Noise Traders, Rational Traders, and the Rest: A New Approach to Efficient Financial Markets

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Abstract

We extend a baseline asset pricing model with noise traders by adding the assumption of rhizomatic beliefs. In a model with market-wide waves of sentiment or noise, those agents who are affected by sentiment may be aware that they are not alone in the market, and that others are affected by the same type of misperceptions as they do. This behavioural assumption allows us to reconcile the hypothesis of market-wide noise with fundamental equilibria that would be predicted by the efficient markets hypothesis. Our model shows that, in spite of lower expected returns, noise traders often have welfare incentives to behave rhizomatically, taking into account that a fraction of other investors may be acting in the same way as they do.
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1 Introduction & Motivation

The rationality of the behaviour of market participants and of markets themselves is a central issue in economics and finance. In the field of capital markets, the fairly homogeneous nature of the goods traded, the seemingly transparent nature of transactions, and the economic returns at stake encouraged the view that markets tended to be efficient and reflect, at the aggregate level, the actions of rational actors responding to incentives anchored on fundamental and market moves. Noise traders, as those trading on beliefs that were not based on market realizations or sensible expectations, were discarded as non-existent or irrelevant. In a notable essay on flexible exchange rates, Friedman (1953) postulated that irrational traders, those moved by noise and sentiment, cannot have a significant impact on asset prices. If an investor undertakes irrational moves in the market, he or she is immediately met by rational arbitrageurs who eliminate any pressure over prices. Rational arbitrageurs make a profit while making irrational noise traders irrelevant at the same time. Moreover, Friedman argued that, since destabilising speculation involves, on average, selling low and buying high, noise traders must, on average, lose money and through this fact push themselves out of the market. The main corollary of this argument is that destabilising speculation cannot be rational. This rationale was a flagship to the argument developed in the Efficient Market Hypothesis developed in Fama (1965).

The Efficient Markets Hypothesis, henceforth referred to as EMH, enjoyed a clear dominance in the field of asset pricing, at least up to the mid-1980s. By 1978, Jensen went as far as stating that "(...) the efficient markets hypothesis is the best established fact in all of the social sciences."¹ After decades of clear dominance, pockets of dissonance arose from within the scientific community. From the second half of the 80’s onwards, new empirical observations as well as and advances in the theory and econometrics of asset pricing opened the way for the emergence of an increasing number of puzzles or anomalies, empirical facts that were not consistent with the EMH predictions. Among the most famous, we find the equity premium puzzle (Mehra and Prescott, 1985), the excess volatility puzzle (Shiller, 1981) and the several day of the week and year end effects which have been persistently reported and statistically confirmed in stock markets. These observations were too anomalous to be easily discarded. Instead, they motivated the rise of less conventional approaches to asset pricing that attempted to align theory with evidence. One of the most successful new theories was The Noise Trader Approach to Finance, put forward in

¹See Jensen (1978).
Shleifer and Summers (1990). Noise traders are a class of investors who defy the well-established paradigm of rationality and rational expectations in asset pricing models. Though behaving rationally from an individual point of view, in the sense of being utility maximisers, their expectations are noisy in the sense that they are affected by assumptions and beliefs that find no support in the observable and predictable behavior of markets. These noisy expectations translate into noisy actions.\footnote{In the bounded rationality sense.}

Noise traders can be arbitraged away by rational traders or disappear in the wake of financial losses, as suggested in Friedman (1953). DeLong et al. (1990a) showed that if a sufficient large number of noise traders act on the basis of a similar bias, arbitrage cannot discard the effects of noise trading on aggregate asset markets. But can noise traders remain completely unaware of the fact that a considerable proportion of the (market) population shares the same sentiment or noise? The possibility that noise traders do not take into account that their biases may be shared by a large number of similar investors, likely to act in a similar way, is behind the suggestion that noise traders are either non-existent or irrelevant. Our main new assumption is that noise traders, or sentiment traders, cannot be irrational to the extent of ignoring the fact that other traders are affected by that very same sentiment, at the same time. Think of specialists forecasts, news in the press, and other facts at the origin of the waves of pessimism or optimism. Even an irrational noise trader will not ignore the fact that others will be tempted to act in the same direction, inspired by the same ”news”. This is specially important when noise traders are in significant numbers, and move according to similar biases, as in DeLong et al. (1990a). And noise traders should take into account this fact when deciding when and how much to buy. Considering this ”second order” information and incorporating it into noise traders behaviour is central to our model of capital markets. In addition, because it mitigates the irrationality of noise traders, it is a response to Friedman’s remarks while still justifying an aggregate impact on markets that cannot be arbitraged away. \textbf{Table 1} shows how our paper fits in the current literature on this field. It becomes clear that when assuming a non-negligible mass of investors with shared biases, it is only natural to explicitly model, in agents’ behaviour, a belief about the size of that mass. Furthermore, as we will formally demonstrate, a noise trader almost always gains in terms of utility by believing that he or she is not a pure noise trader, that is, some others act on the same bias as him or herself.

\footnote{A classic example is the so-called ”January Effect”: the tendency of stock markets to rise between December 31 and the first week of January. This trend has been amply documented since the early 40’s, as noted in Keim (1983). Countless other ”good” and ”bad” times of the year have since been featured in the financial press, with little or no explanation based on fundamentals.}

5
Table 1 - Comparison with the Existing Literature on Noise Traders

<table>
<thead>
<tr>
<th>Idiosyncratic Bias</th>
<th>No Beliefs on Others</th>
<th>Beliefs on Others’ Actions</th>
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<tr>
<td></td>
<td>Not Relevant: Friedman Effect</td>
<td>Not Relevant</td>
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<tr>
<td>Market-wide Bias</td>
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Models of noisy asset pricing consider the usual game-theoretic assumption of expected utility maximisation given the actions by other agents. This is a natural assumption as each individual agent is, in the context of these models, a mere atom of measure zero in a large market. However, consider the case where a noise trader, given the public source of their individual biases - the news, the commentary of specialists, and so on - , suddenly realise that he or she is surrounded by a population of traders who share similar same beliefs and will act on them in a similar way\(^4\). If a sizeable - or non-negligible - portion of the market acts in the same direction, this information should feed naturally back to individual actions. In other words, the fact that noise traders act on noisy information, does not imply they are necessarily inattentive and unaware of the wider relevance of their personal motivations. This inference creates an incentive for the agent \textit{not to do} what he or she would to do as a classical noise trader, or do so in a more muted fashion. Consider, as an example, a bullish agent who is optimistic regarding the future price of a particular stock. He or she realizes that other bullish agents are likely to be present in the market, probably moved to action by similar ”news”, and those will choose to buy the stock. The price will contemporaneously increase, eliminating part of the potential return that the noise trader expected to realise with the purchase, in the first place. This agent may, therefore, have the incentives to either moderate the initial position, or even take the opposite action! This occurs because the second order effects of market-wide noise are now taken into account in the decision of buying or selling the asset. The idea of an inference on similar behaviour by others affecting own behaviour stems from the concept of rhizomatic thinking, as applied to a portfolio allocation problem. Originally discussed by the French philosophers Deleuze and Guattari (1982), rhizomatic thinking finds its first application to economic theory in the work of Furtado and Córte-Real (2011). In their words, ”an agent will think rhizomatically if she believes that a proportion of like-minded others will take the same action as she does, i.e., the agent perceives a connection (that is actually ”inexistent”) among herself and like-minded others.” By thinking rhizomatically, agents are endowed with

\(^4\)This may give the agent an illusion of control - since several people are undertaking the same actions, prices may seem to be influenced by the agent’s actions.
a belief on the proportion of agents that is going to think and act like them.\textsuperscript{5}

In our model, noise traders can think rhizomatically and, in this process, act in a way that partially protects themselves from actions that would make them consistently underperform. The belief on the proportion of agents that think and act similarly may be noisy and need not to coincide with the true distribution of types. Our model takes account of that fact, parameterizing results on the actual and believed proportion of rhizomatic noise traders. The main advantage of incorporating rhizomatic thinking in an asset pricing model is rationalising trading under noisy expectations, especially in the case where a non-negligible proportion of traders shares the same bias, as in DeLong et al. (1990a). In our model, noise traders will impact aggregate market performance, but we will be able to parametrize such influence on both the actual share of noise traders and the associated belief as to what this proportion is likely to be. Our setup is especially relevant for cases where, as in DeLong et al. (1990a), biased beliefs affect a significant number of traders in the same manner, such as with news stories and public forecasts, and traders have a sense of the shared information and use it to infer the proportion of traders that will act on it. As it will become clear, these effects cannot be arbitrated away by ”rational” investors.

The paper is organized as follows. Section 2 conducts a brief survey of models of efficient markets, and the relevance of noise and noise traders. Section 3 presents the benchmark model with noise traders, along the lines of DeLong et al. (1990a). Section 4 introduces rhizomatic noise traders which consider the proportion of traders that share similar biases. Section 5 discusses two possible extensions: idiosyncratic beliefs, and the nature of stock market crashes. Section 6 concludes.

\textsuperscript{5}In economics, rhizomatic thinking has been used to explain voter turn-out by Furtado and Corte-Real (2011) and Chaves and Peralta (2011). The idea underlying these models is that voters, by thinking rhizomatically, may perceive themselves to be pivotal in determining the outcome of an election. Rational voters would abstain because they have zero mass and perceive that they cannot influence the outcome.
2 Literature Review: Efficient Markets, Noise Traders and Arbitrage

In the wake of several empirical challenges to the Efficient Market Hypothesis, DeLong et al. (1990a) formalised the first asset pricing model with noise traders. In their context, noise traders behave rationally but misperceive the future price of the asset and form noisy biased expectations. Biased expectations are assumed to be market-wide, affecting in the same way a non-negligible fraction of contemporaneous noise traders, and thus translate into aggregate market behavior. The result is more extreme price movements and greater volatility, accounting for some of the aforementioned asset pricing puzzles. In the setup of DeLong et al. (1990a) noise persists and cannot be arbitrated away due to two factors: the relative size of the population of noise traders, and the existence of limits to arbitrage which are ignored in the context of the Efficient Market Hypothesis\(^6\). The noise trader model in DeLong et al. (1990a) provided a new impetus to the behavioural asset pricing literature focusing on non-Walrasian economies populated by investors who are either "fundamentalist" - rational-, or "chartist", - following price trends\(^7\).

Noisy beliefs and expectations are easily associated with the popular idea of investor sentiment, defined in Shefrin (2007) as "broad erroneous beliefs about future cash flows and risks". Most investor sentiment models incorporate waves of time-varying optimism and/or pessimism, that is erroneous "biases" that affect a significant amount of investors in a similar manner\(^8\). These waves of optimism and pessimism affect investor preferences, usually through a stochastic discount factor,

\(^6\)An important element limiting arbitrage by rational traders is their degree of risk-aversion, which may be sufficient to impede their countering noise traders’ positions. Shleifer and Vishny (1997) describe other important factors that limit arbitrage behavior in the presence of arbitrage opportunities: capital requirements, excessive risk, and agency problems top the list. Agency problems, for instance, can stem from the fact that arbitrageurs are, in reality, a small class of highly specialised individuals managing other people’s assets. DeLong et al. (1990b) propose a model in which limits to arbitrage coupled with price feedback effects can lead to rational arbitrageurs amplifying, rather than muting, asset pricing biases originating in noise traders.

\(^7\)The first of such models was presented in a seminal article by Beja and Goldman (1980).

\(^8\)The main message from Friedman’s remark applies here: for speculators to destabilise efficient prices, they will be losing money, on average. For prices to exceed the fundamental value of stocks, people are buying too much of them and are likely to lose money. For prices to go below fundamental values, people are selling in excess and also likely to lose money. This logic is especially pervasive in the case of noise and investor sentiment: if one assumes the hypothesis of time-varying waves of investor sentiment to be true, then noise traders will all be buying and selling the same stocks at the same time, buying with high prices selling with low prices. There is some empirical evidence in favor of the Friedman Effect. A notable, and early, attempt to justify the prevalence of apparently
and can be traced back to the well-known Keynesian view of investors as "animal spirits", responding to "spontaneous motivation". Several studies have analyzed the phenomenon. Cecchetti et al. (2000) introduce time-varying mood fluctuations in a classical Lucas asset pricing model, to find a good match between the model’s moment predictions and the data. Abel (2002) mimics the equity premium puzzle by introducing waves of pessimism on the subjective distribution of consumption growth rates, which leads to a reduction of the risk free rate. Baker and Wurgler (2007), besides examining several historical instances in which emotion appears to have defeated rationality, construct a "sentiment index" based on proxies such as surveys of investor mood and measures of expected profitability. The authors find these "sentiment waves", which display a level of persistency, to affect mostly stocks that are difficult to arbitrage. Hirshleifer and Teoh (2008) have put forward the argument that thought and behaviour cause contagion in capital markets. While most asset pricing models focus on the transmission of information through market prices, Hirshleifer and Teoh (2008) argue that social influences, the media, and verbal arguments are important vehicles to disseminate sentiment and beliefs. If a prestigious commentator or analyst consistently broadcasts his or her personal views on the path of a given stock, the latter is likely to spread and generate a market-wide sentiment in a certain direction. Hirshleifer (2001) suggests that macroeconomic forecasts from reputed institutions such as Central Banks can have a similar impact on beliefs.

There is a plethora of empirical studies documenting the impact and of noise traders in a variety of spot markets. Most studies use disaggregate survey data, and relate opinions and sentiments to the evolution of aggregate data. Barkham and Ward (1999) conduct a thorough study of property companies in the United Kingdom, to find that, even after controlling for agency costs, contingent capital gains tax liabilities, fundamentals and other firm-specific factors, market capitalisation is still significantly less than the net asset value of the companies analyzed. The authors show that noise trading appears to be a significant factor motivating this discount. Barber et al. (2006) use disaggregate data on individual investors in the US between 1983 and 2001. They uncover strong evidence of herding and time-varying investor sentiment. Individual investors tend to buy and sell the same stocks contemporaneously, and such "coordinated" movements do have a significant impact on stock prices. In their study, stocks heavily bought by individual investors tend to underperform those heavily sold by 4.4 percentage points over the course of the irrational and myopic behaviour is present in Shefrin (1985), who uses prospect theory to develop a model in which agents have a general disposition to "sell winners too early and hold losers too long". In this model, agents develop investment strategies that are consistent with buying high and selling low, trading at the worst possible times, at the worst possible prices.
following year. Verma and Verma (2006) use data from the American Association of Individual Investors between 1988 and 2004 to show that investor sentiment, uncorrelated with other factors, plays a significant role in determining asset volatility. They also find that optimistic (bullish) and pessimistic (bearish) sentiment generates asymmetric spillover effects. From a survey of institutional investors, similar results are uncovered. Finally, Tsiaplias (2009) studies equity markets from 18 developed countries between 1980 and 2004, finding a strong impact of investor sentiment and noise trading on both returns and volatility. A common criticism of behavioural asset pricing is that even if cognitive biases and noise exist at the individual investor level, these can cancel each other and have no impact at the aggregate market level. Given that most approaches to modelling noise rely on a "bottom up", microfounded approach, they are vulnerable to this critique. Baker and Wurgler (2007) have provided a key contribution, as they rely on a "top down", macro approach to detect waves of sentiment that explain market behaviour beyond the effect of fundamentals.

2.1 Cognitive Biases

Given that noise and sentiment are inherently behavioural phenomena, there has been much effort in the literature to relate them to well-known cognitive biases. Shiller (2003) and Stracca (2002) provide extensive lists of well-documented cognitive biases that may have an impact on market aggregates. Here we focus on biases that are more relevant to the subject of our work.

Shiller (2003) focuses on feedback models: investors observe the rising price on an asset and herd to purchase that asset, leading to further price increases. In the end, these investors are buying at the worst possible price. This mechanism is naturally at the origin of bubbles and crashes: as the price reaches very high levels and demand finally decreases, the feedback channel from prices to higher prices is broken - the bias is "undone, in a sense", and investors are tempted to sell fast and furious before the price falls further, thus contributing to its collapse and a market crash. Feedback effects seem inherent to cognitive psychology, and evidence that "human judgments of the probability of future events show systematic biases"\(^9\). Several factors account for systematic biases: news, the popular press, the opinions therein contained, and how they are mediated by specialists and opinion-makers to finally impact the beliefs of individual investors. Any effect that is generated and is not, in any tangible way, connected to the fundamental factors of the asset at stake can be said to fall under the

branch of investor sentiment, or noise. There is much anecdotal evidence of feedback effects from historical observations, from the Dutch Tulipmania on the 17th century to the stock market bubble of 2000\textsuperscript{10}.

Other well-documented cognitive biases that can justify the emergence of noisy expectations is put forward in Livio (2002). Among these are over-inference from small samples; the belief that positive (or negative) returns are more persistent that in fact they are; anchoring, the belief that current observed prices are the ”normal” or ”equilibrium” prices of an asset; limited attention, when agents are incapable of processing and ”digesting” all information received; and credulity, as agents fail to filter the information received and incorporate biased and noisy beliefs of others as their own\textsuperscript{11}.

A different set of cognitive biases, loosely related to over-inference, consists of overconfidence and the illusion of control. These belong to what Stracca (2002) describes as emotional or visceral biases. Overconfidence is well documented in professional and inexperienced investors alike, for instance in Zindel et al. (2010), who use questionnaire techniques. This may feed overconfidence and the illusion of control through what is known as the confirmatory bias (Rabin and Schrag, 1999). This feedback effect in emotions leads agents and investors to believe that they are more capable than what they really are, potentially leading individuals to forget that they are atoms in the market.\textsuperscript{12}

\textsuperscript{10}It is interesting to note that Shiller’s popular book Irrational Exuberance was published in the beginning of March, 2000. It argued that the so-called ’dot-com bubble’ was a market boom created by feedback effects. That bubble peaked on the 10th of March and the crash soon followed, at the end of the month.

\textsuperscript{11}Credulity can be readily related to the aforementioned role of social influence.

\textsuperscript{12}As shown in the following section, noise traders who are sufficiently bullish and have sufficiently noisy expectations so as to adopt excessively long positions in high risk markets can indeed earn higher returns than their rational counterparts.
3 The Benchmark Model

The benchmark model is inspired by the noise trader asset pricing model developed in DeLong et al. (1990a), in which the coexistence of noise traders with rational investors disrupts an otherwise efficient market equilibrium (in the EMH sense) for an asset with no fundamental risk.

3.1 Set-up and Assumptions

The basic framework is an Overlapping Generations economy in which agents live for two periods. There is no first period consumption, no labour supply decision and no bequests. Agents are born with some exogenous income $y$ (one can think of it as labour income derived from an inelastic labour supply), and solve a portfolio allocation problem when young. They are asked to allocate this exogenous income among two assets with different characteristics. The first is a risk-free asset, $L$, that exists in infinite supply and is traded at a fixed price, normalised to 1. This bond has maturity equal to one period, after which it pays a fixed risk-free dividend $r$ (it can be seen as risk-free government debt). The second asset, $D$, exists in a finite quantity $A < \infty$, implying that its price will be allowed to fluctuate. Young agents at period $t$ purchase it at $p_t$, and sell it when old, at $p_{t+1}^t$. Given that this "risky" asset exists in finite quantity, the equilibrium price is determined by aggregate demand by young agents. The risky asset pays a non-stochastic dividend equal to the risk-free rate $r$. This assumption may be relaxed, and serves the purpose of normalising the equilibrium price of the risky asset with respect to that of the riskless bond.

Each generation is populated by a continuum of agents $I$, indexed by $i \in (0, 1]$, who may come in two different types. Rational traders, $R$, behave rationally in the sense of maximising expected utility subject to rational expectations regarding the future price of the risky asset. On the other hand, noise traders, $N$, being identical to rational traders in every other aspect, have "noisy" expectations regarding the risky asset’s future price. In particular, they are subject to some cognitive bias of the class of those discussed before, so that they misperceive the expected price in the following period by a term $\eta_t$, which may take positive or negative values. In case this term is positive, $\eta_t > 0$, they believe the rational one-period ahead expectation for the future price to be an underestimate of its true future realisation. This, as we will see, boosts their demand for the asset, hence they are said to be bullish. If the term is negative, $\eta_t < 0$, the reverse effect occurs, and they are said to engage
in \textit{bearish} behaviour. When \( \eta_t = 0 \), the expectation of the future price becomes rational, hence their behaviour coincides with that of an element of group \( R \).

Naturally, \( R \cup N = I \). The continuum of agents has measure 1, \( \mu (I) = 1 \). Group \( N \) has measure \( \alpha \), while group \( R \) has measure \( 1 - \alpha \), for \( \alpha \in [0, 1] \). In the context of this model, these measures can be taken as the market proportion, or market "share" for each type of agents\footnote{These proportions can, for example, reflect the share of endowments, or wealth, of each of the groups.}. All traders of a given type are assumed to be perfectly identical in terms of preferences, endowments and optimal behaviour. This allows us to restrict the analysis to a limited set of portfolio allocation problems, one for each type of agent that populates the market.

The key dynamics of the model can be summarised as follows:

1. Young agents are born, endowed with an exogenous income. They decide on how to allocate their wealth between the risky and the risk-free assets;
2. The decision is taken based on maximisation of expected utility when old. Utility depends not only on expected wealth, but also on the variance (risk) of wealth;
3. The solution to the portfolio allocation problem yields the demand for the risky asset, for each player type;
4. The market-clearing condition determines the equilibrium price each period, therefore realising returns.

\subsection*{3.1.1 Preferences, Expectations and Demand}

Preferences are common across types, and based on CARA utility

\[ U = - \exp (-2 \gamma w) \]

where \( \gamma \) stands for the (constant) coefficient of absolute risk aversion and \( w \) denotes wealth. Rational agents believe the \textit{ex-ante} one-period ahead distribution of the price of the risky asset to be normal

\[ p_{t+1} \sim N \left( E_t (p_{t+1}) , \sigma^2_{p,t} \right) \]

where \( E_t (p_{t+1}) \) stands for the (rational) expectation of the future price, conditional on information available at \( t \). \( \sigma^2_{p,t} \) is the one-step ahead (conditional) variance of the
price. Noise traders misperceive the price of the risky asset in the following period by a random i.i.d. normal variable $\eta_t$. More specifically,

$$E_{t,N}(p_{t+1}) = E_t(p_{t+1}) + \eta_t$$

where

$$\eta_t \sim N(\eta^*, \sigma^2_\eta)$$

Notice that even though different misperceptions of the future price arise across generations, the same is not true within generations. As remarked in DeLong et al. (1990), it is crucial to assume that this misperception be market-wide (within the set of noise traders, naturally). If noise were idiosyncratic, it could be argued that rational arbitrageurs would immediately eliminate any pressures that could push the price off its (efficient) equilibrium level. It can also be argued that different, uncorrelated misperceptions over the future price symmetrically cancel at the aggregate level. A formal argument based on this latter argument is presented further ahead.

Given normally distributed expectations, CARA expected utility can be written in mean-variance form

$$E_t(U) = E_t(w_{t+1}) - \gamma \sigma^2_{w,t}$$

where $E_t(w)$ denotes the (conditional) expected value of future wealth at $t$ and $\sigma^2_{w,t}$ is the one-period ahead conditional variance.

The budget constraint for a young agent of type $\bar{j} \in \{R, N\}$ at period $t$ is given by

$$L^j_t + p_t D^j_t = y$$

The expected value of wealth for a young rational agent, when old, is

$$E_{t,R}(w_{t+1}) = (1 + r) L^R_t + [r + E_t(p_{t+1})] D^R_t$$

while for a noise agent, the only difference lies in the presence of the $\eta_t$ term that complements the rational expectation concerning the future price

$$E_{t,N}(w_{t+1}) = (1 + r) L^N_t + [r + E_t(p_{t+1}) + \eta_t] D^N_t$$

Given that the demand for the risky asset is a control variable, and that $\eta_t$ is considered by noise traders before they form their demands, the ex-ante expected variance of wealth is common to both types

$$\sigma^2_{w,t} = \text{var}_t(w_{t+1})$$

$$= \text{var}_t[(1 + r) L^j_t + (r + p_{t+1}) D^j_t]$$

$$= (D^j_t)^2 \sigma^2_{p,t}$$
Furthermore, from the budget constraint, one can eliminate the decision variable \( L_i^j \) as

\[
L_i^j = y - p_i D_i^j
\]

The problem for each type becomes

\[
R : \max_{D_i^R} \left\{ (1 + r) y + [r + E_t (p_{t+1}) - (1 + r) p_t] D_i^R - \gamma \left( D_i^R \right)^2 \sigma_{p,t}^2 \right\}
\]

\[
N : \max_{D_i^N} \left\{ (1 + r) y + [r + E_t (p_{t+1}) + \eta_t - (1 + r) p_t] D_i^N - \gamma \left( D_i^N \right)^2 \sigma_{p,t}^2 \right\}
\]

The necessary and sufficient first order conditions yield the following demands for the risky asset:

\[
D_i^R = \frac{r + E_t (p_{t+1}) - (1 + r) p_t}{2 \gamma \sigma_{p,t}^2}
\]

(1)

\[
D_i^N = \frac{r + E_t (p_{t+1}) - (1 + r) p_t + \eta_t}{2 \gamma \sigma_{p,t}^2}
\]

(2)

As one would expect, demand functions are increasing on future payoffs yielded by the risky asset, \( r + E_t (p_{t+1}) \); decreasing on the current price (at its future value, capitalised by the dividend rate); and decreasing on an interaction term between the coefficient of absolute risk aversion and the one-step ahead variance of the price. Demand by noise traders is further positively influenced by the noise term \( \eta_t \): the greater the degree of bullishness (bearishness), the longer (shorter) will their positions be. Further note that no restrictions are being imposed on short-selling, and agents are free to take negative positions at will.

### 3.2 Equilibrium

Equilibrium at each period is attained by equating aggregate demand of the risky asset to aggregate supply. By construction, old agents sell all of their holdings in quantity \( A \), thus forming aggregate supply. Aggregate demand, \( D_t \), is generated by young agents. The market-clearing condition is, therefore:

\[
D_t = A, \forall t
\]

where

\[
D_t = \int_1^\infty d_i^t di
\]

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Given that \( I = N \cup R \), is the union of two mutually exclusive sets:

\[
D_t = \int_{N} D_t^i + \int_{R} D_t^j
\]

In the absence of idiosyncrasies, optimal demand is the same for all agents of each type, so that

\[
D_t = \mu (N) D_t^N + \mu (R) D_t^R
\]

\[
= \alpha D_t^N + (1 - \alpha) D_t^R
\]

Plugging the expressions for \( D_t^N \) and \( D_t^R \) in the market-clearing condition

\[
\alpha \left[ \frac{r + E_t \{ p_{t+1} \} - (1 + r) p_t + \eta_t}{2 \gamma \sigma^2_{p,t}} \right] + (1 - \alpha) \left[ \frac{r + E_t (p_{t+1}) - (1 + r) p_t}{2 \gamma \sigma^2_{p,t}} \right] = A
\]

Note that \( A \) plays the role of the relative abundance of shares in the market. Solving for \( p_t \) yields

\[
p_t = \frac{1}{1 + r} \left[ r + E_t \{ p_{t+1} \} + \alpha \eta_t - 2 \gamma A \sigma^2_{p,t}\right]
\]

(3)

At this point, it is interesting to note that if all agents were rational and risk-neutral, implying that \( \alpha = 0 \) and \( \gamma = 0 \), respectively, this expression would reduce to the classical Lucas (1978) asset pricing equation

\[
p_t = \frac{r + E_t (p_{t+1})}{1 + r}
\]

In such a situation, a no-bubble market equilibrium would require \( p_t = 1, \forall t \).

3.3 The Pricing Equation

The equilibrium pricing function is a forward-looking recursive equation that depends not only on the expected value of the price in the following period, but also on its variance. It expresses the equilibrium price as a function of the one-period-ahead distribution, the proportion of noise traders in the market \( \alpha \), their noise term \( \eta_t \), the coefficient of risk aversion \( \gamma \) and the market size \( A \). It is worth noting that while the above equation corresponds to a market equilibrium, it does not necessarily represent a steady state due to the presence of both the expected value and variance of the price in the following period. It is analytically convenient to focus the analysis on
steady state equilibria, in which the distribution and, in particular, the variance of \( p_t \) are constant across periods. To do this, assume away bubbles by imposing that

\[
\lim_{T \to \infty} \left( \frac{1}{1 + r} \right)^T E_t (p_{t+T}) = 0
\]

This allows us to solve the price equation recursively and obtain

\[
p_t = 1 + \frac{\alpha (\eta_t - \eta^*)}{1 + r} + \frac{\alpha \eta^*}{r} - \frac{2 \gamma A \sigma^2_{\eta}}{r}
\]

Note that the only time-varying term in the expression is the second one, through \( \eta_t \). This allows us to compute the steady state variance as

\[
\sigma^2_p = \text{var}_t (p_{t+1}) = \frac{\alpha^2 \sigma^2_{\eta}}{(1 + r)^2}
\]

Plugging in the solution for the pricing equation leaves us with its final form

\[
p_t = 1 + \frac{\alpha (\eta_t - \eta^*)}{1 + r} + \frac{\alpha \eta^*}{r} - \frac{2 \gamma A \alpha^2 \sigma^2_{\eta}}{r (1 + r)^2}
\]

(4)

This equation highlights the impact of market-wide noise on the equilibrium steady state price of the asset. The second term is the only time-varying component of the equation, and implies that all price fluctuations should be totally attributed to deviations of the contemporary noise term \( \eta_t \) from its means value \( \eta^* \). That is, price changes are only brought about by "news" regarding the aggregate misperception of future prices by noise traders. As one should expect, the more bullish the current generation of noise traders is relative to what is expected \( (\eta^*) \), the more does the asset price increase in the current period. The reverse logic applies to bearish behaviour on the part of young noise traders.

The presence of the third term evidences that, even in the absence of news and unexpected behaviour by noise traders, they still impact the price of the asset through their mean noise. The market acknowledges the existence of widespread, market-wide noise through the incorporation of this term in the asset price. If the mean noise is zero, so that different generations of noise traders are allowed to "compensate" for others’ behaviour, this term vanishes from the equation.

Finally, the fourth term embodies one of the crucial results of the noise asset pricing model. This rests on the fact that the presence of noise traders endogenously
generates price volatility, even in the absence of stochastic fundamentals. In fact, in this current framework, fundamentals could not be more deterministic - the risky asset pays a constant dividend equal to the risk-free, \( r \). The price variance in the steady state is not equal to zero, and depends on the variance of the noise term. Rational traders would not want to hold an asset whose price is biased upwards (as in the case of noise traders which are, on average, bullish, \( \eta^* > 0 \)), thus running the risk of facing, when selling, a generation of bearish traders that push the price down. Thus this term plays the crucial role of compensating for that risk, by decreasing the price of the risky asset and allowing for higher returns to be earned, on average. This term embodies the risk premium of the asset - we repeat - in the absence of fundamental risk.

### 3.3.1 Why is Market-wide Noise Important

The equilibrium pricing equation can be used to formally illustrate the intuitive argument, already presented, of why market-wide noise can generate endogenous risk and price volatility, as opposed to idiosyncratic noise.

To see this, consider that each young noise trader, \( i \in N \), at \( t \) is endowed with one realisation of the noise variable \( \eta_i \). Further assume that these variables are i.i.d. and drawn from some distribution \( F \) with finite mean \( \eta^* \) and variance \( \sigma_\eta^2 \)

\[
\eta_i \sim F (\eta^*, \sigma_\eta^2)
\]

It is assumed that this distribution is stationary and does not change over time.

In this context, each noise trader \( i \in N \) will display a demand function which depends on the idiosyncratic noise term \( \eta_i \)

\[
D_t^i = \left[ \frac{r + E_t (p_{t+1}) - (1 + r) p_t + \eta_i}{2 \gamma \sigma^2_{p,t}} \right] dt + (1 - \alpha) D_t^R
\]

The presence of this individual term renders derivation of aggregate demand, and thus of the market-clearing condition and the main aggregates, a non-trivial task:

\[
D_t = \int_N D_t^i dt + \int_R D_t^R dt
\]

\[
= \int_N \left[ \frac{r + E_t (p_{t+1}) - (1 + r) p_t + \eta_i}{2 \gamma \sigma^2_{p,t}} \right] dt + (1 - \alpha) D_t^R
\]

\[
= \left[ \frac{r + E_t (p_{t+1}) - (1 + r) p_t}{2 \gamma \sigma^2_{p,t}} \right] \int_N dt + \frac{1}{2 \gamma \sigma^2_{p,t}} \int_N \eta_i dt + (1 - \alpha) D_t^R
\]
From the fact that \( \int_N d_i = \mu (N) = \alpha \), we are left with

\[
D_t = \alpha \left[ \frac{r + E_t (p_{t+1}) - (1 + r) p_t}{2 \gamma \sigma_{p,t}^2} \right] + \frac{1}{2 \gamma \sigma_{p,t}^2} \int_N \eta_i d_i + (1 - \alpha) D_t^R
\]

(5)

The problem is non-trivial due to the presence of the second term in the expression. This term integrates (à la Lebesgue) a random variable over a continuum of agents of measure \( \mu (Z) = \alpha \). The problem is that there is no Law of Large Numbers for economies with a continuum of agents, as remarked by Feldman and Gilles (1985). This problem is extensively addressed in Judd (1985) and Uhlig (1987) who, alternatively, propose weaker forms of the Law of Large Numbers to solve this issue. Given that \( \eta_i \) is independent and uncorrelated over \( i \), we can use the following fact\(^{14}\):

\[
\int_N \eta_i d_i = \mu (N) E (\eta_i) = \alpha \eta^*
\]

Replacing in 5,

\[
D_t = \alpha \left[ \frac{r + E_t (p_{t+1}) - (1 + r) p_t}{2 \gamma \sigma_{p,t}^2} \right] + \frac{\alpha \eta^*}{2 \gamma \sigma_{p,t}^2} + (1 - \alpha) D_t^R
\]

The market-clearing condition takes us to the following price equation

\[
p_t = \frac{1}{1 + r} \left[ r + E_t (p_{t+1}) + \alpha \eta^* - 2 \gamma A \sigma_{p,t}^2 \right]
\]

which, assuming no bubbles, can be recursively solved in order to yield

\[
p_t = 1 + \frac{\alpha \eta^*}{r} - \frac{2 \gamma A \sigma_{p,t}^2}{r}
\]

Clearly, due to the absence of time-varying terms:

\[
\sigma_{p,t}^2 = \text{var}_t (p_{t+1}) = 0
\]

\(^{14}\)Alternatively, one could construct a sequence of discrete economies with \( N \) agents and constant fractions of noise and rational traders. By letting \( N \rightarrow \infty \), we are able to apply the conventional Law of Large Numbers, thereby getting the desired result. This is a technical detail that has been extensively discussed in the literature.
The final pricing equation thus becomes:

\[ p_t = 1 + \frac{\alpha \eta^*}{r} \]

So that, as it was originally argued, uncertainty disappears at the aggregate level\(^{15}\). The price is nevertheless affected by the mean bullishness of noise traders, \( \eta^* \). In case these traders are bearish, then they will be consistently underestimating, on average, the price of the asset, thus opening an arbitrage opportunity for rational traders. The reverse logic applies to the case in which \( \eta^* > 0 \). Due to the fact that rational traders arbitrage way individual misperceptions of the price, the asset is left with no trading risk at the aggregate level and, consequently, no risk premium (that would always push returns upwards). In comparison with the baseline case in which the economy is exclusively populated by rational traders, this group becomes better off by revealed preference, given that its choice set has expanded (recall that the risk-free asset is still available, and would trade at the same price and earn the same return as the risky asset in the baseline efficient markets case).

### 3.4 Returns and Utility - A Brief Analysis

It can be shown, in this model, that noise traders may earn higher returns than rational investors. The return differential (between noise and rational traders) can be shown to depend nonlinearly on \( \eta^* \) and negatively on \( \sigma^2 \). A necessary condition for the differential to be positive is to have positive mean noise, \( \eta^* > 0 \). Given the absence of restrictions on the type of positions that can be taken by agents (either long or short), one would, in principle, expect some symmetry in what concerns this condition. In other words, one would expect that for some negative values of \( \eta^* \), noise traders could, under certain circumstances, also earn higher returns than rational investors (through short positions, as opposed to long ones). This absence of symmetry is related to the existence of an alternative investment: the risk-free bond that yields a constant return at zero risk. To see this, think of the case when \( \eta^* > 0 \), so that noise traders are, on average, bullish. In this case, the equilibrium price of the risky asset will suffer an upwards bias, which, in principle, lowers its average return. Even though this upwards bias is, in usual circumstances, compensated to some extent by the presence of the risk premium term, it still discourages rational traders from investing in the asset. Given that noise traders become, therefore, on average more exposed to the risky asset, they are the ones who are likely to earn

\(^{15}\)As long as the distribution of noise \( F \) is stationary and noise is i.i.d.
greater returns if the next generation of noise traders happens to be even more bullish (thus raising the selling price for the previous generation, the one whose welfare we are discussing). This effect is not felt if noise traders are bearish: in this case, given that they misperceive the future price downwards, they are more likely to take refuge in the risk-free asset. Given that rational investors hold rational expectations regarding the price of the risky asset, and are aware of the fact that it is mispriced due to noise, they are more likely to expose themselves more to this asset in order to take advantage of the mispricing. This explains why only noise traders can earn higher returns when they are bullish: because it is only in this situation that they expose themselves more to risk.

By revealed preference arguments, it can also be shown that rational traders are better off. This argument was already exposed in the previous section: in an efficient markets economy populated only by rationals, the two assets are perfect substitutes (due to the absence of fundamental risk). Therefore, the presence of noise traders brings about a new type of asset. With more possibilities (portfolios) to choose from, and with rational expectations, this type of traders becomes (weakly) better off. The same is not true of noise traders: even though they may earn higher returns, in some circumstances, they do so by incurring in much higher risk than rational traders (because their expectations are biased), and realised returns do not compensate for this amount of risk (even though their \textit{ex-ante} expected returns appeared to compensate for it - after all, they still maximised expected utility).
4 A Model of Asset Pricing with Rhizomatic Noise Traders

We now add a further layer of complexity to the benchmark by considering that noise traders, besides having noisy beliefs regarding the future price of the asset as in the benchmark, also behave rhizomatically. That is, they have exogenous beliefs on the proportion of players that think and act like them. They believe that a proportion $q$ of agents behaves exactly like them, while the remaining $1 - q$ behave differently. The assumption of rhizomatic behaviour is captured through the parameter $q \in [0, 1]$. More specifically, if $q = 1$, a rhizomatic agent believes that everyone in the market behaves exactly like him or herself, thus adjusting own behaviour accordingly. Conversely, if $q = 0$, no assumptions are imposed on other agents’ behaviour and, as we will see, the benchmark noise trader case is recovered. If the proportion $q$ is large enough, the rhizomatic noise agent is naturally led to believe that his or her own actions influence the equilibrium price. By "large enough", we mean $q > 0$.

The benchmark model is altered by assuming that noise traders have rhizomatic beliefs. The set of rhizomatic noise traders, $Z$, has measure $\alpha$, while the remaining $1 - \alpha$ agents are rational, behaving in the same way as in the benchmark.

4.1 Rhizomatic Preferences and Demand

Preferences for elements of group $Z$ are assumed to be identical to those of noise traders in every respect, including the time-varying noise term $\eta_t$. Thus the relevant problem to be solved is given by

$$Z : \max_{D^Z_i} \left\{ (1 + r) y + \left[ r + E_t (p_{t+1}) + \eta_t - (1 + r) p_t \left( D^Z_i \right) \right] D^Z_i - \gamma \left( D^Z_i \right)^2 \sigma^2_{p,t} \right\}$$

The only difference is highlighted by the presence of $p_t \left( D^Z_i \right)$ instead of simply $p_t$. In particular, due to the fact that rhizomatic noise traders believe that a proportion $q$ of agents behaves as they do, they implicitly consider a relationships between their behaviour and market performance. Even though this market power does not exist, at the individual level, due to atomicity of agents, the sole existence of its perception does influence equilibrium results and allocations.
The necessary and sufficient first order condition now yields the following demand function

\[
D^Z_t = \frac{r + E_t (p_{t+1}) - (1 + r) p_t + \eta_t}{2\gamma \sigma^2_{\rho,t} + (1 + r) \frac{\partial p_t}{\partial D^Z_t}}
\]

where the novelty is the presence of the term \(\frac{\partial p_t}{\partial D^Z_t}\). By considering that they somehow influence the equilibrium price, rhizomatic noise agents take into account this influence in choosing their optimal market position. In particular, by assuming that a proportion \(q\) of agents does the same as they do, they perceive the market clearing condition as being given by

\[
qD^Z_t + (1 - q) E_t \left[ D^R_t (p_t) \right] = A
\]

In the absence of individual randomness, demands can simply be taken from the Lebesgue integrals and replaced by the perceived measure of each type of agents. A proportion \(q\) of agents, not \(\alpha\), is perceived to do precisely the same as they do, hence is represented by own demand \(D^Z_t\). The individual rhizomatic noise trader believes that all traders who behave rhizomatically belong to the proportion \(q\)\(^{16}\).

The remaining \(1 - q\) elements of the population are expected to be rational traders. However, rhizomatic noise traders are not exactly sure of whether the other players will behave strictly in a rational manner, hence the expected value is taken over those players’ actions. Note that the model is robust to the possibility of the "other side of the market" being composed of noise traders, or even a linear combination of noise and rational traders. The way rational and noise traders react to changes in price is the same (as one can see from the demand functions in the benchmark case), and that is what is relevant for this discussion. The rhizomatic noise trader is not directly interested in this equilibrium condition per se, but on what underlying relationship it generates between his or her own demand and the equilibrium price.

To see this, define the homogeneous function \(G\)

\[
G = qD^Z_t + (1 - q) E_t \left[ D^R_t (p_t) \right] - A
\]

and implicitly differentiate, assuming that derivatives can be taken within the expected value

\[
\frac{\partial p_t}{\partial D^Z_t} = -\frac{1}{\frac{\partial G}{\partial p_t}} \frac{\partial G}{\partial D^Z_t}
\]

\[
= -\frac{q}{(1 - q) E_t \left[ \frac{\partial D^R_t}{p_t} \right]} \tag{6}
\]

\(^{16}\)It is assumed that there is only one such \(q\) in the economy, an assumption which will be relaxed later on by considering that this parameter may be idiosyncratic

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We now assume that rhizomatic noise traders are aware of the functional form of other players’ demands. One can motivate this assumption by thinking that rhizomatic noise traders were initially rational, or even noise traders. They are aware that other players, who do not behave rhizomatically, take others’ actions as given. Rhizomatic noise traders are regular investors who believe "they know best" the inner workings of the market structure. Thus, it is as if rhizomatic noise traders were eventually "enlightened" by the belief that other players could be doing exactly the same as they do. This provides the rationale for rhizomatic noise agents’ knowledge of structure and specification of other players’ demands, especially when one considers that the objective function maximised by a rhizomatic noise agent is identical to that of a noise trader, save for the assumption that \( p_t (D_t^Z) \), that demand can influence the equilibrium price. From this assumption, and from equation 1 we get that

\[
\frac{\partial D_t^R}{p_t} = -\frac{1 + r}{2\gamma \sigma_{p,t}^2}
\]

All these variables and parameters are known at time \( t \), allowing us to remove the conditional expectation operator from expression 6

\[
\frac{\partial p_t}{\partial D_t^Z} = q \frac{1 + r}{(1 - q)2\gamma \sigma_{p,t}^2}
\]

Note that this approach is robust to measurement errors of other types’ sensitivity to demand, as long as the error has zero expected value, conditional at \( t \).

Plugging this newly calculated derivative in the expression for rhizomatic noise traders’ demand allows us to determine the final expression

\[
D_t^Z = \frac{r + E_t (p_{t+1}) - (1 + r)p_t + \eta_t}{2\gamma \sigma_{p,t}^2 + (1 + r)q \frac{1 + r}{(1 - q)2\gamma \sigma_{p,t}^2}}
\]

\[
= \left[ \frac{r + E_t (p_{t+1}) - (1 + r)p_t + \eta_t}{2\gamma \sigma_{p,t}^2} \right] (1 - q)
\]

(7)

It is clear that the demand of rhizomatic noise agents equal the demand of noise traders in the benchmark model, weighted by \((1 - q)\). This new term scales down demand by the proportion of agents that are believed to act in the same way. This already highlights what is bound to be one of the main roles of the term \( q \): limiting the Friedman effect. If traders believe a large proportion of the market to behave as they do, they are not likely to expose themselves too much to the risky asset even if the

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noise term \( \eta_t \) is high. Rhizomatic noise agents, in this case, believe that a significant share \( q \) of the market also expects high future prices, thus having the incentives to buy and pressuring the price downwards, therefore destroying any price-difference returns that may be extracted from the risky asset. We are therefore introducing individual awareness of the second-order effects of the impact of widespread aggregate noise and reducing the noise trader model’s vulnerability to the original argument put forward in Friedman (1953). An observation that will often be repeated but that is particularly striking in expression 7 is the fact that as \( q \to 0 \), the rhizomatic noise demand expression converges to the optimal action taken by (pure) noise traders.

4.2 Equilibrium

As in the benchmark model, aggregate demand can be written as the weighted average of representative agents’ demand expressions, due to the absence of individual randomness. In this case:

\[
D_t = \alpha D_t^Z + (1 - \alpha) D_t^R
\]

The market clearing condition remains

\[
\alpha D_t^Z + (1 - \alpha) D_t^R = A
\]

Plugging the expressions for each type’s demand, we get

\[
\alpha \left[ \frac{r + E_t(p_{t+1}) - (1 + r) p_t + \eta_t}{2 \gamma \sigma_{p,t}^2} \right] (1 - q) + (1 - \alpha) \left[ \frac{r + E_t(p_{t+1}) - (1 + r) p_t}{2 \gamma \sigma_{p,t}^2} \right] = A
\]

And, solving for the equilibrium price in recursive form

\[
p_t = \frac{1}{1 + r} \left[ r + E_t(p_{t+1}) + \frac{\alpha (1 - q) \eta_t}{1 - \alpha q} - \frac{2 \gamma A \sigma_{p,t}^2}{1 - \alpha q} \right]
\] (8)

Note that for \( q = 0 \), this equation reverts to expression 3. Another equilibrium conclusion that is already hinted at by this new expression is the fact that as \( q \to 1 \), the impact of noise disappears in the main market aggregates (price and variance). Another interesting fact is that whereas the original pricing condition, as in DeLong et al. (1990a), was linear in both noise and the proportion of noise traders, the new condition is nonlinear in both \( \alpha \) and \( q \). This is due to the aforementioned fact that \( q \) allows us to introduce second-order effects of noise in individual optimisation, which are then passed on to the market aggregates.
4.3 Equilibrium Dynamics

Assuming away bubbles, expression 8 can be solved recursively for the steady state price with a constant variance:

\[ p_t = 1 + \frac{\alpha}{1 - \alpha q} \left( \frac{\eta_t - \eta^*}{1 + r} + \frac{\eta^*}{r} \right) - \frac{2\gamma \Lambda \sigma^2_{p,t}}{r (1 - \alpha q)} \]

Once again, the only time-varying term in the expression is \( \eta_t \), so that the equilibrium variance can be expressed as:

\[ \sigma^2_{p,t} = \text{var}_t (p_{t+1}) = \frac{\alpha^2 (1 - q)^2 \sigma^2_\eta}{(1 - \alpha q)^2 (1 + r)^2} \]

We can now write the final expression for the equilibrium price in the steady state

\[ p_t = 1 + \frac{\alpha}{1 - \alpha q} \left( \frac{\eta_t - \eta^*}{1 + r} + \frac{\eta^*}{r} \right) - \frac{2\gamma \Lambda \alpha^2 (1 - q)^2 \sigma^2_\eta}{r (1 - \alpha q)^3 (1 + r)^2} \quad (9) \]

Note, once again, that the expression reduces to equation 4 for \( q = 0 \). The noise term operates in a way similar to the benchmark pricing equation, but the interaction with \( q \) must now be taken into account. As \( q \) increases, the impact of noise is greatly reduced. In particular, as \( q \) approaches 1, the effects of the mean distortion caused by the presence of noise traders in the market, and of the price volatility that is induced by deviations of \( \eta_t \) from its mean, tend to disappear. As \( q \) increases, noise is driven away from the market, independently of the value of \( \alpha \): this effect is, naturally, reflected in the steady state variance that also approaches zero as \( q \to 1 \). The level of market-wide rhizomatic beliefs mutes noise and the volatility it generated endogenously. This is due to the second-order effect of noise which is introduced by our behavioural assumption: as noise traders take into account the aggregate impact of their misperceptions and actions, they tend to be more careful in their trades and thus avoid the Friedman effect. This moderation of their positions shuts the transmission channel from individual noisy expectations to market prices. This highlights one of the main roles of the introduction of the assumption of rhizomatic beliefs in the model: without assuming away the presence of market-wide noise, as in DeLong et al. (1990a), it is possible to obtain an efficient price à la EMH. Thus our assumption of rhizomatic beliefs provides a benchmark where both EMH and noise traders are integrated into a broader asset pricing model. It is worth noting

\[ 17 \text{As in the benchmark model.} \]
that this effect is, apparently, a nonlinear one, as \( q \) appears both in the numerator and in the denominator of the "weighting" coefficient of noise in expression 9.

Let us first focus on the analysis of this coefficient, as it provides a good means to think of the higher order effects that are induced by the presence of rhizomatic beliefs in the market. Suppose that a generation of rhizomatic noise traders is extremely bullish, so that \( \eta_i > \eta^* > 0 \), for example. In the case of pure noise traders, the price would be immediately driven up by the increased demand, with only the risk premium moderating this price increase. Rhizomatic noise traders, on the other hand, will tend to moderate their positions, which is reflected in smaller price increases. This, however, decreases the risk of the asset, through the fact that \( \sigma_n^2 \) is now weighted by \( (1 - q) \) as well, so that it may create incentives for rational traders to increase the magnitude of their positions in the risky asset. This happens, naturally, if expected returns more than compensate the risk of investing in the asset.

It is relevant to notice that, regardless of the values of \( \alpha \) and \( q \), the sensitivity of price to noise can never be greater than in the benchmark case. Recall that, in the latter, it was simply given by \( \alpha \). It is easy to prove that

\[
\frac{\alpha (1 - q)}{1 - \alpha q} \leq \alpha \iff \alpha \leq 1
\]

which is always true, given the nature of the parameter \( \alpha \). The inequality is strict for \( q > 0 \). In what concerns the third and final term, the risk premium, the presence of nonlinear effects is once again evident. The logic is the same as the one pointed out above: while more rhizomatic noise traders tend to limit their positions (thus limiting the price variance), this also provides incentives for rational traders to boost their own demand (increasing the price variance). However, due to the stronger nonlinearity of these interaction effects (through a quadratic term in the numerator and a cubic term in the denominator), it is not possible, in principle, to establish a result analogous to the previous one. This motivates the following sub-section, which looks into the equilibrium variance term/risk premium in closer detail.

4.3.1 Equilibrium Variance

As presented above, the steady state equilibrium variance of the asset price is given by

\[
\sigma_p^2 = \frac{\alpha^2 (1 - q)^2 \sigma_n^2}{(1 - \alpha q)^2 (1 + r)^2}
\]
For $\alpha < 1$, it is clear that, in the limit, when $q \to 1$, the impact of noise in the variance of the asset disappears. This is the closing of the noise transmission channel mentioned above. As rhizomatic noise traders become more cautious in their positions, the asset becomes less risky. However, for this very same reason, returns tend to increase relative to the risk of the asset, which may induce additional demand, not only affecting rational traders, but rhizomatic noise traders themselves. As positions increase, and unless they are perfectly symmetric, risk should increase as well. These effects illustrate the nonlinearity of $q$ in the variance term.

In spite of this effect, it is possible to establish that the variance of the asset price in an environment with rhizomatic noise traders is never greater than the variance with pure noise traders. This condition is once again equivalent to $\alpha \leq 1$, thus being trivially satisfied. In fact, we have that

$$\frac{\partial (\sigma_p^2)}{\partial q} = \frac{2\alpha^2 (1 - q) \sigma_q^2 (\alpha - 1)}{(1 - \alpha q)^3 (1 + r)^2}$$

which is negative as long as $\alpha < 1$. Thus a greater $q$ always leads to a decrease in the variance term, as long as the economy is not entirely populated by rhizomatic noise traders. If the latter is the case, then the impact of $q$ on noise disappears. In fact:

$$\sigma_p^2|_{q=1} = \frac{(1 - q)^2 \sigma_q^2}{(1 - q)^2 (1 + r)^2} = \frac{\sigma_q^2}{(1 + r)^2}$$

If the economy is entirely populated by noise traders then there are no rational arbitrageurs taking advantage of the noise. Their presence is the reason why, in the benchmark case, the expression for the asset price variance is equal to the one above, but weighted by $\alpha^2 \leq 1$. Rational investors take advantage of noise by betting against it and thus contributing to the moderation of price fluctuations. In their absence, there is no moderating force. Price fluctuations thus fully reflect the degree of noise of the current generation of traders. It is also worth noting that, in this particular case, the influence of $q$ vanishes in equilibrium.

**Figure 1** represents the equilibrium variance of the risky asset as a function of $q$ for different levels of $\alpha$: 

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Figure 1 - Variance of the asset as a function of $q$

It also becomes clear that, as the proportion of rhizomatic noise traders increases, the equilibrium variance also increases. Note that for $q = 0$, we are in the benchmark case of pure noise versus rational traders, and this effect is also felt. In line with what we have seen before, as $q \to 1$ variance vanishes. The only exception is the aforementioned case where $\alpha = 1$. These dynamics are highlighted in the following figure, which plots the steady state equilibrium variance against both $q$ and $\alpha$, simultaneously.
The DeLong et al. (1990a) noise trader model is represented by looking at the cross section of the above graph for $q = 0$: what happens to the variance as $\alpha$ increases. The $q$-axis (for a fixed $\alpha$) embodies the dimension that we introduce in the model. In fact, Figure 1 represents cross-sections of Figure 2 for a fixed $\alpha^{18}$.

4.3.2 Equilibrium Price

Now that the variance (the risk premium) of the asset has been analysed in greater detail, let us return to the solution for the equilibrium price equation in the steady

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18 A pathological case arises when $q$ and $\alpha$ approach 1 simultaneously. While the variance is ensured to be finite as long as $\sigma_{\eta}^2 < \infty$ for all other cases, the same is not true for this one. Appendix 1 shows that the limit for the variance does not exist in this case. The case of $(q, \alpha) = (1, 1)$ is, however, an uninteresting one, hence we can take this as a mere technical detail.
state. As we have seen, and excluding the pathological and potentially uninteresting case of $\alpha = 1 \land q = 1$, both the impact of noise and the risk-premium are decreasing in $q$. This suggests opposing effects of $q$ on the equilibrium price. Think of the case in which rhizomatic noise traders are, on average, bullish so that $\eta^* > 0$. In such a situation, a greater $q$ attenuates the (positive) impact of noise on the price, so that the second term decreases. On the other hand, as we have seen, it also leads to lower equilibrium variance and thus to a lower risk premium that enters the price equation with a minus sign, leading to a price increase. Therefore, the net impact of $q$ in the equilibrium price is, in principle, ambiguous. While the extreme cases, $q = 0$ and $q = 1$, provide us with the noise trader benchmark and with the efficient markets case, respectively, the interior dynamics are far from trivial. In fact, from expression 9, the derivative of the equilibrium price with respect to $q$ is given by

$$\frac{\partial p_t}{\partial q} = \alpha (\alpha - 1) \left( \frac{\eta_t - \eta^*}{1 + r} + \frac{\eta^*}{r} \right)_{<0} + 2\gamma A \alpha^2 \sigma^2 \left( \frac{(1 - q)}{r (1 + r)^2 (1 - \alpha q)}_{\geq 0} \right) [2 + \alpha q - 3\alpha]$$

(10)

highlighting the role of nonlinear dynamics. This expression tells us that changes in $q$ affect the price in two major ways: through the mean noise term (first term) and through the risk premium (second term). Focusing on the interior cases, for $q \in (0, 1)$, one can establish, at a minimum, monotonicity of the price behaviour with respect to $q$ under certain circumstances. In particular, note that regardless of the assumptions imposed on noise behaviour, the first component of the first term in expression 10 is always negative, while the first component of the second term is always positive.

For us to ensure that the relationship between $p_t$ and $q$ is negative, component $A$ should be positive, whereas component $B$ should be negative. The conditions for a positive relationship are symmetric. For component $A$ to be positive:

$$\left( \frac{\eta_t - \eta^*}{1 + r} + \frac{\eta^*}{r} \right) > 0 \iff r\eta_t + \eta^* > 0$$

So that traders should be sufficiently bullish, either on average or in the current period, for the impact of noise on price to be positive. Bearish behaviour is responsible for inducing nonlinearities. This can be shown to be linked to the already mentioned lack of symmetry in returns (with respect to buying and short-selling, which are treated as symmetric activities in this model) and will be further developed in the following sections. For component $B$ to be negative

$$2 + \alpha q - 3\alpha < 0 \iff \frac{2}{3 - q} < \alpha$$

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Thus the proportion of rhizomatic noise traders must be high enough in order to ensure a monotone, negative relationship. Note that we are only pinning down the situation in which we can ensure that $\partial p_t / \partial q < 0$. Symmetric conditions ensure a positive relationship between the price and $q$. However, in general, interior dynamics will vary according to the specific values taken by the parameters and the stochastic term $\eta_t$.

As highlighted in the preceding analysis, these nonlinearities arise not only due to stochastic noise (the first component), but also due to the interaction between the proportion of rhizomatic noise traders $\alpha$ and their rhizomatic beliefs, $q$. The latter effect is due to the fact that, through the interaction terms, $\alpha$ scales the magnitude of the impact of changes in $q$ on the mean noise component of the price and on the risk premium of the asset. Thus, for different levels of $\alpha$, an increase in $q$ may lead to a fast decline of, say, the variance, leading to increased demand for the risky asset by both rational and rhizomatic noise traders. Besides, and as it has been remarked before, rhizomatic noise traders’ demands are highly nonlinear in the sense that, everything else constant, while an increase in $q$ makes them, on average, more cautious and leads them to adopt more moderate positions, this increase also reduces the risk of the asset, thus establishing the incentives for those positions to be reinforced. Given that rational traders attempt to exploit the arbitrage opportunities that are generated by noise, such nonlinear dynamics are transmitted to their own demands. While an increase in $q$ decreases risk, thus boosting their demand, it also reduces the transmission of noise to the price (due to rhizomatic noise trader cautiousness), which effectively limits the extent to which they can earn positive returns by arbitrage. All this considered, it is not surprising to observe that aggregate price dynamics are so dependent on the interaction between these two terms.

Figure 3 plots the equilibrium price as a function of $q$ for different levels of $\alpha$ and for $\eta_t^* > 0$. The solid lines correspond to $\eta_t = \eta_t^*$, while the dashed lines correspond to negative and positive shocks on $\eta_t$ equal to one standard deviation. This is done to provide some idea of where will equilibrium prices most likely lie in the presence of stochastic, time-varying noise.
Figure 3 - Equilibrium price as a function of $q$ for $\eta^* > 0$

Given the present calibration, price is always decreasing in $q$. As the proportion of rhizomatic noise traders rises, the dashed intervals become wider, given that this increase in $\alpha$ results in a higher variance for the asset price. However, as $q$ approaches one, not only does the asset price converge to the efficient fundamental price, but variance also decreases (the intervals between dashed lines get thinner). This picture describes many of the dynamics that have already been mentioned in the analysis of the price and variance expressions. Further note that the price at $q = 0$ corresponds to the pure noise benchmark price. The figure confirms that the introduction of rhizomatic beliefs provides the asset pricing model with a parameter that is able to connect the original benchmark with the EMH (for $q = 1$).

Figure 4 consists of the same exercise, but for $\eta^* < 0$: 

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Figure 4 - Equilibrium price as a function of $q$ for $\eta^* < 0$

In this case, the relationship between $p_t$ and $q$ is positive. The asset is priced under its fundamental value due to the fact that the mean noise term is now negative. There is a "bearish" premium over the risky asset that can be exploited by rational traders. As $q$ increases and bearish traders become more careful (through less short-selling of the asset, for example), the noise channel is muted, thus the price increases - converging to its fundamental value, which happens in the limit as in the case with $\eta^* > 0$.

It is also interesting to observe the plot of the equilibrium price versus $\alpha$ and $q$, represented in Figure 5 for $\eta^* > 0$ and $\eta_t = \eta^*$.
Once again, as in Figure 2, the cross-section for \( q = 0 \) corresponds to the standard DeLong et al. (1990a) noise trader model, and it can be seen that the equilibrium price is increasing, as one would expect, with the proportion of rhizomatic noise traders in the market. For a given \( q \), and from the fact that \( \alpha < 1 \), an increase in \( \alpha \) leads to an increase in price due to the fact that while the price rises at rate \( \alpha \) from the noise transmission channel (which pulls the price up, given that \( \eta^* > 0 \)), the risk premium only rises at rate \( \alpha^2 \) (which pulls the price down). Thus the net effect over the price is positive. Figure 3 essentially plots cross-sections of Figure 5 for fixed values of \( \alpha \).

### 4.4 Returns and Utility

At this point, it is interesting to analyse what happens in terms of utility and returns. Can rhizomatic noise traders, in spite of their noisy beliefs, earn greater returns than
rational traders? We show that this can indeed be true under certain circumstances. However, given that rational traders are maximising their positions with respect to the true distribution of the asset price, and with respect to the correct structure of market behaviour, one would expect that even under the possibility of lower returns than their rhizomatic noise counterparts, utility is always greater for rational agents. This is due to the fact that high returns are associated to a greater degree of risk that penalises welfare given the utility specification that is adopted in this model.

4.4.1 Equilibrium Demands

Demand expressions for rational and rhizomatic noise traders have been derived and presented as expressions 1 and 7. Recall that it was argued that since the rational traders’ objective function does not change when noise traders become rhizomatic, nothing changes in terms of their optimal demand. Equation 8, the recursive expression for equilibrium price, can be rewritten so that, in equilibrium:

\[ r + E_t (p_{t+1}) - (1 + r) p_t = \frac{1}{1-\alpha q} \left[ 2\gamma A\sigma_{p,t}^2 - \alpha (1-q) \eta_t \right] \]  

(11)

This expression can be plugged in the demand equations, in order to determine their equilibrium values:

\[ D_t^R (p_t^*) = \frac{1}{1-\alpha q} \left[ A - \frac{\alpha (1-q) \eta_t}{2\gamma \sigma_{p,t}^2} \right] \]

\[ D_t^Z (p_t^*) = \frac{1}{1-\alpha q} \left[ (1-q) A + \frac{(1-\alpha) (1-q) \eta_t}{2\gamma \sigma_{p,t}^2} \right] \]

where \( \sigma_{p,t}^2 \) stands for the constant price variance in the steady state. Clearly, rhizomatic noise traders’ demand varies positively with the noise term (as one would expect), while rational traders’ demand varies negatively. This reflects the role of rational traders as stabilising arbitrageurs who invest against the noise: if rhizomatic noise agents are extremely bullish, then rational traders acknowledge that the asset eventually becomes extremely overpriced and tend to demand less of it. In an extreme situation, this may lead to short-selling of the asset. Conversely, in the presence of a bearish generation of rhizomatic noise traders, rational investors recognise that the asset is underpriced, therefore purchasing it. However, for both types of traders the impact of noise is weighted by the variance term, reflecting the fact that how demands react to "news" (new noise) is crucially dependent on how risk-averse they are. Analysis of how these terms behave with respect to \( q \) must take into account the
variance term. Assuming that the economy is in the steady state, so that variance equals its steady state value yields the following expressions

\[
D_t^R(p_t^*) = \frac{1}{1 - \alpha q} A - \frac{(1 - \alpha q)(1 + r)^2 \eta_t}{2\gamma \alpha (1 - q) \sigma^2_\eta} \tag{12}
\]

\[
D_t^L(p_t^*) = \frac{1 - q}{1 - \alpha q} A + \frac{(1 - \alpha)(1 - \alpha q)(1 + r)^2 \eta_t}{2\gamma \alpha^2 (1 - q) \sigma^2_\eta} \tag{13}
\]

As with price and variance, demand expressions will be affected nonlinearly by \( q \). Once again, the interaction between the rhizomatic parameter and \( \alpha \) will matter. In particular, it will weigh the way how traders react to current noise. Note that demands have an exogenous, or constant, term (the first), which depends only on \( (A, \alpha, q) \); and an endogenous term, the second, which depends on both current noise and the variance of the price. It is the second term that interests us, mostly. As \( q \) tends to 1, for a fixed \( \alpha \), demands tend to explode either upwards or downwards, depending on whether \( \eta_t \) is positive or negative. Furthermore, they explode in opposite directions for different types of investors. Recall that as \( q \) converges to one, the impact of noise disappears from the market. As this effect takes place, variance (the risk premium) converges to zero. Therefore, traders, both rational and rhizomatic, start perceiving less risky arbitrage opportunities. Thinking of the original demand expressions, this is somehow analogous to the expected returns term (the numerator) being kept constant, while the risk term (the denominator) converges to zero.

As a consequence, demands explode: if \( \eta_t > 0 \), rhizomatic noise traders will take extremely long positions on the risky asset, with rational traders short-selling the asset to infinity. This happens because rhizomatic noise traders perceive a higher expected (selling) price at no cost of risk. Rational traders perceive a riskless arbitrage opportunity realised by short selling the asset. Traders move away from the riskless bond and take extreme positions in the risky asset. The reverse case occurs for \( \eta_t < 0 \), in which case pessimistic rhizomatic noise traders short-sell the asset while rational traders go infinitely long. It is clear from the demand expression that, for rhizomatic noise traders, the "riskless arbitrage" effect overcomes the cautiousness effect as \( q \to 1 \).

Once again, for \( \alpha = 1 \wedge q = 1 \) a pathological case arises, given that the endogenous component of equilibrium demand disappears, while the exogenous component explodes. Therefore, it is useful to restrict the analysis to the cases in which \( \alpha < 1 \), so that the exogenous component of equilibrium demand remains finite (and only the endogenous part is allowed to explode). The reason for why equilibrium demand functions are not well-defined for \( q = 1 \) is the same reason for why demands are
not defined either in the case in which all traders are rational. In the absence of endogenous volatility, and given that there is no fundamental volatility, the price of the asset reverts to its fundamental value, $p_t = 1$. This converts the risky asset into a perfect substitute for the riskless bond, so that, in the context of this model, it does not make sense neither conceptually nor analytically to define a demand function for it\(^{19}\).

**Figures 6** and **7** show how equilibrium demands evolve as a function for $q$, and for different levels of $\alpha$. Rational traders’ demand is represented as a dashed line, while rhizomatic noise traders have their equilibrium demand represented by the solid line. The above analysis is evident by considering the case in which $\eta_t > 0$ (**Figure 6**) and $\eta_t < 0$ (**Figure 7**).

**Figure 6 - Equilibrium demands as a function of $q$ for $\eta_t > 0$**

\(^{19}\)This question, along with that of exploding demands, can be conveniently solved with the assumption of fundamental noise. A simple exercise is to set a stochastic dividend for the risky asset, equal to $r + \varepsilon_t$, where $\varepsilon_t$ is i.i.d. white noise. In this case, neither demands nor returns will explode and most aggregates behave in a much smoother manner. By introducing exogenous volatility in the model, the risky asset no longer becomes a perfect substitute to the riskless bond.
Another aspect that is evident from Figures 6 and 7, but which has not been discussed in detail is demand asymmetry. In particular, for small populations of rhizomatic noise traders, their equilibrium demands appear to display larger magnitude when compared to rationals’ demands (for the same level of $q$). This would, in principle, not be the case, given that rhizomatic noise traders have their exogenous demand weighted by $1 - q$. However the variance of noise is discounted at $\alpha^2$ for them, while rationals discount it by $\alpha$ (in equilibrium). Therefore, for $\alpha < 1$, this generates less sensitivity to the variance of the asset. Essentially, rhizomatic noise traders appear, on average, to be willing to take greater risks than rational traders. This is a natural consequence from their biased expectations: they do not perceive to be engaging in greater risks even if, in equilibrium, they do so!

4.4.2 Expected Returns

The study of equilibrium demands allows us to advance towards a more relevant measure of the different types’ positions in the market: equilibrium returns. It is
particularly relevant to check whether it is possible to define the conditions under which rhizomatic noise traders can earn higher returns than rational traders. The net expected return from investing in the risky asset for a player of type $j$ when old, conditional at the time of investment, is given by

$$E_t^j (\pi_{t+1}^j) = D_t^j [r + E_t^j (p_{t+1}) - (1 + r) p_t]$$

Agents optimise over expected, not actual returns, hence their relevance. Using the relation in 11, one can write the expected return for a player of type $j$ as

$$E_t^j (\pi_{t+1}^j) = D_t^j \frac{1}{1 - \alpha q} \left[ 2 \gamma A \sigma_{p,t}^2 - \alpha (1 - q) \eta_t \right]$$

$$\sigma_{p,t}^2 = \frac{\alpha^2 (1 - q)^2 \sigma_n^2}{(1 - \alpha q)^2 (1 + r)^2}$$

replacing $\sigma_{p,t}^2$ for the steady state equilibrium variance

$$E_t^j (\pi_{t+1}^j) = D_t^j \left[ \frac{2 \gamma A \alpha^2 (1 - q)^2 \sigma_n^2}{(1 - \alpha q)^3 (1 + r)^2} - \frac{\alpha (1 - q) \eta_t}{1 - \alpha q} \right]$$

and using the equilibrium demand relations in 12 and 13, one gets

$$E_t^j (\pi_{t+1}^{R}) = \frac{2 \gamma A^2 \alpha^2 (1 - q)^2 \sigma_n^2}{(1 - \alpha q)^3 (1 + r)^2} - \frac{2 \alpha (1 - q) A \eta_t}{(1 - \alpha q)^2} + \frac{(1 + r)^2 \eta_t^2}{2 \gamma \alpha \sigma_n^2} \quad (14)$$

$$E_t^j (\pi_{t+1}^{Z}) = \frac{2 \gamma A^2 \alpha^2 (1 - q)^3 \sigma_n^2}{(1 - \alpha q)^4 (1 + r)^2} + \frac{(1 - q) (1 - 2 \alpha + \alpha q) A \eta_t}{(1 - \alpha q)^2}$$

$$- \frac{(1 - \alpha) (1 + r)^2 \eta_t^2}{2 \gamma \alpha \sigma_n^2} \quad (15)$$

Interpretation of these conditions is not straightforward and a more intuitive analysis follows from the expected return differential. Using 14 and 15

$$E_t^j (\pi_{t+1}^{Z}) - E_t^j (\pi_{t+1}^{R}) = \frac{(1 - q) (1 + \alpha q) A \eta_t}{(1 - \alpha q)^2} - \frac{(1 + r)^2 \eta_t^2}{2 \gamma \alpha \sigma_n^2} - q^2 \frac{2 \gamma A^2 \alpha^2 (1 - q)^2 \sigma_n^2}{(1 - \alpha q)^4 (1 + r)^2} \quad (16)$$

Note that the second and third terms are always negative. Therefore, whether rhizomatic noise traders can earn greater expected returns or not will depend on the sign of the first term. In particular, a necessary (but not sufficient) condition
for rhizomatic traders to display greater expected returns is for them to be bullish in the current period, i.e., $\eta_t > 0$. Only if they perceive the rational expectation for the future price as being an underestimate can they earn greater returns. The reason for this is fairly intuitive: an agent can only, on average, earn greater returns by investing in the risky asset. Rhizomatic noise traders only invest more in the risky asset than rational traders if their misperception regarding the future price is positive. The second term refers embodies the Friedman effect: as the variance (and the risk) of the asset increases, rational traders move away from it, leaving rhizomatic noise traders alone in the market for the asset. Given that noise is market-wide, and each generation of rhizomatic noise traders is subject to the same noise term, they all undertake the optimal action at the same time. This means everyone selling at the same time or everyone buying at the same time. The fact that rational traders flee from the increased risk means that there is no counter-weight in the market to eliminate the market pressure that is caused by the rhizomatic noise traders’ actions, thereby leading to a reduction in their own expected returns.

The third term along with the first absorb all nonlinearities that are added by the assumption of rhizomatic beliefs. Note that the third term is always negative, and disappears when $q = 0$. Intuitively, this term embodies a cost for rhizomatic noise traders - the cost of being too cautious regarding the risky asset. The reason for this is that by taking into account the effects of market-wide noise, and trying to evade the Friedman effect, rhizomatic noise traders tend to reduce the magnitude of their positions over the risky asset. This means that they will tend to hold less, on average, of the risky asset when compared to a pure noise trader (this follows immediately from their demand function), thus foregoing the opportunity to earn high returns when $\eta_t$ is high and positive. Note, however, that while this term is negative, it depends non-linearly on $q$ and the way it depends on $q$ is contingent, once again, on the interaction of this parameter with $\alpha$. Numerically, it can be shown that this term depends positively (and monotonically) on $q$ under reasonable calibrations for the model parameters. Therefore, the expected return differential is decreasing on $q$ through this term. The level of rhizomatic beliefs also affects the first term, though. It can be shown that the derivative of this term will be positive if and only if

$$\eta_t > 0 \land \frac{3\alpha - 1}{3\alpha (1 - \alpha)} > q$$

or

$$\eta_t < 0 \land \frac{3\alpha - 1}{3\alpha (1 - \alpha)} < q$$

Thus under circumstances, for certain values of $(\eta_t, \alpha, q)$, the first term can actually
be increasing in $q$. It can also be shown that this may reflect in a net positive variation of the expected return differential with $q$. The most interesting case is the one in which expected returns from rhizomatic noise traders exceed those of rational traders and the differential still varies positively with $q$. Given that $\eta_t > 0$ is a necessary condition for the differential to be positive, for this to happen we must have $q < \frac{3\alpha-1}{3\alpha(1-\alpha)}$; that is, a low enough $q$. In this case, rhizomatic noise traders are cautious, but not too cautious as to jeopardise the return opportunities that can be earned by trading the risky asset.

Finally, what happens as $q \to 1$? Expression 16 tells us that the first and third term would vanish. This would leave us with

$$E_t(\pi_{t+1}) - E_t(\pi_{t+1}^R)|_{q=1} = -\frac{(1+r)^2 \eta_t^2}{2\gamma \alpha \sigma_y^2}$$

which is negative as long as $\eta_t < 0$. Thus, in the limit, as rhizomatic noise traders (wrongly) perceive that the whole market behaves like them, rational traders will beat them in terms of expected returns. As we have seen, in this limit, the perception of riskless arbitrage overcomes the cautiousness effect, prompting infinite positions on both sides of the market. Rational traders beat rhizomatic ones because they hold the correct, rational expectation regarding the future price. Rational traders take infinite positions based on correct beliefs, while those of rhizomatic noise traders are based on biased expectations. Thus the only case in which the differential becomes non-negative is when $\eta_t = 0$ - when rhizomatic noise traders abandon their noisy beliefs and approach rational expectations (and, even then, the return differential becomes zero, as both types of agents are investing with the same type of expectations, thus essentially adopting the same strategies).

Expected returns for $\eta_t > 0$ are plotted in Figure 8. The above described conditions for a positive return differential increasing in $q$ ($\eta_t > 0$ and low enough $q$) can be immediately observed in the plot.
Figure 8 - Expected returns as a function of $q$ for $\eta_t > 0$

Figure 9 plots expected returns for $\eta_t < 0$. Once again, the intuitive results of the above analysis hold, by noting that the expected return differential never attains the positive orthant, and that it is increasing for high enough values of $q$. 

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Figure 9 - Expected returns as a function of $q$ for $\eta_t < 0$

Finally, Figure 10 and Figure 11 provide a hopefully more intuitive interpretation of the dynamics of returns as a function of $q$, by using expression 16 to plot the (conditional) expected return differential between rhizomatic noise and rational traders for both cases, $\eta_t > 0$ and $\eta_t < 0$. Once again, it becomes evident that the return differential can only be positive in the case where $\eta_t > 0$. 
Figure 10 - Expected return Differential as a function of $q$ for $\eta_t > 0$
4.4.3 Realised Returns

Possibly more relevant to the welfare analysis is the discussion of actual realised returns. Even though agents decide based on expected returns, their utility (when old) depends on realised ones. For a current generation of young traders, $j \in \{R, Z\}$, realised returns when old are given by

$$\pi_t^{j} = D_j^i [r + p_{t+1} - (1 + r) p_t]$$

while equilibrium demands are given, as before, by expressions 12 and 13, the expression for realised returns is no longer totally determined by the current equilibrium price (as in the case of expected returns). Instead, and assuming that the price follows its steady state equilibrium path, as in equation 9, we have that

$$r + p_{t+1} - (1 + r) p_t = \frac{\alpha(1-q)}{(1-\alpha q)(1+r)} [\eta_t^{j} - \eta^* - (1+r) \eta_d] + \frac{2\gamma A \alpha^2 (1-q)^2 \sigma_{\eta}^2}{(1-\alpha q)^3 (1+r)^2}$$

(17)
Actual returns depend positively on the risk-premium of the asset through the second term. The first term relates to returns that can be earned through arbitrage of intergenerational noise: if the next generation of noise traders is more bullish than the current one (and this is not accounted by the mean noise), this term becomes positive, as the asset price tends to increase in the "selling" period (when the current generation of young traders become old, at \(t + 1\)).

Using 12, 13 and 17, this means that realised returns for each type of agents are given by

\[
\pi_{t+1}^R = \frac{\alpha (1 - q) A}{(1 - \alpha q)^2 (1 + r)} \left[ \eta_{t+1} - \eta^* - 2 (1 + r) \eta_t + \frac{2 \gamma A \alpha (1 - q) \sigma^2_{\eta}}{(1 - \alpha q)^2 (1 + r)} \right] \\
- \frac{1 + r}{2 \gamma \sigma^2_{\eta}} \left[ (1 + r) \eta_t - \eta^* - (1 + r) \eta_t \right]
\]

\[
\pi_{t+1}^Z = \frac{\alpha (1 - q) A}{(1 - \alpha q)^2 (1 + r)} \left[ \eta_{t+1} - \eta^* - (1 + r) \eta_t + \frac{2 \gamma A \alpha (1 - q) \sigma^2_{\eta}}{(1 - \alpha q)^2 (1 + r)} \right] \\
+ \frac{1 + r}{2 \gamma \alpha \sigma^2_{\eta}} \left[ (1 + r) \eta_t - (1 + r) \eta_t \right]
\]

With the realised return differential being given by

\[
\pi_{t+1}^Z - \pi_{t+1}^R = \frac{\alpha (1 - q) A}{(1 - \alpha q)^2 (1 + r)} \left[ \left( \frac{1 + \alpha q}{\alpha} \right) (1 + r) \eta_t - q \left( \eta_{t+1} - \eta^* \right) - q \frac{2 \gamma A \alpha (1 - q) \sigma^2_{\eta}}{(1 - \alpha q)^2 (1 + r)} \right] \\
+ \frac{1 + r}{2 \gamma \alpha \sigma^2_{\eta}} \left[ (1 + r) \eta_t - \eta^* - (1 + r) \eta_t \right]
\]

The presence of \(\eta_{t+1}\) and its interaction with \(\eta_t\) and \(\eta^*\) complicate interpretation of this expression. In order to remove these elements, one can analyse a more general version of this object by taking unconditional expectations:

\[
E \left( \pi_{t+1}^Z - \pi_{t+1}^R \right) = \frac{(1 + \alpha q) (1 - q) A}{(1 - \alpha q)^2} \frac{(1 + r) (2 + r)}{2 \gamma \alpha \sigma^2_{\eta}} \left( \eta^* \right)^2 - \frac{(1 + r)^2}{2 \gamma \alpha} \frac{2 \gamma \alpha^2 A^2 (1 - q)^2 \sigma^2_{\eta}}{(1 - \alpha q)^4 (1 + r)^2}
\]

As before, taking unconditional expectations evidences that greater returns can only be attained by rhizomatic noise traders if the noise term \(\eta^*\) is positive and sufficiently high to overcome the greater risk that is associated to greater expected returns. Also note that in the benchmark case \(q = 0\), this differential would be unambiguously increasing on \(\sigma^2_{\eta}\), the variance of noise. In this case, the effect is clear: as the
variance of noise increases, rational traders refrain from arbitraging noise trader positions, hence allowing the latter to earn greater returns whenever the following generation of traders is more bullish. For \( q > 0 \), the fourth term appears, so that the impact of the variance of noise becomes nonlinear due to the caution that rhizomatic noise traders now exhibit: while the lack of arbitrage from rational traders boosts their (unconditional) expected returns, awareness of the Friedman Effect may also contract their own positions, thereby forfeiting the opportunity of earning greater returns in cases such as the one described before.

4.5 Utility

The previous subsection highlighted that it is possible, under certain circumstances, for rhizomatic noise traders to earn higher returns than rational traders. The question now is whether this translates in greater utility or not. Note that while earning greater returns, rhizomatic noise traders are also incurring in more risk. That is the reason why rational traders do not earn those greater returns in the first place: while balancing the risk-return trade-off through their optimal portfolio allocation, they realise that the additional risk is not worth the potentially higher returns. As in the benchmark case, rational traders see their utility increase by the presence of traders who are not rational. In a perfectly efficient and rational market, the riskless bond and the risky asset are perfect substitutes - rational traders would have access to what would effectively be a single asset. The introduction of noise, or rhizomatic noise traders effectively creates a new asset, thus the opportunity set of rational traders is expanded, increasing their utility by a revealed preference argument.

It can be argued that the analysis and comparison of expected utility is an uninteresting exercise, given that each type maximises expected utility by construction. A more insightful approach is to compute a composite measure that combines \textit{ex-post} utility (equal to the realised returns) with the \textit{ex-ante} penalty for risk. This measure depends positively on realised returns and negatively on the expected variance of these returns. For each type of traders, this measure is expressed by

\[
U_{t+1}^R = \pi_{t+1}^R - \gamma \left[ D_t^R (p_t^*) \right]^2 \sigma_{p,t}^2,
\]

\[
U_{t+1}^Z = \pi_{t+1}^Z - \gamma \left[ D_t^Z (p_t^*) \right]^2 \sigma_{p,t}^2,
\]

where \( D_t^R (p_t^*) \) and \( D_t^Z (p_t^*) \) are the equilibrium demands computed in 12 and 13. It ought to be noted that this involves manipulation of the realised returns expressions that, as we have seen in the previous subsections, are quite cumbersome and difficult
to interpret. Given that we are more interested in relative welfare between rhizomatic noise and rational traders (and, in particular, if rhizomatic noise traders can realise higher utility than rational ones), we can work with the realised utility differential:

\[ U_{t+1}^Z - U_{t+1}^R = (\pi_{t+1}^Z - \pi_{t+1}^R) - \left\{ \left[ D_t^Z (p_t^*) \right] - \left[ D_t^R (p_t^*) \right] \right\} \gamma \sigma_{p,t}^2 \]

where \((\pi_{t+1}^Z - \pi_{t+1}^R)\) has been computed in 19. Plugging in the expression, and replacing for equilibrium demands and variance:

\[ U_{t+1}^Z - U_{t+1}^R = \frac{\alpha q (1 - q) A}{(1 - \alpha q)^2 (1 + r)} \left[ \frac{(1 + r) \eta_t}{\alpha} - \eta_{t+1} + \eta^* - \frac{q \gamma A \alpha (1 - q) \sigma^2_{\eta}}{(1 - \alpha q)^2 (1 + r)} \right] \]

\[ + \frac{(1 + r) \eta_t}{2 \gamma \alpha \sigma^2_{\eta}} \left[ \eta_{t+1} - \eta^* - \frac{(1 + r)}{2 \alpha} \eta_t \right] \]

Taking unconditional expectations, we get

\[ E \left( U_{t+1}^Z - U_{t+1}^R \right) = \frac{q (1 - q) A}{(1 - \alpha q)^2} \left[ \eta^* - \frac{\gamma A \alpha^2 q (1 - q)}{(1 - \alpha q)^2 (1 + r)} \sigma^2_{\eta} \right] - \frac{(1 + r)}{2 \gamma \alpha \sigma^2_{\eta}} (\eta^*)^2 - \frac{(1 + r)^2}{4 \gamma \alpha^2 \sigma^2_{\eta}} (\eta^*)^2 + \sigma^2_{\eta} \]

It is interesting to note that if \(q = 0\), the whole expression becomes negative. However, as long as \(q > 0\), the first term is positive, thereby allowing, in principle, for the possibility of rhizomatic noise traders attaining higher utility than rational traders, in expectational terms. Further note that the first two terms are the only ones dependent on \(q\) - these are the terms that arise from rhizomatic beliefs and through which the Friedman Effect is taken into account. On the one hand, the first term boosts returns, by accounting for the fact that rhizomatic noise traders do, after all, trade on noise - but that this can be a good thing if they are cautious about the effects of noise. The second term embodies the bad thing about carefulness, by weighting the variance of noise itself with a non-linear effect of \(q\). The last two terms correspond to the standard utility losses that arise from the fact that rational traders optimise given the true distribution of prices and returns, while rhizomatic noise and (pure) noise traders do not. Note that these two terms are independent from \(q\) and also appear in the benchmark noise-trading model. In the benchmark, noise traders would always realise lower utility than rational traders, while the outcome is not as straightforward once rhizomatic beliefs are taken into account.

The key explanation for this qualitative deviation from the benchmark lies in the way that rhizomatic noise traders handle the risk-return trade-off. Being rhizomatic leads them to limit their positions to an extent that greatly minimises the impact of risk in utility, while not completely disregarding the possibility of greater returns that arises from the existence of noise in the market.
4.6 Does it Pay to be Rhizomatic?

Does a regular noise trader have the incentives, in terms of utility and/or returns, to behave rhizomatically? In principle, this question does not have a straightforward answer: on the one hand, rhizomatic noise traders do take into account second-order effects of noise, which should, in principle, minimise their utility losses. However, this is done at the cost of introducing another cognitive bias, which further disrupts the regular process of expected utility maximisation.

First, let us see what happens in terms of realised returns. For the aforementioned reasons and due to their trading cautiously, one should expect rhizomatic noise traders to display, on average, lower returns. Using 18, the unconditional expectation of realised returns for a rhizomatic noise trader is

\[
E\left(\pi_{i+1}^Z\right) = \frac{(1 - q) A\eta^*}{(1 - \alpha q)^2} (1 - 2\alpha + \alpha q) + \frac{2\gamma A^2 \alpha^2 (1 - q)^3 \sigma^2}{(1 - \alpha q)^4 (1 + r)^2} \frac{(1 - \alpha) (1 + r)}{2} \frac{(\eta^*)^2 - (1 - \alpha) (1 + r)^2}{2\gamma \alpha \sigma^2} [\eta^*] + \frac{2\gamma A^2 \alpha^2 \sigma^2}{(1 - \alpha q)^4 (1 + r)^2} \left[\frac{\gamma (1 - q)^2}{2\gamma \alpha \sigma^2} \right]
\]

The dynamics of \( q \) are not easily interpreted through this expression. The last two terms do not depend on \( q \) and are unambiguously negative. In fact, taking the derivative of the above expression with respect to \( q \) can be shown to be

\[
\frac{\partial E\left(\pi_{i+1}^Z\right)}{\partial q} = \frac{A\eta^*}{(1 - \alpha q)^3} \left\{ -1 + \alpha \left[ 5 - 4\alpha - 3q (1 - \alpha) \right] \right\} + \frac{2\gamma A^2 \alpha^2 \sigma^2}{(1 + r)^2} \frac{(1 - q)^2}{(1 - \alpha q)^5} \left[\alpha (4 - q) - 3 \right]
\]

Showing that, once again, the direction of the nonlinearities will depend on the relative values of \( \alpha \) and \( q \). For high levels of \( \alpha \), for example, the sign of the first term is likely to be negative, whereas the sign of the second term will probably be positive. In order to better understand the underlying dynamics, Figure 12 plots the unconditional expectation of realised return for a rhizomatic noise trader as a function of \( q \), for \( \eta^* > 0 \) and \( \alpha = 50\% \).
Figure 12 - Realised Return of Rhizomatic Noise Traders as a function of $q$

For non-extreme values of $\alpha$, such as in the case of the above figure, realised return is a decreasing function of $q$. This effect is robust to the cases in which $\eta^* < 0$: even though additional nonlinearities are generated in that case (even for intermediate levels of $\alpha$, maximum expected return is always attained at $q = 0$)\textsuperscript{20}. In particular, realised returns attain a maximum at $q = 0$. The fact that realised returns attain this maximum highlights that pure noise traders, by exposing themselves more to market risk earn, on average, higher returns than their rhizomatic noise counterparts. It should be emphasised, however, that this is a particular case that arises from a

\textsuperscript{20}The only exception to this, for both $\eta^* > 0$ and $\eta^* < 0$ is when $\alpha$ attains exceptionally high values.
specific choice of numerical values for the relevant parameters. In general, virtually anything can happen.

Does this conclusion also extend to utility? This may not be the case due to the increasingly risky positions as \( q \) approaches 0. We take advantage of the fact that \( q = 0 \) corresponds to the pure noise trader case to shed some light over this question. From the already presented fact that

\[
U_{t+1}^Z = \pi_{t+1}^Z - \gamma \left[ D_t^Z (p_t^*) \right]^2 \sigma_{p,t}^2
\]

replacing for the equilibrium values of \( \pi_{t+1}^Z, D_t^Z (p_t^*) \) and \( \sigma_{p,t}^2 \) leaves us with the following expression:

\[
U_{t+1}^Z = \frac{(1 - \alpha)(1 + r) \eta_t}{2 \gamma \alpha \sigma_\eta^2} \left[ \eta_{t+1} - \eta^* - (1 + r) \eta_t \right] - \frac{(1 - q)(1 - \alpha) \eta_t}{2(1 - \alpha q)} + \frac{(1 - q)(1 - \alpha) A \eta_t}{(1 - \alpha q)^2}
\]

\[
+ \frac{\alpha}{(1 - \alpha q)^2 (1 + r)} \left[ \eta_{t+1} - \eta^* - (1 + r) \eta_t \right] - \frac{\gamma \alpha^2 A (1 - q)^3 \sigma_\eta^2}{(1 - \alpha q)^3 (1 + r)^2} + \frac{2 \gamma A^2 \alpha^2 (1 - q)^3 \sigma_\eta^2}{(1 - \alpha q)^4 (1 + r)^2}
\]

Its unconditional expectation being given by

\[
E \left( U_{t+1}^Z \right) = \frac{(1 - q)(1 - 2 \alpha + \alpha q) \eta^*}{(1 - \alpha q)^2} + \frac{\alpha^2 A \gamma (1 - q)^3 (2A - 1 + \alpha q)}{(1 - \alpha q)^3 (1 + r)^2} \sigma_\eta^2 - \frac{(1 - q)(1 - \alpha)}{2(1 - \alpha q)} \eta^*
\]

\[
- \frac{(1 - \alpha)(1 + r)(2 + r)}{2 \gamma \alpha \sigma_\eta^2} (\eta^*)^2 - \frac{(1 - \alpha)(1 + r)^2}{2 \gamma \alpha}
\]

(20)

The last two terms are independent of \( q \), whereas in the first three nonlinear effects are paramount. Once again, the way that expected utility varies with \( q \) depends greatly on the interaction between this parameter and others, namely \( \alpha \) and \( \eta^* \). To analyse utility differences between rhizomatic noise and (pure) noise traders in a more intuitive manner, note that by setting \( q = 0 \) one generates the ex-post composite utility for a pure noise trader (in equilibrium)

\[
E \left( U_{t+1}^Z \right)_{q=0} = (1 - 2 \alpha) A \eta^* + \frac{\alpha^2 A \gamma (2A - 1)}{(1 + r)^2} \sigma_\eta^2 - \frac{(1 - \alpha)(1 + r)(2 + r)}{2 \gamma \alpha \sigma_\eta^2} (\eta^*)^2
\]

\[
- \frac{(1 - \alpha)}{2} \eta^* - \frac{(1 - \alpha)(1 + r)^2}{2 \gamma \alpha}
\]

\[
\equiv E \left( U_{t+1}^N \right)
\]

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Taking the difference between 20 and this expression we have just derived:

\[
E \left( U_{i+1}^Z \right) - E \left( U_{i+1}^N \right) = \frac{q (1 - \alpha)^2}{2 (1 - \alpha q)} \eta^* + \frac{(1 - q) (1 - 2\alpha + \alpha q) - (1 - 2\alpha) (1 - \alpha q)^2}{(1 - \alpha q)^2} Aq^*
\]
\[
+ \frac{(1 - q)^3 (2A - 1 + \alpha q) - (2A - 1) (1 - \alpha q)^4}{(1 - \alpha q)^4} \frac{\alpha^2 A \gamma}{(1 + r)^2 \sigma^2 y}
\]

While the first term is always positive for interior values of \( q \) and \( \alpha \), as well as for \( \eta^* > 0 \), the sign of the second and third terms is ambiguous and depends on the interaction between the different parameters, namely \( \alpha \) and \( q \). It can be shown that, in general, rhizomatic noise traders are capable of realising, on average, greater utility than pure noise ones as long as \( \eta^* > 0 \). This effect arises from the increased cautiousness which makes them less exposed to price risk, while still enjoying the benefits of greater returns if the generations of non-rational traders are, on average, bullish. This effect is cancelled, however, as \( q \to 1 \), given that rhizomatic noise traders become excessively risk-averse and forfeit these greater returns due to the involved risk. Furthermore, as explained in previous sections, as \( q \) approaches 1 rhizomatic noise traders misperceive a riskless arbitrage opportunity as they believe the whole market to behave like them. Given that their expectations are biased, and they do not maximise over the true distribution of returns, their extreme positions in the limit are arbitrated by rational traders, thus resulting in considerable losses.

**Figure 13** plots the unconditional expectation of utility for a rhizomatic noise trader, expression 20. This calibration considers \( \eta^* > 0 \) and \( \alpha = 50\% \).
Figure 13 - Expected Utility for Rhizomatic Noise Traders as a function of $q$

Note that for the parameter values that were chosen to perform this calibration, it becomes clear that a positive $q$ can, in fact, yield greater utility to rhizomatic noise traders, when compared to their pure noisy counterparts ($q = 0$). It should be noted, however, that this is an equilibrium analysis, in the sense that we are comparing the utility of a representative rhizomatic noise trader in a world of rhizomatic noise (and rational) traders with the utility of a noise trader in a world of noise (and rational) traders. We are not checking whether a noise trader in a world populated by noise traders has the incentives to become a rhizomatic noise trader. This obviously assumes that the possibility of making such a choice exists, a hypothesis that may be inconsistent with the original theory of rhizomatic thinking, which in turn revolves around an innate cognitive bias or capacity.
Finally, a potentially useful and interesting exercise is to compute the $q^*$ such that:

$$q^* = \arg \max_{q \in [0,1]} \left\{ E \left( U_{t+1}^Z \right) \right\}$$

That is the value of $q$ that maximises expected (ex-post) utility, as a function of the parameters of the model, namely $\alpha$, whose interaction with $q$ is, as we have seen, extremely relevant for the present analysis. Appendix 2 sketches a proof in which it is shown that $q^* = 0$ holds for a particular set of parameters (i.e., it is optimal not to have rhizomatic beliefs). For all sets of parameters $(\alpha, \gamma, \eta^*, \sigma_B^2, A)$ that do not satisfy that condition, $q^* \neq 0$, hence it is optimal to behave rhizomatically to a certain extent, provided that $q^* \in [0,1]$.
5 Extensions and Future Work

After a thorough discussion of the dynamic and welfare properties of the model, this section presents and proposes two extensions for the original model that may be the subject of further work.

5.1 A Model of Idiosyncratic q’s

The introduction of heterogeneous levels of rhizomatic beliefs is an assumption that may, in some sense, precede the model itself. In fact, economic applications of rhizomatic thinking have always been implemented by allowing for individual levels of thinking for each agent, idiosyncratic q’s (Furtado and Cörte-Real, 2011; Chaves and Peralta, 2011). It is, indeed, a much more sensible assumption: to impose that each agent may be perceiving its own "market power", and the proportion of agents who behave like him or herself in a different way.

It is shown, however, that the conclusions of the previous model are robust to the structure of rhizomatic thinking in the economy: it is irrelevant, under certain conditions, whether this is an aggregate, market-wide or individual, heterogeneity-inducing phenomenon. The argument is very similar to the one presented in section 3.3.1, making use of the same technical tools.

To see this, assume that each rhizomatic noise agent of group Z is endowed with a different $q_i$. This means that there will be a continuum of random variables, $\{q_i\}$, which are assumed to follow some i.i.d. distribution $F$ with well-defined mean and variance. The expected value of this random variable $q_i$ is $\bar{q}$. It is irrelevant whether this distribution is stationary or not, as long as the random draws are independent and identically distributed at each period. Note that while the problem for rational traders remains unchanged, the only difference for rhizomatic noise traders is that the general market-wide $q$ is replaced, for the purpose of their own optimisation problem, for their individual $q_i$. This means that we now allow for heterogeneous rhizomatic noise demand functions

$$D^Z_t = \left[ \frac{r + E_t(p_{t+1}) - (1 + r) p_t + \eta_t}{2\gamma\sigma^2_{p,t}} \right] (1 - q_i)$$

How does this affect the market-clearing condition? As before, aggregate demand
equals aggregate supply

\[
\int_Z D_{t+1}^{Z,i} \, di + \int_R D_{t}^{R,i} \, di = A
\]

\[
\left[ \frac{r + E_t (p_{t+1}) - (1 + r) p_t + \eta_t}{2\gamma \sigma_{p,t}^2} \right] \int_Z (1 - q_i) \, di + \mu (R) D_{t}^{R} = A
\]

\[
\left[ \frac{r + E_t (p_{t+1}) - (1 + r) p_t + \eta_t}{2\gamma \sigma_{p,t}^2} \right] \left( \alpha - \int_Z q_i \, di \right) (1 - \alpha) D_{t}^{R} = A
\]

at this stage, we once again appeal to a law of large numbers for a continuum of random variables. The argument is the same, and this conclusion could be easily drawn by discretising the model, leaving all main results unchanged. Therefore, from the fact that \( \int_Z q_i \, di = \mu (Z) \bar{q} = \alpha \bar{q} \),

\[
\alpha \left[ \frac{r + E_t (p_{t+1}) - (1 + r) p_t + \eta_t}{2\gamma \sigma_{p,t}^2} \right] (1 - \bar{q}) + (1 - \alpha) D_{t}^{R} = A
\]

Note that once this crucial step is done, this particular case reverts to the general model. The equilibrium price and variance will depend on \( \bar{q} \) and not on the individual \( q_i \)'s, so that the above analysis is valid for the general aggregate variables of interest. The only potential difference lies, obviously, with individual demand functions and returns. There will now be a plethora of heterogeneous returns within the family of rhizomatic noise traders, due to the different perceptions of the market structure. In the end, however, any attempt at aggregation or establishment of a representative agent, who is endowed with the average or mean \( q_i \), results in a reversion to the original model. One should not forget, however, that some of the aggregates, such as the \textit{ex-post} distribution of wealth may display totally different dynamics and behaviour in this extension.

It is worth emphasising that the only prerequisite for this correspondence to be established is the distribution of \( q_i \) having a well-defined expected value - no other distributional assumptions are required. More than for presenting an extension, this section is used to demonstrate that the model which serves as the main focus of this paper can be seen as a generalised approach to the question of rhizomatic asset pricing.

### 5.2 \( q \) as a State Variable - a Model of Dynamic Rhizomes

So far, \( q \), either as an aggregate market-wide or as the expected value of idiosyncratic levels of rhizomatic beliefs, has been treated as a time-invariant parameter that does
not change with the succession of generations in the original framework. But, what if these exogenous beliefs held by rhizomatic noise traders on the proportion of agents taking the same action may vary over time? That is, young rhizomatic traders at time $t$ may have a certain $q$ which may evolve according to some law of motion, which is linked to how they (mis)perceive the interaction between their own actions and the aggregate market outcomes.

The parameter $q$ has, thus far, measured the proportion of agents in the market that a certain rhizomatic noise agents believes to behave like her or him. It is plausible to assume that this belief may change due to observation of market outcomes, and on the observations that the agent makes regarding his or her own perceived impact over these very same outcomes. In particular, suppose that the agent decided to purchase the risky asset, and aggregate demand pressure caused the price to increase. Faced with the increase, and the synchronicity of the agent’s action with the market outcome, he or she may be led to believe that the initial exogenous beliefs on the proportion of agents doing the same were indeed correct. Not only this, but the agent may feel tempted to reinforce his or her own beliefs regarding the proportion of the market that adopts a similar (demand) behaviour. Conversely, if the agent adopted a long position on the risky asset, but the equilibrium price decreases, there may be incentives to revise these exogenous (now endogenous) beliefs downwards - probably, most of the market is behaving in other, different manner and the considered $q$ was an overestimate.

The introduction of a dynamic level of rhizomatic beliefs does introduce a new layer of dynamics and interactions over the original model, as presented and developed in the preceding sections. From the conceptual point of view, some doubts may be raised concerning the introduction of this idea in the basic framework that is adopted. In fact, one may be tempted to argue that the introduction of a behavioural state variable is inconsistent with the basic premise that the model is populated by unconnected overlapping generations, whose agents are not linked in any way, either through bequests or social security systems. It is impossible to introduce a behavioural state variable without connecting, to some extent, two consecutive generations. This would, quite possibly and probably, lead to huge differences in the qualitative conclusions that have been presented so far. However, the case for a dynamic and persistent behaviour variable can nevertheless be made. Once can think, for example, that new generations of rhizomatic noise traders try to learn from the glories and failures of past generations. One can even think that generations are myopic, thus they only look at behaviour undertaken by the immediately preceding generation. They can also incorporate the notion that current old players’ behaviour results from
learning over many generations. If behaviour is persistent enough, all past knowledge is incorporated in the behaviour of the current generations of traders. This can, in fact, be seen as a model of learning, in which rhizomatic noise agents tend to refine their behaviour when faced with different market conditions and circumstances, by evaluating the consistency of their own individual actions with the aggregate market outcomes.

Asset pricing models of learning are abundant in the literature, even in OLG frameworks such as the one that serves as setting for this paper’s baseline model. In their seminal paper, DeLong et al. (1990a) draft a model of learning in which agents have the choice of becoming rational or noise depending on observed returns. In this setting, agents cannot choose their own type, but are able to adjust the parameter that governs their rhizomatic behaviour. Merz (2003), develops a two-period OLG model in which irrational young traders can learn to become rational old traders. However, in this context, learning is not a persistent intergenerational process. Zhang (2004) builds an asset pricing OLG model with Bayesian learning and belief updating. Using a baseline model very similar to this paper’s benchmark, Branch and Evans (2010) outline a model of asset pricing in which successive generations of traders learn how to forecast risk, and how can that lead to bubbles and crashes in asset prices.

The key underlying idea to the behavioural dynamics in this model should be the following: if a player buys (sells) and the price goes up (down), then the agent reinforces his or her own beliefs on the proportion of agents doing the same. If the player buys (sells) and the price goes down (up), then there are incentives for the agent to tend towards the benchmark belief of market atomicity, thereby reducing their $q$. The "Law of Motion of Rhizomatic Beliefs" is assumed to be some function $f$ such that

$$q_{t+1} = f(q_t, D_t^Z \Delta p_t) \in [0, 1]$$

$$\frac{\partial f}{\partial q_t} \frac{\partial f}{\partial (D_t^Z \Delta p_t)} > 0$$

where $\Delta p_t = p_t - p_{t-1}$ and its interaction with equilibrium demand $D_t^Z$ tries to capture the aforementioned effect: if the product between these two terms is positive (positive demand plus rising price, or negative demand plus falling price), then the agent reinforces its belief regarding the market structure, so that $q_{t+1} \uparrow$ with respect to $q_t$. There are some delicate aspects concerning the choice of a functional form for $f$. First of all, this function should only take values in the interval $[0, 1]$, where the
variable $q_t$ is defined. Second, the functional form should be such that it attributes more weight to the sign of the term $D_t^2 \Delta p_t$ than to its magnitude. As we have seen, in the original model, positions may explode in some limit cases (such as when $q \to 1$). The agent should be more concerned with whether he or she was right or wrong concerning the evolution of the aggregate price, than with how wrong he or she was. This term could also be weighted by some parameter $\varepsilon$ that measures the sensitivity of $q_t$ to the comparison of market outcomes to individual actions.

The basic problem for rational and rhizomatic noise traders remains unchanged, with the difference that instead of $q$, we now observe the presence of $q_t$ in the demand function of the latter.

$$D_t^2 = \left[ \frac{r + E_t (p_{t+1}) - (1 + r) p_t + \eta_t}{2\gamma \sigma_{p,t}^2} \right] (1 - q_t)$$

This means that the basic form of the market-clearing condition becomes unchanged, thus allowing us to write the equilibrium price in recursive form as

$$p_t = \frac{1}{1 + r} \left[ r + E_t (p_{t+1}) + \frac{\alpha (1 - \alpha q_t) \eta_t}{1 - \alpha q_t} - \frac{2\gamma \sigma_{p,t}^2}{1 - \alpha q_t} \right]$$

So that the model is now totally described by the following system of stochastic difference equations

$$\begin{cases}
q_{t+1} &= f \left( q_t, D_t^2 \Delta p_t \right) \\
p_t &= \frac{1}{1 + r} \left[ r + E_t (p_{t+1}) + \frac{\alpha (1 - \alpha q_t) \eta_t}{1 - \alpha q_t} - \frac{2\gamma \sigma_{p,t}^2}{1 - \alpha q_t} \right] \\
\eta_t &\sim N \left( \eta^*, \sigma_{\eta}^2 \right)
\end{cases}$$

which can then be solved to study the dynamic properties of the model. Solving this system is a non-trivial task, in the sense that: (i) The difference equation for the price is nonlinear on $q_t$; (ii) Even if a linear specification is chosen for $f$, the interaction between demand and the level change in prices yields more nonlinearities, furthermore generating a second-order difference equation in prices (depending on $E_t (p_{t+1}), p_t$ and $p_{t-1}$); (iii) Unless a steady state for $q_t$ is determined, it is in principle impossible to find a steady state for $p_t$ in which the distribution is time-invariant. This complicates determination of the one-period ahead variance $\sigma_{p,t}^2$, and it is likely that it becomes impossible to find a stationary distribution for $p_t$.

A non-stochastic steady state for this dynamic model would correspond to a triplet $(q^*, p^*, \eta^*)$. In particular, the noise term would have to be constant over time,
\( \eta_t = \eta^*, \forall t. \) In the steady state, \( \Delta p_t = 0, \) hence the time-invariant level of rhizomatic beliefs would satisfy \( q^* = f(q^*, 0). \) In what concerns the price, notice that due to the non-stochastic nature of this instance, its variance will be zero, \( \sigma_{p,t}^2 = 0, \) naturally eliminating the risk-premium. Solving for the steady state price in this context:

\[
p^* = 1 + \frac{\alpha (1 - q^*) \eta^*}{r (1 - \alpha q^*)}.
\]

However, due to the dynamic nature of the model, the economy will rarely find itself in this state. \( \eta_t \) is allowed to change every period, triggering changes in the price \( p_t \) and inducing the next generation of young rhizomatic noise traders to adjust their beliefs in the following period, \( q_{t+1}. \) In spite of not developing this extension analytically, an intuitive description of the model’s possible dynamics follows in the next paragraph.

Suppose that \( \eta_t > \eta^* \) at period \( t, \) so that a generation of particularly bullish traders is born. In this case, for a constant \( q_t, \) the increased demand pressure pushes the price upwards. Given that rhizomatic noise traders, for a constant \( q_t, \) are, on average, more bullish, they demand more of the asset (which is what causes the price to go up in the first place). Thus the interaction between demand and the price change at \( t \) is positive, prompting the next generation of traders to revise their beliefs upwards, \( q_{t+1} \uparrow. \) This increase in \( q_{t+1}, \) as we know from the analysis of the original model, results in a decrease in the equilibrium price. Further note that this decrease is felt in the current period, \( t, \) given the forward-looking nature of asset prices. Whether the final effect generates an upward or downward adjustment of the price will depend on how the model is calibrated, and on how the functional form of \( f \) is specified.

The interesting case is that in which, in the end, \( p_t \) does increase. Under specific assumptions for \( f, \) successive generations of traders with \( \eta_t > \eta^* \) may give birth to a feedback effect that culminates with an asset price bubble being generated. This comes in contrast to the original model, in which bubbles could be ruled out due to the i.i.d. nature of noise - price only changed if noise was unexpected, and even then the effects were only contemporary. By introducing a state variable in the model, \( q_t, \) we are inducing persistency of noise, which may account for abnormal and persistent movements in asset prices: bubbles and crashes. Thus a model of dynamic rhizomatic beliefs can be developed to provide cognitive and psychological explanations to some aggregate market phenomena that are not accounted by neither the EMH nor by the noise trading hypothesis.
6 Conclusion

In this paper we propose an extension of a noise asset pricing model, by explicitly considering the extent to which noise traders are aware that their bias is shared with other traders. While pure noise traders have been discarded as irrelevant or meaningless at least since Friedman (1953), DeLong et al. (1990a) have shown that, given a non-negligible mass of noise traders with common biases, their effect on the market does not disappear through arbitrage. We take a step further and introduce the assumption that noise traders that share a bias have at least some sense of the proportion of other traders whose behavior is motivated by the same bias. These rhizomatic noise traders take this inference into account when deciding when and how to trade. We thus account for the equilibrium impact of price feedback effects. Standard models of noise trading, while accounting for market sentiment as noise, do not incorporate the investors’ awareness of the second order effects of that noise. In particular, we model explicitly how people react to waves of market sentiment, and try to take advantage of them through trades. The main assumptions of this model are motivated by individual investor behaviour well-documented in the literature.

We uncover interesting effects of our assumption, namely on the price and variance of assets: as individuals believe that a larger share of traders shares the same bias, the standard noisy equilibrium, with price far from its fundamentals value and excess volatility, converges to a pattern that would be predicted by the Efficient Market Hypothesis. In other words, accounting for the newly assumed beliefs approaches market behavior under the Efficient Market Hypothesis: price converges to the value of fundamentals, and excess volatility disappears. The introduction of the parameter $q$, proportion of other traders believed to act under the same bias, allows us to establish an explicit continuum of equilibria that connects the standard noise trading equilibrium to the efficient rational expectations equilibrium. Our model thus includes the two opposing models of market behavior as particular cases.

While rhizomatic noise traders are capable, under certain circumstances, of earning greater returns than rational traders, this is not possible in terms of utility. We do show, however, that rhizomatic noise traders are, in general, better off than pure noise traders in equilibrium, thus providing a rationale for noise traders to start thinking rhizomatically. We also propose and explore the implementation of possible extensions, namely idiosyncratic and dynamic levels of rhizomatic thinking. This parameter, and our model, can be calibrated to potentially help explaining puzzles such as the equity premium and the excess volatility ones, while still considering equilibria whose workings are integrated into the standard Efficient Market Hypothesis.
framework.
7 Appendix

7.1 Non-existence of equilibrium variance limit when \((\alpha, q) \to (1, 1)\)

It can be easily shown that different approaches to this limit yield different results. In particular, approaching by \(\alpha \to 1\) first

\[
\lim_{(\alpha, q)\to(1,1)} \frac{\alpha^2 (1-q)^2 \sigma^2_{\eta}}{(1-\alpha q)^2 (1+r)^2} = \lim_{q^{-1}} \left[ \lim_{\alpha^{-1}} \frac{\alpha^2 (1-q)^2 \sigma^2_{\eta}}{(1-\alpha q)^2 (1+r)^2} \right]
\]

\[
= \lim_{q^{-1}} \frac{(1-q)^2 \sigma^2_{\eta}}{(1-q)^2 (1+r)^2}
\]

\[
= \frac{\sigma^2_{\eta}}{(1+r)^2}
\]

while approaching by \(q \to 1\) first yields

\[
\lim_{(\alpha, q)\to(1,1)} \frac{\alpha^2 (1-q)^2 \sigma^2_{\eta}}{(1-\alpha q)^2 (1+r)^2} = \lim_{\alpha^{-1}} \left[ \lim_{q^{-1}} \frac{\alpha^2 (1-q)^2 \sigma^2_{\eta}}{(1-\alpha q)^2 (1+r)^2} \right]
\]

\[
= \lim_{\alpha^{-1}} \frac{0}{(1+r)^2}
\]

\[
= 0
\]

which is sufficient to prove that the limit does not exist, unless \(\sigma^2_{\eta} = 0\). ■

7.2 Optimal Level of Rhizomatic Beliefs

We are interested in finding

\[
q^* = \arg \max_{q \in [0,1]} \{ E (U^Z_{i+1}) \}
\]

From the fact that

\[
E (U^Z_{i+1}) = \frac{(1-q)(1-2\alpha + \alpha q) A}{(1-\alpha q)^2} \eta^* + \frac{\alpha^2 A \gamma (1-q)^3 (2A - 1 + \alpha q)}{(1-\alpha q)^4 (1+r)^2} \sigma^2_{\eta} - \frac{(1-q)(1-\alpha)}{2(1-\alpha q)} \eta^*
\]

\[
- \frac{(1-\alpha)(1+r)(2+r)}{2\gamma a \sigma^2_{\eta}} (\eta^*)^2 - \frac{(1-\alpha)(1+r)^2}{2\gamma a}
\]

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We want to solve
\[
\max_q \left\{ E \left( U_{t+1}^Z \right) \right\}
\]
s.t.
\[
q \leq 1
\]
\[
q \geq 0
\]
We can construct the Lagrangian
\[
L = E \left( U_{t+1}^Z \right) (q) + \mu (1 - q)
\]
where \( \mu \) denotes the multiplier associated with the feasibility constraint \( q \leq 1 \). The Kuhn-Tucker conditions for this problem are
\[
\frac{\partial L}{\partial q} \leq 0 \land q \geq 0 \land \frac{\partial E \left( U_{t+1}^Z \right)}{\partial q} = 0
\]
\[
\frac{\partial L}{\partial \mu} \geq 0 \land \mu \geq 0 \land \frac{\partial L}{\partial \mu} = 0
\]
Normalising \( A = 1 \), we have that
\[
\frac{\partial L}{\partial q} = \frac{\partial E \left( U_{t+1}^Z \right)}{\partial q} - \mu \leq 0
\]
\[
\frac{\partial L}{\partial \mu} \geq 0
\]
Explicitly:
\[
\eta^* \frac{\alpha (5 - 4\alpha) - 3\alpha (1 - \alpha) q - 1}{(1 - \alpha q)^3} + \frac{\alpha^2 \gamma \sigma_q^2 (1 - q)^2}{(1 + r)^2} \frac{q \alpha (3\alpha - 5) + 5\alpha - 3}{(1 - \alpha q)^3} + \frac{\eta^* (1 - \alpha)^2}{2 (1 - \alpha q)^2} - \mu \leq 0
\]
\[
1 - q \geq 0
\]
Due to the nonlinear nature of this expression, it is difficult to compute the value of \( q^* \) analytically. We can, however, check when do we have \( q^* = 0 \). In this case, \( \mu = 0 \) given that the constraint is not binding. This will happen when the parameters are such that the following condition is satisfied:
\[
\eta^* \left[ \alpha (5 - 4\alpha) - 1 \right] + \frac{\alpha^2 \gamma \sigma_q^2 (5\alpha - 3)}{(1 + r)^2} + \frac{\eta^* (1 - \alpha)^2}{2} \leq 0
\]
or

\[ \eta^* \leq \frac{\alpha^2 \sigma_n^2 (3 - 5\alpha)}{(1 + r)^2 \left[ \alpha (5 - 4\alpha) - 1 + \frac{(1-\alpha)^2}{2} \right]} \]

For any value of \( \eta^* \) distinct from the ones above, \( q^* > 0 \). What we have generated is a condition such that, if satisfied, it does not pay to be rhizomatic. For any other value of \( \eta^*, q^* \), the level of rhizomatic thinking that maximises expected utility, is different from zero and lies in the interval \((0, 1]\), hence it does pay to be rhizomatic.
References


