On some linear recurrences

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Abstract

The solution for a type of linear homogeneous recurrence relation with constant coefficients is present with the help of Chebyshev polynomials of the second kind. An application to ladder networks is provided.

Keywords: Chebyshev polynomials, recurrence relations, ladder networks

2010 Mathematics Subject Classification: 65Q30

1 Introduction

A fundamental concept in Discrete Mathematics is the notion of *recurrence relation*. These recursion formulas appear frequently in numerous areas of knowledge such as Biology (population growth), Finance (compound interest), Computer Science (analysis of algorithms) and Electronics (ladder networks and electric line theory), among others. Although techniques exist to solve general linear recurrence relations with constant coefficients, it is always of great interest to have explicit formulas to express their solutions. In this short note, we deduce closed-form expressions for the solution of a second and fourth order linear homogeneous recurrence relation with constant coefficients using Chebyshev polynomials of the second kind, taking advantage of a well-known identity involving these polynomials. These formulas reveal to be not only of compact shape when compared, for instance, with the traditional approach that requires the roots of the characteristic equation derived from the recurrence relation, but also they lead to a Hankel matrix having (a sum of) Chebyshev polynomials of the second kind as entries in the course of the computation that employs the initial conditions.

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2 Solution via Chebyshev polynomials

Throughout, we shall denote by $U_p(x)$, $p \geqslant 0$, the pth degree Chebyshev polynomial of the second kind, i.e., the polynomials defined by $U_0(x) = 1$, $U_1(x) = 2x$, and for $n \geqslant 2$,

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x).$$

Our first statement gives the solution of some second order linear homogeneous recurrence relations with constant coefficients by using these polynomials. Here, $[\mathbf{v}]_k$ stands for the kth entry of a given column vector $\mathbf{v}$.

Theorem 2.1. *Let a and b be complex numbers such that $b \neq 0$ and $\{x_n\}_{n\geqslant 0}$, a sequence of complex numbers satisfying*

$$x_{n+2} = -\frac{a}{b}x_{n+1} - x_n, \quad n \geqslant 0. \tag{2.1}$$

If $a + 2b \neq 0$ and $a - 2b \neq 0$, then

$$x_n = \left[\mathbf{H}^{-1}\mathbf{x}\right]_1 U_n\left(-\frac{a}{2b}\right) + \left[\mathbf{H}^{-1}\mathbf{x}\right]_2 U_{n+1}\left(-\frac{a}{2b}\right)$$

with

$$\mathbf{H} = \begin{bmatrix} U_0\left(-\frac{a}{2b}\right) & U_1\left(-\frac{a}{2b}\right) \\ U_1\left(-\frac{a}{2b}\right) & U_2\left(-\frac{a}{2b}\right) \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}.$$

Proof. Suppose ξ is a root of $Q(x) = bx^2 + ax + b$. Hence, $1/\xi$ is also a root of $Q(x)$ because $Q(x)$ is two self-reciprocal (see [5], page 431). Thus,

$$b\xi^{k+3} + a\xi^{k+2} + b\xi^{k+1} = 0 \tag{2.2}$$

and

$$b\left(\frac{1}{\xi}\right)^{k+3} + a\left(\frac{1}{\xi}\right)^{k+2} + b\left(\frac{1}{\xi}\right)^{k+1} = 0 \tag{2.3}$$

for each $k \geqslant 0$. Setting $\theta = \frac{1}{2}\left(\xi + \frac{1}{\xi}\right)$, we have $2b\theta + a = 0$. Because $a + 2b \neq 0$ and $a - 2b \neq 0$, we obtain $\xi - 1/\xi \neq 0$. Thereby, (2.2) and (2.3) ensure

$$bU_{k+2}\left(-\frac{a}{2b}\right) + aU_{k+1}\left(-\frac{a}{2b}\right) + bU_k\left(-\frac{a}{2b}\right) = 0$$

because, for every nonzero complex number x,

$$\left(x - \frac{1}{x}\right)U_k\left[\frac{1}{2}\left(x + \frac{1}{x}\right)\right] = x^{k+1} - \frac{1}{x^{k+1}}, \quad k \geqslant 0 \tag{2.4}$$

(see, for instance, [1]). Therefore, the sequence

$$x_n = \alpha U_n\left(-\frac{a}{2b}\right) + \beta U_{n+1}\left(-\frac{a}{2b}\right)$$

satisfies (2.1) for every complex number α and β. The conclusion follows by solving the system of linear equations

$$\begin{cases} \alpha U_0\left(-\frac{a}{2b}\right) + \beta U_1\left(-\frac{a}{2b}\right) = x_0, \\ \alpha U_1\left(-\frac{a}{2b}\right) + \beta U_2\left(-\frac{a}{2b}\right) = x_1, \end{cases}$$

in the variables α and β, which has unique solution $\alpha = \left[\mathbf{H}^{-1}\mathbf{x}\right]_1$ and $\beta = \left[\mathbf{H}^{-1}\mathbf{x}\right]_2$. The thesis is established. ☐

Parallel to Theorem 2.1, the next result gives us the solution for a type of fourth order linear homogeneous recurrence relation with constant coefficients. The proof is analogous.

Theorem 2.2. *Let a, b, c be complex numbers such that $c \neq 0$ and $\{x_n\}_{n\geqslant 0}$, a sequence of complex numbers satisfying*

$$x_{n+4} = -\frac{b}{c}x_{n+3} - \frac{a}{c}x_{n+2} - \frac{b}{c}x_{n+1} - x_n, \quad n \geqslant 0. \tag{2.5}$$

If $b^2 + 8c^2 - 4ac \neq 0$, $a + 2c + 2b \neq 0$, and $a + 2c - 2b \neq 0$, then

$$x_n = \left[\mathbf{H}^{-1}\mathbf{x}\right]_1 H_n + \left[\mathbf{H}^{-1}\mathbf{x}\right]_2 H_{n+1} + \left[\mathbf{H}^{-1}\mathbf{x}\right]_3 H_{n+2} + \left[\mathbf{H}^{-1}\mathbf{x}\right]_4 H_{n+3}, \quad n \geqslant 0 \tag{2.6}$$

where

$$H_n = U_n\left(-\frac{b - \sqrt{b^2 + 8c^2 - 4ac}}{4c}\right) + U_n\left(-\frac{b + \sqrt{b^2 + 8c^2 - 4ac}}{4c}\right), \tag{2.7}$$

and

$$\mathbf{H} = \begin{bmatrix} H_0 & H_1 & H_2 & H_3 \\ H_1 & H_2 & H_3 & H_4 \\ H_2 & H_3 & H_4 & H_5 \\ H_3 & H_4 & H_5 & H_6 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}. \tag{2.8}$$

Proof. Let ξ be a root of $Q(x) = cx^4 + bx^3 + ax^2 + bx + c$. Thus, $1/\xi$ is also a root of $Q(x)$ because $Q(x)$ is four self-reciprocal. Hence,

$$c\xi^{k+5} + b\xi^{k+4} + a\xi^{k+3} + b\xi^{k+2} + c\xi^{k+1} = 0 \tag{2.9}$$

and

$$c\left(\frac{1}{\xi}\right)^{k+5} + b\left(\frac{1}{\xi}\right)^{k+4} + a\left(\frac{1}{\xi}\right)^{k+3} + b\left(\frac{1}{\xi}\right)^{k+2} + c\left(\frac{1}{\xi}\right)^{k+1} = 0 \tag{2.10}$$

for each $k \geqslant 0$. Setting $\theta = \frac{1}{2}\left(\xi + \frac{1}{\xi}\right)$, we have

$$4c\theta^2 + 2b\theta + (a - 2c) = 0. \tag{2.11}$$

From $a + 2c + 2b \neq 0$ and $a + 2c - 2b \neq 0$, it follows that $\xi - 1/\xi \neq 0$. Hence, (2.4), (2.9), and (2.10) give

$$cU_{k+4}(\theta) + bU_{k+3}(\theta) + aU_{k+2}(\theta) + bU_{k+1}(\theta) + cU_k(\theta) = 0$$

where θ is a solution of (2.11). Therefore, the sequence

$$x_n = \alpha H_n + \beta H_{n+1} + \mu H_{n+2} + \nu H_{n+3}$$

with H_n defined by (2.7) verifies (2.5) for all complex numbers α, β, μ, ν. By noting that the determinant of $\mathbf{H}$ in (2.8) is $(b^2 + 8c^2 - 4ac)^2/c^4$, the following system of linear equations in the variables α, β, μ, ν

$$\begin{cases} \alpha H_0 + \beta H_1 + \mu H_2 + \nu H_3 = x_0 \\ \alpha H_1 + \beta H_2 + \mu H_3 + \nu H_4 = x_1 \\ \alpha H_2 + \beta H_3 + \mu H_4 + \nu H_5 = x_2 \\ \alpha H_3 + \beta H_4 + \mu H_5 + \nu H_6 = x_3 \end{cases}$$

has the unique solution $\alpha = \left[\mathbf{H}^{-1}\mathbf{x}\right]_1$, $\beta = \left[\mathbf{H}^{-1}\mathbf{x}\right]_2$, $\mu = \left[\mathbf{H}^{-1}\mathbf{x}\right]_3$, and $\nu = \left[\mathbf{H}^{-1}\mathbf{x}\right]_4$. The proof is complete. □

Remark 2.3. It should be observed that for computational purposes, the inversion of Hankel matrix $\mathbf{H}$ in the above theorems can be performed by using an algorithm from Trench in [9].

For sequences $\{x_n\}_{n\geqslant 0}$ of complex numbers satisfying

$$x_{n+2m} = -\frac{1}{d_m} \sum_{k=1-m}^{m-1} d_{|k|} x_{n+k+m} - x_n, \quad n \geqslant 0,$$

where m is a positive integer and $d_0, d_1, \ldots, d_m$ are complex numbers such that $d_m \neq 0$, the preceding statements seem to suggest the following general formula

$$x_n = \sum_{k=1}^{2m} \left[\mathbf{H}^{-1}\mathbf{x}\right]_k H_{n+k-1}$$

with H_{n+k-1} involving a sum of Chebyshev polynomials of the second kind, $\mathbf{H} = [H_{k+\ell-2}]_{k,\ell=1}^{2m}$ and $\mathbf{x}$, the corresponding $2m$-dimensional vector of the initial conditions. It would be interesting to find the appropriate assumptions on the coefficients $d_0, d_1, \ldots, d_m$ that allow us to establish such a formula, and the explicit form of components H_{n+k-1} that forms it.

3 Ladder networks

Historically, Morgan-Voyce polynomials were introduced by A.M. Morgan-Voyce in 1959 [4] having, as a major motivation, ladder networks of n resistors (see Chapter 41 of [2] for expository reference). These polynomials are defined recursively as follows:

$$b_n(t) = tB_{n-1}(t) + b_{n-1}(t), \tag{3.1}$$
$$B_n(t) = (t+1)B_{n-1}(t) + b_{n-1}(t), \tag{3.2}$$

where $b_0(t) = 1 = B_0(t)$ and $n \geqslant 1$ (see [8]). For any $n \geqslant 1$, it is straightforward to see that

$$b_n(t) \underset{(3.1)}{=} tB_{n-1}(t)+b_{n-1}(t) \underset{(3.2)}{=} tB_{n-1}(t)+[B_n(t) - (t+1)B_{n-1}(t)] = B_n(t)-B_{n-1}(t). \tag{3.3}$$

Because

$$B_{n+1}(t) - (t+1)B_n(t) \underset{(3.2)}{=} b_n(t) \underset{(3.3)}{=} B_n(t) - B_{n-1}(t), \quad n \geqslant 1,$$

we obtain the Morgan-Voyce polynomials of the first kind

$$\begin{cases} B_0(t) = 1, \\ B_1(t) = t + 2, \\ B_n(t) = (t + 2)\, B_{n-1}(t) - B_{n-2}(t), \quad n \geqslant 2. \end{cases} \tag{3.4}$$

On the other hand, (3.1) can be rewritten as

$$tB_n(t) = b_{n+1}(t) - b_n(t), \quad n \geqslant 0, \tag{3.1'}$$

and for each $n \geqslant 0$, we have

$$\begin{aligned} [b_{n+2}(t) - b_{n+1}(t)] - tb_{n+1}(t) &\underset{(3.1')}{=} tB_{n+1}(t) - tb_{n+1}(t) \\ &\underset{(3.3)}{=} t\,[b_{n+1}(t) + B_n(t)] - tb_{n+1}(t) \\ &\underset{(3.1')}{=} b_{n+1}(t) - b_n(t), \end{aligned}$$

which leads to the Morgan-Voyce polynomials of the second kind

$$\begin{cases} b_0(t) = 1, \\ b_1(t) = t + 1, \\ b_n(t) = (t + 2)\, b_{n-1}(t) - b_{n-2}(t), \quad n \geqslant 2. \end{cases} \tag{3.5}$$

The Morgan-Voyce polynomials of first and second kind were extensively studied in the past, namely by Swamy, who established many of their identities and properties (see [8], [7] and [6]).

An application of the formulas presented in previous section can be seen in the calculus of electric magnitudes in ladder networks and electric line theory. Following closely [3], the electric magnitudes in general ladder networks are obtained by introducing a parameter P that includes the ratio between their longitudinal and lateral elements. In these computations, Lahr [3] considered the Morgan-Voyce polynomials of the first kind

$$\begin{cases} B_1 = 1, \\ B_2 = P, \\ B_{n+2} = P\, B_{n+1} - B_n, \quad n \geqslant 1, \end{cases}$$

and the Morgan-Voyce polynomials of the second kind

$$\begin{cases} b_1 = 1, \\ b_2 = P - 1, \\ b_{n+2} = P\, b_{n+1} - b_n, \quad n \geqslant 1, \end{cases}$$

distinguishing several different particular cases for the real number P. $P = t + 2$ in (3.4) and (3.5); let us also point out that from mathematical point of view, it does not matter if the indexation of these polynomials starts with 0 or 1. Our Theorem 2.1 allows us to exempt the aforementioned case division; indeed, supposing $P \neq 2$ and $P \neq -2$, we get

$$B_n = U_{n-1}\left(\frac{P}{2}\right), \quad n \geqslant 1, \tag{3.6}$$

and

$$b_n = (1-P)U_{n-1}\left(\frac{P}{2}\right) + U_n\left(\frac{P}{2}\right), \quad n \geqslant 1 \tag{3.7}$$

because

$$\begin{bmatrix} U_0\left(\frac{P}{2}\right) & U_1\left(\frac{P}{2}\right) \\ U_1\left(\frac{P}{2}\right) & U_2\left(\frac{P}{2}\right) \end{bmatrix}^{-1} = \begin{bmatrix} 1 & P \\ P & P^2-1 \end{bmatrix}^{-1} = \begin{bmatrix} 1-P^2 & P \\ P & -1 \end{bmatrix},$$

$$\begin{bmatrix} 1-P^2 & P \\ P & -1 \end{bmatrix}\begin{bmatrix} 1 \\ P \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} 1-P^2 & P \\ P & -1 \end{bmatrix}\begin{bmatrix} 1 \\ P-1 \end{bmatrix} = \begin{bmatrix} 1-P \\ 1 \end{bmatrix}.$$

Notice that both expressions (3.6) and (3.7) remain valid when $P = 2$ or $P = -2$ because $U_n(\pm 1) = (\pm 1)^n(n+1)$, $n \geqslant 0$, confirming the expressions already stated in [3] (page 157).

If P is a complex number, i.e., $P = a + bi$ with i denoting the imaginary unit, then its real and imaginary parts must be taken into account on the computation of the complex Morgan-Voyce polynomials of first and second kind required in the determination of currents and voltages. This leads to a fourth-order linear homogeneous recurrence relation of the form (2.5) for the real and imaginary parts of such polynomials. According to Theorem 2.2, a nice and lightweight solution using Chebyshev polynomials of the second kind can be reached, avoiding the use of many auxiliary constants as those admitted in [3]. For instance, the real part M_n of the complex Morgan-Voyce polynomials of the first kind is given by

$$\begin{cases} M_1 = 1, \\ M_2 = a, \\ M_3 = a^2 - b^2 - 1, \\ M_4 = a\left(a^2 - 3b^2 - 2\right), \\ M_{n+4} = 2aM_{n+3} - \left(2 + a^2 + b^2\right)M_{n+2} + 2aM_{n+1} - M_n, \quad n \geqslant 1, \end{cases}$$

with a and b real numbers. Assuming $b \neq 0$ and setting $H_k = U_k\left(\frac{a+bi}{2}\right) + U_k\left(\frac{a-bi}{2}\right)$, $k \geqslant 0$, we get, after some calculations,

$$\begin{bmatrix} H_0 & H_1 & H_2 & H_3 \\ H_1 & H_2 & H_3 & H_4 \\ H_2 & H_3 & H_4 & H_5 \\ H_3 & H_4 & H_5 & H_6 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ a \\ a^2 - b^2 - 1 \\ a\left(a^2 - 3b^2 - 2\right) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

and from Theorem 2.2,

$$M_n = \frac{1}{2}U_{n-1}\left(\frac{a+bi}{2}\right) + \frac{1}{2}U_{n-1}\left(\frac{a-bi}{2}\right), \quad n \geqslant 1.$$

The imaginary part N_n of the complex Morgan-Voyce polynomials of the first kind can also be derived through Chebyshev polynomials; we have

$$\begin{cases} N_1 = 0, \\ N_2 = b, \\ N_3 = 2ab, \\ N_4 = b\left(3a^2 - b^2 - 2\right), \\ N_{n+4} = 2aN_{n+3} - \left(2 + a^2 + b^2\right) N_{n+2} + 2aN_{n+1} - N_n, \quad n \geqslant 1, \end{cases}$$

and Theorem 2.2 still yields, for every $n \geqslant 1$,

$$N_n = \frac{1}{2b}\left[U_{n+2}\left(\frac{a+bi}{2}\right) + U_{n+2}\left(\frac{a-bi}{2}\right)\right] - \frac{a}{b}\left[U_{n+1}\left(\frac{a+bi}{2}\right) + U_{n+1}\left(\frac{a-bi}{2}\right)\right] + \frac{1+a^2+b^2}{2b}\left[U_n\left(\frac{a+bi}{2}\right) + U_n\left(\frac{a-bi}{2}\right)\right] - \frac{a}{2b}\left[U_{n-1}\left(\frac{a+bi}{2}\right) + U_{n-1}\left(\frac{a-bi}{2}\right)\right]$$

by noting that

$$\begin{bmatrix} H_0 & H_1 & H_2 & H_3 \\ H_1 & H_2 & H_3 & H_4 \\ H_2 & H_3 & H_4 & H_5 \\ H_3 & H_4 & H_5 & H_6 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ b \\ 2ab \\ b\left(3a^2 - b^2 - 2\right) \end{bmatrix} = \begin{bmatrix} -\frac{a}{2b} \\ \frac{1+a^2+b^2}{2b} \\ -\frac{a}{b} \\ \frac{1}{2b} \end{bmatrix}.$$

Acknowledgements. This work is a contribution to the Project UID/GEO/04035/2013, funded by FCT - Fundação para a Ciência e a Tecnologia, Portugal.

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