-1,j} \geq c_{r,l} + c_{r-1,l}$ and $c_{i,j} = c_{i,l} = p$, for $i = 1, \ldots, r-2$. So, we get (2).

Assume now that $r > s$. Since $c_{t,j} = p$, $t = 1, \ldots, r-2$, if $r \geq s+2$ then we get (2). If $r = s+1$ and $c_{s,j} = p$, then (2) holds. If $r = s+1$ and $c_{s,j} < p$ then $c_{s,j} = 0$, and once again (2) holds.

2. Let $1 \leq j < l < m$. We will prove that

$$\sum_{h=1}^{n} c_{jh} \geq \sum_{h=1}^{n} c_{lh}. \tag{3}$$

Using the algorithm we know that when $1 \leq k < m$ and $c_{ki} < a_{ki}$ implies that $c_{ti} = 0 \leq a_{ti}$, for $t = k+1, \ldots, m-1$. Since $\sum_{h=1}^{n} a_{jh} \geq \sum_{h=1}^{n} a_{lh}$ we conclude (3).

3. Using the proof of the second statement and statements 1. and 2.,the third statement is easy to prove. □

Theorem 3.10 *Let R and S be partitions of t such that $R \preceq_p S^*$. Then the algorithm constructs a matrix $A \in \mathcal{A}^{(p)}(R,S)$.*

Proof: If $m = 1$, since $\overline{A_S}$ has column-sum partition S and $R_1 = S_1 + \ldots + S_n$, then $\overline{A_S} \in \mathcal{A}^{(p)}(R,S)$.

We now assume that $m > 1$. Since $S \preceq_p R^*$ we have $R \preceq_p S^*$. By Lemmas 3.4 and 3.7 and Remark 3.6, we conclude that

$$v_m^* \leq R_m \leq v_1^*.$$

Thus, the step 4 of the algorithm can be carried out until we obtain a matrix C having the mth row-sum equal to R_m. By lemma 3.9, there is a partition $S' = (u_1, \ldots, u_n)$ such that the matrix left in rows $1, 2, \ldots, m-1$ of C is $\overline{A_{S'}}$. Now we will show that $(R_1, \ldots, R_{m-1}) \preceq_p (S')^*$. Let $(S')^* = (u_1^*, \ldots, u_{u_1}^*)$.

We first observe that

$$R_1 + \ldots + R_{m-1} = (S_1 + \ldots + S_n) - R_m = u_1^* + \ldots + u_{u_1}^*.$$

Let (f, l) be the last entry of $\overline{A_S}$ changed by the algorithm. Using Definition 3.1 (the definition of matrix $\overline{A_S}$) we know that $d_l = f$ or $d_l = f - 1$.

Let k be an integer with $1 \leq k \leq m - 1$.

We consider two cases.

Case 1: If $f > k$ then the algorithm did not change rows $1, \ldots, k$ of $\overline{A_S}$. Using Definition 3.1 we have

$$\sum_{i=1}^{kp} u_i^* = \sum_{i=1}^{n} \min(kp, u_i) = \sum_{i=1}^{n} \min(kp, S_i) = \sum_{i=1}^{kp} S_i^*.$$

Since $R \preceq_p S^*$, we conclude that

$$\sum_{i=1}^{kp} u_i^* \geq \sum_{i=1}^{k} R_i.$$

Case 2: $f \leq k$. If $v_{k+1}^* \geq R_m$ then the last row of $\overline{A_S}$ changed by the algorithm would be greater than k. So, $f > k$ (impossible). Consequently, $v_{k+1}^* < R_m$.

Suppose that $d_l = f - 1$. Using Definition 3.1 we get

$$\begin{aligned} \sum_{i=1}^{kp} u_i^* &= \sum_{i=1}^{n} \min(kp, u_i) \\ &= \sum_{i \in \{r:\ d_r \geq f+1\}} \min(kp, S_i - p) + \sum_{i \in \{r:\ d_r = f\}} \min(kp, S_i - b_i) + \\ &\quad + \sum_{i \in \{r:\ d_r = f-1,\ r<l\}} \min(kp, S_i) + \sum_{i \in \{r:\ d_r = f-1,\ r>l\}} \min(kp, S_i - b_i) + \\ &\quad + \min(kp, S_l - a) + \sum_{i \in \{r:\ d_r \leq f-2\}} \min(kp, S_i), \end{aligned} \tag{4}$$

where

$$a = R_m - \left(\sum_{i \in \{r:\ d_r \geq f+1\}} p + \sum_{i \in \{r:\ d_r = f\}} b_i + \sum_{i \in \{r:\ d_r = f-1,\ r>l\}} b_i \right).$$

Now we are going to rewrite each summand of (4) in terms of $\min(kp, S_i)$:

- $$\sum_{i\in\{r:\ d_r\geq f+1\}} min(kp, S_i - p) =$$

$$= \sum_{i\in\{r:\ d_r\geq k+1\}} min(kp, S_i - p) + \sum_{i\in\{r:\ f+1\leq d_r\leq k\}} \min(kp, S_i - p)$$

$$= \sum_{i\in\{r:\ d_r\geq f+1\}} \min(kp, S_i) - p|\{i:\ k\geq d_i\geq f+1\}| + \sum_{i\in\{r:\ d_r=k\geq f+1\}} b_i,$$

- $$\sum_{i\in\{r:\ d_r=f\}} \min(kp, S_i - b_i) = \begin{cases} \sum_{i\in\{r:\ d_r=f\}} \min(kp, S_i) - \sum_{i\in\{r:\ d_r=f\}} b_i & \text{if} \quad k > f \\ \sum_{i\in\{r:\ d_r=f\}} \min(kp, S_i), & \text{if} \quad k = f, \end{cases}$$

- $$\sum_{i\in\{r:\ d_r=f-1,\ r>l\}} \min(kp, S_i - b_i) = \sum_{i\in\{r:\ d_r=f-1,\ r>l\}} \min(kp, S_i) - \sum_{i\in\{r:\ d_r=f-1,\ r>l\}} b_i,$$

and

- $\min(kp, S_l - a) = \min(kp, S_l) - a.$

Consequently,

$$\sum_{i=1}^{kp} u_i^* = \sum_{i=1}^{n} \min(kp, S_i) - R_m + p|\{i:\ d_i\geq k+1\}| + \sum_{i\in\{r:\ d_r=k\}} b_i,$$

either $k > f$ or $k = f$.

Thus,

$$\sum_{i=1}^{kp} u_i^* = \sum_{i=1}^{n} \min(kp, S_i) - R_m + v_{k+1}^*.$$

Suppose that $d_l = f$. Using Definition 3.1 we have

$$\begin{aligned}
\sum_{i=1}^{kp} u_i^* &= \sum_{i=1}^{n} \min(kp, u_i) \\
&= \sum_{i\in\{r:\ d_r\geq f+1\}} \min(kp, S_i - p) + \sum_{i\in\{r:\ d_r=f,\ r<l\}} \min(kp, S_i - b_i) + \\
&\quad + \sum_{i\in\{r:\ d_r=f,\ r>l\}} \min(kp, S_i - p) + \sum_{i\in\{r:\ d_r=f-1,\ r>l\}} \min(kp, S_i - b_i) + \\
&\quad + \min(kp, S_l - b_l - a) + \sum_{i\in\{r:\ d_r\leq f-2\}} \min(kp, S_i) \qquad (5)
\end{aligned}$$

where

$$\begin{aligned}
a &= R_m - \left(\sum_{i\in\{r:\ d_i\geq f+1\}} p + \sum_{i\in\{r:\ d_r=f,\ r<l\}} b_i + \sum_{i\in\{r:\ d_r=f,\ r>l\}} p \right. \\
&\quad \left. + \sum_{i\in\{r:\ d_r=f-1,\ r>l\}} b_i + b_l \right).
\end{aligned}$$

As before, we are going to rewrite each summand of (5) in terms of $\min(kp, S_i)$:

- $$\sum_{i\in\{r:\ d_r\geq f+1\}} \min(kp, S_i - p) =$$

  $$= \sum_{i\in\{r:\ d_r\geq k+1\}} \min(kp, S_i - p) + \sum_{i\in\{r:\ k\geq d_r\geq f+1\}} \min(kp, S_i - p)$$

  $$= \sum_{i\in\{r:\ d_r\geq f+1\}} \min(kp, S_i) - p|\{i : k \geq d_i \geq f+1\}| + \sum_{i\in\{r:\ d_r=k\geq f+1\}} b_i,$$

- $$\sum_{i\in\{r:\ d_r=f,\ r<l\}} \min(kp, S_i - b_i) =$$

$$= \begin{cases} \sum_{i\in\{r:\ d_r=f,\ r<l\}} \min(kp, S_i) - \sum_{i\in\{r:\ d_r=f,\ r<l\}} b_i, & \text{if} \quad k > f \\ \sum_{i\in\{r:\ d_r=f,\ r<l\}} \min(kp, S_i), & \text{if} \quad k = f, \end{cases}$$

- $$\sum_{i\in\{r:\ d_r=f,\ r>l\}} \min(kp, S_i - p) =$$

$$= \begin{cases} \sum_{i\in\{r:\ d_r=f,\ r>l\}} \min(kp, S_i) - p|\{i : i > l,\ d_i = f\}|, & \text{if} \quad k > f \\ \sum_{i\in\{r:\ d_r=f,\ r>l\}} \min(kp, S_i) - p|\{i : i > l,\ d_i = f\}| + \\ + \sum_{i\in\{r:\ d_r=f=k,\ r>l\}} b_i, & \text{if} \quad k = f, \end{cases}$$

- $$\sum_{i\in\{r:\ d_r=f-1,\ r>l\}} \min(kp, S_i - b_i) = \sum_{i\in\{r:\ d_r=f-1,\ r>l\}} \min(kp, S_i) - \sum_{i\in\{r:\ d_r=f-1,\ r>l\}} b_i,$$

and

- $$\min(kp, S_l - b_l - a) = \begin{cases} \min(kp, S_l) - a - b_l, & \text{if} \quad k > f \\ \min(kp, S_l) - a, & \text{if} \quad k = f. \end{cases}$$

Consequently,

$$\sum_{i=1}^{kp} u_i^* =$$

$$= \begin{cases} \displaystyle\sum_{i=1}^{n} \min(kp, S_i) - R_m + p|\{i : d_i \geq k+1\}| + \\ \displaystyle\sum_{i \in \{r:\ d_r = k \geq f+1\}} b_i, & \text{if} \quad k > f \\ \\ \displaystyle\sum_{i=1}^{n} \min(kp, S_i) - R_m + p|\{i : d_i \geq k+1\}| + \\ \\ \displaystyle\sum_{i \in \{r:\ d_r = f,\ r < l\}} b_i + \sum_{i \in \{r:\ d_r = f,\ r > l\}} b_i + b_l, & \text{if} \quad k = f. \end{cases}$$

Thus,

$$\begin{aligned} \sum_{i=1}^{kp} u_i^* &= \sum_{i=1}^{n} \min(kp, S_i) - R_m + v_{k+1}^* \\ &= \sum_{i=1}^{kp} S_i^* - R_m + v_{k+1}^* \\ &= \sum_{i=1}^{(k+1)p} S_i^* - R_m \\ &\geq \sum_{i=1}^{k+1} R_i - R_m \\ &\geq \sum_{i=1}^{k} R_i, \end{aligned}$$

and we conclude that $(R_1, \ldots, R_{m-1}) \preceq_p (S')^*$. Therefore, the theorem follows. $\square$

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