# Change in Vertex Status after Removal of Another Vertex in the General Setting

Charles R. Johnson[*] Carlos M. Saiago[†] Kenji Toyonaga[‡]

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**Abstract**

In the theory of multiplicities for eigenvalues of symmetric matrices whose graph is a tree, it proved very useful to understand the change in status (Parter, neutral, or downer) of one vertex upon removal of another vertex of given status (both in case the two vertices are adjacent or non-adjacent). As the subject has evolved toward the study of more general matrices, over more general fields, with more general graphs, it is appropriate to resolve the same type of question in the more general settings. "Multiplicity" now means geometric multiplicity. Here, we give a complete resolution in three more general

[*]Department of Mathematics, College of William and Mary, P.O. Box 8795, Williamsburg, VA 23187-8795, USA (crjohn@wm.edu).

[†]Corresponding author. Centro de Matemática e Aplicações (CMA), FCT/UNL, and Departamento de Matemática, Faculdade de Ciências e Tecnologia da Universidade NOVA de Lisboa, 2829-516 Caparica, Portugal (cls@fct.unl.pt). This work was carried out within the activities of CMA/FCT/UNL and it was partially supported by the Fundação para a Ciência e a Tecnologia (Portuguese Foundation for Science and Technology) through the project UIDB/00297/2020. Part of this work was done while visiting the National Institute of Technology, Kitakyushu College, Japan.

[‡]Department of Creative Engineering, National Institute of Technology, Kitakyushu College, Kokuraminami-ku, Kitakyushu, 802-0985, Japan (toyonaga@kct.ac.jp).

settings and compare these with the classical case (216 “Yes” or “No” results). As a consequence, several unexpected insights are recorded.

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# 1 Introduction

In the study of the possible lists of multiplicities for the eigenvalues of a matrix with a given graph, it proved useful (in the case of real symmetric matrices whose graph is a tree) to understand the change in status of one vertex upon removal of another [JL-DMc, JSu, JS18]. The *status* of a vertex is defined relative to a graph, a matrix and a designated eigenvalue. The removal of a vertex (extraction of a principal submatrix) can increase or decrease (by 1 each) or leave unchanged the geometric multiplicity of the eigenvalue. If there is an increase, the vertex is called *Parter*, a decrease it is called *downer*, and unchanged it is called *neutral*. For simplicity, these are abbreviated as P, D and N, respectively. See [JS18] for all the general background. In the case that has been studied most (the graph is a tree, the matrix is real symmetric or, equivalently, complex Hermitian, and multiplicity is unambiguous, as algebraic and geometric multiplicity are the same), a complete catalog of the 54 possible situations (distinguishing between adjacent and non-adjacent, which can be different) was given in [JL-DMc], following the Honors Thesis of Paul McMichael [Mc].

In the past few years, multiplicity research has begun to consider more general situations by looking at either more general graphs and/or non-Hermitian matrices, possibly over general fields. In the case of general matrices, “multiplicity” becomes ambiguous, but the most fruitful definition is geometric multiplicity [JS17, JST17b, JL-DS18, JS18]. Here, we take up the change in status question more generally and give a complete table of the possible occurrences in the four natural settings: (1) trees and (real) symmetric matrices (already done); (2) trees and general real (combinatorially symmetric) matrices with geometric multiplicity; (3) general undirected graphs and real symmetric matrices; and, finally, (4) general undirected graphs and general real (combinatorially symmetric) matrices, with geometric multiplicity, all in both the case that the two vertices are adjacent or non-adjacent. This is a total of 216 possible outcomes, which are either "Yes" (the outcome can occur) or "No" (it cannot occur).

Many of the table entries are implied by other entries. For instance, an example, implying "Yes" in a more restrictive situation, also implies "Yes" in less restrictive ones. Or a counting argument that implies "No" in one situation may be equally valid in others. The more general geometric theory, developed in [JS17] shows that the counting arguments are equally valid in more general settings. Some table entries, however, are more subtle when an example is difficult to construct or a possibility cannot be excluded by counting (but only by some more subtle application of the theory). The completed table revealed some interesting phenomena, that we comment upon, and that we would not have predicted.

We establish the validity of the table's entries as follows. In the next section, the necessary background is summarized. In Section 3, we discuss why prior counting arguments that produced "No" answers remain valid and yield "No" answers throughout the row. Counting arguments also show equivalence of some cells with others, perhaps limiting necessary examples or special arguments. In Section 4 we present some auxiliary results on (geometric) multiplicity theory for combinatorially symmetric matrices, over a field, whose graph is a tree. Then we explain why the table entries, corresponding to trees and real combinatorially symmetric matrices, are identical to those corresponding to trees and real symmetric matrices. In Section 5, the examples necessary to verify all the "Yes" table entries are given. As it turns out, a single example is sufficient to clarify all the "Yes" entries in a row, and some of these already exist in the tree/symmetric case [JL-DMc, table 1]. However, in some cases, where there was a "No" entry in the prior table [JL-DMc, table 1], new examples are needed. In Section 6, we give a key result, involving a special type of neutral vertex/index, that resolves remaining table entries. Then in Section 7, the table, with some comment, is presented. In the last section, we mention some facts that emerge from the table and that we would not otherwise predict.

# 2 Background

An $n$-by-$n$ combinatorially symmetric matrix is a matrix $A = (a_{ij})$ such that $a_{ij} \neq 0$ if and only if $a_{ij} \neq 0$. The pattern of $A$ is described by the graph $\mathrm{G}(A)$ of $A$, an undirected simple graph on vertices $1, \ldots, n$ in which $\{i, j\}$ is an edge if and only if $a_{ij} \neq 0$. Given a graph $G$ on $n$ vertices and a field $\mathbb{F}$, we denote by $\mathcal{F}(G)$ the set of all $n$-by-$n$ combinatorially symmetric matrices, over $\mathbb{F}$, whose graph is $G$; no restriction is placed on the diagonal entries. We denote by $\mathcal{S}(G)$ the set of all real symmetric matrices, whose graph is $G$. For $A \in \mathcal{F}(G)$ we denote by $\mathrm{gm}_A(\lambda)$ the geometric multiplicity of $\lambda$ as an eigenvalue of $A$, and we denote by $\sigma(A)$ the spectrum of $A$.

Given a graph $G$ on $n$ vertices, $A \in \mathcal{F}(G)$ and an index subset $\alpha$ of $\{1, \ldots, n\}$, we denote by $A(\alpha)$ (resp. $G - \alpha$) the principal submatrix of $A$ (resp. induced subgraph of $G$) resulting from deletion of the rows and columns (resp. vertices) indexed by $\alpha$, and we denote by $A[\alpha]$ (resp. $G[\alpha]$) the principal submatrix of $A$ (resp. induced subgraph of $G$) resulting from keeping only the rows and columns (resp. vertices) indexed by $\alpha$. If $G' = G[\alpha]$ we often write $A[G']$, meaning the principal submatrix $A[\alpha]$. We abbreviate $A(\{i\})$ (resp. $G - \{i\}$) by $A(i)$ (resp. $G - i$). When $G$ is a tree, $A(i)$ is a direct sum, whose summands correspond to components of $G - i$, which we call *branches* of $G$ at $i$. Generally, when indices/vertices are deleted, we refer to the remaining indices/vertices via their original labels.

The following result states a characterization of the status of a vertex as Parter, neutral or downer, which justifies that, for a graph $G$, $A \in \mathcal{F}(G)$ and $\lambda \in \mathbb{F}$, we have $\left|\mathrm{gm}_A(\lambda) - \mathrm{gm}_{A(v)}(\lambda)\right| \leq 1$.

**Lemma 1 ([JS17])** *Let $G$ be a graph, $v$ a vertex of $G$, $A \in \mathcal{F}(G)$ and $\lambda \in \mathbb{F}$. Then, in $A(v)$ there are 3 possibilities, the third occurring only in case* $\mathrm{gm}_A(\lambda) \geq 1$*:*

1. $\mathrm{gm}_{A(v)}(\lambda) = \mathrm{gm}_A(\lambda) + 1$, *which occurs if and only if*

$$\mathrm{rank}\,(A(v) - \lambda I) = \mathrm{rank}\,(A - \lambda I) - 2;$$

2. $\mathrm{gm}_{A(v)}(\lambda) = \mathrm{gm}_A(\lambda)$, *which occurs if and only if*

$$\mathrm{rank}\,(A(v) - \lambda I) = \mathrm{rank}\,(A - \lambda I) - 1; \quad \textit{and}$$

3. $\mathrm{gm}_{A(v)}(\lambda) = \mathrm{gm}_A(\lambda) - 1$, *which occurs if and only if*

$$\mathrm{rank}\,(A(v) - \lambda I) = \mathrm{rank}\,(A - \lambda I)\,.$$

Depending on the context it is useful to use alternate, equivalent characterizations of the status of a vertex. For this purpose we present some definitions and notation.

Let $G$ be a graph, $v$ a vertex of $G$, $A = (a_{ij}) \in \mathcal{F}(G)$ and $\lambda \in \mathbb{F}$. After permutation similarity, matrix $A$ may be displayed as

$$A = \begin{bmatrix} a_{vv} & \mathrm{x}^{\mathrm{T}} \\ \mathrm{y} & A(v) \end{bmatrix} \tag{1}$$

so that, *wlog*, focussing upon a vertex/index $v$ we assume that $A$ appears as in (1). We denote the column space and the row space of $A$ by $\mathrm{CS}(A)$ and $\mathrm{RS}(A)$, respectively. For convenience, we consider $\mathrm{CS}(A)$ a space of column vectors and $\mathrm{RS}(A)$ a space of row vectors.

**Lemma 2 ([JST17b])** *Let $G$ be a graph, $v$ a vertex of $G$, $A \in \mathcal{F}(G)$ and $\lambda \in \mathbb{F}$. Let*

$$A - \lambda I = \begin{bmatrix} a_{vv} - \lambda & \mathrm{x}^{\mathrm{T}} \\ \mathrm{y} & A(v) - \lambda I \end{bmatrix} = \begin{bmatrix} a & \mathrm{x}^{\mathrm{T}} \\ \mathrm{y} & B \end{bmatrix}.$$

1. *Vertex $v$ is Parter for $\lambda$ in $A$ if and only if*

$$\mathrm{x}^{\mathrm{T}} \notin \mathrm{RS}(B) \quad \textit{and} \quad \mathrm{y} \notin \mathrm{CS}(B).$$

2. *Vertex $v$ is neutral for $\lambda$ in $A$ if and only if*

$$\left(\begin{bmatrix} a & \mathrm{x}^{\mathrm{T}} \end{bmatrix} \notin \mathrm{RS}(\begin{bmatrix} \mathrm{y} & B \end{bmatrix}) \quad \textit{and} \quad \mathrm{y} \in \mathrm{CS}(B)\right)$$

*or*

$$\left(\begin{bmatrix} a \\ \mathrm{y} \end{bmatrix} \notin \mathrm{CS}\left(\begin{bmatrix} \mathrm{x}^{\mathrm{T}} \\ B \end{bmatrix}\right) \quad \textit{and} \quad \mathrm{x}^{\mathrm{T}} \in \mathrm{RS}(B)\right).$$

3. *If $\lambda \in \sigma(A)$ then vertex $v$ is downer for $\lambda$ in $A$ if and only if*

$$\begin{bmatrix} a & \mathrm{x}^{\mathrm{T}} \end{bmatrix} \in \mathrm{RS}(\begin{bmatrix} \mathrm{y} & B \end{bmatrix}) \quad \textit{and} \quad \mathrm{y} \in \mathrm{CS}(B)$$

*if and only if*

$$\begin{bmatrix} a \\ \mathrm{y} \end{bmatrix} \in \mathrm{CS}\left(\begin{bmatrix} \mathrm{x}^{\mathrm{T}} \\ B \end{bmatrix}\right) \quad \textit{and} \quad \mathrm{x}^{\mathrm{T}} \in \mathrm{RS}(B).$$

We note that Lemmas 1 and 2 may both be viewed as statements about a matrix and an index, irrespective of the underlying graph.

Under the conditions of Lemma 2, if

$$\mathrm{y} \in \mathrm{CS}(B) \quad \text{and} \quad \mathrm{x}^{\mathrm{T}} \in \mathrm{RS}(B) \quad \text{and} \quad \begin{bmatrix} a & \mathrm{x}^{\mathrm{T}} \end{bmatrix} \notin \mathrm{RS}(\begin{bmatrix} \mathrm{y} & B \end{bmatrix})$$

or, equivalently,

$$\mathrm{y} \in \mathrm{CS}(B) \quad \text{and} \quad \mathrm{x}^{\mathrm{T}} \in \mathrm{RS}(B) \quad \text{and} \quad \begin{bmatrix} a \\ \mathrm{y} \end{bmatrix} \notin \mathrm{CS}\left(\begin{bmatrix} \mathrm{x}^{\mathrm{T}} \\ B \end{bmatrix}\right),$$

then $v$ is neutral for $\lambda$. Note that this "neutrality" of $v$ depends on the value of entry $a_{vv}$ of $A$. (In fact, there is a unique perturbation of the diagonal entry $a_{vv}$ that increases the geometric multiplicity of $\lambda$ and makes vertex $v$ a downer in the perturbed matrix [JST17b, theorem 20].) We call such a neutral vertex a *type-I neutral* vertex.

Again, under the conditions of Lemma 2, if

$$\big(\mathrm{y} \in \mathrm{CS}(B) \quad \text{and} \quad \mathrm{x}^{\mathrm{T}} \notin \mathrm{RS}(B)\big) \quad \text{or} \quad \big(\mathrm{y} \notin \mathrm{CS}(B) \quad \text{and} \quad \mathrm{x}^{\mathrm{T}} \in \mathrm{RS}(B)\big)$$

then $v$ is still a neutral vertex for $\lambda$. Note that this "neutrality" of $v$ does not depend on the value of the diagonal entry $a_{vv}$. We call such a neutral vertex a *type-II neutral* vertex.

The existence of these two types of neutral vertices was noted by the authors in [JST17b, theorem 20]. Note that, in particular, neither a symmetric matrix over $\mathbb{F}$ nor a complex Hermitian matrix may have a type-II neutral vertex/index.

# 3 Counting arguments

The following results are stated relative to an identified eigenvalue of a matrix $A \in \mathcal{F}(G)$, $G$ a graph. For convenience we often refer interchangeably to the graph $G$ and matrix $A$. In particular, given an induced subgraph $G'$ of $G$, sometimes we refer to the "eigenvalues of $G'$", meaning the "eigenvalues of $A[G']$". We also write "$\mathrm{gm}_{G'}(\lambda)$" meaning "$\mathrm{gm}_{A[G']}(\lambda)$".

**Argument 1 ([JL-DMc])** *The (geometric) multiplicity of $\lambda$ as an eigenvalue of $G - \{i, j\}$ has a unique value; thus, if we remove $i$ and then $j$, the multiplicity of $\lambda$ in the resulting subgraph should be the same as if we remove $j$ and then $i$.*

This is a simple but important observation that allows us (as was noted by the authors in [JL-DMc]), jointly with Lemma 1, to justify 10 rows of the Table 1, the rows **6**–**10** and **18**–**22** (the "Yes**" in rows **7** and **21** are justified, in Section 4, by the "downer branch mechanism", Theorem 5 below, for trees).

Argument 1 and Lemma 1 also justify, in particular, the following result whose proof we omit. The proof may be the one presented in [JS18, theorem 2.5.2] by replacing "$T$" by "$G$", "interlacing" by "Lemma 1" and "$\mathcal{S}(T)$" by "$\mathcal{F}(G)$".

**Theorem 3** *Let $G$ be a graph and $i$, $j$ be two distinct vertices of $G$. Then $i$ is Parter in $G$ and $j$ is downer in $G$ if and only if $i$ is Parter in $G-j$ and $j$ is downer in $G-i$.*

From Theorem 3 we conclude, in particular, that any Parter vertex of $G$ remains Parter after the removal of a downer vertex from $G$ and, conversely, that any downer vertex of $G$ remains downer after the removal of a Parter vertex from $G$. Thus, by using Theorem 3, we could conclude the rows **7**–**9** and **19**–**21** already justified by Argument 1 and Lemma 1.

It is useful to use a short notation to describe the outcome corresponding to each row of Table 1. For a particular row, if the initial status of vertex $i$ is $S_i$, the initial status of vertex $j$ is $S_j$ and the status of $i$ when $j$ is removed becomes $S_i'$, then we denote this outcome by $\overset{\curvearrowleft}{S_i'[S_i}, \dot{S}_j]$.

We say that row $m$ and row $n$ of Table 1 are *equivalent* when the outcome described in row $m$ can occur if and only if the outcome described in row $n$ can occur. In that case we say that $(m, n)$ is a *pair of equivalent rows* of Table 1.

For example, row **16** and row **23** are equivalent and we may conclude this in the following way. Row **16** has the outcome $\overset{\curvearrowleft}{\mathrm{P}[\mathrm{N}}, \dot{\mathrm{D}}]$, i.e., $i$ is N in $G$, $j$ is D in $G$ and $i$ is P in $G-j$. If $\lambda$ is the relevant eigenvalue and $\mathrm{gm}_G(\lambda) = k$, then the multiplicity of $\lambda$ in $G - \{i, j\}$, in order to have the desired outcome, is

$$k \xrightarrow[\text{remove } j]{} k - 1 \xrightarrow[\text{remove } i]{} k.$$

Since $i$ is originally N, by Argument 1 we conclude that $j$ must be N in $G-i$ in order to have $\mathrm{gm}_{G-\{i,j\}}(\lambda) = k$. Thus, this row is equivalent to row **23**, $\overset{\curvearrowleft}{\mathrm{N}[\mathrm{D}}, \dot{\mathrm{N}}]$, in which $i$ is D in $G$, $j$ is N in $G$ and $i$ is N in $G-j$ (simply switch the labels on $i$ and $j$).

In fact, as was noted in [JL-DMc], there are five pairs of equivalent rows in Table 1.

**Argument 2 ([JL-DMc])** *The following are pairs of equivalent rows of Table 1:* $(\mathbf{4}, \mathbf{11})$, $(\mathbf{5}, \mathbf{12})$, $(\mathbf{7}, \mathbf{21})$, $(\mathbf{16}, \mathbf{23})$ *and* $(\mathbf{17}, \mathbf{24})$.

# 4 $\mathcal{F}(T)$, $T$ a tree

In [JS17], the authors have shown that a substancial part of the Parter-Wiener, etc. theory [JL-DS03a] for real symmetric (Hermitian) matrices (*symmetric case*) whose graph is a tree generalizes for geometric multiplicities of combinatorially symmetric matrices (*general setting*) whose graph is a tree. Not all do, however.

It is well known [JL-DS03a, JL-DSSuWi] that any real symmetric (Hermitian) matrix whose graph is a tree on more than one vertex has at least two of its (real) eigenvalues of multiplicity 1. However, there do exist trees for which there are combinatorially symmetric matrices with no geometric multiplicity 1 eigenvalues [JL-DS18]. Another example of a difference is the minimum number of distinct eigenvalues of a matrix whose graph is a tree. In the symmetric case [JL-D02a] and in the general setting (for diagonalizable matrices) [JS17], this minimum number is known to be at least the "diameter" (measured as the number of vertices in a longest path) of the tree. However, there are trees of diameter 7 for which any real symmetric matrix with such a graph has, at least, 8 distinct eigenvalues [BF04, JS16]. But, for any tree of diameter 7 there is always a diagonalizable combinatorially symmetric matrix whose graph is that tree, with only 7 distinct eigenvalues [S19].

Our purpose in this section is (1) to show that the results and arguments of [JL-DMc] that were used to construct [JL-DMc, table 1], all generalize to combinatorially symmetric matrices, over a field, whose graph is also a tree, when "multiplicity" is replaced by "geometric multiplicity" and, consequently, (2) the table entries in Table 1, corresponding to trees and real combinatorially symmetric matrices, are exactly the same as the ones corresponding to trees and real symmetric matrices [JL-DMc, table 1].

The recognition of the existence of Parter vertices plays an important role in the multiplicity theory. The following [JS17] is the analog, for combinatorially symmetric matrices whose graph is a tree and geometric multiplicities, of the fundamental "Parter-Wiener, etc. Theorem" [JL-DS03a].

**Theorem 4 ([JS17])** *Let $\mathbb{F}$ be a field, $T$ a tree and $A \in \mathcal{F}(T)$. Suppose that there is a vertex $v$ of $T$ such that $\lambda \in \sigma(A) \cap \sigma(A(v))$. Then,*

1. *there is a vertex $u$ of $T$ such that $\mathrm{gm}_{A(u)}(\lambda) = \mathrm{gm}_A(\lambda) + 1$, i.e., $u$ is Parter for $\lambda$ (with respect to $A$ and $T$);*
2. *if $\mathrm{gm}_A(\lambda) \geq 2$, then $u$ may be chosen so that $\deg_T(u) \geq 3$ and so that there are at least 3 components $T_1$, $T_2$, and $T_3$ of $T - u$ such that $\mathrm{gm}_{A[T_i]}(\lambda) \geq 1$, $i = 1, 2, 3$; and*
3. *if $\mathrm{gm}_A(\lambda) = 1$, then $u$ may be chosen so that $\deg_T(u) \geq 2$ and so that there are 2 components $T_1$ and $T_2$ of $T - u$ such that $\mathrm{gm}_{A[T_i]}(\lambda) = 1$, $i = 1, 2$.*

If a vertex $v$ of degree $k$ is removed from a tree $T$ then $T - v$ is formed by the $k$ branches $T_1, \ldots, T_k$ at $v$ and each neighbor $u_i$ of $v$ will be in exactly one of these branches $T_i$. If $A \in \mathcal{F}(T)$ and $u_i$ is a downer vertex in $T_i$ (for the eigenvalue $\lambda$ of $A[T_i]$) then we call $T_i$ a *downer branch* at $v$ and $u_i$ a *downer vertex* at $v$ or a *downer neighbor* of $v$ (for the eigenvalue $\lambda$ relative to $A$). As we will see, a downer neighbor of $v$ may or may not be a downer vertex in $T$ (exactly the same happens in the symmetric case).

The following result [JS17] establishes the "downer branch mechanism" for Parter vertices in the general setting, a key result for the characterization of Parter vertices, which is exactly analogous to the symmetric case.

**Theorem 5 ([JS17])** *Let $T$ be a tree, $A \in \mathcal{F}(T)$ and $\lambda \in \mathbb{F}$. Then a vertex $v$ of $T$ is Parter for $\lambda$ if and only if there is at least one downer branch at $v$ for $\lambda$.*

The status of a vertex $v$ as Parter only depends on the nature of its neighbors in their branches. If there is only one (resp. more than one) downer branch at $v$ then we call $v$ a *singly Parter* (resp. *multiply Parter*) vertex.

We will see, in particular, that the only downer neighbor of a singly Parter vertex cannot be downer in $T$ and no neighbor of a singly Parter vertex can be downer in $T$. However, the downer neighbors of a multiply Parter vertex are all downers in $T$. The structural properties of the vertices of a tree that we present here for the general setting are exactly analogous to the symmetric case.

The remaining results of this section are stated relative to an identified eigenvalue of a matrix in $\mathcal{F}(T)$, $T$ a tree. The first result is a characterization of a multiply Parter vertex and of a singly Parter vertex.

**Theorem 6** *Let $T$ be a tree and $u$ be a vertex of $T$. We have in $T$:*

1. *$u$ is multiply Parter if and only if $u$ is Parter with a neighbor that is a downer in $T$.*

   *Moreover, the downer neighbors of a multiply Parter vertex $u$ of $T$ are the neighbors of $u$ in $T$ that are downers in $T$.*

2. *$u$ is singly Parter if and only if $u$ is Parter with no neighbors that are downers in $T$.*

***Proof***. This theorem (and its proof) is the analog, for $\mathcal{F}(T)$, of [JS18, theorem 2.5.3 and proof], by replacing in the latter "$\mathcal{S}(T)$", "multiplicity m" and "interlacing" by "$\mathcal{F}(T)$", "geometric multiplicity gm" and "Lemma 1", respectively. (Note that in [JL-DMc, JS18], the fact $\mathrm{m}_A(\lambda) - \mathrm{m}_{A(v)}(\lambda) \in \{-1, 0, 1\}$ is justified by the interlacing inequalities for the eigenvalues of Hermitian matrices. For general matrices and geometric multiplicities, the mentioned fact may be justified by Lemma 1. For short, we simply say that "interlacing" should be replaced by "Lemma 1" in the original proof.) $\square$

From Theorem 6 we conclude that a Parter vertex may have two different kinds of downer neighbors. If $v$ is a downer neighbor of $u$ but $v$ is not a downer in $T$, then we call $v$ a *local downer neighbor* of $u$. If $v$ is still a downer in $T$, then we call $v$ a *global downer neighbor*.

**Corollary 7** *Let $T$ be a tree.*

1. *The neighbors of a singly Parter vertex must be either Parter (singly Parter or multiply Parter) or neutral in $T$.*

2. *No vertex has more than one local downer neighbor (relative to an identified eigenvalue).*

3. *The neighbors of a downer vertex must be either downer or multiply Parter in $T$.*
   *(Or, equivalently, if $u$ is a downer in $T$ and $v$ is either singly Parter or neutral in $T$ then $u$ and $v$ are not adjacent.)*

4. *In the unique path connecting a neutral vertex to a (necessarily non-adjacent) downer vertex in $T$ there is a multiply Parter vertex.*

***Proof***. This corollary (and its proof) is the analog, for $\mathcal{F}(T)$, of [JS18, corollary 2.5.4 and proof], by replacing in the latter "$\mathcal{S}(T)$", "multiplicity m" and "interlacing" by "$\mathcal{F}(T)$", "geometric multiplicity gm" and "Lemma 1", respectively. □

The following result, states an important feature of multiply Parter vertices.

**Theorem 8** *Let $T$ be a tree and $u$ be a multiply Parter vertex. Upon removal of $u$ from $T$, no other vertex of $T$ changes its status in $T-u$ (relative to an identified eigenvalue of a matrix in $\mathcal{F}(T)$). Moreover, the number of downer neighbors of each Parter vertex in $T-u$ is the same as in $T$.*

***Proof***. This theorem (and its proof) is the analog, for $\mathcal{F}(T)$, of [JS18, theorem 2.5.1 and proof], by replacing in the latter "$\mathcal{S}(T)$", "multiplicity m" and "interlacing" by "$\mathcal{F}(T)$", "geometric multiplicity gm" and "Lemma 1", respectively. □

Next we show that the removal of a neutral vertex does not change the status of a downer vertex and conversely.

**Theorem 9** *Let $T$ be a tree and $u, v$ be two non-adjacent vertices of $T$. If $u$ is neutral in $T$ and $v$ is a downer in $T$ then $u$ is neutral in $T-v$ and $v$ is a downer in $T-u$ (relative to an identified eigenvalue of a matrix in $\mathcal{F}(T)$).*

***Proof*.** This theorem (and its proof) is the analog, for $\mathcal{F}(T)$, of [JS18, theorem 2.5.7 and proof], by replacing in the latter "$\mathcal{S}(T)$", "multiplicity m" and "interlacing" by "$\mathcal{F}(T)$", "geometric multiplicity gm" and "Lemma 1", respectively. □

Given the results and definitions presented in this section, which are exactly the analogs of the corresponding results and definitions that were needed to justify [JL-DMc, table 1], we may conclude that [JL-DMc, table 1] also describes the resolution to the analogous problem when we consider real combinatorially symmetric matrices instead of real symmetric or Hermitian matrices. The proof (of "Adjacent cases" and "Non-adjacent cases") is analogous to the proofs that may be found in [JL-DMc, pages 36–41] or [JS18, pages 37–42]. (Since an entry with a "Yes" in the original table 1 of [JL-DMc] means a "Yes" in the case of real combinatorially symmetric matrices, only those entries with a "No", or an annotated "Yes" (with a "*" or a "**") need be analyzed to transist to the more general setting.)

# 5 Examples

In this section we give the examples necessary to verify all the "Yes" table entries. Several table entries in a row are implied by another entry of the same row. For instance, an example justifying a "Yes" in a more restrictive situation, also implies "Yes" in less restrictive ones (for example, a "Yes" in the adjacent case for $\mathcal{S}(T)$ also implies a "Yes" for the remaining adjacent cases of the same row). For the "Adjacent case" (resp. "Non-adjacent case") only one example is sufficient to clarify also all the "Yes" entries to the right in the same row for the "Adjacent case" (resp. "Non-adjacent case"), and some of these already exist in the tree/symmetric case [JL-DMc, table 1]. Thus we only need to consider examples to justify a "Yes" entry, in the new table of Section 7, where there is a "No" entry on the left corresponding to the prior table [JL-DMc, table 1].

**Example 10** In each of the following examples, if the graph $G$ has $n$ vertices, the mentioned vertices $i$ and $j$ are $i = 1$ and $j = n$. The classification of each vertex $i$ and $j$ relative to the displayed matrix (whose graph is $G$) is with respect to the eigenvalue $\lambda = 0$.

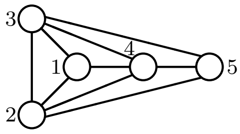
**Row 2 in $\mathcal{F}(G)$** $\mathrm{N}[\mathrm{P}, \dot{\mathrm{P}}]$

- **Adjacent case:**

$$G = K_5 \text{ and } A = \begin{bmatrix} 0 & 2 & 1 & 1 & 3 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 \\ 3 & 1 & 1 & 1 & 0 \end{bmatrix} \in \mathcal{F}(G).$$

$\mathrm{gm}_A(\lambda) = 1$, $\mathrm{gm}_{A(i)}(\lambda) = \mathrm{gm}_{A(j)}(\lambda) = 2$ and $\mathrm{gm}_{A(\{j,i\})}(\lambda) = 2$.

- **Non-adjacent case:**

$G =$ 

$$\text{and } A = \begin{bmatrix} 0 & 2 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \in \mathcal{F}(G).$$

$\mathrm{gm}_A(\lambda) = 1$, $\mathrm{gm}_{A(i)}(\lambda) = \mathrm{gm}_{A(j)}(\lambda) = 2$ and $\mathrm{gm}_{A(\{j,i\})}(\lambda) = 2$.

Note that, in both examples (adjacent case and non-adjacent case), the relevant neutral vertices are type-II.

**Row 13 in $\mathcal{S}(G)$** $\mathrm{P}[\mathrm{N}, \dot{\mathrm{N}}]$

- **Adjacent case:**

$$G = K_5 \text{ and } A = \begin{bmatrix} 0 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 0 \end{bmatrix} \in \mathcal{S}(G).$$

$\mathrm{gm}_A(\lambda) = 1$, $\mathrm{gm}_{A(i)}(\lambda) = \mathrm{gm}_{A(j)}(\lambda) = 1$ and $\mathrm{gm}_{A(\{j,i\})}(\lambda) = 2$.

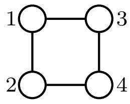
### Row 15 in $\mathcal{F}(G)$ $\mathrm{D}[\mathrm{N}, \dot{\mathrm{N}}]$

- **Adjacent case:**

$$G = K_3 \text{ and } A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \in \mathcal{F}(G).$$

$\mathrm{gm}_A(\lambda) = 1$, $\mathrm{gm}_{A(i)}(\lambda) = \mathrm{gm}_{A(j)}(\lambda) = 1$ and $\mathrm{gm}_{A(\{j,i\})}(\lambda) = 0$.

- **Non-adjacent case:**

$G =$ 


$$\text{and } A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix} \in \mathcal{F}(G).$$

$\mathrm{gm}_A(\lambda) = 1$, $\mathrm{gm}_{A(i)}(\lambda) = \mathrm{gm}_{A(j)}(\lambda) = 1$ and $\mathrm{gm}_{A(\{j,i\})}(\lambda) = 0$.

Note that, in both examples (adjacent case and non-adjacent case), the relevant neutral vertices are type-II.

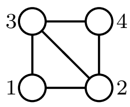
### Row 16 in $\mathcal{F}(G)$ (Equivalent to row **23**) $\mathrm{P}[\mathrm{N}, \dot{\mathrm{D}}]$

- **Adjacent case:**

$$G = K_4 \text{ and } A = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & 2 & 3 & 2 \end{bmatrix} \in \mathcal{F}(G).$$

$\mathrm{gm}_A(\lambda) = 1$, $\mathrm{gm}_{A(i)}(\lambda) = 1$, $\mathrm{gm}_{A(j)}(\lambda) = 0$ and $\mathrm{gm}_{A(\{j,i\})}(\lambda) = 1$.

- **Non-adjacent case:**

$G =$ 


$$\text{and } A = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 1 & 1 & 2 \\ 2 & 1 & 1 & 2 \\ 0 & 1 & -1 & 0 \end{bmatrix} \in \mathcal{F}(G).$$

$\mathrm{gm}_A(\lambda) = 1$, $\mathrm{gm}_{A(i)}(\lambda) = 1$, $\mathrm{gm}_{A(j)}(\lambda) = 0$ and $\mathrm{gm}_{A(\{j,i\})}(\lambda) = 1$.

Note that, in both examples (adjacent case and non-adjacent case), the relevant neutral vertices are type-II.

**Row 17 in $\mathcal{S}(G)$** (Equivalent to row **24**) $\mathrm{N}[\mathrm{N}, \dot{\mathrm{D}}]$

- **Adjacent case:**

$$G = K_4 \text{ and } A = \begin{bmatrix} 3 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \in \mathcal{S}(G).$$

$\mathrm{gm}_A(\lambda) = 2$, $\mathrm{gm}_{A(i)}(\lambda) = 2$, $\mathrm{gm}_{A(j)}(\lambda) = 1$ and $\mathrm{gm}_{A(\{j,i\})}(\lambda) = 1$.

**Row 23 in $\mathcal{F}(G)$** (Equivalent to row **16**) $\mathrm{N}[\mathrm{D}, \dot{\mathrm{N}}]$

By Argument 2, row **23** is equivalent to row **16**.

- **Adjacent case:** Simply switch the labels for $i$ and $j$ on the corresponding example for row **16** and obtain an example here.
- **Non-adjacent case:** Simply switch the labels for $i$ and $j$ on the corresponding example for row **16** and obtain an example here.

Note that, in both examples (adjacent case and non-adjacent case), the relevant neutral vertices are type-II.

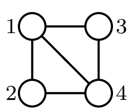
**Row 24 in $\mathcal{S}(G)$** (Equivalent to row **17**) $\mathrm{D}[\mathrm{D}, \dot{\mathrm{N}}]$

By Argument 2, row **24** is equivalent to row **17**.

- **Adjacent case:** Simply switch the labels for $i$ and $j$ on the corresponding example for row **17** and obtain an example here.

**Row 25 in $\mathcal{S}(G)$** $\mathrm{P}[\mathrm{D}, \dot{\mathrm{D}}]$

- **Adjacent case:**

$G =$ 

and $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \in \mathcal{S}(G).$

$\mathrm{gm}_A(\lambda) = 1$, $\mathrm{gm}_{A(i)}(\lambda) = \mathrm{gm}_{A(j)}(\lambda) = 0$ and $\mathrm{gm}_{A(\{j,i\})}(\lambda) = 1$.

**Row 27 in $\mathcal{S}(G)$** $\quad$ $\overset{\curvearrowleft}{\mathrm{D}[\mathrm{D}}, \dot{\mathrm{D}}]$

- **Adjacent case:**

  $$G = K_3 \text{ and } A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \in \mathcal{S}(G).$$

  $\mathrm{gm}_A(\lambda) = 2$, $\mathrm{gm}_{A(i)}(\lambda) = \mathrm{gm}_{A(j)}(\lambda) = 1$ and $\mathrm{gm}_{A(\{j,i\})}(\lambda) = 0$.

---

# 6 The importance of type-II neutral vertices

In this section we justify, in particular, that the outcomes of rows **2**, **15**, **16** and **23** of Table 1, for $\mathcal{F}(G)$, only occur when the relevant neutral vertices are type-II. We record here these results in Theorem 12 below, a key result that supports the annotated #-entries in Table 1 and resolve the remaining entries not justified so far.

For the proof of Theorem 12, the following auxiliary result is needed.

**Lemma 11** *Let* $A = \begin{bmatrix} \mathrm{b}^{\mathrm{T}} \\ B \end{bmatrix}$*, in which* $B$ *is a matrix and* b *is a column vector, and let* $\mathrm{y} \in \mathrm{CS}(B)$*. Then, for any scalar* $w$*, we have* $\begin{bmatrix} w \\ \mathrm{y} \end{bmatrix} \in \mathrm{CS}(A)$*, unless* $\mathrm{b}^{\mathrm{T}} \in \mathrm{RS}(B)$.

***Proof.*** Since $\mathrm{y} \in \mathrm{CS}(B)$ per assumption, we have $\mathrm{rank}\left(\begin{bmatrix} \mathrm{y} & B \end{bmatrix}\right) = \mathrm{rank}(B)$, which implies $\mathrm{rank}\left(\begin{bmatrix} w & \mathrm{b}^{\mathrm{T}} \\ \mathrm{y} & B \end{bmatrix}\right) \leq \mathrm{rank}(B) + 1$.

If $\mathrm{b}^{\mathrm{T}} \notin \mathrm{RS}(B)$ then, for any scalar $w$, $\begin{bmatrix} w & \mathrm{b}^{\mathrm{T}} \\ \mathrm{y} & B \end{bmatrix}$ and $\begin{bmatrix} \mathrm{b}^{\mathrm{T}} \\ B \end{bmatrix}$ have the same rank ($\mathrm{rank}(B) + 1$) and, thus, $\begin{bmatrix} w \\ \mathrm{y} \end{bmatrix} \in \mathrm{CS}\left(\begin{bmatrix} \mathrm{b}^{\mathrm{T}} \\ B \end{bmatrix}\right)$. $\square$

**Theorem 12** *For each outcome of rows* **2**, **15**, **16** *and* **23** *of Table 1 (resp.* $\overset{\curvearrowleft}{\mathrm{N}[\mathrm{P}}, \dot{\mathrm{P}}]$, $\overset{\curvearrowleft}{\mathrm{D}[\mathrm{N}}, \dot{\mathrm{N}}]$, $\overset{\curvearrowleft}{\mathrm{P}[\mathrm{N}}, \dot{\mathrm{D}}]$, $\overset{\curvearrowleft}{\mathrm{N}[\mathrm{D}}, \dot{\mathrm{N}}]$*) there is a graph* $G$ *and a matrix in* $\mathcal{F}(G)$ *realizing such outcome only when the relevant neutral vertices are type-II.*

***Proof*.** Let $G$ be a graph. Given the matrices presented in Example 10 we only need to show that, in any matrix of $\mathcal{F}(G)$ realizing the outcome of one of the rows **2**, **15**, **16** or **23** of Table 1, the relevant neutral vertices must be type-II.

Let $i, j$ be vertices of $G$, $A \in \mathcal{F}(G)$, and let $\lambda$ be the relevant eigenvalue. *Wlog* we consider $\lambda = 0$ and suppose that $i = 1$ and $j = 2$, so that we see matrix $A$ with the block decomposition

$$A = \left[\begin{array}{c|c|c} a_{11} & a_{12} & \mathrm{b}_1^{\mathrm{T}} \\ \hline a_{21} & a_{22} & \mathrm{b}_2^{\mathrm{T}} \\ \hline \mathrm{c}_1 & \mathrm{c}_2 & B \end{array}\right],$$

in which $B = A(\{i, j\})$ and $\mathrm{b}_1, \mathrm{b}_2, \mathrm{c}_1, \mathrm{c}_2$ are column vectors.

$\boxed{\text{Row } \mathbf{2} \ \ \dot{\mathrm{N}}[\mathrm{P}, \dot{\mathrm{P}}]}$: Suppose that $A$ realizes this outcome. Then, $i$ and $j$ are both Parter in $G$, and $i$ is neutral in $G - j$. In order to reach a contradiction we suppose that $i$ is type-I neutral in $G - j$. Under this hypothesis we have

$$\mathrm{b}_1^{\mathrm{T}} \in \mathrm{RS}(B), \quad \mathrm{c}_1 \in \mathrm{CS}(B), \quad \begin{bmatrix} a_{11} & \mathrm{b}_1^{\mathrm{T}} \end{bmatrix} \notin \mathrm{RS}\left(\begin{bmatrix} \mathrm{c}_1 & B \end{bmatrix}\right) \quad \text{and} \quad \begin{bmatrix} a_{11} \\ \mathrm{c}_1 \end{bmatrix} \notin \mathrm{CS}\left(\begin{bmatrix} \mathrm{b}_1^{\mathrm{T}} \\ B \end{bmatrix}\right). \tag{2}$$

Because $i$ is Parter in $G$ we have

$$\begin{bmatrix} a_{12} & \mathrm{b}_1^{\mathrm{T}} \end{bmatrix} \notin \mathrm{RS}\left(\begin{bmatrix} a_{22} & \mathrm{b}_2^{\mathrm{T}} \\ \mathrm{c}_2 & B \end{bmatrix}\right) \quad \text{and} \quad \begin{bmatrix} a_{21} \\ \mathrm{c}_1 \end{bmatrix} \notin \mathrm{CS}\left(\begin{bmatrix} a_{22} & \mathrm{b}_2^{\mathrm{T}} \\ \mathrm{c}_2 & B \end{bmatrix}\right). \tag{3}$$

Since $j$ is Parter in $G$ we also have

$$\begin{bmatrix} a_{21} & \mathrm{b}_2^{\mathrm{T}} \end{bmatrix} \notin \mathrm{RS}\left(\begin{bmatrix} a_{11} & \mathrm{b}_1^{\mathrm{T}} \\ \mathrm{c}_1 & B \end{bmatrix}\right) \quad \text{and} \quad \begin{bmatrix} a_{12} \\ \mathrm{c}_2 \end{bmatrix} \notin \mathrm{CS}\left(\begin{bmatrix} a_{11} & \mathrm{b}_1^{\mathrm{T}} \\ \mathrm{c}_1 & B \end{bmatrix}\right).$$

By counting, we conclude that $j$ must also be neutral in $G - i$, so that

$$\begin{bmatrix} a_{22} & \mathrm{b}_2^{\mathrm{T}} \end{bmatrix} \notin \mathrm{RS}\left(\begin{bmatrix} \mathrm{c}_2 & B \end{bmatrix}\right) \quad \text{or} \quad \begin{bmatrix} a_{22} \\ \mathrm{c}_2 \end{bmatrix} \notin \mathrm{CS}\left(\begin{bmatrix} \mathrm{b}_2^{\mathrm{T}} \\ B \end{bmatrix}\right).$$

*Wlog* we suppose

$$\begin{bmatrix} a_{22} & \mathrm{b}_2^{\mathrm{T}} \end{bmatrix} \notin \mathrm{RS}\left(\begin{bmatrix} \mathrm{c}_2 & B \end{bmatrix}\right). \tag{4}$$

(If we suppose $\begin{bmatrix} a_{22} \\ \mathrm{c}_2 \end{bmatrix} \notin \mathrm{CS}\left(\begin{bmatrix} \mathrm{b}_2^{\mathrm{T}} \\ B \end{bmatrix}\right)$ the argument is analogous.) Since $\mathrm{c}_1 \in \mathrm{CS}(B)$ we have $\mathrm{c}_1 \in \mathrm{CS}\left(\begin{bmatrix} \mathrm{c}_2 & B \end{bmatrix}\right)$ so, by Lemma 11, we also have $\begin{bmatrix} a_{21} \\ \mathrm{c}_1 \end{bmatrix} \in \mathrm{CS}\left(\begin{bmatrix} a_{22} & \mathrm{b}_2^{\mathrm{T}} \\ \mathrm{c}_2 & B \end{bmatrix}\right)$, a contradiction to (3), unless $\begin{bmatrix} a_{22} & \mathrm{b}_2^{\mathrm{T}} \end{bmatrix} \in \mathrm{RS}\left(\begin{bmatrix} \mathrm{c}_2 & B \end{bmatrix}\right)$. But $\begin{bmatrix} a_{22} & \mathrm{b}_2^{\mathrm{T}} \end{bmatrix} \in \mathrm{RS}\left(\begin{bmatrix} \mathrm{c}_2 & B \end{bmatrix}\right)$ contradicts (4). This contradiction means that vertex $i$ cannot be type-I neutral in $G - j$. Similarly, vertex $j$ cannot be type-I neutral in $G - i$. Therefore the outcome of row **2** occurs (adjacent case and non-adjacent case) only when $i$ is type-II neutral in $G - j$ and $j$ is type-II neutral in $G - i$.

$\boxed{\text{Row } \mathbf{15} \;\; \mathrm{D}[\mathrm{N}, \dot{\mathrm{N}}]}$: Suppose that $A$ realizes this outcome. Then, $i$ and $j$ are both neutral in $G$, and $i$ is downer in $G - j$. By counting, we conclude that $j$ is also downer in $G - i$, so that

$$\begin{bmatrix} a_{22} & \mathrm{b}_2^{\mathrm{T}} \end{bmatrix} \in \mathrm{RS}\left(\begin{bmatrix} \mathrm{c}_2 & B \end{bmatrix}\right) \quad \text{and} \quad \begin{bmatrix} a_{22} \\ \mathrm{c}_2 \end{bmatrix} \in \mathrm{CS}\left(\begin{bmatrix} \mathrm{b}_2^{\mathrm{T}} \\ B \end{bmatrix}\right). \tag{5}$$

Thus, there are column vectors x and y such that

$$\begin{bmatrix} a_{22} & \mathrm{b}_2^{\mathrm{T}} \\ \mathrm{c}_2 & B \end{bmatrix} = \begin{bmatrix} \mathrm{x}^{\mathrm{T}} B \mathrm{y} & \mathrm{x}^{\mathrm{T}} B \\ B \mathrm{y} & B \end{bmatrix}.$$

In order to reach a contradiction we suppose that $i$ is type-I neutral in $G$. Under this hypothesis we have

$$\begin{bmatrix} a_{12} & \mathrm{b}_1^{\mathrm{T}} \end{bmatrix} \in \mathrm{RS}\left(\begin{bmatrix} a_{22} & \mathrm{b}_2^{\mathrm{T}} \\ \mathrm{c}_2 & B \end{bmatrix}\right) \quad \text{and} \quad \begin{bmatrix} a_{21} \\ \mathrm{c}_1 \end{bmatrix} \in \mathrm{CS}\left(\begin{bmatrix} a_{22} & \mathrm{b}_2^{\mathrm{T}} \\ \mathrm{c}_2 & B \end{bmatrix}\right).$$

Since from (5) we have

$$\mathrm{RS}\left(\begin{bmatrix} a_{22} & \mathrm{b}_2^{\mathrm{T}} \\ \mathrm{c}_2 & B \end{bmatrix}\right) = \mathrm{RS}\left(\begin{bmatrix} \mathrm{c}_2 & B \end{bmatrix}\right) \text{ and } \mathrm{CS}\left(\begin{bmatrix} a_{22} & \mathrm{b}_2^{\mathrm{T}} \\ \mathrm{c}_2 & B \end{bmatrix}\right) = \mathrm{CS}\left(\begin{bmatrix} \mathrm{b}_2^{\mathrm{T}} \\ B \end{bmatrix}\right),$$

if $i$ is type-I neutral in $G$ then this means that

$$A = \begin{bmatrix} a_{11} & \mathrm{z}^{\mathrm{T}} B \mathrm{y} & \mathrm{z}^{\mathrm{T}} B \\ \mathrm{x}^{\mathrm{T}} B \mathrm{w} & \mathrm{x}^{\mathrm{T}} B \mathrm{y} & \mathrm{x}^{\mathrm{T}} B \\ B \mathrm{w} & B \mathrm{y} & B \end{bmatrix},$$

in which z and w are column vectors. Because $i$ is downer in $G-j$ we have $\operatorname{rank}\left(\left[\begin{array}{cc} a_{11} & \mathrm{z}^{\mathrm{T}}B \\ B\mathrm{w} & B \end{array}\right]\right) = \operatorname{rank} B$ and, therefore, $a_{11} = \mathrm{z}^{\mathrm{T}}B\mathrm{w}$. But then

$$A = \left[\begin{array}{ccc} \mathrm{z}^{\mathrm{T}}B\mathrm{w} & \mathrm{z}^{\mathrm{T}}B\mathrm{y} & \mathrm{z}^{\mathrm{T}}B \\ \mathrm{x}^{\mathrm{T}}B\mathrm{w} & \mathrm{x}^{\mathrm{T}}B\mathrm{y} & \mathrm{x}^{\mathrm{T}}B \\ B\mathrm{w} & B\mathrm{y} & B \end{array}\right]$$

and $i$ is downer in $G$, not neutral, which is a contradiction. This contradiction means that the neutral vertex $i$ in $G$ cannot be type-I. Similarly, the neutral vertex $j$ in $G$ cannot be type-I. Therefore the outcome of row **15** occurs (adjacent case and non-adjacent case) only when $i$ and $j$ are both type-II neutral in $G$.

$\boxed{\text{Row } \mathbf{16}\ \ \mathrm{P}[\overset{\frown}{\mathrm{N}},\dot{\mathrm{D}}]}$: Suppose that $A$ realizes this outcome. Then, $i$ is neutral in $G$, $j$ is downer in $G$, and $i$ is Parter in $G-j$. By counting, we conclude that $j$ must be neutral in $G-i$, so that we have, at least,

$$\mathrm{b}_2^{\mathrm{T}} \in \mathrm{RS}(B) \quad \text{or} \quad \mathrm{c}_2 \in \mathrm{CS}(B)$$

(we have both if $j$ is type-I in $G-i$; only one of them if $j$ is type-II in $G-i$). Suppose *wlog* that $\mathrm{b}_2^{\mathrm{T}} \in \mathrm{RS}(B)$. In order to reach a contradiction we suppose that $i$ is type-I neutral in $G$. Under this hypothesis we have

$$\left[\begin{array}{cc} a_{12} & \mathrm{b}_1^{\mathrm{T}} \end{array}\right] \in \mathrm{RS}\left(\left[\begin{array}{cc} a_{22} & \mathrm{b}_2^{\mathrm{T}} \\ \mathrm{c}_2 & B \end{array}\right]\right) \quad \text{and} \quad \left[\begin{array}{c} a_{21} \\ \mathrm{c}_1 \end{array}\right] \in \mathrm{CS}\left(\left[\begin{array}{cc} a_{22} & \mathrm{b}_2^{\mathrm{T}} \\ \mathrm{c}_2 & B \end{array}\right]\right).$$

Since $\mathrm{b}_2^{\mathrm{T}} \in \mathrm{RS}(B)$ and $\left[\begin{array}{cc} a_{12} & \mathrm{b}_1^{\mathrm{T}} \end{array}\right] \in \mathrm{RS}\left(\left[\begin{array}{cc} a_{22} & \mathrm{b}_2^{\mathrm{T}} \\ \mathrm{c}_2 & B \end{array}\right]\right)$, this implies $\mathrm{b}_1^{\mathrm{T}} \in \mathrm{RS}(B)$, a contradiction to the fact of $i$ being Parter in $G-j$ and, thus, $\mathrm{b}_1^{\mathrm{T}} \notin \mathrm{RS}(B)$. This contradiction means that vertex $i$ cannot be type-I neutral in $G$.

We show now that also the neutral vertex $j$ in $G-i$ cannot be type-I. In order to reach a contradiction we suppose that $j$ is type-I neutral in $G-i$. Under this hypothesis we have

$$\mathrm{b}_2^{\mathrm{T}} \in \mathrm{RS}(B) \quad \text{and} \quad \mathrm{c}_2 \in \mathrm{CS}(B).$$

Since $i$ is type-II neutral in $G$, we have

$$\begin{bmatrix} a_{12} & \mathrm{b}_1^{\mathrm{T}} \end{bmatrix} \in \mathrm{RS}\left(\begin{bmatrix} a_{22} & \mathrm{b}_2^{\mathrm{T}} \\ \mathrm{c}_2 & B \end{bmatrix}\right) \tag{6}$$

or

$$\begin{bmatrix} a_{21} \\ \mathrm{c}_1 \end{bmatrix} \in \mathrm{CS}\left(\begin{bmatrix} a_{22} & \mathrm{b}_2^{\mathrm{T}} \\ \mathrm{c}_2 & B \end{bmatrix}\right) \tag{7}$$

(not both). If (6), since $\mathrm{b}_2^{\mathrm{T}} \in \mathrm{RS}(B)$, this implies $\mathrm{b}_1^{\mathrm{T}} \in \mathrm{RS}(B)$, a contradiction to the fact of $i$ being Parter in $G - j$. If (7), since $\mathrm{c}_2 \in \mathrm{CS}(B)$, this implies $\mathrm{c}_1 \in \mathrm{CS}(B)$, again a contradiction to the fact of $i$ being Parter in $G - j$. This contradiction means that the neutral vertex $j$ in $G - i$ cannot be type-I. Therefore the outcome of row **16** occurs (adjacent case and non-adjacent case) only when $i$ is type-II neutral in $G$ and $j$ is type-II neutral in $G - i$.

Moreover, $i$ is type-II neutral in $G$ satisfying

$$\begin{bmatrix} a_{12} & \mathrm{b}_1^{\mathrm{T}} \end{bmatrix} \in \mathrm{RS}\left(\begin{bmatrix} a_{22} & \mathrm{b}_2^{\mathrm{T}} \\ \mathrm{c}_2 & B \end{bmatrix}\right) \quad \text{and} \quad \begin{bmatrix} a_{21} \\ \mathrm{c}_1 \end{bmatrix} \notin \mathrm{CS}\left(\begin{bmatrix} a_{22} & \mathrm{b}_2^{\mathrm{T}} \\ \mathrm{c}_2 & B \end{bmatrix}\right).$$

if and only if $j$ is type-II neutral in $G - i$ satisfying

$$\mathrm{b}_2^{\mathrm{T}} \notin \mathrm{RS}(B) \quad \text{and} \quad \mathrm{c}_2 \in \mathrm{CS}(B).$$

$\boxed{\text{Row } \mathbf{23} \quad \mathrm{N}[\mathrm{D}, \dot{\mathrm{N}}]}$: Change the proof corresponding to row **16** by simply switching the labels for $i$ and $j$. $\square$

Since, in particular, a symmetric matrix does not have type-II neutral vertices, an immediate consequence of Theorem 12, regarding the resolution of remaining entries of Table 1, is that the outcomes in rows **2**, **15**, **16** and **23** cannot occur in $\mathcal{S}(G)$.

# 7 The table

Table 1: The possible changes in status of a vertex

| | Status of | | | $\mathcal{S}(T)$ | | $\mathcal{F}(T)$ | | $\mathcal{S}(G)$ | | $\mathcal{F}(G)$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | $i$ | $j$ | $i$ when $j$ is removed | Possible? | | Possible? | | Possible? | | Possible? | |
| | | | | ADJ. | NON. | ADJ. | NON. | ADJ. | NON. | ADJ. | NON. |
| **1** | P | P | P | Yes | Yes | Yes | Yes | Yes | Yes | Yes | Yes |
| **2** | P | P | N | No | No | No | No | No | No | Yes# | Yes# |
| **3** | P | P | D | Yes* | Yes* | Yes* | Yes* | Yes | Yes | Yes | Yes |
| **4** | P | N | P | Yes | Yes | Yes | Yes | Yes | Yes | Yes | Yes |
| **5** | P | N | N | Yes* | Yes* | Yes* | Yes* | Yes | Yes | Yes | Yes |
| **6** | P | N | D | No | No | No | No | No | No | No | No |
| **7** | P | D | P | Yes** | Yes | Yes** | Yes | Yes | Yes | Yes | Yes |
| **8** | P | D | N | No | No | No | No | No | No | No | No |
| **9** | P | D | D | No | No | No | No | No | No | No | No |
| **10** | N | P | P | No | No | No | No | No | No | No | No |
| **11** | N | P | N | Yes | Yes | Yes | Yes | Yes | Yes | Yes | Yes |
| **12** | N | P | D | Yes* | Yes* | Yes* | Yes* | Yes | Yes | Yes | Yes |
| **13** | N | N | P | No | Yes* | No | Yes* | Yes | Yes | Yes | Yes |
| **14** | N | N | N | Yes | Yes | Yes | Yes | Yes | Yes | Yes | Yes |
| **15** | N | N | D | No | No | No | No | No | No | Yes# | Yes# |
| **16** | N | D | P | No | No | No | No | No | No | Yes# | Yes# |
| **17** | N | D | N | No | Yes | No | Yes | Yes | Yes | Yes | Yes |
| **18** | N | D | D | No | No | No | No | No | No | No | No |
| **19** | D | P | P | No | No | No | No | No | No | No | No |
| **20** | D | P | N | No | No | No | No | No | No | No | No |
| **21** | D | P | D | Yes** | Yes | Yes** | Yes | Yes | Yes | Yes | Yes |
| **22** | D | N | P | No | No | No | No | No | No | No | No |
| **23** | D | N | N | No | No | No | No | No | No | Yes# | Yes# |
| **24** | D | N | D | No | Yes | No | Yes | Yes | Yes | Yes | Yes |
| **25** | D | D | P | No | Yes* | No | Yes* | Yes | Yes | Yes | Yes |
| **26** | D | D | N | Yes | Yes | Yes | Yes | Yes | Yes | Yes | Yes |
| **27** | D | D | D | No | Yes | No | Yes | Yes | Yes | Yes | Yes |

*Occurs only when the relevant Parter vertices are singly Parter.

**Only occurs if the Parter vertex is multiply Parter in $T$.

#Occurs only when the relevant neutral vertices are type-II neutral.

# 8 Deductions from the table

We can deduce some observations from Table 1. Let $i$ and $j$ be distinct vertices in a tree $T$ or, a more general graph $G$. Suppose that the status of vertex $i$ for an eigenvalue is known before and after removing a distinct vertex $j$ from the graph, then the status of $j$ is deduced in Table 2 below for a matrix $A$ in either $\mathcal{S}(T)$, $\mathcal{F}(T)$ or $\mathcal{S}(G)$.

Table 2: Status of $j$ in $A \in \mathcal{S}(T)$, $\mathcal{F}(T)$ or $\mathcal{S}(G)$

| Status of $i$ in A \ Status of $i$ in $A(j)$ | D | N | P |
|---|---|---|---|
| D | P, N*, D* | D | D* |
| N | P | P, N, D* | N* |
| P | P | N | P, N, D |

*Does not occur when $j$ is adjacent to $i$ in $T$.

From Table 2, we can see that when the status of $i$ in a matrix $A$ in either $\mathcal{S}(T)$, $\mathcal{F}(T)$ or $\mathcal{S}(G)$ changes to another status by removing a distinct vertex $j$, then the status of $j$ is uniquely determined (see the non-diagonal cells). For example, when the status of $i$ in $A$ changes from downer to neutral by removing a distinct vertex $j$, then $j$ must be downer in $A$, which is obtained from row **26** in Table 1. In this case, $j$ cannot be neutral or Parter in $A$ according to rows **20** and **23**.

However, the diagonal entries of Table 2 show that the status of $j$ in $A$ has three possibilities, Parter, neutral or downer, if the status of $i$ does not change by removing a distinct vertex $j$, except that there are fewer cases when $j$ is adjacent to $i$ in a tree $T$. For example, when the status of $i$ in $A$ stays downer by removing a distinct vertex $j$, then the status of $j$ has three possibilities in $A$ using rows **21**, **24** and **27** in Table 1, except for the case in which $j$ is adjacent to $i$ in a tree $T$.

**Proposition 13** *Let $T$ be a tree, $G$ a graph, $A$ be a matrix in either $\mathcal{S}(T)$, $\mathcal{F}(T)$ or $\mathcal{S}(G)$, and $i$, $j$ be two distinct vertices of $T$ or $G$. Then, if the status* *of $i$ for an eigenvalue of $A$ changes to another status in $A(j)$, then the status of $j$ is uniquely determined in $A$, as stated in Table 2.*

For $A \in \mathcal{F}(G)$, there are more possibilities for the status of $j$, when the status of vertex $i$ for an eigenvalue is known before and after removing $j$ from the graph. These extra possibilities are due to the existence of type-II neutral vertices in $\mathcal{F}(G)$.

Table 3: Status of $j$ in $A \in \mathcal{F}(G)$

| Status of $i$ in A \ Status of $i$ in $A(j)$ | D | N | P |
|---|---|---|---|
| D | P, N*, D* | N#, D | D* |
| N | P, N# | P, N, D* | N*, D# |
| P | P | P#, N | P, N, D |

* Does not occur when $j$ is adjacent to $i$ in $T$.

#Does not occur in either $\mathcal{S}(T)$, $\mathcal{F}(T)$ or $\mathcal{S}(G)$, and the relevant neutral vertices must be type-II neutral.

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