

Matrices in $\mathcal{A}(R, S)$ with minimum t -term ranks

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Abstract

Let R and S be two sequences of nonnegative integers in nonincreasing order which have the same sum, and let $\mathcal{A}(R, S)$ be the class of all $(0, 1)$ -matrices which have row sums given by R and column sums given by S . For a positive integer t , the t -term rank of a $(0, 1)$ -matrix A is defined as the maximum number of 1's in A with at most one 1 in each column and at most t 1's in each row. In this paper, we address conditions for the existence of a matrix in $\mathcal{A}(R, S)$ that realizes all the minimum t -term ranks, for $t \geq 1$.

Keywords: t -term rank, $(0,1)$ -matrix, Gale-Ryser Theorem, Network flows

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1 Introduction

Consider the following problem: “We are organizing a social dinner for the participants of a mathematical meeting. The participants are from n different countries. We intend to seat the participants in m tables, each one with a fixed number of seats, so that in each table there are no two participants with the same nationality. The number of seats is equal to the number of participants and there is a possible distribution in these conditions. After dinner, there will be a show. Before the show, the host will call some participants for a joke. The host will choose the maximum number of participants selecting, at most, one per table and, at most, one per country. As we do not know when the artists will be ready for the show, the host will repeat the joke with another group, with the maximum number of participants, again choosing, at most, one per table and one per country, but the countries are now different from the countries selected in the first group. The host will repeat this joke until the show begins, or all countries were chosen. How can we distribute the participants in the tables so that after t rounds of jokes the total number of involved participants is as small as possible?”

The above problem is a classical problem of distributing n elements into m sets with some constraints. An important tool for solving these kind of problems are matrices whose entries are just 0's and 1's, the $(0, 1)$ -matrices. In fact, Ford and Fulkerson noted that a $(0, 1)$ -matrix can be regarded as distributing n elements into m sets: the 1's in row i designate the elements that occur in the i th set, and the 1's in column j designate the sets that contain the j th element (see [10]). Therefore, the $(0, 1)$ -matrices are essential tools in many combinatorial investigations, and hence these matrices are among those which have received more attention in the last years (see [1, 2, 6, 7, 8]). Consider a $(0, 1)$ -matrix which models a certain distribution of the participants into the tables. The number of selected participants after t rounds is a combinatorial parameter (properties which are invariant under arbitrary permutations of its rows and columns) of a $(0, 1)$ -matrix called *t -term rank*. The t -term rank of a $(0, 1)$ -matrix A , denoted $\rho_t(A)$, is the maximum number of 1's in A with at most one 1 in each column and at most t 1's in each row. When $t = 1$, we have the well-known term rank of A , denoted $\rho(A)$. Seeing a $(0, 1)$ -matrix as the incidence matrix of sets vs. elements, the t -term rank of a $(0, 1)$ -matrix is the maximum number of distinct elements that we have if we choose at most t elements in each set. So, a solution for the above problem is a $(0, 1)$ -matrix with prescribed row and column sum vectors, that realizes all the minimum t -term ranks,

for all $t \geq 1$, among all $(0, 1)$ -matrices with the same prescribed row and column sum vectors. In this paper, we focus our attention on this kind of $(0, 1)$ -matrices, that is, $(0, 1)$ -matrices with prescribed row sum and column sum vectors having the t -term rank equal to the minimum t -term rank, for all $t \geq 1$, among all $(0, 1)$ -matrices with the same constraints.

2 Background on $\mathcal{A}(R, S)$

Let $R = (R_1, \dots, R_m)$, and $S = (S_1, \dots, S_n)$ be two partitions of the same weight (this is, R and S are integral vectors such that $R_1 \geq \dots \geq R_m > 0$, $S_1 \geq \dots \geq S_n > 0$ and $R_1 + \dots + R_m = S_1 + \dots + S_n$). The *conjugate vector* of R is the partition $R^* = (R_1^*, R_2^*, \dots, R_{R_1}^*)$ defined by

$$R_j^* = |\{i : m \geq i \geq 1, R_i \geq j\}|.$$

The vector S is *majorized* by the vector R^* , [15], when

$$S_1 + \dots + S_i \leq R_1^* + \dots + R_i^*, \quad \text{for } 1 \leq i \leq \min\{R_1, n\}.$$

The class of all $(0, 1)$ -matrices with row sum vector R and column sum vector S is denoted by $\mathcal{A}(R, S)$. This class has been heavily investigated (see [1, 2, 5] for details) since the fifties, and many notable results have been obtained. The aim of this section is to make a brief sketch of these results.

A natural question is to determine when $\mathcal{A}(R, S)$ is nonempty. This problem was solved independently by Gale (see [11]) and by Ryser (see [14]). In fact, they proved a theorem, now called the *Gale-Ryser theorem*, which states that the class $\mathcal{A}(R, S)$ is nonempty if and only if S is majorized by R^* .

In the Ryser's paper, [14], an algorithm for the construction of a matrix in $\mathcal{A}(R, S)$ is also presented. This algorithm, called *Ryser's algorithm*, starts with an m -by- n $(0, 1)$ -matrix \bar{A} whose row sum vector is R and whose column sum vector is the conjugate vector R^* . Thus the 1's occupy the initial positions in each row of \bar{A} . The construction begins by shifting S_n of the last 1's of certain rows of \bar{A} to column n . The 1's in column n are in rows of \bar{A} with largest sum, giving preference to the bottommost positions in case of ties. Reducing by 1 those R_i corresponding to the rows that contain the 1's placed in column n , we obtain a vector R' which also satisfies the monotonicity assumption. We now proceed inductively to construct the columns $n-1, n-2, \dots, 1$. Ryser proved that if S is majorized by R^* , then this algorithm can be carried out in order to produce a matrix in $\mathcal{A}(R, S)$.

When $\mathcal{A}(R, S) \neq \emptyset$, the matrix \tilde{A} constructed by Ryser's algorithm is often called the *canonical matrix* of $\mathcal{A}(R, S)$.

Another fundamental result due to Ryser is the so called *Ryser's interchange theorem*, [2], which states that given two matrices $A, B \in \mathcal{A}(R, S)$, A can be transformed in B by a finite sequence of *interchanges*. An interchange is an operation which replaces a 2-by-2 submatrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{into} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

or vice-versa. Therefore, every matrix in $\mathcal{A}(R, S)$ can be transformed by a finite sequence of interchanges into the canonical matrix \tilde{A} of $\mathcal{A}(R, S)$.

Ford and Fulkerson, presented another criterium for the nonemptiness of $\mathcal{A}(R, S)$. Let $T(R, S) = T = [t_{i,j}]$ be the $(m+1)$ -by- $(n+1)$ matrix (the rows are indexed in $\{0, \dots, m\}$ and the columns are indexed in $\{0, \dots, n\}$), such that

$$t_{k,l} = kl - \sum_{j=1}^l S_j + \sum_{i=k+1}^m R_i, \quad (\text{for } 0 \leq k \leq m, \quad 0 \leq j \leq n).$$

This matrix is called the *structure matrix* associated with R and S , and Ford and Fulkerson proved that $\mathcal{A}(R, S) \neq \emptyset$ if and only if each entry of T is nonnegative.

Let t be a positive integer. Motivated by the study of combinatorial batch codes (see [3, 12, 13]), the authors of [5] defined the t -term rank of a $(0, 1)$ -matrix A , denoted $\rho_t(A)$.

By the König-Egerváry theorem (see [2], p.6) $\rho(A)$, the 1-term rank, equals the minimum number of lines that cover all the 1's of A :

$$\rho(A) = \min\{e + f : \exists \text{ a cover of } A \text{ with } e \text{ rows and } f \text{ columns}\}.$$

In [5] the authors established a generalization of this theorem:

Proposition 1 [5] *Let A be an m -by- n , $(0, 1)$ -matrix and let t be a positive integer. Then*

$$\rho_t(A) = \min\{te + f : \exists \text{ a cover of } A \text{ with } e \text{ rows and } f \text{ columns}\}.$$

The term rank and the t -term rank are two of several combinatorial parameters of a $(0, 1)$ -matrix. In this paper we turn our attention to the minimal t -term rank of a nonempty class $\mathcal{A}(R, S)$, denoted by $\tilde{\rho}_t(R, S)$. So,

$$\tilde{\rho}_t(R, S) = \min\{\rho_t(A) : A \in \mathcal{A}(R, S)\}.$$

A formula for computing the $\tilde{\rho}_1(R, S)$ was derived by Haber and simplified by Brualdi [2]. A generalization for $t > 1$ was stated by Fernandes and da Cruz in [9]. The formula for computing the $\tilde{\rho}_t(R, S)$, with $t \geq 1$, is obtained using the matrix Φ defined as follows: Let $[t_{i,j}]$ be the $(m+1)$ -by- $(n+1)$ structure matrix associated with R and S . Then the matrix $\Phi = [\phi_{k,l}]$, also denoted by $\Phi(R, S)$, is such that

$$\phi_{k,l} = \min\{t_{i_1, l+j_2} + t_{k+i_2, j_1} + (k-i_1)(l-j_1)\},$$

for all $0 \leq k \leq m$, $0 \leq l \leq n$, and the minimum is taken over all integers i_1, i_2 and j_1, j_2 that satisfy

$$0 \leq i_1 \leq k \leq k+i_2 \leq m, \quad \text{and} \quad 0 \leq j_1 \leq l \leq l+j_2 \leq n.$$

Proposition 2 [9] *Let R and S be partitions of the same weight such that the class $\mathcal{A}(R, S)$ is nonempty. Let t be a positive integer. Then,*

$$\tilde{\rho}_t(R, S) = \min\{te + f : 0 \leq e \leq m, 0 \leq f \leq n, \phi_{e,f} = t_{e,f}\}.$$

In [9] the authors proved the following theorem which, in particular, implies the existence of a special matrix in $\mathcal{A}(R, S)$ with t -term rank $\tilde{\rho}_t(R, S)$.

Theorem 3 [9] *Let R and S be partitions of the same weight such that the class $\mathcal{A}(R, S)$ is nonempty. Then, there is a matrix in $\mathcal{A}(R, S)$ all of whose 1's are contained in the union of its first e rows and first f columns if and only if $\phi_{e,f} = t_{e,f}$*

Our main purpose in this paper is to know if in any nonempty class $\mathcal{A}(R, S)$ there is a matrix that realizes all the minimum t -term ranks, for $t \geq 1$. The analogous problem for the maximum t -term rank in $\mathcal{A}(R, S)$ was solved in [5]. In fact, the authors proved that in any nonempty class $\mathcal{A}(R, S)$ there always exists a matrix A which realizes all the maximum t -term ranks, for $t \geq 1$, and the authors of [9] conjectured that the same happens for all the minimum t -term ranks, with $t \geq 1$.

Conjecture 4 [9] *If R and S are partitions of the same weight such that $\mathcal{A}(R, S)$ is nonempty, then there is a matrix $A \in \mathcal{A}(R, S)$ such that $\rho_t(A) = \tilde{\rho}_t(R, S)$, for all $t \geq 1$*

This paper is organized as follows: As we wrote in the Introduction, the $(0, 1)$ -matrices can be regarded as incidence matrices of sets vs. elements.

So, in the next section, we compute the t -term rank of a matrix using the Ford-Fulkerson algorithm. Despite what happens with the maximum t -term rank, in Section 4, we present a class $\mathcal{A}(R, S)$ where there is no matrix A that realizes all the minimum t -term ranks, for $t \geq 1$. However, with some restrictions on R and S , we prove that there are classes $\mathcal{A}(R, S)$ where such matrices exist. This is the subject of Section 5. The existence of these classes depends on the existence of a special matrix with prescribed zeros blocks. In Section 6, we present an algorithm for constructing $(0, 1)$ -matrices with these constraints. In Section 7, we give the conclusions of this paper, and we describe an open problem.

3 Network flows and the t -term rank

Another way to obtain the t -term rank of a $(0, 1)$ -matrix $A = [a_{ij}]$ is using network flows. Let $G = (X \cup Y, U)$ be the bipartite direct graph where $X = \{1, \dots, m\}$, $Y = \{m + 1, \dots, m + n\}$ and there is the edge (x, y) of G , with $x \in X$ and $y \in Y$, if and only if $a_{xy} = 1$.

Let $G_0 = (Z_0, U_0)$ be the graph obtained from G putting two new vertices, s and w , and the edges (s, p) , (f, w) , for $p = 1, \dots, m$, $f = m + 1, \dots, m + n$.

Let v_0 be the map from U_0 to $\{0, 1, \dots, t\}$ such that

$$v_0((c, d)) = \begin{cases} t & \text{if } c = s \\ 1 & \text{otherwise.} \end{cases}$$

The t -term rank can also be obtained using the Ford-Fulkerson algorithm [10]:

1. Let $i = 0$.
2. If there is a path P_i , in G_0 , from s to w ,

$$s, (s, x_{i_1}), x_{i_1}, (x_{i_1}, y_{i_1}), y_{i_1}, (x_{i_2}, y_{i_1}), x_{i_2}, \dots, y_{i_h}, (y_{i_h}, w), w,$$

such that

$$\begin{aligned} v_i((s, x_{i_1})) &> 0, \\ v_i((x_{i_f}, y_{i_f})) &= 1, \quad \text{for } f = 1, \dots, h, \\ v_i((x_{i_{f+1}}, y_{i_f})) &= 0, \quad \text{for } f = 1, \dots, h - 1, \\ v_i((y_{i_h}, w)) &= 1, \end{aligned}$$

then go to step 3. Otherwise, go to step 4.

3. Let v_{i+1} be the map from U_0 to $\{0, 1, \dots, t\}$ such that

$$v_{i+1}((c, d)) = \begin{cases} v_i((c, d)) - 1 & \text{if } (c, d) \in P_i \text{ and, if there is} \\ & l \in \{1, \dots, h-1\} \text{ such that} \\ & c = x_{i+1} \text{ then } d \neq y_i \\ 1 & \text{if } (c, d) \in P_i \text{ and, there is} \\ & l \in \{1, \dots, h-1\} \text{ such that} \\ & c = x_{i+1} \text{ and } d = y_i \\ v_i((c, d)) & \text{otherwise.} \end{cases}$$

Go to step 2, with $i + 1$.

4. Stop. $\rho_t(A) = i$.

Example 5 Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. Using the algorithm we can obtain

$\rho_2(A)$ as described:

Let $X = \{1, 2, 3\}$ be the set of rows of A and $Y = \{4, 5, 6, 7\}$ be the set of columns of A . Let G_0 be the direct graph whose set of vertices, Z_0 , is $\{s\} \cup X \cup Y \cup \{w\}$ and set of edges U_0 is $\{(s, 1), (s, 2), (s, 3), (1, 4), (1, 5), (1, 6), (1, 7), (2, 4), (3, 4)\} \cup \{(4, w), (5, w), (6, w), (7, w)\}$.

Let v_0 be the map from U_0 to $\{0, 1, 2\}$ such that

$$v_0((x, y)) = \begin{cases} 2 & \text{if } x = s \\ 1 & \text{otherwise.} \end{cases}$$

We begin the algorithm with $i = 0$ and the path

$$s, (s, 1), 1, (1, 5), 5, (5, w), w.$$

Let v_1 be the map from U_0 to $\{0, 1, 2\}$ such that

$$v_1((x, y)) = \begin{cases} 2 & \text{if } x = s \text{ and } y \neq 1 \\ 0 & \text{if } (x, y) = (1, 5) \text{ or } (x, y) = (5, w) \\ 1 & \text{otherwise.} \end{cases}$$

With $i = 1$, the path

$$s, (s, 1), 1, (1, 6), 6, (6, w), w$$

is a path which meets the conditions of step 2. Let v_2 be the map from U_0 to $\{0, 1, 2\}$ such that

$$v_2((x, y)) = \begin{cases} 2 & \text{if } x = s \text{ and } y \neq 1 \\ 0 & \text{if } (x, y) \in \{(1, 5), (1, 6), (5, w), (6, w), (s, 1)\} \\ 1 & \text{otherwise.} \end{cases}$$

With $i = 2$, the path

$$s, (s, 2), 2, (2, 4), 4, (4, w), w$$

is a path which meets the conditions of step 2. Let v_3 be the map from U_0 to $\{0, 1, 2\}$ such that

$$v_3((x, y)) = \begin{cases} 2 & \text{if } (x, y) = (s, 3) \\ 0 & \text{if } (x, y) \in \{(1, 5), (1, 6), (5, w), (6, w), (s, 1), (2, 4), (4, w)\} \\ 1 & \text{otherwise.} \end{cases}$$

Since with $i = 3$ there is no path which meets the conditions of step 2, then

$$\rho_2(A) = i = 3.$$

4 A counterexample for conjecture 4

Let $R = (6, 5, 4, 3, 3, 2, 2, 1, 1)$ and $S = (7, 3, 3, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$. The structure matrix, T , associated with R and S is

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	27	20	17	14	12	10	9	8	7	6	5	4	3	2	1	0
1	21	15	13	11	10	9	9	9	9	9	9	9	9	9	9	9
2	16	11	10	9	9	9	10	11	12	13	14	15	16	17	18	19
3	12	8	8	8	9	10	12	14	16	18	20	22	24	26	28	30
4	9	6	7	8	10	12	15	18	21	24	27	30	33	36	39	42
5	6	4	6	8	11	14	18	22	26	30	34	38	42	46	50	54
6	4	3	6	9	13	17	22	27	32	37	42	47	52	57	62	67
7	2	2	6	10	15	20	26	32	38	44	50	56	62	68	74	80
8	1	2	7	12	18	24	31	38	45	52	59	66	73	80	87	94
9	0	2	8	14	21	28	36	44	52	60	68	76	84	92	100	108

and the matrix Φ is

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	9	9	9	9	9	9
2	0	2	4	6	8	9	10	11	12	13	14	15	16	17	18	19
3	0	2	5	8	9	10	12	14	16	18	20	22	24	26	28	30
4	0	2	6	8	10	12	15	18	21	24	27	30	33	36	39	42
5	0	2	6	8	11	14	18	22	26	30	34	38	42	46	50	54
6	0	2	6	9	13	17	22	27	32	37	42	47	52	57	62	67
7	0	2	6	10	15	20	26	32	38	44	50	56	62	68	74	80
8	0	2	7	12	18	24	31	38	45	52	59	66	73	80	87	94
9	0	2	8	14	21	28	36	44	52	60	68	76	84	92	100	108

The grey entries are the first entries, in each column, that are equal in matrices T and Φ .

So, by Proposition 2

$$\tilde{\rho}_1(R, S) = 1 * 3 + 3 = 6,$$

$$\tilde{\rho}_2(R, S) = 2 * 3 + 3 = 2 * 2 + 5 = 9,$$

$$\tilde{\rho}_3(R, S) = 3 * 2 + 5 = 11,$$

$$\tilde{\rho}_4(R, S) = 4 * 2 + 5 = 4 * 1 + 9 = 13,$$

$$\tilde{\rho}_5(R, S) = 5 * 1 + 9 = 14,$$

$$\tilde{\rho}_6(R, S) = 6 * 1 + 9 = 6 * 0 + 15 = 15.$$

Suppose there is a matrix $A \in \mathcal{A}(R, S)$ such that $\rho_i(A) = \tilde{\rho}_i(R, S)$, for $i = 1, 2, 3, 4, 5, 6$. Since $\phi_{1,9} = t_{1,9}$, if we compare the $5 * 1 + 9$ with $5 * i + j$, for all pairs (i, j) , $0 \leq i \leq 9$ and $0 \leq j \leq 15$, for which $\phi_{i,j} = t_{i,j}$, then we conclude that $5 * i + j > 5 * 1 + 9$. By Proposition 2, $\tilde{\rho}_5(R, S) = 5 * 1 + 9 = 14$. By Theorem 3, the 1's of A can be covered by one row and nine columns.

Using the fact that all columns of A have at least a nonzero entry and A has five columns with at least two nonzero entries, then with one row we must cover all 1's of six columns of A . The unique row of A with at least 6 nonzero entries is the first. Consequently, we can assume that the first row of A is

$$[0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1].$$

Let C be the matrix obtained from A by removing the first row and the last six columns. Since $\phi_{2,5} = t_{2,5}$, if we compare $3 * 2 + 5$ with $3 * i + j$, for all pairs (i, j) , $0 \leq i \leq 9$ and $0 \leq j \leq 15$ for which $\phi_{i,j} = t_{i,j}$, then we conclude $3 * i + j > 3 * 2 + 5$. By Proposition 2, $\tilde{\rho}_3(R, S) = 3 * 2 + 5 = 11$. By Theorem 3, the 1's of A can be covered by two rows and five columns. Using the above arguments, the 1's of C can be covered with one row and five columns. Note that C has nine columns, and five of them have at least two nonzero entries. This implies that the row that covers the 1's of C also covers the 1's in the last four columns of C . Only the first and the second rows of C have at least four nonzero entries. Let D be the matrix obtained from C by removing the last four columns.

The matrix D has five columns and eight rows. Moreover, the row sum vector of D is $G_1 = (1, 4, 3, 3, 2, 2, 1, 1)$ or $G_2 = (5, 0, 3, 3, 2, 2, 1, 1)$ and the column sum vector of D is $H = (7, 3, 3, 2, 2)$.

Since $\tilde{\rho}_1(R, S) = 1 * 3 + 3 = 6$, as above, the 1's of A can be covered with three rows and three columns. We have two cases:

If the row sum vector of D is $G_2 = (5, 0, 3, 3, 2, 2, 1, 1)$ and its column sum is $H = (7, 3, 3, 2, 2)$, using the above arguments, all 1's of D will be covered with one row and three columns. This is impossible because five columns of D have at least two nonzero entries.

If the row sum vector of D is $G_1 = (1, 4, 3, 3, 2, 2, 1, 1)$, and its column sum is $H = (7, 3, 3, 2, 2)$, using the above arguments, all 1's of D will be covered with two rows (the first row of D is one of these two rows) and three columns. This is impossible because four columns of the matrix obtained from D by removing the first row have at least two nonzero entries.

Therefore, no matrix in $\mathcal{A}(R, S)$ realizes all the minimum i -term ranks, for $i \geq 1$.

5 Special partitions R and S

Let $R = (R_1, \dots, R_m)$ and $S = (S_1, \dots, S_n)$ be partitions of the same weight such that $\mathcal{A}(R, S)$ is nonempty. In spite of what we saw in the last

section, there are special partitions R and S for which there is a matrix in $\mathcal{A}(R, S)$ that realizes all the minimum t -term ranks, for $t \geq 1$. In this section, we describe some of those partitions. So, the next result is our main result:

Theorem 6 *Let $R = (R_1, \dots, R_m)$ and $S = (S_1, \dots, S_n)$, with $m > 2$, $n > 2$, be two partitions of the same weight such that $\mathcal{A}(R, S)$ is nonempty. Let t be a positive integer. If the minimum integers f, f' such that $\phi_{2,f} = t_{2,f}$ and $\phi_{1,f'} = t_{1,f'}$ satisfy $1 \leq f < f' < n$, $S_f = S_{f+1} = \dots = S_n = 1$, and for all $1 \leq k \leq t$, $\tilde{\rho}_k(R, S) \in \{k + f', 2k + f\}$, then there exists a matrix $A \in \mathcal{A}(R, S)$ such that*

$$\rho_k(A) = \tilde{\rho}_k(R, S), \quad \text{for } k = 1, \dots, t.$$

We start with the following lemma:

Lemma 7 [1] *Let (R_1, \dots, R_m) and (S_1, \dots, S_n) be two partitions of the same weight, and let $[c_{i,j}]$ be an m -by- n nonnegative integral matrix. There is a nonnegative integral matrix $A = [a_{i,j}]$, satisfying*

$$0 \leq a_{i,j} \leq c_{i,j}, \quad \text{for } i \in \{1, \dots, m\} \text{ and } j \in \{1, \dots, n\},$$

$$\sum_{j=1}^n a_{i,j} = R_i, \quad \text{for } i \in \{1, \dots, m\},$$

and

$$\sum_{i=1}^m a_{i,j} = S_j, \quad \text{for } j \in \{1, \dots, n\},$$

if and only if, for all $I \subseteq \{1, \dots, m\}$ and $J \subseteq \{1, \dots, n\}$,

$$\sum_{i \in I, j \in J} c_{i,j} \geq \sum_{j \in J} S_j - \sum_{i \notin I} R_i.$$

Let $T = [t_{i,j}]$ be the $(m+1)$ -by- $(n+1)$ structure matrix associated with R and S . For $0 \leq a < b \leq m$, and $0 \leq c < d \leq n$, define the nonnegative integer $\psi_{a,b;c,d}$ as

$$\psi_{a,b;c,d} = \min\{t_{i_1, d+j_3} + t_{a+i_2, c+j_2} + t_{b+i_3, j_1} + (a-i_1)(d-c-j_2) + (b-a-i_2)(c-j_1) + (a-i_1)(c-j_1)\},$$

where the minimum is taken over all integers i_1, i_2, i_3 and j_1, j_2, j_3 that satisfy

$$0 \leq i_1 \leq a \leq a + i_2 \leq b \leq b + i_3 \leq m$$

and

$$0 \leq j_1 \leq c \leq c + j_2 \leq d \leq d + j_3 \leq n.$$

To compute a matrix in the conditions of Theorem 3 we can use the modified Ryser algorithm (see [4] or Section 6) or the known algorithms from the network flows (see [10]). The following proposition is a generalization of Theorem 3.

Proposition 8 *Let $R = (R_1, \dots, R_m)$ and $S = (S_1, \dots, S_n)$ be two partitions of the same weight such that $\mathcal{A}(R, S)$ is nonempty. Let e, e', f, f' be integers such that $0 \leq e' < e \leq m$ and $0 \leq f < f' \leq n$. Then there is a matrix in $\mathcal{A}(R, S)$ of the form*

$$\begin{bmatrix} X & Y & V \\ Z & W & O_{e-e', n-f'} \\ U & O_{m-e, f'-f} & O_{m-e, n-f'} \end{bmatrix}. \quad (1)$$

if and only if

$$\psi_{e', e; f, f'} \geq t_{e, f} + t_{e', f'}.$$

Proof. The proof follows the steps of the proof of theorem 3.5.8 in [2]. Let C be the m -by- n matrix such that

$$C = [c_{i,j}] = \begin{bmatrix} J_{e', f} & J_{e', f'-f} & J_{e', n-f'} \\ J_{e-e', f} & J_{e-e', f'-f} & O_{e-e', n-f'} \\ J_{m-e, f} & O_{m-e, f'-f} & O_{m-e, n-f'} \end{bmatrix},$$

where $J_{u,v}$ denotes the u -by- v matrix whose entries are all equal to 1. By Lemma 7, there is a matrix in $\mathcal{A}(R, S)$ of the form (1) if and only if

$$\sum_{i \in I, j \in J} c_{i,j} \geq \sum_{j \in J} S_j - \sum_{i \notin I} R_i, \quad (2)$$

for all $I \subseteq \{1, \dots, m\}$ and $J \subseteq \{1, \dots, n\}$.

For $I \subseteq \{1, \dots, m\}$ and $J \subseteq \{1, \dots, n\}$, we write $I = I_1 \cup I_2 \cup I_3$ where $I_1 \subseteq \{1, \dots, e'\}$, $I_2 \subseteq \{e' + 1, \dots, e\}$ and $I_3 \subseteq \{e + 1, \dots, m\}$, and we write $J = J_1 \cup J_2 \cup J_3$ where $J_1 \subseteq \{1, \dots, f\}$, $J_2 \subseteq \{f + 1, \dots, f'\}$ and $J_3 \subseteq \{f' + 1, \dots, n\}$. We agree to take complements of $I_1, I_2, I_3, J_1, J_2,$

and J_3 with respect to $\{1, \dots, e'\}$, $\{e'+1, \dots, e\}$, $\{e+1, \dots, m\}$, $\{1, \dots, f\}$, $\{f+1, \dots, f'\}$, and $\{f'+1, \dots, n\}$, respectively. Then (2) is equivalent to

$$\begin{aligned} & |I_1||J_1| + |I_1||J_2| + |I_1||J_3| + |I_2||J_1| + |I_2||J_2| + |I_3||J_1| \geq \\ & \geq \sum_{j \in J_1 \cup J_2 \cup J_3} S_j - \sum_{i \in \bar{I}_1 \cup \bar{I}_2 \cup \bar{I}_3} R_i, \end{aligned}$$

for all $I_1 \subseteq \{1, \dots, e'\}$, $I_2 \subseteq \{e'+1, \dots, e\}$, $I_3 \subseteq \{e+1, \dots, m\}$, $J_1 \subseteq \{1, \dots, f\}$, $J_2 \subseteq \{f+1, \dots, f'\}$ and $J_3 \subseteq \{f'+1, \dots, n\}$. Let $k_1 = |I_1|$, $k_2 = |I_2|$, $k_3 = |I_3|$, $l_1 = |J_1|$, $l_2 = |J_2|$ and $l_3 = |J_3|$. Since R and S are nonincreasing, it follows that the last inequality is equivalent to

$$\begin{aligned} & k_1 l_1 + k_1 l_2 + k_1 l_3 + k_2 l_1 + k_2 l_2 + k_3 l_1 \geq \\ & \geq \sum_{j=1}^{l_1} S_j + \sum_{j=f+1}^{f+l_2} S_j + \sum_{j=f'+1}^{f'+l_3} S_j - \sum_{i=e'-k_1+1}^{e'} R_i - \sum_{i=e-k_2+1}^e R_i - \sum_{i=m-k_3+1}^m R_i, \end{aligned}$$

holding for all integers k_1, k_2 , and k_3 , with $0 \leq k_1 \leq e' \leq e'+k_2 \leq e \leq e+k_3 \leq m$ and all integers l_1, l_2 , and l_3 , with $0 \leq l_1 \leq f \leq f+l_2 \leq f' \leq f'+l_3 \leq n$. The last inequality is equivalent to

$$t_{K,L} - k_2 l_3 - k_3 l_2 - k_3 l_3 \geq 0, \quad (3)$$

where $K_1 = \{1, \dots, k_1\}$, $K_2 = \{e'+1, \dots, e'+k_2\}$, $K_3 = \{e+1, \dots, e+k_3\}$, $K = K_1 \cup K_2 \cup K_3$, $L_1 = \{1, \dots, l_1\}$, $L_2 = \{f+1, \dots, f+l_2\}$, $L_3 = \{f'+1, \dots, f'+l_3\}$, $L = L_1 \cup L_2 \cup L_3$. Let A be any matrix in $\mathcal{A}(R, S)$ and partition A according to the diagram

	L_1	\bar{L}_1	L_2	\bar{L}_2	L_3	\bar{L}_3
K_1	(0)		(0)		(0)	
\bar{K}_1		(1)		(1)		(1)
K_2	(0)		(0)		-(1)	
\bar{K}_2		(1)		(1)		(1)
K_3	(0)		-(1)		-(1)	
\bar{K}_3		(1)		(1)		(1)

Then (3) counts 0's and 1's in submatrices of A as shown above, where $-(1)$ means the negative of the number of 1's in the submatrices of A indicated.

On the other hand, the expression $t_{k_1, f'+l_3} + t_{e'+k_2, f+l_2} + t_{e+k_3, l_1}$ counts 0's and 1's as shown below:

	L_1	\overline{L}_1	L_2	\overline{L}_2	L_3	\overline{L}_3
K_1	3(0)	2(0)	2(0)	(0)	(0)	
\overline{K}_1	2(0)	(0)	(0)			(1)
K_2	2(0)	(0)	(0)			(1)
\overline{K}_2	(0)			(1)	(1)	2(1)
K_3	(0)			(1)	(1)	2(1)
\overline{K}_3		(1)	(1)	2(1)	2(1)	3(1)

In the matrix, 2(0) means twice the number of 0's in the corresponding submatrices of A , 3(0) means three times the number of 0's in the corresponding submatrices of A , 2(1) means twice the number of 1's in the corresponding submatrices of A , and 3(1) means three times the number of 1's in the corresponding submatrices of A .

It now follows that

$$t_{k_1, f'+l_3} + t_{e'+k_2, f+l_2} + t_{e+k_3, l_1} - (t_{K,L} - k_2l_3 - k_3l_2 - k_3l_3) \quad (4)$$

counts 0's and 1's in submatrices of A as indicated in the matrix diagram.

	L_1	\overline{L}_1	L_2	\overline{L}_2	L_3	\overline{L}_3
K_1	2(0)	2(0)	(0)	(0)		
\overline{K}_1	2(0)	(0) - (1)	(0)	-(1)		
K_2	(0)	(0)			(1)	(1)
\overline{K}_2	(0)	-(1)			(1)	(1)
K_3			(1)	(1)	2(1)	2(1)
\overline{K}_3			(1)	(1)	2(1)	2(1)

Hence, (4) equals

$$t_{e,f} + t_{e',f'} - (e' - k_1)(f - l_1) - (e' - k_1)(f' - f - l_2) - (e - e' - k_2)(f - l_1).$$

Therefore (3) holds if and only if $\psi_{e',e;f,f'} \geq t_{e,f} + t_{e',f'}$. \blacksquare

Proposition 9 *Let $R = (R_1, \dots, R_m)$ and $S = (S_1, \dots, S_n)$, with $m > 2$, $n > 2$, be two partitions of the same weight such that $\mathcal{A}(R, S)$ is nonempty. Let f, f' be integers such that $1 \leq f < f' < n$ and $S_f = S_{f+1} = \dots = S_n = 1$. If $\phi_{2,f} = t_{2,f}$ and $\phi_{1,f'} = t_{1,f'}$, then*

$$\psi_{1,2;f,f'} \geq t_{1,f'} + t_{2,f}.$$

Proof. Let p be an integer such that $f \leq p \leq n$. Then,

$$t_{1,p} = p + \sum_{i=2}^m R_i - \sum_{j=1}^p S_j = t_{1,f} + (p - f) + \sum_{j=f+1}^p S_j.$$

Since $S_f = S_{f+1} = \dots = S_n = 1$ then

$$t_{1,p} = t_{1,f}.$$

Let s be an integer such that $f \leq s \leq n$. Then,

$$t_{2,s} = 2s + \sum_{i=3}^m R_i - \sum_{j=1}^s S_j = t_{2,f} + 2(s - f) + \sum_{j=f+1}^s S_j.$$

Since $S_f = S_{f+1} = \dots = S_n = 1$ then

$$t_{2,s} = t_{2,f} + (s - f).$$

By definition,

$$\begin{aligned} \psi_{1,2;f,f'} &= \min\{t_{i_1,f'+j_3} + t_{1+i_2,f+j_2} + t_{2+i_3,j_1} + \\ &+ (1 - i_1)(f' - f - j_2) + (2 - 1 - i_2)(f - j_1) + (1 - i_1)(f - j_1)\}, \end{aligned}$$

where the minimum is taken over all integers i_1, i_2, i_3 and j_1, j_2, j_3 that satisfy

$$0 \leq i_1 \leq 1 \leq 1 + i_2 \leq 2 \leq 2 + i_3 \leq m$$

and

$$0 \leq j_1 \leq f \leq f + j_2 \leq f' \leq f' + j_3 \leq n.$$

Therefore, we conclude that:

- If $i_1 = 0$, then $t_{i_1,f'+j_3} = t_{0,n} = 0$;
- If $i_1 = 1$, then $t_{i_1,f'+j_3} = t_{1,f}$;
- If $i_2 = 0$, then $t_{1+i_2,f+j_2} = t_{1,f}$;
- If $i_2 = 1$, then $t_{1+i_2,f+j_2} = t_{2,f} + j_2$.

So, we obtain four cases:

- **Case 1.** If $i_1 = 0$ and $i_2 = 0$, then

$$\psi_{1,2;f,f'} = 0 + t_{1,f'} + t_{2+i_3,j_1} + 0 + (f - j_1) + (f - j_1).$$

Since $t_{2,f} = \phi_{2,f} \leq 0 + t_{2+i_3,j_1} + 2(f - j_1)$, we get

$$\psi_{1,2;f,f'} \geq t_{1,f'} + t_{2,f}.$$

- **Case 2.** If $i_1 = 0$ and $i_2 = 1$, then

$$\psi_{1,2;f,f'} = 0 + t_{2,f} + j_2 + t_{2+i_3,j_1} + (f' - f - j_2) + 0 + (f - j_1).$$

Since $t_{1,f'} = \phi_{1,f'} \leq 0 + t_{2+i_3,j_1} + (f' - j_1)$, we get

$$\psi_{1,2;f,f'} \geq t_{1,f'} + t_{2,f}.$$

- **Case 3.** If $i_1 = 1$ and $i_2 = 0$, then

$$\psi_{1,2;f,f'} = t_{1,f'} + t_{1,f'} + t_{2+i_3,j_1} + 0 + (f - j_1) + 0.$$

Since $t_{2,f} = \phi_{2,f} \leq t_{1,f'} + t_{2+i_3,j_1} + (f - j_1)$, we get

$$\psi_{1,2;f,f'} \geq t_{1,f'} + t_{2,f}.$$

- **Case 4.** If $i_1 = 1$ and $i_2 = 1$, then

$$\psi_{1,2;f,f'} = t_{1,f'} + t_{2,f} + j_2 + t_{2+i_3,j_1} + 0 + 0 + 0.$$

Consequently,

$$\psi_{1,2;f,f'} \geq t_{1,f'} + t_{2,f}.$$

■

Lemma 10 *Let e, e', f, f', k , and l be nonnegative integers such that $1 \leq k < l$, $ke + f < ke' + f'$, and $le + f > le' + f'$. Then $e' < e$ and $f < f'$.*

Proof. Since $ke + f < ke' + f'$, and $le + f > le' + f'$, we have

$$k(e - e') < f' - f < l(e - e').$$

If $e \leq e'$ then $e - e' \leq 0$. Using the inequality $1 \leq k < l$ we get

$$k(e - e') \geq l(e - e').$$

Contradiction. So, $e' < e$. Consequently, $e - e' > 0$ and

$$0 < k(e - e') < f' - f.$$

Therefore, $f < f'$. ■

Proof of Theorem 6. If $t = 1$, the result follows. Let $t > 1$. Suppose there is no matrix $A \in \mathcal{A}(R, S)$ such that

$$\rho_k(A) = \tilde{\rho}_k(R, S), \quad \text{for } k = 1, \dots, t.$$

Let l be the greatest integer such that $1 \leq l \leq t$ and there is a matrix D in $\mathcal{A}(R, S)$ with

$$\rho_1(D) = \tilde{\rho}_1(R, S), \dots, \rho_{l-1}(D) = \tilde{\rho}_{l-1}(R, S), \text{ and } \rho_l(D) \neq \tilde{\rho}_l(R, S).$$

Let B be a matrix in $\mathcal{A}(R, S)$ such that

$$\rho_l(B) = \tilde{\rho}_l(R, S) < \rho_l(D).$$

So, there is an integer k , with $1 \leq k < l$, such that

$$\rho_{k+1}(B) = \tilde{\rho}_{k+1}(R, S), \dots, \rho_l(B) = \tilde{\rho}_l(R, S),$$

and

$$\rho_k(B) \neq \tilde{\rho}_k(R, S) = \rho_k(D).$$

Consequently,

$$\rho_k(B) > \rho_k(D), \text{ and } \rho_l(D) > \rho_l(B).$$

By Proposition 2, there are nonnegative integers a, b, c, d , such that $\tilde{\rho}_k(R, S) = kb + c$, $\phi_{b,c} = t_{b,c}$, and $\tilde{\rho}_l(R, S) = la + d$, $\phi_{a,d} = t_{a,d}$.

Then $1 \leq k < l$, $kb + c < ka + d$, and $lb + c > la + d$. Using Lemma 10, we get $a < b$ and $c < d$. By hypothesis, $a = 1$, $b = 2$, $c = f$ and $d = f'$.

Using Proposition 9, we have

$$\psi_{1,2;f,f'} \geq t_{1,f'} + t_{2,f}.$$

By Proposition 8, we conclude that there exists a matrix Q in $\mathcal{A}(R, S)$ all of whose 1's are contained in the union of its first 2 rows and first f columns and, in the union of its first row and first f' columns. Thus, by Proposition 1

$$\rho_1(Q) = \tilde{\rho}_1(R, S), \dots, \rho_l(Q) = \tilde{\rho}_l(R, S).$$

Contradiction. Therefore, there exists a matrix $A \in \mathcal{A}(R, S)$ such that $\rho_k(A) = \tilde{\rho}_k(R, S)$, for all $k = 1, \dots, t$. ■

6 An algorithm for constructing matrices with fixed zeros blocks

Let $R = (R_1, \dots, R_n)$ and $S = (S_1, \dots, S_n)$ be partitions of the same weight. Let e', e, f and f' positive integers such that $0 \leq e' < e \leq m$ and $0 \leq f < f' \leq n$. Denote by $\mathcal{A}_{e', e; f, f'}(R, S)$ the set of all matrices of $\mathcal{A}(R, S)$ all of whose 1's are contained in the union of the first e' rows and first f' columns and, in the union of the first e rows and first f columns. In the last section we studied the existence of a matrix in $\mathcal{A}_{e', e; f, f'}(R, S)$ then it would be of interest to find a way to construct matrices in $\mathcal{A}_{e', e; f, f'}(R, S)$. Assuming that $\mathcal{A}_{e', e; f, f'}(R, S)$ is nonempty, in this section we present an algorithm for constructing a matrix in $\mathcal{A}_{e', e; f, f'}(R, S)$. This algorithm generalizes the algorithm stated by Brualdi and Dahl in [4] for constructing a matrix in $\mathcal{A}_{e, f}(R, S)$ (the set of all matrices of $\mathcal{A}(R, S)$ all of whose 1's are contained in the union of the first e rows and first f columns), when $\mathcal{A}_{e, f}(R, S) \neq \emptyset$.

We first describe the modified Ryser algorithm.

Let $R^{(e)} = (R_1, \dots, R_e)$ and $S^{(f)} = (S_1, \dots, S_f)$. Let $F = (F_1, \dots, F_p)$ be an integer vector with p coordinates. Let σ_F be a permutation of $\{1, \dots, p\}$ such that the vector $F_{\sigma} = (F_{\sigma(1)}, \dots, F_{\sigma(p)})$ satisfies $F_{\sigma(1)} \geq \dots \geq F_{\sigma(p)}$. We denote by P_{σ_F} the p -by- p permutation matrix associated with σ_F , and by $P_{\sigma_F}^{-1}$ we denote its inverse.

The modified Ryser algorithm:

1. Start with an m -by- n $(0, 1)$ -matrix B_m whose row sum vector is R and whose column sum vector is R^* . Thus the 1's occupy the initial positions in each row. Let $B_{e, n}$ be the submatrix obtained from B_m deleting rows $e + 1, \dots, m$. Let \bar{B}_0 be the e -by-0 empty matrix.
2. For $k = n, n - 1, \dots, f + 1$, do:

Shift to column k the final 1's in those S_k rows of $B_{e,k}$ with the largest sum, with preference given to the lowest rows (those with the largest index) in case of ties. This results in a matrix

$$\left[B_{e,k-1} \mid \overline{B}_{n-k+1} \right],$$

where $B_{e,k-1}$ has $k - 1$ columns.

3. Let C_n be the n -by- m $(0, 1)$ -matrix whose row sum vector is S and whose column sum vector is S^* . Thus the 1's occupy the initial positions in each row. Let $C_{f,m}$ be the submatrix obtained from C_n by deleting rows $f + 1, \dots, n$. Let \overline{C}_0 be the f -by-0 empty matrix.
4. For $h = m, m - 1, \dots, e + 1$, do:

Shift to column h the final 1's in those R_h rows of $C_{f,h}$ with the largest sum, with preference given to the lowest rows (those with the largest index) in case of ties. This results in a matrix

$$\left[C_{f,h-1} \mid \overline{C}_{m-h+1} \right],$$

where $C_{f,h-1}$ has $h - 1$ columns.

5. Let \hat{R} be the row sum sequence of \overline{B}_{n-f} and \hat{S} be the row-sum sequence of \overline{C}_{m-e} . Let $\bar{R} = R^{(e)} - \hat{R}$ and $\bar{S} = S^{(f)} - \hat{S}$.
6. Let A_1 be the canonical matrix of $\mathcal{A}(\bar{R}_{\sigma_{\bar{R}}}, \bar{S}_{\sigma_{\bar{S}}})$.

Output:

$$A = \left[\begin{array}{c|c} P_{\sigma_{\bar{R}}}^{-1} A_1 P_{\sigma_{\bar{S}}}^{-1} & \overline{B}_{n-f} \\ \hline \overline{C}_{m-e}^T & 0 \end{array} \right],$$

where \overline{C}_{m-e}^T is the transpose of \overline{C}_{m-e} .

The matrix \overline{B}_{n-f} is the canonical column f -submatrix relative to R , S and f . The matrix \overline{C}_{m-e} is the canonical column e -submatrix relative to S , R and e .

Example 11 Let $R = (4, 2, 2, 2, 1, 1, 1)$ and $S = (2, 2, 2, 2, 1, 1, 1, 1)$. Then $R^* = (7, 4, 1, 1)$. Let $T = [t_{ij}]$ be the structure matrix associated with R and S . Since $t_{2,4} = 7 = \phi_{2,4}$, by Theorem 3, we get $\mathcal{A}_{2,4}(R, S) \neq \emptyset$.

Using the last algorithm with $e = 2$ and $f = 4$, let

$$B_7 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$B_{2,9} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The following matrices $[B_{2,i} \mid \overline{B}_{9-i}]$ are produced using step 2,

$$\begin{aligned} & \left[\begin{array}{cccccccc|cccc} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right], \quad \left[\begin{array}{cccccccc|cccc} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right], \\ & \left[\begin{array}{cccccccc|cccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right], \quad \left[\begin{array}{cccccccc|cccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{array} \right], \\ & \left[\begin{array}{cccccccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{array} \right], \quad \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{array} \right]. \end{aligned}$$

In step 3,

$$C_{4,7} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The following matrices $[C_{4,i} \mid \overline{C}_{7-i}]$ are produced using step 4,

$$\begin{aligned} & \left[\begin{array}{cccccc|cccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right], \quad \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right], \\ & \left[\begin{array}{cccccc|cccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{array} \right], \quad \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right], \end{aligned}$$

$$\left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right], \quad \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$$

In this case, in step 5, we have

$$\bar{B}_5 = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix},$$

with row sum sequence $\hat{R} = (3, 2)$ and

$$\bar{C}_5 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix},$$

with row sum sequence $\hat{S} = (1, 2, 2, 2)$.

So, $\bar{R} = R^{(e)} - \hat{R} = (4, 2) - (3, 2) = (1, 0)$ and $\bar{S} = S^{(f)} - \hat{S} = (2, 2, 2, 2) - (1, 2, 2, 2) = (1, 0, 0, 0)$.

The canonical matrix of $\mathcal{A}(\bar{R}, \bar{S})$ is

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, the desired matrix in $\mathcal{A}_{2,4}(R, S)$ is

$$A = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

Theorem 12 Let $R = (R_1, \dots, R_m)$ and $S = (S_1, \dots, S_n)$ be partitions of the same weight. Let e', e, f and f' positive integers with $0 \leq e' < e \leq m$, $0 \leq f < f' \leq n$ and $\mathcal{A}_{e', e; f, f'}(R, S) \neq \emptyset$. Let $A_{f'}$ be the canonical column f' -submatrix relative to R, S , and f' and, A_e^T be the transpose of the canonical column e -submatrix relative to S, R and e . Then there are matrices $A_{11}, A_{12}, A_{21}, A_{22}$ and permutation matrices P and Q such that the matrix

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{f'} \\ A_{21} & A_{22} & 0 \\ A_e^T & 0 & 0 \end{bmatrix},$$

belongs to $\mathcal{A}_{e',e;f,f'}(R, S)$ and

$$P \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} Q$$

is the canonical matrix in the class to which it belongs.

Proof. Let

$$D = \begin{bmatrix} A_1 & A_2 & A_3 \\ A_4 & A_5 & 0 \\ A_6 & 0 & 0 \end{bmatrix}$$

be a matrix in $\mathcal{A}_{e',e;f,f'}(R, S)$. From Ryser's interchange theorem we may apply interchanges to the matrix

$$[A_1 \ A_2 \ A_3]$$

to obtain the matrix

$$[B_1 \ B_2 \ A_{f'}].$$

Applying a similar argument to the matrix

$$\begin{bmatrix} B_1 \\ A_4 \\ A_6 \end{bmatrix}$$

we obtain the matrix

$$\begin{bmatrix} C_1 & B_2 & A_{f'} \\ B_3 & A_5 & 0 \\ A_e^T & 0 & 0 \end{bmatrix}.$$

Consider the submatrix

$$C = \begin{bmatrix} C_1 & B_2 \\ B_3 & A_5 \end{bmatrix}.$$

Let R' and S' be the row sum vector and the column sum vector of C , respectively, and let C' be the canonical matrix of $\mathcal{A}(R'_{\sigma_{R'}}, S'_{\sigma_{S'}})$. Then, there are permutation matrices P_1 and Q_1 such that $P_1 C' Q_1$ has row sum vector R' and column sum vector S' . Replacing C by C' we have the desired matrix. ■

Now we can use Theorem 12 to give an algorithm to construct a matrix in $\mathcal{A}_{e',e;f,f'}(R, S)$, when $\mathcal{A}_{e',e;f,f'}(R, S) \neq \emptyset$:

Algorithm to construct a matrix in $\mathcal{A}_{e',e,f,f'}(R, S)$:

Let e', e, f and f' positive integers such that $0 \leq e' < e \leq m$ and $0 \leq f < f' \leq n$.

1. Construct $A_{f'}$, the canonical column f' -submatrix, relative to R, S and f' .
2. Construct A_e , the canonical column e -submatrix, relative to S, R and e .
3. Let \hat{R} be the row sum sequence of $A_{f'}$ and \hat{S} be the row sum sequence of A_e . Let $\bar{R} = R^{(e)} - \hat{R}$ and $\bar{S} = S^{(f')} - \hat{S}$.
4. Let A_1 be the canonical matrix of $\mathcal{A}(\bar{R}_{\sigma_{\bar{R}}}, \bar{S}_{\sigma_{\bar{S}}})$.
5. Let P and Q be permutation matrices such that PA_1Q has row sum sequence \bar{R} and column sum sequence \bar{S} .

Output:

$$A = \left[\begin{array}{c|c} PA_1Q & \frac{A_{f'}}{0} \\ \hline A_e^T & 0 \end{array} \right],$$

where A_e^T is the transpose of A_e .

Example 13 Let $R = (4, 2, 2, 2, 1, 1, 1)$ and $S = (2, 2, 2, 2, 1, 1, 1, 1, 1, 1)$ as in the last example. Let $T = [t_{ij}]$ be the structure matrix associated with R and S . Since

$$\psi_{2,3;3,4} = 15 = 7 + 8 = t_{2,4} + t_{3,3},$$

by Proposition 8, $\mathcal{A}_{2,3;3,4}(R, S) \neq \emptyset$.

In this case, $e' = 2, e = 3, f = 3$ and $f' = 4$.

In the last example we constructed the canonical column f' -submatrix, relative to R, S and f' ,

$$A_{f'} = \bar{B}_5 = \left[\begin{array}{ccccc} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{array} \right].$$

This matrix has row sum sequence $\hat{R} = (3, 2)$.

The canonical column e -submatrix, relative to S , R and e is

$$A_e = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

This matrix has row sum sequence $\hat{S} = (1, 2, 2)$.

So, $\bar{R} = R^{(e)} - \hat{R} = (4, 2, 2) - (3, 2, 0) = (1, 0, 2)$ and $\bar{S} = S^{(f')} - \hat{S} = (2, 2, 2, 2) - (1, 2, 2, 0) = (1, 0, 0, 2)$

The canonical matrix of $\mathcal{A}(\bar{R}_{\sigma_{\bar{R}}}, \bar{S}_{\sigma_{\bar{S}}}) = \mathcal{A}((2, 1, 0), (2, 1, 0, 0))$ is

$$A_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, the matrix in $\mathcal{A}_{2,3,3,4}(R, S)$ is

$$A = \left[\begin{array}{cccc|cccc} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

7 Conclusions

Using two partitions $R = (R_1, \dots, R_m)$ and $S = (S_1, \dots, S_n)$ of the same weight such that $\mathcal{A}(R, S)$ is nonempty and four integers e', e, f, f' such that $1 \leq e' < e < m$ and $1 \leq f < f' < n$, we have introduced the class $\mathcal{A}_{e',e,f,f'}(R, S)$. An existence theorem for matrices in $\mathcal{A}_{e',e,f,f'}(R, S)$ was given. We have also described an algorithm to construct a matrix in $\mathcal{A}_{e',e,f,f'}(R, S)$.

We have shown a nonempty class $\mathcal{A}(R, S)$ where no matrix realizes all the minimum t -term ranks, for all $t \geq 1$, while in other classes this matrix exists. An interesting question would be the study of these two types of classes.

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