Convergence in p-mean for arrays of random variables

João Lita da Silva[1]

Department of Mathematics and GeoBioTec
Faculty of Sciences and Technology
NOVA University of Lisbon
Quinta da Torre, 2829-516 Caparica, Portugal

Abstract

In this paper, conditions are given to ensure the convergence in mean of order p ($1 \leqslant p < 2$) for arrays of random variables. Recent results about convergence in p-mean for arrays of row-wise pairwise negative quadrant dependent random variables will be improved.

Key words and phrases: convergence in p-mean, row-wise pairwise positive quadrant dependent array, row-wise pairwise negative quadrant dependent array

2010 Mathematics Subject Classification: 60F25

1 Introduction

The convergence in mean of order p (or, for short, p-mean) of arrays of random variables has been studied lately by many authors under weaker assumptions of dependence for the random variables (see, for instance, [1], [4], [5], [6], [8], among others). The main goal of this paper is to establish the convergence in mean of order p for arrays of random variables in a general setting, where no specific assumptions concerning to dependence of the random variables are made. To accomplish this aim, we will admit only asymptotic conditions in the main statement.

Associated to a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we shall consider the space $\mathscr{L}_p$ ($p > 0$) of all measurable functions X (necessarily random variables) for which $\mathbb{E}\,|X|^p < \infty$. Throughout, we shall denote the indicator random variable of an event A by I_A, and the symbol C shall indicate a generic positive constant whose value may vary from one place to another.

---

[1] *E-mail address:* `jfls@fct.unl.pt`; `joao.lita@gmail.com`

2 Main results

For triangular arrays of random variables verifying some type of negative dependence, there are classical inequalities which can be properly extended to those structures (see, for instance, Lemma 2 of [4]). However, for generic triangular arrays it is no longer possible to take advantage of such inequalities. A way to deal with this problem is to find suitable covariance quantities defined for this scenario of triangular arrays. By controlling asymptotically these quantities, we shall be able to prove the convergence in p-mean for general triangular arrays of random variables assuming additionally weak asymptotic integrability conditions.

For each $\ell > 0$, let us define the nondecreasing real valued function $g_\ell(t) = \max(\min(t, \ell), -\ell)$ which performs the truncation at level ℓ, and introduce, for any $n \geqslant 1$,

$$\Delta_n(x, y) := \sum_{1 \leqslant k < j \leqslant n} \big[\mathbb{P}\left\{X_{n,k} \geqslant x, X_{n,j} \geqslant y\right\} - \mathbb{P}\left\{X_{n,k} \geqslant x\right\} \mathbb{P}\left\{X_{n,j} \geqslant y\right\}\big].$$

Obviously, if $X_{n,k}$ and $X_{n,j}$ are independent random variables for every $k \neq j$ then $\Delta_n(x, y) = 0$. For every $t > 0$, consider also the following quantities

$$G_n(t) := \int_{-t}^{t} \int_{-t}^{t} \Delta_n(x, y) \mathrm{d}x \mathrm{d}y$$

and $G_n^+(t) := \max(0, G_n(t))$.

Now, we state and proof the main result of this paper.

Theorem 1 *Let* $1 \leqslant p < 2$ *and* $\{X_{n,k},\ 1 \leqslant k \leqslant n,\ n \geqslant 1\}$ *be a triangular array of random variables. If* $\{b_n\}$ *is a sequence of positive constants, and*

(a) $\displaystyle\sum_{k=1}^{n} \int_{0}^{\varepsilon b_n^p} \mathbb{P}\left\{|X_{n,k}|^p > t\right\} \mathrm{d}t = O\left(b_n^p\right)$, *$n \to \infty$ for any* $\varepsilon > 0$,

(b) $\displaystyle\sum_{k=1}^{n} \int_{\varepsilon b_n^p}^{\infty} \mathbb{P}\left\{|X_{n,k}|^p > t\right\} \mathrm{d}t = o\left(b_n^p\right)$, *$n \to \infty$ for any* $\varepsilon > 0$ *when* $1 < p < 2$ *or* (b') $\displaystyle\sum_{k=1}^{n} \mathbb{E}|X_{n,k}| I_{\{|X_{n,k}| > \varepsilon b_n\}} = o\left(b_n\right)$, *$n \to \infty$ for any* $\varepsilon > 0$ *whenever* $p = 1$,

(c) $\displaystyle\int_{\varepsilon b_n}^{\infty} t^{p-3} G_n^+(t) \mathrm{d}t = o(b_n^p)$, *$n \to \infty$ for any* $\varepsilon > 0$,

then

$$\frac{1}{b_n} \sum_{k=1}^{n} \left(X_{n,k} - \mathbb{E}\, X_{n,k}\right) \xrightarrow{\mathscr{L}_p} 0.$$

Proof. Fix $\varepsilon > 0$ and define

$$X'_{n,k} := g_{t^{1/p}}(X_{n,k}) = X_{n,k} I_{\{|X_{n,k}| \leqslant t^{1/p}\}} + t^{1/p} I_{\{X_{n,k} > t^{1/p}\}} - t^{1/p} I_{\{X_{n,k} < -t^{1/p}\}},$$

$$X''_{n,k} := X_{n,k} - g_{t^{1/p}}(X_{n,k}) = X_{n,k} I_{\{|X_{n,k}| > t^{1/p}\}} + t^{1/p} I_{\{X_{n,k} < -t^{1/p}\}} - t^{1/p} I_{\{X_{n,k} > t^{1/p}\}}.$$

Thus, $X'_{n,k} + X''_{n,k} = X_{n,k}$ and

$$\begin{aligned}
\mathbb{E}\left|\frac{1}{b_n}\sum_{k=1}^{n}(X_{n,k}-\mathbb{E}\,X_{n,k})\right|^p &= \frac{1}{b_n^p}\int_0^\infty \mathbb{P}\left\{\left|\sum_{k=1}^{n}(X_{n,k}-\mathbb{E}\,X_{n,k})\right|^p > t\right\}\mathrm{d}t \\
&\leqslant \varepsilon + \frac{1}{b_n^p}\int_{\varepsilon b_n^p}^\infty \mathbb{P}\left\{\left|\sum_{k=1}^{n}(X_{n,k}-\mathbb{E}\,X_{n,k})\right| > t^{1/p}\right\}\mathrm{d}t \\
&\leqslant \varepsilon + \frac{1}{b_n^p}\int_{\varepsilon b_n^p}^\infty \mathbb{P}\left\{\left|\sum_{k=1}^{n}\left(X'_{n,k}-\mathbb{E}\,X'_{n,k}\right)\right| > \frac{t^{1/p}}{2}\right\}\mathrm{d}t \\
&\quad + \frac{1}{b_n^p}\int_{\varepsilon b_n^p}^\infty \mathbb{P}\left\{\left|\sum_{k=1}^{n}\left(X''_{n,k}-\mathbb{E}\,X''_{n,k}\right)\right| > \frac{t^{1/p}}{2}\right\}\mathrm{d}t.
\end{aligned} \tag{2.1}$$

Now, the last two terms in the right-hand side of (2.1) must be estimated. From Chebyshev inequality, Theorem 2.3 of [7] and Lemma 1 of [3] we have

$$\begin{aligned}
&\int_{\varepsilon b_n^p}^\infty \mathbb{P}\left\{\left|\sum_{k=1}^{n}\left(X'_{n,k}-\mathbb{E}\,X'_{n,k}\right)\right| > \frac{t^{1/p}}{2}\right\}\mathrm{d}t \\
&\quad\leqslant C\int_{\varepsilon b_n^p}^\infty t^{-2/p}\,\mathbb{E}\left|\sum_{k=1}^{n}\left(X'_{n,k}-\mathbb{E}\,X'_{n,k}\right)\right|^2\mathrm{d}t \\
&\quad\leqslant C\int_{\varepsilon b_n^p}^\infty t^{-2/p}\Bigg[\sum_{k=1}^{n}\mathbb{E}(X'_{n,k}-\mathbb{E}\,X'_{n,k})^2 \\
&\qquad\qquad + \sum_{1\leqslant k<j\leqslant n}\mathrm{Cov}\left(X'_{n,k}-\mathbb{E}\,X'_{n,k};X'_{n,j}-\mathbb{E}\,X'_{n,j}\right)\Bigg]\mathrm{d}t \\
&\quad= C\int_{\varepsilon b_n^p}^\infty t^{-2/p}\left[\sum_{k=1}^{n}\mathbb{E}(X'_{n,k}-\mathbb{E}\,X'_{n,k})^2 + \sum_{1\leqslant k<j\leqslant n}\mathrm{Cov}\left(X'_{n,k};X'_{n,j}\right)\right]\mathrm{d}t \\
&\quad\leqslant C\sum_{k=1}^{n}\int_{\varepsilon b_n^p}^\infty\left(t^{-2/p}\,\mathbb{E}X_{n,k}^2 I_{\left\{|X_{n,k}|\leqslant t^{1/p}\right\}} + \mathbb{P}\left\{|X_{n,k}|>t^{1/p}\right\}\right)\mathrm{d}t \\
&\qquad + C\sum_{1\leqslant k<j\leqslant n}\int_{\varepsilon b_n^p}^\infty t^{-2/p}\int_{-t^{1/p}}^{t^{1/p}}\int_{-t^{1/p}}^{t^{1/p}}\mathrm{Cov}\left(I_{\left\{X_{n,k}\leqslant x\right\}};I_{\{X_{n,j}\leqslant y\}}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}t \\
&\quad= C\sum_{k=1}^{n}\int_{\varepsilon b_n^p}^\infty t^{-2/p}\int_0^{t^{1/p}} s\,\mathbb{P}\left\{|X_{n,k}|>s\right\}\mathrm{d}s\,\mathrm{d}t + C\int_{\varepsilon b_n^p}^\infty t^{-2/p}G_n\left(t^{1/p}\right)\mathrm{d}t \\
&\quad= C\sum_{k=1}^{n}\int_0^\infty s\,\mathbb{P}\left\{|X_{n,k}|>s\right\}\int_{\max(\varepsilon b_n^p,s^p)}^\infty t^{-2/p}\mathrm{d}t\,\mathrm{d}s + Cp\int_{\varepsilon^{1/p}b_n}^\infty s^{p-3}G_n(s)\mathrm{d}s \\
&\quad\leqslant \frac{Cp\varepsilon^{1-2/p}b_n^{p-2}}{2-p}\sum_{k=1}^{n}\int_0^{\varepsilon^{1/p}b_n} s\,\mathbb{P}\left\{|X_{n,k}|>s\right\}\mathrm{d}s \\
&\qquad + \frac{Cp}{2-p}\sum_{k=1}^{n}\int_{\varepsilon^{1/p}b_n}^\infty s^{p-1}\mathbb{P}\left\{|X_{n,k}|>s\right\}\mathrm{d}s + Cp\int_{\varepsilon^{1/p}b_n}^\infty s^{p-3}G_n^+(s)\mathrm{d}s
\end{aligned} \tag{2.2}$$

$$\begin{aligned}
&\leqslant \frac{Cp\varepsilon^{2/p-1}}{2-p}\sum_{k=1}^{n}\int_0^{\varepsilon^{2/p}b_n} s^{p-1}\,\mathbb{P}\left\{|X_{n,k}|>s\right\}\mathrm{d}s+\\
&\qquad+\frac{Cp}{2-p}\sum_{k=1}^{n}\int_{\varepsilon^{2/p}b_n}^{\max(\varepsilon^{1/p},\varepsilon^{2/p})b_n} s^{p-1}\,\mathbb{P}\left\{|X_{n,k}|>s\right\}\mathrm{d}s\\
&\qquad+\frac{Cp}{2-p}\sum_{k=1}^{n}\int_{\varepsilon^{1/p}b_n}^{\infty} s^{p-1}\mathbb{P}\left\{|X_{n,k}|>s\right\}\,\mathrm{d}s+Cp\int_{\varepsilon^{1/p}b_n}^{\infty}s^{p-3}G_n^{+}(s)\mathrm{d}s\\
&\leqslant \frac{Cp\varepsilon^{2/p-1}}{2-p}\sum_{k=1}^{n}\int_0^{\varepsilon^{2/p}b_n} s^{p-1}\,\mathbb{P}\left\{|X_{n,k}|>s\right\}\mathrm{d}s+Cp\int_{\varepsilon^{1/p}b_n}^{\infty}s^{p-3}G_n^{+}(s)\mathrm{d}s\\
&\qquad\qquad+\frac{Cp}{2-p}\sum_{k=1}^{n}\int_{\min(\varepsilon^{1/p},\varepsilon^{2/p})b_n}^{\infty} s^{p-1}\mathbb{P}\left\{|X_{n,k}|>s\right\}\,\mathrm{d}s\\
&=\frac{C\varepsilon^{2/p-1}}{2-p}\sum_{k=1}^{n}\int_0^{\varepsilon^{2}b_n^p}\mathbb{P}\left\{|X_{n,k}|^p>y\right\}\mathrm{d}y+Cp\int_{\varepsilon^{1/p}b_n}^{\infty}u^{p-3}G_n^{+}(u)\mathrm{d}u\\
&\qquad\qquad+\frac{C}{2-p}\sum_{k=1}^{n}\int_{\min(\varepsilon,\varepsilon^{2})b_n^p}^{\infty}\mathbb{P}\left\{|X_{n,k}|^p>y\right\}\,\mathrm{d}y
\end{aligned}$$

since $\int_0^{\varepsilon^{1/p}b_n} s\,\mathbb{P}\left\{|X_{n,k}|>s\right\}\mathrm{d}s$ is upper bounded as follows

$$\begin{aligned}
&\int_0^{\varepsilon^{1/p}b_n} s\,\mathbb{P}\left\{|X_{n,k}|>s\right\}\mathrm{d}s\\
&\qquad\leqslant\int_0^{\max(\varepsilon^{1/p},\varepsilon^{2/p})b_n} s\,\mathbb{P}\left\{|X_{n,k}|>s\right\}\mathrm{d}s\\
&\qquad=\int_0^{\varepsilon^{2/p}b_n} s\,\mathbb{P}\left\{|X_{n,k}|>s\right\}\mathrm{d}s+\int_{\varepsilon^{2/p}b_n}^{\max(\varepsilon^{1/p},\varepsilon^{2/p})b_n} s\,\mathbb{P}\left\{|X_{n,k}|>s\right\}\mathrm{d}s\\
&\qquad\leqslant\varepsilon^{4/p-2}b_n^{2-p}\int_0^{\varepsilon^{2/p}b_n} s^{p-1}\,\mathbb{P}\left\{|X_{n,k}|>s\right\}\mathrm{d}s\\
&\qquad\qquad+\varepsilon^{2/p-1}b_n^{2-p}\int_{\varepsilon^{2/p}b_n}^{\max(\varepsilon^{1/p},\varepsilon^{2/p})b_n} s^{p-1}\,\mathbb{P}\left\{|X_{n,k}|>s\right\}\mathrm{d}s.
\end{aligned}$$

Noting that $|X''_{n,k}|\leqslant|X_{n,k}|\,I_{\left\{|X_{n,k}|>t^{1/p}\right\}}$, we have for every $1<p<2$,

$$\begin{aligned}
&\int_{\varepsilon b_n^p}^{\infty}\mathbb{P}\left\{\left|\sum_{k=1}^{n}\left(X''_{n,k}-\mathbb{E}\,X''_{n,k}\right)\right|>\frac{t^{1/p}}{2}\right\}\mathrm{d}t\\
&\leqslant C\int_{\varepsilon b_n^p}^{\infty}t^{-1/p}\sum_{k=1}^{n}\mathbb{E}\,|X''_{n,k}|\,\mathrm{d}t\\
&\leqslant C\int_{\varepsilon b_n^p}^{\infty}t^{-1/p}\sum_{k=1}^{n}\mathbb{E}\,|X_{n,k}|I_{\left\{|X_{n,k}|>t^{1/p}\right\}}\mathrm{d}t\\
&=C\sum_{k=1}^{n}\left(\int_{\varepsilon b_n^p}^{\infty}t^{-1/p}\int_{t^{1/p}}^{\infty}\mathbb{P}\left\{|X_{n,k}|>s\right\}\mathrm{d}s\,\mathrm{d}t+\int_{\varepsilon b_n^p}^{\infty}\mathbb{P}\left\{|X_{n,k}|>t^{1/p}\right\}\mathrm{d}t\right)\qquad(2.3)
\end{aligned}$$

$$
\begin{aligned}
&= C\sum_{k=1}^{n}\left(\int_{b_n\varepsilon^{1/p}}^{\infty}\mathbb{P}\left\{|X_{n,k}|>s\right\}\int_{\varepsilon b_n^p}^{s^p}t^{-1/p}\mathrm{d}t\,\mathrm{d}s+\int_{\varepsilon b_n^p}^{\infty}\mathbb{P}\left\{|X_{n,k}|>t^{1/p}\right\}\mathrm{d}t\right)\\
&\leqslant C\sum_{k=1}^{n}\left(\frac{p}{p-1}\int_{b_n\varepsilon^{1/p}}^{\infty}s^{p-1}\mathbb{P}\left\{|X_{n,k}|>s\right\}\mathrm{d}s+\int_{\varepsilon b_n^p}^{\infty}\mathbb{P}\left\{|X_{n,k}|>t^{1/p}\right\}\mathrm{d}t\right)\\
&= \frac{C}{p-1}\sum_{k=1}^{n}\int_{\varepsilon b_n^p}^{\infty}\mathbb{P}\left\{|X_{n,k}|>t^{1/p}\right\}\mathrm{d}t.
\end{aligned}
$$

According to (2.1), (2.2) and (2.3), we obtain

$$
\begin{aligned}
\mathbb{E}\left|\frac{1}{b_n}\sum_{k=1}^{n}\left(X_{n,k}-\mathbb{E}\,X_{n,k}\right)\right|^p &\leqslant \varepsilon+\frac{C\varepsilon^{2/p-1}}{(2-p)b_n^p}\sum_{k=1}^{n}\int_0^{\varepsilon^2 b_n^p}\mathbb{P}\left\{|X_{n,k}|^p>y\right\}\mathrm{d}y\\
&+\frac{C}{(2-p)(p-1)b_n^p}\sum_{k=1}^{n}\int_{\min(\varepsilon,\varepsilon^2)b_n^p}^{\infty}\mathbb{P}\left\{|X_{n,k}|^p>y\right\}\,\mathrm{d}y+\frac{Cp}{b_n^p}\int_{\varepsilon^{1/p}b_n}^{\infty}u^{p-3}G_n^{+}(u)\mathrm{d}u.
\end{aligned}
$$

Hence, assumptions (a), (b), (c) and the arbitrariness of ε guarantee

$$
\mathbb{E}\left|\frac{1}{b_n}\sum_{k=1}^{n}\left(X_{n,k}-\mathbb{E}\,X_{n,k}\right)\right|^p\longrightarrow 0
$$

as $n\to\infty$, completing the proof for the case $1<p<2$. It remains to show the case $p=1$, that is, the convergence in mean (to zero) of $\sum_{k=1}^{n}\left(X_{n,k}-\mathbb{E}\,X_{n,k}\right)/b_n$. For each $\varepsilon>0$, we have

$$
\begin{aligned}
\sup_{t\geqslant\varepsilon b_n}\left|\frac{1}{t}\sum_{k=1}^{n}\mathbb{E}\,X''_{n,k}\right| &\leqslant \sup_{t\geqslant\varepsilon b_n}\frac{1}{t}\sum_{k=1}^{n}\mathbb{E}|X''_{n,k}|\\
&\leqslant \sup_{t\geqslant\varepsilon b_n}\frac{1}{t}\sum_{k=1}^{n}\mathbb{E}|X_{n,k}|I_{\{|X_{n,k}|>t\}}\\
&\leqslant \frac{1}{\varepsilon b_n}\sum_{k=1}^{n}\mathbb{E}|X_{n,k}|I_{\{|X_{n,k}|>\varepsilon b_n\}}\\
&\leqslant \frac{1}{\varepsilon b_n}\sum_{k=1}^{n}\mathbb{E}|X_{n,k}|I_{\{|X_{n,k}|>\varepsilon b_n\}}\longrightarrow 0
\end{aligned}
$$

as $n\to\infty$. Therefore, for n large enough we obtain

$$
\left|\sum_{k=1}^{n}\mathbb{E}\,X''_{n,k}\right|\leqslant\frac{t}{4}
$$

and setting $\Gamma_n(t):=\bigcup_{k=1}^{n}\left\{|X_{n,k}|>t\right\}$ we get

$$
\begin{aligned}
&\int_{\varepsilon b_n}^{\infty}\mathbb{P}\left\{\left|\sum_{k=1}^{n}\left(X''_{n,k}-\mathbb{E}\,X''_{n,k}\right)\right|>\frac{t}{2}\right\}\mathrm{d}t\\
&\quad\leqslant\int_{\varepsilon b_n}^{\infty}\mathbb{P}\left\{\left|\sum_{k=1}^{n}X''_{n,k}\right|>\frac{t}{4}\right\}\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \int_{\varepsilon b_n}^{\infty} \mathbb{P}\left\{\sum_{k=1}^{n} |X''_{n,k}| > \frac{t}{4}\right\} \mathrm{d}t \\
&= \int_{\varepsilon b_n}^{\infty} \mathbb{P}\left[\left(\left\{\sum_{k=1}^{n} |X''_{n,k}| > \frac{t}{4}\right\} \cap \Gamma_n(t)\right) \cup \left(\left\{\sum_{k=1}^{n} |X''_{n,k}| > \frac{t}{4}\right\} \cap \Gamma_n(t)^{\complement}\right)\right] \mathrm{d}t \qquad (2.4) \\
&\leqslant \int_{\varepsilon b_n}^{\infty} \mathbb{P}\left[\Gamma_n(t)\right] \mathrm{d}t \\
&\leqslant \int_{\varepsilon b_n}^{\infty} \sum_{k=1}^{n} \mathbb{P}\left\{|X_{n,k}| > t\right\} \mathrm{d}t \\
&\leqslant \sum_{k=1}^{n} \mathbb{E}|X_{n,k}| I_{\left\{|X_{n,k}| > \varepsilon b_n\right\}}.
\end{aligned}
$$

From (2.1), (2.2) and (2.4) it follows

$$
\begin{aligned}
&\mathbb{E}\left|\frac{1}{b_n} \sum_{k=1}^{n} \left(X_{n,k} - \mathbb{E}\, X_{n,k}\right)\right| \\
&\qquad \leqslant \varepsilon + \frac{C\varepsilon}{b_n} \sum_{k=1}^{n} \int_{0}^{\varepsilon^2 b_n} \mathbb{P}\left\{|X_{n,k}| > y\right\} \mathrm{d}y + \frac{C}{b_n} \sum_{k=1}^{n} \int_{\min(\varepsilon,\varepsilon^2) b_n}^{\infty} \mathbb{P}\left\{|X_{n,k}| > y\right\} \mathrm{d}y \\
&\qquad\qquad + \frac{1}{b_n} \sum_{k=1}^{n} \mathbb{E}|X_{n,k}| I_{\left\{|X_{n,k}| > \varepsilon b_n\right\}} + \frac{C}{b_n} \int_{\varepsilon b_n}^{\infty} \frac{G_n^{+}(u)}{u^2} \mathrm{d}u \\
&\qquad \leqslant \varepsilon + \frac{C\varepsilon}{b_n} \sum_{k=1}^{n} \int_{0}^{\varepsilon^2 b_n} \mathbb{P}\left\{|X_{n,k}| > y\right\} \mathrm{d}y \\
&\qquad\qquad + \frac{C}{b_n} \sum_{k=1}^{n} \mathbb{E}|X_{n,k}| I_{\left\{|X_{n,k}| > \min(\varepsilon,\varepsilon^2) b_n\right\}} + \frac{C}{b_n} \int_{\varepsilon b_n}^{\infty} \frac{G_n^{+}(u)}{u^2} \mathrm{d}u
\end{aligned}
$$

for n large and assumptions (a), (b'), (c) as well as the arbitrariness of ε establish the thesis. The proof is complete. □

Analyzing assumptions of Theorem 1, we conclude that condition (b') is stronger than $\sum_{k=1}^{n} \int_{\varepsilon b_n}^{\infty} \mathbb{P}\left\{|X_{n,k}| > t\right\} \mathrm{d}t = o(b_n)$, $n \to \infty$ (i.e. condition (b) with $p = 1$) because

$$
\int_{u^p}^{\infty} \mathbb{P}\left\{|X_{n,k}|^p > t\right\} dt \leqslant \mathbb{E}\left|X_{n,k}\right|^p I_{\left\{|X_{n,k}| > u\right\}} \qquad (p, u > 0).
$$

Thereby, assumption (b), being sufficient to get the convergence in mean of order p of $\sum_{k=1}^{n} \left(X_{n,k} - \mathbb{E}\, X_{n,k}\right)/b_n$ when $1 < p < 2$, must be strength for $p = 1$. Note also that assumption (c) of Theorem 1 is a summation of covariance quantities which, up to a integration, is controlled asymptotically and allows us to consider triangular arrays of random variables having infinite variance.

Remark 1 Let us observe that if $\{u_n\}$ and $\{v_n\}$ are two any (finite) sequences of integers such that $u_n < v_n$ for all $n \geqslant 1$ and $v_n - u_n \to \infty$ as $n \to \infty$ then Theorem 1 holds true for arrays $\{X_{n,k},\, u_n \leqslant k \leqslant v_n,\, n \geqslant 1\}$, that is, $\sum_{k=u_n}^{v_n} \left(X_{n,k} - \mathbb{E}\, X_{n,k}\right)/b_n \xrightarrow{\mathscr{L}_p} 0$ provided that $\{b_n\}$ is a sequence of positive constants, and

(A) $\sum_{k=u_n}^{v_n} \int_0^{\varepsilon b_n^p} \mathbb{P}\left\{|X_{n,p}|^p > t\right\} \mathrm{d}t = O\left(b_n^p\right)$, $n \to \infty$ for any $\varepsilon > 0$,

(B) $\sum_{k=u_n}^{v_n} \int_{\varepsilon b_n^p}^{\infty} \mathbb{P}\left\{|X_{n,k}|^p > t\right\} \mathrm{d}t = o\left(b_n^p\right)$, $n \to \infty$ for any $\varepsilon > 0$ when $1 < p < 2$ or (B') $\sum_{k=u_n}^{v_n} \mathbb{E}|X_{n,k}| I_{\left\{|X_{n,k}| > \varepsilon b_n\right\}} = o\left(b_n\right)$, $n \to \infty$ for any $\varepsilon > 0$ whenever $p = 1$,

(C) $\int_{\varepsilon b_n}^{\infty} t^{p-3} G^+_{u_n,v_n}(t) \mathrm{d}t = o(b_n^p)$, $n \to \infty$ for any $\varepsilon > 0$,

where $G^+_{u_n,v_n}(t)$ stands for

$$\max\left(0, \int_{-t}^{t}\int_{-t}^{t} \sum_{u_n \leqslant k < j \leqslant v_n} \left[\mathbb{P}\left\{X_{n,k} \geqslant x, X_{n,j} \geqslant y\right\} - \mathbb{P}\left\{X_{n,k} \geqslant x\right\} \mathbb{P}\left\{X_{n,j} \geqslant y\right\}\right] \mathrm{d}x\mathrm{d}y\right).$$

The proof can be performed *mutatis mutandis* the above one for Theorem 1.

Remark 2 In [1], the authors introduced the concept of h-integrability with respect to constants $\{a_{n,k}\}$ for an array of random variables $\{Y_{n,k},\ u_n \leqslant k \leqslant v_n,\ n \geqslant 1\}$ (see [1], page 647). Although our statement does not consider explicitly weights $\{a_{n,k}\}$, we can always take $X_{n,k} = b_n a_{n,k} Y_{n,k}$ to obtain them in our assumptions. For simplicity, supposing $u_n = 1$ and $v_n = n$, condition (a) of Theorem 1 can be written as $\sum_{k=1}^{n} \int_0^{\varepsilon} \mathbb{P}\left\{|a_{n,k}Y_{n,k}|^p > t\right\} \mathrm{d}t = O(1)$, $n \to \infty$. Thus, condition $\sup_{n\geqslant 1} \sum_{k=1}^{n} |a_{n,k}| \mathbb{E}\,|Y_{n,k}| < \infty$ of [1] is stronger than $\sum_{k=1}^{n} \int_0^{\varepsilon} \mathbb{P}\left\{|a_{n,k}Y_{n,k}| > t\right\} \mathrm{d}t = O(1)$, $n \to \infty$ (i.e. condition (a) for $p = 1$) since

$$\mathbb{E}|a_{n,k}Y_{n,k}|^p = \int_0^{\infty} \mathbb{P}\left\{|a_{n,k}Y_{n,k}|^p > t\right\} \mathrm{d}t \geqslant \int_0^{\varepsilon} \mathbb{P}\left\{|a_{n,k}Y_{n,k}|^p > t\right\} \mathrm{d}t \qquad (p, \varepsilon > 0). \tag{2.5}$$

On the other hand, the second condition of h-integrability presented in [1],

$$\sum_{k=1}^{n} |a_{n,k}|\, \mathbb{E}\,|Y_{n,k}|\, I_{\left\{|Y_{n,k}| > h(n)\right\}} = o(1), \quad n \to \infty,$$

is similar to our assumption (b'); indeed, (b') can be written as

$$\sum_{k=1}^{n} |a_{n,k}|\, \mathbb{E}\,|Y_{n,k}| I_{\left\{|a_{n,k}Y_{n,k}| > \varepsilon\right\}} = o(1), \quad n \to \infty.$$

Without additional information on the sequence $h(n)$ and weights $\{a_{n,k}\}$, we cannot compare them.

2.1 Convergence in p-mean for arrays of row-wise pairwise PQD random variables

In particular, our Theorem 1 will be useful on triangular arrays of random variables which are row-wise pairwise positive quadrant dependent (PQD), i.e. arrays $\{X_{n,k},\ 1 \leqslant k \leqslant n,\ n \geqslant 1\}$ satisfying for each $n \geqslant 1$ and every $1 \leqslant k, j \leqslant n$ such that $k \neq j$,

$$\mathbb{P}\left\{X_{n,k} \leqslant x_{n,k}, X_{n,j} \leqslant y_{n,j}\right\} \geqslant \mathbb{P}\left\{X_{n,k} \leqslant x_{n,k}\right\} \mathbb{P}\left\{X_{n,j} \leqslant y_{n,j}\right\}$$

for all real numbers $x_{n,k}$ and $y_{n,j}$ (see [2] for the notion of positive quadrant dependence).

Theorem 2 *Let* $1 \leqslant p < 2$, $\{X_{n,k},\ 1 \leqslant k \leqslant n,\ n \geqslant 1\}$ *be a triangular array of row-wise pairwise PQD random variables and* $\{b_n\}$ *be a sequence of positive constants. If conditions* (a), (b) *when* $1 < p < 2$ *or* (b') *whenever* $p = 1$ *of Theorem 1 are satisfied, and*

(c') $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Delta_n(x, y)\, \mathrm{d}x\mathrm{d}y = o(b_n^2)$, $n \to \infty$,

then $\sum_{k=1}^{n} \left(X_{n,k} - \mathbb{E}\, X_{n,k}\right)/b_n \xrightarrow{\mathscr{L}_p} 0$.

Proof. According to Theorem 1 it suffices to show its condition (c). We have

$$\begin{aligned}
\int_{\varepsilon b_n}^{\infty} t^{p-3} G_n^{+}(t)\mathrm{d}t &= \int_{\varepsilon b_n}^{\infty} t^{p-3} G_n(t)\mathrm{d}t \\
&= \iint_{\{\max(|x|,|y|)>\varepsilon b_n\}} \Delta_n(x, y) \int_{\max(|x|,|y|)}^{\infty} t^{p-3}\mathrm{d}t\, \mathrm{d}x\mathrm{d}y + \\
&\qquad \iint_{[-\varepsilon b_n, \varepsilon b_n]^2} \Delta_n(x, y) \int_{\varepsilon b_n}^{\infty} t^{p-3}\mathrm{d}t\, \mathrm{d}x\mathrm{d}y \\
&= \frac{1}{2-p} \iint_{\{\max(|x|,|y|)>\varepsilon b_n\}} \Delta_n(x, y) \big[\max(|x|, |y|)\big]^{p-2} \mathrm{d}x\mathrm{d}y + \\
&\qquad \frac{\varepsilon^{p-2} b_n^{p-2}}{2-p} \iint_{[-\varepsilon b_n, \varepsilon b_n]^2} \Delta_n(x, y)\, \mathrm{d}x\mathrm{d}y \\
&\leqslant \frac{\varepsilon^{p-2} b_n^{p-2}}{2-p} \iint_{\{\max(|x|,|y|)>\varepsilon b_n\}} \Delta_n(x, y)\, \mathrm{d}x\mathrm{d}y + \\
&\qquad \frac{\varepsilon^{p-2} b_n^{p-2}}{2-p} \iint_{[-\varepsilon b_n, \varepsilon b_n]^2} \Delta_n(x, y)\, \mathrm{d}x\mathrm{d}y \\
&= \frac{\varepsilon^{p-2} b_n^{p-2}}{2-p} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Delta_n(x, y)\, \mathrm{d}x\mathrm{d}y
\end{aligned}$$

since $\Delta_n(x, y) \geqslant 0$ for all x, y. From (c') it follows $\int_{\varepsilon b_n}^{\infty} t^{p-3} G_n^{+}(t)\mathrm{d}t = o(b_n^p)$ as $n \to \infty$ completing the proof. □

Remark 3 Let us observe that if $\mathrm{Cov}(X_{n,k}; X_{n,j})$ exists for every $1 \leqslant k \leqslant n$, $n \geqslant 1$ then condition (c') of Theorem 2 is equivalent to $\sum_{1 \leqslant k < j \leqslant n} \mathrm{Cov}(X_{n,k}; X_{n,j}) = o(b_n^2)$, $n \to \infty$ via Hoeffding identity (see Lemma 2 of [2]).

2.2 Convergence in p-mean for arrays of row-wise pairwise NQD random variables

Recall that an array $\{X_{n,k},\, 1 \leqslant k \leqslant n,\, n \geqslant 1\}$ of random variables is said to be row-wise pairwise negative quadrant dependent (NQD) if for each $n \geqslant 1$ and every $1 \leqslant k, j \leqslant n$ such that $k \neq j$,

$$\mathbb{P}\{X_{n,k} \leqslant x_{n,k}, X_{n,j} \leqslant y_{n,j}\} \leqslant \mathbb{P}\{X_{n,k} \leqslant x_{n,k}\}\, \mathbb{P}\{X_{n,j} \leqslant y_{n,j}\}$$

for all real numbers $x_{n,k}$ and $y_{n,j}$.

Next, we shall prove that for arrays of row-wise pairwise NQD random variables the assumption (c) in Theorem 1 can be dropped.

Theorem 3 *Let* $1 \leqslant p < 2$*,* $\{X_{n,k},\, 1 \leqslant k \leqslant n,\, n \geqslant 1\}$ *be a triangular array of row-wise pairwise NQD random variables and* $\{b_n\}$ *be a sequence of positive constants. If conditions* (a)*,* (b) *when* $1 < p < 2$ *or* (b') *whenever* $p = 1$*, of Theorem 1 are satisfied then* $\sum_{k=1}^{n} (X_{n,k} - \mathbb{E}\, X_{n,k})/b_n \xrightarrow{\mathscr{L}_p} 0$.

Proof. From Theorem 1, it remains to prove that its condition (c) holds. If $\{X_{n,k},\, 1 \leqslant k \leqslant n,\, n \geqslant 1\}$ is a triangular array of row-wise pairwise NQD random variables then, for each $n \geqslant 1$, $\Delta_n(x,y) \leqslant 0$ for all x, y entailing $G_n^+(t) = 0$ for every t and $\int_{\varepsilon b_n}^{\infty} t^{p-3} G_n^+(t)\mathrm{d}t = 0$ for any $n \geqslant 1$. ☐

We now compare the convergence in p-mean for weighted sums of NQD random variables obtained in [5] with Theorem 3. By taking $X_{n,k} = b_n a_{n,k} Y_{n,k}$ with $\{Y_{n,k},\, 1 \leqslant k \leqslant n,\, n \geqslant 1\}$ an array of row-wise pairwise NQD random variables we can introduce the weights in our set of assumptions. As pointed out in [4] (page 350), condition (b) of Theorem 3 reveals to be weaker than assumption (ii) of Theorem 2.1 in [5]; in fact, condition (b) can be written as $\sum_{k=1}^{n} \int_{\varepsilon}^{\infty} \mathbb{P}\{|a_{n,k}Y_{n,k}|^p > t\}\, \mathrm{d}t = o(1)$, $n \to \infty$ and

$$\int_{\varepsilon}^{\infty} \mathbb{P}\{|a_{n,k}Y_{n,k}|^p > t\}\, dt \leqslant |a_{n,k}|^p\, \mathbb{E}\, |Y_{n,k}|^p\, I_{\{|a_{n,k}Y_{n,k}|^p > \varepsilon\}} \qquad (p, \varepsilon > 0).$$

Moreover, from (2.5) we can still verify that our condition (a) is weaker than assumption (i) of Theorem 2.1 in [5].

Looking at assumptions (a) and (b) of Theorem 1 in [4] and the corresponding conditions (a) and (b) of our Theorem 1, it is straightforward to see that the former are stronger; note that condition (b') in Theorem 3 is equivalent to $\sum_{k=1}^{n} \int_{\varepsilon b_n}^{\infty} \mathbb{P}\{|X_{n,k}| > t\}\, \mathrm{d}t = o(b_n)$, $n \to \infty$ and $\sum_{k=1}^{n} \mathbb{P}\{|X_{n,k}| > \varepsilon b_n\} = o(1)$, $n \to \infty$ for any $\varepsilon > 0$. However, recall that assumption (c) of our main statement Theorem 1 is not guaranteed for triangular arrays of row-wise extended negatively dependent random variables.

Acknowledgements. This work is a contribution to the Project UID/GEO/04035/2013, funded by FCT - Fundação para a Ciência e a Tecnologia, Portugal.

References

[1] M.O. Cabrera, A.I. Volodin, Mean convergence theorems and weak laws of large numbers for weighted sums of random variables under a condition of weighted integrability, *J. Math. Anal. Appl.* **305** (2005), 644–658.

[2] E.L. Lehmann, Some concepts of dependence, *Ann. Math. Statist.* **37** (1966), 1137–1153.

[3] J. Lita da Silva, Almost sure convergence for weighted sums of extended negatively dependent random variables, *Acta Math. Hungar.* **146** (2015), 56–70.

[4] J. Lita da Silva, Convergence in p-mean for arrays of row-wise extended negatively dependent random variables, *Acta Math. Hungar.* **150**(2) (2016), 346–362.

[5] S.H. Sung, Convergence in r-mean of weighted sums of NQD random variables, *Appl. Math. Lett.* **26** (2013), 18–24.

[6] Y. Wu, M. Guan, Mean convergence theorems and weak laws of large numbers for weighted sums of dependent random variables, *J. Math. Anal. Appl.* **377** (2011), 613–623.

[7] H. Yu, A Glivenko-Cantelli lemma and weak convergence for empirical processes of associated sequences, *Probab. Theory Related Fields* **95** (1993), 357–370.

[8] D. Yuan, B. Tao, Mean Convergence theorems for weighted sums of arrays of residually h-integrable random variables concerning the weights under dependence assumptions, *Acta Appl. Math.* **103** (2008) 221–234.