Journal of Algebra 533 (2019) 142–173

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Conjugacy in inverse semigroups

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ARTICLE INFO

Article history:
Received 6 October 2018
Available online 31 May 2019
Communicated by Volodymyr Mazorchuk

MSC:
20M10
20M18
20M05
20M20

Keywords:
Inverse semigroups
Conjugacy
Symmetric inverse semigroups
Free inverse semigroups
McAllister P-semigroups
Factorizable inverse monoids
Clifford semigroups
Bicyclic monoid
Stable inverse semigroups

ABSTRACT

In a group G, elements a and b are conjugate if there exists $g \in G$ such that $g^{-1}ag = b$. This conjugacy relation, which plays an important role in group theory, can be extended in a natural way to inverse semigroups: for elements a and b in an inverse semigroup S, a is conjugate to b, which we will write as $a \sim_i b$, if there exists $g \in S^1$ such that $g^{-1}ag = b$ and $gbg^{-1} = a$. The purpose of this paper is to study the conjugacy $\sim_i$ in several classes of inverse semigroups: symmetric inverse semigroups, McAllister P-semigroups, factorizable inverse monoids, Clifford semigroups, the bicyclic monoid, stable inverse semigroups, and free inverse semigroups.

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https://doi.org/10.1016/j.jalgebra.2019.05.022
0021-8693/© 2019 Published by Elsevier Inc.

1. Introduction

The conjugacy relation $\sim_G$ in a group G is defined as follows: for $a, b \in G$, $a \sim_G b$ if there exists $g \in G$ such that $g^{-1}ag = b$ and $b = gag^{-1}$.

A semigroup S is said to be *inverse* if for every $x \in S$, there exists exactly one $x^{-1} \in S$ such that $x = xx^{-1}x$ and $x^{-1} = x^{-1}xx^{-1}$. Thus one can extend the definition of group conjugacy verbatim to inverse semigroups. (As usual S^1 denotes the semigroup S extended by an identity element when none is present.)

Definition 1.1. Let S be an inverse semigroup. Elements $a, b \in S$ are said to be *i-conjugate*, denoted $a \sim_i b$, if there exists $g \in S^1$ such that $g^{-1}ag = b$ and $gbg^{-1} = a$. In short,

$$a \sim_i b \iff \exists_{g\in S^1} \left(g^{-1}ag = b \text{ and } gbg^{-1} = a \right). \tag{1.1}$$

We call the relation $\sim_i$ *i-conjugacy* ("i" for "inverse").

At first glance, this notion of conjugacy for inverse semigroups seems simultaneously both natural and naive: natural because it is an obvious way to extend $\sim_G$ formally to inverse semigroups, and naive because one might not initially expect a formal extension to exhibit much structure. But surprisingly, it turns out that this conjugacy coincides with one that Mark Sapir considered the best notion for inverse semigroups. The aim of this paper is to carry out an in depth study of $\sim_i$, and to show that its naturality goes far beyond its definition; $\sim_i$ is, in fact, as hinted by Sapir, a highly structured and interesting notion of conjugacy.

Our first three results (Section 2) generalize the known theorems for permutations on a set X to partial injective transformations on X.

(1) Elements α and β of the symmetric inverse semigroup $\mathcal{I}(X)$ are i-conjugate if and only if they have the same cycle-chain-ray type (Theorem 2.10).
(2) If X is a finite set with n elements, then $\mathcal{I}(X)$ has $\sum_{r=0}^{n} p(r)p(n-r)$ i-conjugacy classes (Theorem 2.16).
(3) If X is an infinite set with $|X| = \aleph_\varepsilon$, then $\mathcal{I}(X)$ has $\kappa^{\aleph_0}$ i-conjugacy classes, where $\kappa = \aleph_0 + |\varepsilon|$ (Theorem 2.18).

The following results (Sections 3, 4, 5 and 6) concern other important classes of inverse semigroups.

(4) If $S = P(G, \mathcal{X}, \mathcal{Y})$ is a McAlister P-semigroup, then for all $(A, g), (B, h) \in S$, $(A, g) \sim_i (B, h)$ if and only if there exists $(C, k) \in S$ such that (i) $A = kB = C \wedge gC \wedge A$ and (ii) $g = khk^{-1}$ (Theorem 3.1).

(5) If S is an inverse semigroup, then for all $a, b \in S$, the set of all $g \in S^1$ such that $g^{-1}ag = b$ and $gbg^{-1} = a$ is upward closed in the natural partial order on S^1 (Theorem 4.1).
(6) If S is a factorizable inverse monoid, then two elements are $\sim_i$-related if and only if they are conjugate under a unit element. (Corollary 4.2).
(7) If G is a group, then for any Ha, Kb in the coset monoid $\mathcal{CM}(G)$, $Ha \sim_i Kb$ if and only if there exists $g \in G$ such that $g^{-1}Hag = Kb$ and $gKbg^{-1} = Ha$ (Corollary 4.4).
(8) Elements a and b in a Clifford semigroup S are i-conjugate if and only if a and b are group conjugate in some subgroup of S (Theorem 5.7).
(9) If S is an inverse semigroup, then S is a Clifford semigroup if and only if no two different idempotents of S are i-conjugate (Theorem 5.8).
(10) In the bicyclic monoid $\mathcal{B}$, i-conjugacy coincides with the minimal group congruence (Theorem 6.1).
(11) An inverse semigroup S is stable if and only if i-conjugacy and the natural partial order intersect trivially as relations in $S \times S$ (Theorem 6.3).

The next four results (Section 7) concern free inverse semigroups.

(12) Elements $w\tau$ and $v\tau$ of the free inverse semigroup $\mathcal{FI}(X)$ are i-conjugate if and only if their birooted word trees (T_w, α_w, β_w) and (T_v, α_v, β_v) are equal and $w(\Pi(\alpha_w, \alpha_v)) = w(\Pi(\beta_w, \beta_v))$ (Theorem 7.9).
(13) Every i-conjugacy class in $\mathcal{FI}(X)$ is finite (Corollary 7.7).
(14) The number of elements of i-conjugacy class of $w\tau \in \mathcal{FI}(X)$ is equal to the number of vertices of a certain word subtree of (T_w, α_w, β_w) (Corollary 7.11).
(15) The relation $\sim_i$ in $\mathcal{FI}(X)$ is decidable (Corollary 7.15).

We give now a quick overview of the state of the art regarding notions of conjugacy for semigroups. As recalled above, the conjugacy relation $\sim_G$ in a group G is defined by

$$a \sim_G b \quad \iff \quad \exists_{g \in G} \, (g^{-1}ag = b \text{ and } gbg^{-1} = a) . \tag{1.2}$$

(In fact, $\sim_G$ is traditionally defined less symmetrically, but the symmetric form of (1.2) follows since $g^{-1}ag = b$ if and only if $gbg^{-1} = a$.) This definition does not make sense in general semigroups, so conjugacy has been generalized to semigroups in a variety of ways.

For a monoid S with identity element 1, let $U(S)$ denote its group of units. *Unit conjugacy* in S is modeled on $\sim_G$ in groups by

$$a \sim_u b \quad \iff \quad \exists_{g \in U(S)} \, (g^{-1}ag = b \text{ and } gbg^{-1} = a) . \tag{1.3}$$

(We will write "$\sim$" with various subscripts for possible definitions of conjugacy in semigroups. In this case, the subscript u stands for "unit.") See, for instance, [14,15]. However,

$\sim_u$ does not make sense in a semigroup without an identity element. Requiring $g \in U(S^1)$ in unit conjugacy does not help for arbitrary semigroups because $U(S^1) = \{1\}$ if S is not a monoid.

Conjugacy in a group G can, of course, be rewritten without using inverses: $a \sim_G b \in G$ are conjugate if and only if there exists $g \in G$ such that $ag = gb$. Using this formulation, *left conjugacy* $\sim_l$ has been defined for a semigroup S [24,33,34]:

$$a \sim_l b \quad \iff \quad \exists_{g \in S^1}\ ag = gb. \tag{1.4}$$

In a general semigroup S, the relation $\sim_l$ is reflexive and transitive, but not symmetric. In addition, if S has a zero, then $\sim_l$ is the universal relation $S \times S$, so $\sim_l$ is not useful for such semigroups.

The relation $\sim_l$, however, is an equivalence on any free semigroup. Lallement [16] defined two elements of a free semigroup to be conjugate if they are related by $\sim_l$, and then showed that $\sim_l$ is equal to the following relation in a free semigroup S:

$$a \sim_p b \quad \iff \quad \exists_{u,v \in S^1}\ (\, a = uv \ \text{ and } \ b = vu\,). \tag{1.5}$$

In a general semigroup S, $\sim_p \neq \sim_l$ and in fact, the relation $\sim_p$ is reflexive and symmetric, but not necessarily transitive. Kudryavtseva and Mazorchuk [14,15] considered the transitive closure $\sim_p^*$ of $\sim_p$ as a conjugacy relation in a general semigroup. (See also [9].)

Otto [24] studied the relations $\sim_l$ and $\sim_p$ in the monoids S presented by finite Thue systems, and then symmetrized $\sim_l$ to give yet another definition of conjugacy in such an S:

$$a \sim_o b \quad \iff \quad \exists_{g,h \in S^1}\ (\, ag = gb \ \text{ and } \ bh = ha\,)\,. \tag{1.6}$$

The relation $\sim_o$ is an equivalence relation in an arbitrary semigroup S, but, again, it is the universal relation for any semigroup with zero.

This deficiency of $\sim_o$ was remedied in [2], where the following relation was defined on an arbitrary semigroup S:

$$a \sim_c b \quad \iff \quad \exists_{g \in \mathbb{P}^1(a)}\, \exists_{h \in \mathbb{P}^1(b)}\ (\, ag = gb \ \text{ and } \ bh = ha\,), \tag{1.7}$$

where for $a \neq 0$, $\mathbb{P}(a) = \{g \in S : \forall_{m \in S^1}\,(ma \neq 0 \Rightarrow (ma)g \neq 0)\}$, $\mathbb{P}(0) = \{0\}$, and $\mathbb{P}^1(a) = \mathbb{P}(a) \cup \{1\}$. (See [2, §2] for a motivation for this definition.) The relation $\sim_c$ is an equivalence on S, it does not reduce to $S \times S$ if S has a zero, and it is equal to $\sim_o$ if S does not have a zero.

In 2018, the third author [13] defined a conjugacy $\sim_\mathfrak{n}$ on any semigroup S by

$$a \sim_\mathfrak{n} b \quad \iff \quad \exists_{g,h \in S^1}\ (\, ag = gb,\ bh = ha,\ hag = b, \ \text{ and } \ gbh = a\,)\,. \tag{1.8}$$

The relation $\sim_{\mathfrak{n}}$ is an equivalence relation on any semigroup and it does not reduce to $S \times S$ if S has a zero. In fact, it is the smallest of all conjugacies defined up to this point for general semigroups.

We point out that each of the relations (1.3)–(1.8) reduces to group conjugacy when S is a group. However, assuming we require conjugacy to be an equivalence relation on general semigroups, only $\sim_p^*$, $\sim_o$, $\sim_c$, and $\sim_{\mathfrak{n}}$ can provide possible definitions of conjugacy. We have

$$\sim_{\mathfrak{n}} \ \subseteq \sim_p^* \subseteq \ \sim_o \text{ and } \sim_{\mathfrak{n}} \ \subseteq \sim_c \subseteq \ \sim_o,$$

and, with respect to inclusion, $\sim_p^*$ and $\sim_c$ are not comparable [13, Prop. 2.3]. For detailed comparison and analysis, in various classes of semigroups, of the conjugacies $\sim_p^*$, $\sim_o$, $\sim_c$, and also trace (character) conjugacy $\sim_{tr}$ defined for epigroups, see [1].

A notion of conjugacy for inverse semigroups equivalent to our $\sim_i$ has appeared elsewhere. In fact, part of our motivation for the present study was a MathOverflow post by Sapir [28], in which he claimed that the following is the best notion of conjugacy in inverse semigroups: for a, b in an inverse semigroup S, a is conjugate to b if there exists $t \in S^1$ such that

$$t^{-1}at = b, \;\; a \cdot tt^{-1} = tt^{-1} \cdot a = a, \;\text{ and }\; b \cdot t^{-1}t = t^{-1}t \cdot b = b. \tag{1.9}$$

Sapir notes that this notion of conjugacy is implicit in the work of Yamamura [32]. It is easy to show that Sapir's relation coincides with $\sim_i$ (Proposition 1.3).

Of the conjugacies $\sim_p^*$, $\sim_o$, $\sim_c$, and $\sim_{\mathfrak{n}}$ defined for an arbitrary semigroup S, only $\sim_{\mathfrak{n}}$ reduces to $\sim_i$ if S is an inverse semigroup [13, Thm. 2.6]. Observe also that if S is an inverse monoid,

$$\sim_u \ \subseteq \ \sim_i . \tag{1.10}$$

This inclusion is generally proper, but we will see that equality holds in factorizable inverse monoids (Corollary 4.2).

We conclude this introduction with three general results about i-conjugacy. In an inverse semigroup S, both of the following identities hold: for all $x, y \in S$,

$$(x^{-1})^{-1} = x \;\text{ and }\; (xy)^{-1} = y^{-1}x^{-1} .$$

From these, the following is easy to see.

Lemma 1.2. *The conjugacy $\sim_i$ is an equivalence relation in any inverse semigroup.*

Proof. Let S be an inverse semigroup. Then $\sim_i$ is reflexive (since $1 \in S^1$) and symmetric (since $(g^{-1})^{-1} = g$ for every $g \in S$). For all $a, b, c \in S$, $g, h \in S^1$, if $g^{-1}ag = b$, $gbg^{-1} = a$, $h^{-1}bh = c$, and $hch^{-1} = b$, then $(gh)^{-1} \cdot a \cdot gh = c$ and $gh \cdot c \cdot (gh)^{-1} = a$. Thus $\sim_i$ is also transitive. $\square$

For an inverse semigroup S, the equivalence class of $a \in S$ with respect to $\sim_i$ will be called the *i-conjugacy class* of a and denoted by $[a]_{\sim_i}$.

The following proposition shows that (1.1) is equivalent to Sapir's formulation (1.9), and also shows the specific connection between the conjugacies $\sim_i$ and $\sim_o$. For an inverse semigroup S, $a, b \in S$ and $g \in S^1$, we consider the following equations.

$$
\begin{array}{rlrl}
\text{(i)} & g^{-1}ag = b & \text{(ii)} & gbg^{-1} = a \\
\text{(iii)} & ag = gb & \text{(iv)} & bg^{-1} = g^{-1}a \\
\text{(v)} & a \cdot gg^{-1} = a & \text{(vi)} & gg^{-1} \cdot a = a \\
\text{(vii)} & b \cdot g^{-1}g = b & \text{(viii)} & g^{-1}g \cdot b = b
\end{array}
$$

Proposition 1.3. *Let S be an inverse semigroup. For $a, b \in S$ and $g \in S^1$, the following sets of conditions are equivalent and each set implies all of (i)–(viii).*

(a) {(i), (ii)} (*that is, $a \sim_i b$*);
(b) {(i), (v), (vi)};
(c) {(iii), (v), (viii)};
(d) {(ii), (vii), (viii)};
(e) {(iv), (vi), (vii)}.

Proof. (a) $\Longrightarrow$ (b): $a \cdot gg^{-1} = gbg^{-1}gg^{-1} = gbg^{-1} = a$ and $gg^{-1} \cdot a = gg^{-1}gbg^{-1} = gbg^{-1} = a$.

(b) $\Longrightarrow$ (c): $ag = gg^{-1} \cdot ag = gb$ and $g^{-1}g \cdot b = g^{-1}g \cdot g^{-1}ag = g^{-1}ag = b$.

(c) $\Longrightarrow$ (a): $g^{-1}ag = g^{-1}g \cdot b = b$ and $gbg^{-1} = a \cdot gg^{-1} = a$.

The cycle of implications (a) $\Longrightarrow$ (d) $\Longrightarrow$ (e) $\Longrightarrow$ (a) follows from the cycle already proven by exchanging the roles of a and b and replacing g with g^{-1} (since $(g^{-1})^{-1} = g$). □

Finally, we characterize one of the two extreme cases for i-conjugacy on an inverse semigroup S, namely where $\sim_i$ is the universal relation $S \times S$. In Theorem 5.9 we will consider the opposite extreme, where $\sim_i$ is the identity relation (equality). Similar discussions for other notions of conjugacy can be found in [1].

For an inverse semigroup S, we denote by $E(S)$ the semilattice of idempotents of S [10, p. 146].

Theorem 1.4. *Let S be an inverse semigroup. Then $\sim_i$ is the universal relation $S \times S$ if and only if S is a singleton.*

Proof. Suppose $\sim_i$ is universal. For all $e \in E(S)$ and $g \in S$, we have $g^{-1}eg \in E(S)$, and so every element of S is an idempotent, that is, S is a semilattice. Now for $e, f \in S$, let $g \in S$ be given such that $g^{-1}eg = f$ and $gfg^{-1} = e$. Since $g^{-1} = g$, we have $f = geg = egg = eg$ and so $e = gfg = fgg = fg = (eg)g = eg = f$. Therefore S has only one element. The converse is trivial. □

2. Conjugacy in symmetric inverse semigroups

For a nonempty set X (finite or infinite), denote by $\mathcal{I}(X)$ the *symmetric inverse semigroup* on X, that is, the semigroup of partial injective transformations on X under composition. The semigroup $\mathcal{I}(X)$ is universal for the class of inverse semigroups (see [25] and [10, Ch. 5]) since every inverse semigroup can be embedded in some $\mathcal{I}(X)$ [10, Thm. 5.1.7]. This is analogous to the fact that every group can be embedded in some symmetric group $\mathrm{Sym}(X)$ of permutations on a set X. The semigroup $\mathcal{I}(X)$ has $\mathrm{Sym}(X)$ as its group of units and contains a zero (the empty transformation, which we will denote by 0).

In this section, we will describe i-conjugacy in $\mathcal{I}(X)$ and its ideals, and count the i-conjugacy classes in $\mathcal{I}(X)$ for both finite and infinite X.

2.1. Cycle-chain-ray decomposition of elements of $\mathcal{I}(X)$

The cycle decomposition of a permutation can be extended to the cycle-chain-ray decomposition of a partial injective transformation (see [12]).

We will write functions on the right and compose from left to right; that is, for $f : A \to B$ and $g : B \to C$, we will write xf, rather than $f(x)$, and $x(fg)$, rather than $g(f(x))$. Let $\alpha \in \mathcal{I}(X)$. We denote the domain of α by $\mathrm{dom}(\alpha)$ and the image of α by $\mathrm{im}(\alpha)$. The union $\mathrm{dom}(\alpha) \cup \mathrm{im}(\alpha)$ will be called the *span* of α and denoted $\mathrm{span}(\alpha)$. We say that α and β in $\mathcal{I}(X)$ are *completely disjoint* if $\mathrm{span}(\alpha) \cap \mathrm{span}(\beta) = \emptyset$. For $x, y \in X$, we write $x \overset{\alpha}{\to} y$ if $x \in \mathrm{dom}(\alpha)$ and $x\alpha = y$.

Definition 2.1. Let M be a set of pairwise completely disjoint elements of $\mathcal{I}(X)$. The *join* of the elements of M, denoted $\bigsqcup_{\gamma\in M} \gamma$, is the element of $\mathcal{I}(X)$ whose domain is $\bigcup_{\gamma\in M} \mathrm{dom}(\gamma)$ and whose values are defined by

$$x(\bigsqcup_{\gamma\in M} \gamma) = x\gamma_0,$$

where γ_0 is the (unique) element of M such that $x \in \mathrm{dom}(\gamma_0)$. If $M = \emptyset$, we define $\bigsqcup_{\gamma\in M} \gamma$ to be 0 (the zero in $\mathcal{I}(X)$). If $M = \{\gamma_1, \gamma_2, \dots, \gamma_k\}$ is finite, we may write the join as $\gamma_1 \sqcup \gamma_2 \sqcup \cdots \sqcup \gamma_k$.

Definition 2.2. Let $\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots$ be pairwise distinct elements of X. The following elements of $\mathcal{I}(X)$ will be called *basic* partial injective transformations on X.

- A *cycle* of length k ($k \ge 1$), written $(x_0\, x_1 \dots x_{k-1})$, is an element $\delta \in \mathcal{I}(X)$ with $\mathrm{dom}(\delta) = \{x_0, x_1, \dots, x_{k-1}\}$, $x_i\delta = x_{i+1}$ for all $0 \le i < k-1$, and $x_{k-1}\delta = x_0$.
- A *chain* of length k ($k \ge 1$), written $[x_0\, x_1 \dots x_k]$, is an element $\theta \in \mathcal{I}(X)$ with $\mathrm{dom}(\theta) = \{x_0, x_1, \dots, x_{k-1}\}$ and $x_i\theta = x_{i+1}$ for all $0 \le i \le k-1$.

- A *double ray*, written $\langle \dots x_{-1}\, x_0\, x_1 \dots\rangle$, is an element $\omega \in \mathcal{I}(X)$ with $\mathrm{dom}(\omega) = \{\dots, x_{-1}, x_0, x_1, \dots\}$ and $x_i\omega = x_{i+1}$ for all i.
- A *right ray*, written $[x_0\, x_1\, x_2 \dots\rangle$, is an element $\upsilon \in \mathcal{I}(X)$ with $\mathrm{dom}(\upsilon) = \{x_0, x_1, x_2, \dots\}$ and $x_i\upsilon = x_{i+1}$ for all $i \geq 0$.
- A *left ray*, written $\langle \dots x_2\, x_1\, x_0]$, is an element $\lambda \in \mathcal{I}(X)$ with $\mathrm{dom}(\lambda) = \{x_1, x_2, x_3, \dots\}$ and $x_i\lambda = x_{i-1}$ for all $i > 0$.

By a *ray* we will mean a double, right, or left ray.

We note the following.

- The span of a basic partial injective transformation is exhibited by the notation. For example, the span of the right ray $[1\,2\,3 \dots\rangle$ is $\{1, 2, 3, \dots\}$.
- The left bracket in "$\eta = [x \dots$" indicates that $x \notin \mathrm{im}(\eta)$; while the right bracket in "$\eta = \dots x]$" indicates that $x \notin \mathrm{dom}(\eta)$. For example, for the chain $\theta = [1\,2\,3\,4]$, $\mathrm{dom}(\theta) = \{1, 2, 3\}$ and $\mathrm{im}(\theta) = \{2, 3, 4\}$.
- A cycle $(x_0\, x_1 \dots x_{k-1})$ differs from the corresponding cycle in the symmetric group of permutations on X in that the former is undefined for every $x \in (X \setminus \{x_0, x_1, \dots, x_{k-1}\})$, while the latter fixes every such x.

The following decomposition was proved in [12, Prop. 2.4].

Proposition 2.3. *Let $\alpha \in \mathcal{I}(X)$ with $\alpha \neq 0$. Then there exist unique sets: Δ_α of cycles, Θ_α of chains, Ω_α of double rays, Υ_α of right rays, and Λ_α of left rays such that the transformations in $\Delta_\alpha \cup \Theta_\alpha \cup \Omega_\alpha \cup \Upsilon_\alpha \cup \Lambda_\alpha$ are pairwise completely disjoint and*

$$\alpha = \bigsqcup_{\delta \in \Delta_\alpha} \delta \sqcup \bigsqcup_{\theta \in \Theta_\alpha} \theta \sqcup \bigsqcup_{\omega \in \Omega_\alpha} \omega \sqcup \bigsqcup_{\upsilon \in \Upsilon_\alpha} \upsilon \sqcup \bigsqcup_{\lambda \in \Lambda_\alpha} \lambda. \tag{2.1}$$

We will call the join (2.1) the *cycle-chain-ray decomposition* of α. If $\eta \in \Delta_\alpha \cup \Theta_\alpha \cup \Omega_\alpha \cup \Upsilon_\alpha \cup \Lambda_\alpha$, we will say that η is *contained* in α (or that α *contains* η). If $\alpha = 0$, we set $\Delta_\alpha = \Theta_\alpha = \Omega_\alpha = \Upsilon_\alpha = \Lambda_\alpha = \emptyset$. We note the following.

- If $\alpha \in \mathrm{Sym}(X)$, then $\alpha = \bigsqcup_{\delta \in \Delta_\alpha} \delta \sqcup \bigsqcup_{\omega \in \Omega_\alpha} \omega$ (since $\Theta_\alpha = \Upsilon_\alpha = \Lambda_\alpha = \emptyset$), which corresponds to the usual cycle decomposition of a permutation [29, 1.3.4].
- If $\mathrm{dom}(\alpha) = X$, then $\alpha = \bigsqcup_{\delta \in \Delta_\alpha} \delta \sqcup \bigsqcup_{\omega \in \Omega_\alpha} \omega \sqcup \bigsqcup_{\upsilon \in \Upsilon_\alpha} \upsilon$ (since $\Theta_\alpha = \Lambda_\alpha = \emptyset$), which corresponds to the decomposition given in [18].
- If X is finite, then $\alpha = \bigsqcup_{\delta \in \Delta_\alpha} \delta \sqcup \bigsqcup_{\theta \in \Theta_\alpha} \theta$ (since $\Omega_\alpha = \Upsilon_\alpha = \Lambda_\alpha = \emptyset$), which is the decomposition given in [20, Thm. 3.2].

For example, if $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, then

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 6 & - & 5 & 9 & 8 & - & 2 & - \end{pmatrix} \in \mathcal{I}(X)$$

written in cycle-chain decomposition (no rays since X is finite) is $\alpha = (2\,6\,8) \sqcup [1\,3] \sqcup [4\,5\,9]$. The following β is an example of an element of $\mathcal{I}(\mathbb{Z})$ written in cycle-chain-ray decomposition:

$$\beta = (2\,4) \sqcup [6\,8\,10] \sqcup \langle \ldots -6\ -4\ -2\ -1\ -3\ -5\ \ldots\rangle \sqcup [1\,5\,9\,13\,\ldots\rangle \sqcup \langle \ldots 15\,11\,7\,3].$$

2.2. Characterization of $\sim_i$ in $\mathcal{I}(X)$

We will now characterize $\sim_i$ in $\mathcal{I}(X)$ using the cycle-chain-ray decomposition of partial injective transformations.

Notation 2.4. We will fix an element $\diamond \notin X$. For $\alpha \in \mathcal{I}(X)$ and $x \in X$, we will write $x\alpha = \diamond$ if and only if $x \notin \mathrm{dom}(\alpha)$. We will also assume that $\diamond\alpha = \diamond$. With this notation, it will make sense to write $x\alpha = y\beta$ or $x\alpha \neq y\beta$ ($\alpha, \beta \in \mathcal{I}(X)$, $x, y \in X$) even when $x \notin \mathrm{dom}(\alpha)$ or $y \notin \mathrm{dom}(\beta)$.

Lemma 2.5. *Let $\alpha, \beta, \tau \in \mathcal{I}(X)$ and suppose $\tau^{-1}\alpha\tau = \beta$ and $\tau\beta\tau^{-1} = \alpha$. Then for all $x, y \in X$:*

(1) $\mathrm{span}(\alpha) \subseteq \mathrm{dom}(\tau)$ *and* $\mathrm{span}(\beta) \subseteq \mathrm{im}(\tau)$*;*
(2) *if $x \xrightarrow{\alpha} y$ then $x\tau \xrightarrow{\beta} y\tau$;*
(3) *if $x \notin \mathrm{dom}(\alpha)$ and $x \in \mathrm{dom}(\tau)$, then $x\tau \notin \mathrm{dom}(\beta)$;*
(4) *if $x \notin \mathrm{im}(\alpha)$ and $x \in \mathrm{dom}(\tau)$, then $x\tau \notin \mathrm{im}(\beta)$.*

Proof. By Proposition 1.3, $\alpha = \tau(\tau^{-1}\alpha)$ and $\alpha = (\alpha\tau)\tau^{-1}$. Thus $\mathrm{dom}(\alpha) \subseteq \mathrm{dom}(\tau)$ and $\mathrm{im}(\alpha) \subseteq \mathrm{im}(\tau^{-1}) = \mathrm{dom}(\tau)$, and so $\mathrm{span}(\alpha) \subseteq \mathrm{dom}(\tau)$. By the foregoing argument, $\mathrm{span}(\beta) \subseteq \mathrm{dom}(\tau^{-1}) = \mathrm{im}(\tau)$. We have proved (1).

To prove (2), let $x \xrightarrow{\alpha} y$. Since $\alpha\tau = \tau\beta$ (by Proposition 1.3), $(x\tau)\beta = (x\alpha)\tau = y\tau$. Since $y\tau \neq \diamond$ by (1), it follows that $x\tau \xrightarrow{\beta} y\tau$.

To prove (3), let $x \notin \mathrm{dom}(\alpha)$ and $x \in \mathrm{dom}(\tau)$. Then $(x\tau)\beta = (x\alpha)\tau = \diamond\tau = \diamond$. Thus $(x\tau)\beta = \diamond$, that is, $x\tau \notin \mathrm{dom}(\beta)$.

To prove (4), let $x \notin \mathrm{im}(\alpha)$ and $x \in \mathrm{dom}(\tau)$. Suppose to the contrary that $x\tau \in \mathrm{im}(\beta)$. Then $z\beta = x\tau$ for some $z \in X$. By Proposition 1.3, $\beta\tau^{-1} = \tau^{-1}\alpha$. Thus $x = (x\tau)\tau^{-1} = (z\beta)\tau^{-1} = (z\tau^{-1})\alpha$, and so $x \in \mathrm{im}(\alpha)$, which is a contradiction. Hence $x\tau \notin \mathrm{im}(\beta)$. □

Definition 2.6. Let $\ldots, x_{-1}, x_0, x_1, \ldots$ be pairwise distinct elements of X. Let $\delta = (x_0 \ldots x_{k-1})$, $\theta = [x_0\,x_1\,\ldots\,x_k]$, $\omega = \langle \ldots x_{-1}\,x_0\,x_1 \ldots\rangle$, $\upsilon = [x_0\,x_1\,x_2 \ldots\rangle$, and $\lambda = \langle \ldots x_2\,x_1\,x_0]$. For any $\eta \in \{\delta, \theta, \omega, \upsilon, \lambda\}$ and any $\tau \in \mathcal{I}(X)$ such that $\mathrm{span}(\eta) \subseteq \mathrm{dom}(\tau)$, we define $\eta\tau^*$ to be η in which each x_i has been replaced with $x_i\tau$. Since τ is injective, $\eta\tau^*$ is a cycle of length k [chain of length k, double ray, right ray, left ray] if η is a cycle of length k [chain of length k, double ray, right ray, left ray]. For example,

$$\delta\tau^* = (x_0\tau\, x_1\tau \ldots x_{k-1}\tau) \text{ and } \lambda\tau^* = \langle \ldots x_2\tau\, x_1\tau\, x_0\tau].$$

Notation 2.7. For $0 \neq \alpha \in \mathcal{I}(X)$, let Δ_α be the set of cycles and Θ_α be the set of chains that occur in the cycle-chain-ray decomposition of α (see (2.1)). For $k \geq 1$, we denote by Δ_α^k the set of cycles in Δ_α of length k, and by Θ_α^k the set of chains in Θ_α of length k. If $\alpha = 0$, we set $\Delta_\alpha^k = \Theta_\alpha^k = \emptyset$.

For a function $f : A \to B$ and $A_0 \subseteq A$, $A_0 f = \{af : a \in A_0\}$ denotes the image of A_0 under f.

Proposition 2.8. *Let $\alpha, \beta, \tau \in \mathcal{I}(X)$ be such that $\tau^{-1}\alpha\tau = \beta$ and $\tau\beta\tau^{-1} = \alpha$. Then for every $k \geq 1$, $\Delta_\alpha^k\tau^* = \Delta_\beta^k$, $\Theta_\alpha^k\tau^* = \Theta_\beta^k$, $\Omega_\alpha\tau^* = \Omega_\beta$, $\Upsilon_\alpha\tau^* = \Upsilon_\beta$, and $\Lambda_\alpha\tau^* = \Lambda_\beta$.*

Proof. Let $k \geq 1$. Let $\delta = (x_0\, x_1 \ldots x_{k-1}) \in \Delta_\alpha^k$. Then $\delta\tau^* = (x_0\tau\, x_1\tau \ldots x_{k-1}\tau)$. We have $x_0 \overset{\alpha}{\to} x_1 \overset{\alpha}{\to} \cdots \overset{\alpha}{\to} x_{k-1} \overset{\alpha}{\to} x_0$, and so $x_0\tau \overset{\beta}{\to} x_1\tau \overset{\beta}{\to} \cdots \overset{\beta}{\to} x_{k-1}\tau \overset{\beta}{\to} x_0\tau$ by Lemma 2.5. Thus $\delta\tau^* \in \Delta_\beta^k$. We have proved that $\Delta_\alpha^k\tau^* \subseteq \Delta_\beta^k$. Let $\sigma = (y_0\, y_1 \ldots y_{k-1}) \in \Delta_\beta^k$. By the foregoing argument, $\sigma(\tau^{-1})^* = (y_0\tau^{-1}\, y_1\tau^{-1} \ldots y_{k-1}\tau^{-1}) \in \Delta_\alpha^k$. Further, $(\sigma(\tau^{-1})^*)\tau^* = (y_0\tau^{-1}\tau\, y_1\tau^{-1}\tau \ldots y_{k-1}\tau^{-1}\tau) = (y_0\, y_1 \ldots y_{k-1}) = \sigma$. It follows that $\Delta_\alpha^k\tau^* = \Delta_\beta^k$.

Let $\theta = [x_0\, x_1 \ldots x_k] \in \Theta_\alpha^k$. Then $\theta\tau^* = [x_0\tau\, x_1\tau \ldots x_k\tau]$. We have $x_0 \overset{\alpha}{\to} x_1 \overset{\alpha}{\to} \cdots \overset{\alpha}{\to} x_k$, and so $x_0\tau \overset{\beta}{\to} x_1\tau \overset{\beta}{\to} \cdots \overset{\beta}{\to} x_k\tau$ by Lemma 2.5. Also by Lemma 2.5, $x_0\tau \notin \mathrm{im}(\beta)$ (since $x_0 \notin \mathrm{im}(\alpha)$) and $x_k\tau \notin \mathrm{dom}(\beta)$ (since $x_k \notin \mathrm{dom}(\alpha)$). Thus $\theta\tau^* \in \Theta_\beta^k$. We have proved that $\Theta_\alpha^k\tau^* \subseteq \Theta_\beta^k$. Let $\eta = (y_0\, y_1 \ldots y_{k-1}) \in \Theta_\beta^k$. By the foregoing argument, $\eta(\tau^{-1})^* = [y_0\tau^{-1}\, y_1\tau^{-1} \ldots y_k\tau^{-1}] \in \Theta_\alpha^k$. Further, $(\eta(\tau^{-1})^*)\tau^* = [y_0\tau^{-1}\tau\, y_1\tau^{-1}\tau \ldots y_k\tau^{-1}\tau] = [y_0\, y_1 \ldots y_k] = \eta$. It follows that $\Theta_\alpha^k\tau^* = \Theta_\beta^k$.

The proofs of the remaining equalities are similar. □

Definition 2.9. Let $\alpha \in \mathcal{I}(X)$. The sequence

$$\langle |\Delta_\alpha^1|, |\Delta_\alpha^2|, |\Delta_\alpha^3|, \ldots; |\Theta_\alpha^1|, |\Theta_\alpha^2|, |\Theta_\alpha^3|, \ldots; |\Omega_\alpha|, |\Upsilon_\alpha|, |\Lambda_\alpha| \rangle$$

(indexed by the elements of the ordinal $2\omega + 3$) will be called the *cycle-chain-ray type* of α. This notion generalizes the cycle type of a permutation [5, p. 126].

The cycle-chain-ray type of α is completely determined by the *form* of the cycle-chain-ray decomposition of α. The form is obtained from the decomposition by omitting each occurrence of the symbol "$\sqcup$" and replacing each element of X by some generic symbol, say "$*$." For example, $\alpha = (2\,6\,8) \sqcup [1\,3] \sqcup [4\,5\,9]$ has the form $(*\,*\,*)[*\,*][*\,*\,*]$, and

$$\beta = (2\,4) \sqcup [6\,8\,10] \sqcup \langle \ldots -6\ -4\ -2\ -1\ -3\ -5 \ldots\rangle \sqcup [1\,5\,9\,13 \ldots\rangle \sqcup \langle \ldots 15\,11\,7\,3]$$

has the form $(*\,*)[*\,*\,*]\langle \ldots *\,*\,* \ldots\rangle[*\,*\,* \ldots\rangle\langle \ldots *\,*\,*]$.

It is well known that two elements of the symmetric group $\mathrm{Sym}(X)$ are conjugate if and only if they have the same cycle type [5, Prop. 11, p. 126]. The following description of the i-conjugacy in the symmetric inverse semigroup $\mathcal{I}(X)$ generalizes this result.

Theorem 2.10. *Elements α and β of $\mathcal{I}(X)$ are i-conjugate if and only if they have the same cycle-chain-ray type.*

Proof. Let $\alpha, \beta \in \mathcal{I}(X)$. Suppose $\alpha \sim_i \beta$, that is, there is $\tau \in \mathcal{I}(X)$ such that $\tau^{-1}\alpha\tau = \beta$ and $\tau\beta\tau^{-1} = \alpha$. Then α and β have the same type by Proposition 2.8 and the fact that τ^* restricted to any set from $\{\Delta_\alpha^k : k \geq 1\} \cup \{\Theta_\alpha^k : k \geq 1\} \cup \{\Omega_\alpha, \Upsilon_\alpha, \Lambda_\alpha\}$ is injective.

Conversely, suppose α and β have the same cycle-chain-ray type. Then for every $k \geq 1$, there are bijections $f_k : \Delta_\alpha^k \to \Delta_\beta^k$, $g_k : \Theta_\alpha^k \to \Theta_\beta^k$, $h : \Omega_\alpha \to \Omega_\beta$, $i : \Upsilon_\alpha \to \Upsilon_\beta$, and $j : \Lambda_\alpha \to \Lambda_\beta$. For all $\delta \in \Delta_\alpha^k$, $\theta \in \Theta_\alpha^k$, $\omega \in \Omega_\alpha$, $\upsilon \in \Upsilon_\alpha$, and $\lambda \in \Lambda_\alpha$, we define τ on $\mathrm{span}(\delta) \cup \mathrm{span}(\theta) \cup \mathrm{span}(\omega) \cup \mathrm{span}(\upsilon) \cup \mathrm{span}(\lambda)$ in such a way that $\delta\tau^* = \delta f_k$, $\theta\tau^* = \theta g_k$, $\omega\tau^* = \omega h$, $\upsilon\tau^* = \upsilon i$, and $\lambda\tau^* = \lambda j$. Note that this defines an injective τ with $\mathrm{dom}(\tau) = \mathrm{span}(\alpha)$ and $\mathrm{im}(\tau) = \mathrm{span}(\beta)$.

Let $x \in X$. We will prove that $x(\tau^{-1}\alpha\tau) = x\beta$. If $x \notin \mathrm{span}(\beta)$ then $x \notin \mathrm{dom}(\tau^{-1})$ (since $\mathrm{dom}(\tau^{-1}) = \mathrm{im}(\tau) = \mathrm{span}(\beta)$), and so $x(\tau^{-1}\alpha\tau) = \diamond(\alpha\tau) = \diamond$ and $x\beta = \diamond$. Suppose $x \in (\mathrm{im}(\beta) \setminus \mathrm{dom}(\beta))$. Then there is $\xi = \ldots x]$ that is either a chain or left ray contained in β. By the definition of τ, there is $\eta = \ldots z]$ that is either a chain or left ray contained in α with $z\tau = x$. Then $x(\tau^{-1}\alpha\tau) = z(\alpha\tau) = \diamond\tau = \diamond$ and $x\beta = \diamond$. Finally, suppose $x \in \mathrm{dom}(\beta)$. Then there is $\xi = \ldots x\, y \ldots$ that is a basic partial injective transformation contained in β. By the definition of τ, there is $\eta = \ldots z\, w \ldots$ that is a basic partial injective transformation contained in α with $z\tau = x$ and $w\tau = y$. Then $x(\tau^{-1}\alpha\tau) = z(\alpha\tau) = w\tau = y$ and $x\beta = y$.

We have proved that $\tau^{-1}\alpha\tau = \beta$. By the same argument, applied to τ^{-1}, β, α instead of τ, α, β, we have $\tau\beta\tau^{-1} = \alpha$. Hence $\alpha \sim_i \beta$. □

Theorem 2.10 also follows from [13, Cor. 5.2] and the fact that $\sim_i \; = \; \sim_{\mathfrak{n}}$ in inverse semigroups (see Section 1). However, the proof in [13] is not direct since it relies on a characterization of $\sim_{\mathfrak{n}}$ in subsemigroups of the semigroup $P(X)$ of all partial transformations on X.

Suppose X is finite with $|X| = n$ and let $\alpha \in \mathcal{I}(X)$. Then α contains no rays, no cycles of length greater than n, and no chains of length greater than $n-1$. Therefore, the cycle-chain-ray type of α can be written as

$$\langle |\Delta_\alpha^1|, |\Delta_\alpha^2|, \ldots, |\Delta_\alpha^n|; |\Theta_\alpha^1|, |\Theta_\alpha^2|, \ldots, |\Theta_\alpha^{n-1}| \rangle. \tag{2.2}$$

We will refer to (2.2) as the *cycle-chain type* of α. By Theorem 2.10, for all $\alpha, \beta \in \mathcal{I}(X)$,

$$\alpha \sim_i \beta \quad \iff \quad \langle |\Delta_\alpha^1|, \ldots, |\Delta_\alpha^n|; |\Theta_\alpha^1|, \ldots, |\Theta_\alpha^{n-1}| \rangle = \langle |\Delta_\beta^1|, \ldots, |\Delta_\beta^n|; |\Theta_\beta^1|, \ldots, |\Theta_\beta^{n-1}| \rangle. \tag{2.3}$$

Suppose $\alpha \in \mathcal{I}(X)$ has a finite domain. Then α does not contain any rays. Therefore, we will refer to the cycle-chain-ray type of α as the cycle-chain type of α even when X is infinite.

By (1.1), α and β in $\mathcal{I}(X)$ are i-conjugate if and only if there exists $\tau \in \mathcal{I}(X)$ such that $\tau^{-1}\alpha\tau = \beta$ and $\tau\beta\tau^{-1} = \alpha$. If X is finite, we can replace τ with a permutation on X.

Proposition 2.11. *Let X be a finite set, and let $\alpha, \beta \in \mathcal{I}(X)$. Then the following conditions are equivalent:*

(i) *α and β are i-conjugate;*
(ii) *α and β have the same cycle-chain type;*
(iii) *there exists $\sigma \in \mathrm{Sym}(X)$ such that $\sigma^{-1}\alpha\sigma = \beta$.*

Proof. Conditions (i) and (ii) are equivalent by Theorem 2.10, and (iii) clearly implies (i). It remains to show that (i) implies (iii). Suppose (i) holds, that is, $\tau^{-1}\alpha\tau = \beta$ and $\tau\beta\tau^{-1} = \alpha$ for some $\tau \in \mathcal{I}(X)$. By Proposition 2.8, τ maps $\mathrm{span}(\alpha)$ onto $\mathrm{span}(\beta)$. Thus $|\mathrm{span}(\alpha)| = |\mathrm{span}(\beta)|$, and so, since X is finite, $|X \setminus \mathrm{span}(\alpha)| = |X \setminus \mathrm{span}(\beta)|$. We fix a bijection $f : X \setminus \mathrm{span}(\alpha) \to X \setminus \mathrm{span}(\beta)$ and define $\sigma : X \to X$ by

$$x\sigma = \begin{cases} x\tau & \text{if } x \in \mathrm{span}(\alpha), \\ xf & \text{if } x \in (X \setminus \mathrm{span}(\alpha)). \end{cases}$$

Clearly, $\sigma \in \mathrm{Sym}(X)$. Let $x \in X$. If $x \notin \mathrm{span}(\beta)$, then $x\sigma^{-1} \notin \mathrm{span}(\alpha)$, and so $x(\sigma^{-1}\alpha\sigma) = \diamond\sigma = \diamond = x\beta$. Suppose $x \in (\mathrm{im}(\beta) \setminus \mathrm{dom}(\beta))$. Then $x\tau^{-1} = x\sigma^{-1}$ (by the definition of σ) and $x\tau^{-1} \notin \mathrm{dom}(\alpha)$ (by Lemma 2.5). Thus, $x(\sigma^{-1}\alpha\sigma) = x(\tau^{-1}\alpha\sigma) = \diamond\sigma = \diamond = x\beta$. Suppose $x \in \mathrm{dom}(\beta)$. Then $x\tau^{-1} = x\sigma^{-1}$ and $x\tau^{-1} \in \mathrm{dom}(\alpha)$. Hence, $(x\tau^{-1})\alpha \in \mathrm{im}(\alpha)$, and so $((x\tau^{-1})\alpha)\tau = ((x\tau^{-1})\alpha)\sigma$. Therefore, $x(\sigma^{-1}\alpha\sigma) = x(\tau^{-1}\alpha\tau) = x\beta$.

We have proved that $x(\sigma^{-1}\alpha\sigma) = x\beta$ for all $x \in X$, and so (i) implies (iii). □

The equivalence of (ii) and (iii) is stated in [4, p. 120]. Proposition 2.11 is not true for an infinite set X. Let $X = \{1, 2, 3, \ldots\}$ and consider $\alpha = [2\,3\,4\ldots\rangle$ and $\beta = [1\,2\,3\ldots\rangle$ in $\mathcal{I}(X)$. Then α and β are i-conjugate by Theorem 2.10. Note that $1 \notin \mathrm{dom}(\alpha)$ and $\mathrm{dom}(\beta) = X$. Thus, by Lemma 2.5(3), if $\tau \in \mathcal{I}(X)$ is such that $\tau^{-1}\alpha\tau = \beta$ and $\tau\beta\tau^{-1} = \alpha$, then $1 \notin \mathrm{dom}(\tau)$. Consequently, (iii) is not satisfied.

2.3. Conjugacy in the ideals of $\mathcal{I}(X)$

We have already dealt with i-conjugacy in $\mathcal{I}(X)$ (Theorem 2.10). Here, we will describe i-conjugacy in an arbitrary proper (that is, different from $\mathcal{I}(X)$) ideal of $\mathcal{I}(X)$. For $\alpha \in \mathcal{I}(X)$, the *rank* of α is the cardinality of $\mathrm{im}(\alpha)$. Since α is injective, we have

$\text{rank}(\alpha) = |\operatorname{im}(\alpha)| = |\operatorname{dom}(\alpha)|$. For a cardinal r with $0 < r \le |X|$, let $J_r = \{\alpha \in \mathcal{I}(X) : \text{rank}(\alpha) < r\}$. Then the set $\{J_r : 0 < r \le |X|\}$ consists of all proper ideals of $\mathcal{I}(X)$ [19].

Theorem 2.12. *Let J_r be a proper ideal of $\mathcal{I}(X)$, where r is finite, and let $\alpha, \beta \in J_r$. Then α and β are i-conjugate in J_r if and only if they have the same cycle-chain type and $|\operatorname{span}(\alpha)| < r$.*

Proof. Suppose $\alpha \sim_i \beta$ in J_r. Then $\alpha \sim_i \beta$ in $\mathcal{I}(X)$, and so α and β have the same cycle-chain type by Theorem 2.10. Let $\tau \in J_r$ such that $\tau^{-1}\alpha\tau = \beta$ and $\tau\beta\tau^{-1} = \alpha$. Then, by Lemma 2.5, $\operatorname{span}(\alpha) \subseteq \operatorname{dom}(\tau)$, and so $|\operatorname{span}(\alpha)| \le |\operatorname{dom}(\tau)| = \text{rank}(\tau) < r$.

Conversely, suppose that α and β have the same cycle-chain type and $|\operatorname{span}(\alpha)| < r$. Then $\alpha \sim_i \beta$ in $\mathcal{I}(X)$ by Theorem 2.10. In the proof of Theorem 2.10, we constructed $\tau \in \mathcal{I}(X)$ such that $\operatorname{dom}(\tau) = \operatorname{span}(\alpha)$, $\tau^{-1}\alpha\tau = \beta$, and $\tau\beta\tau^{-1} = \alpha$. Since $\text{rank}(\tau) = |\operatorname{dom}(\tau)| = |\operatorname{span}(\alpha)| < r$, we have $\tau \in J_r$, and so $\alpha \sim_i \beta$ in J_r. □

We note that for all $\alpha, \beta \in J_r$, where r is finite,

$$|\operatorname{span}(\alpha)| = \text{rank}(\alpha) + \text{the number of chains in } \alpha,$$

and that if α and β have the same cycle-chain type, then $\text{rank}(\alpha) = \text{rank}(\beta)$ and $|\operatorname{span}(\alpha)| = |\operatorname{span}(\beta)|$.

As an example, let $X = \{1, \dots, 8\}$ and consider $\alpha = (1\,2)[3\,4][5\,6\,7]$ and $\beta = (5\,9)[1\,6][3\,8\,7]$ in $\mathcal{I}(X)$. Then $\alpha, \beta \in J_6$ but they are not i-conjugate in J_6 since $|\operatorname{span}(\alpha)| = 7 > 6$. Note, however, that $\alpha \sim_i \beta$ in J_8.

If r is infinite, then the conjugacy $\sim_i$ in J_r is the restriction of $\sim_i$ in $\mathcal{I}(X)$, that is, for all $\alpha, \beta \in J_r$, $\alpha \sim_i \beta$ in J_r if and only if $\alpha \sim_i \beta$ in $\mathcal{I}(X)$.

Theorem 2.13. *Let J_r be a proper ideal of $\mathcal{I}(X)$, where r is infinite, and let $\alpha, \beta \in J_r$. Then α and β are i-conjugate in J_r if and only if they have the same cycle-chain-ray type.*

Proof. If $\alpha \sim_i \beta$ in J_r, then $\alpha \sim_i \beta$ in $\mathcal{I}(X)$, and so α and β have the same cycle-chain-ray type by Theorem 2.10. Conversely, suppose that α and β have the same cycle-chain-ray type. Then $\alpha \sim_i \beta$ in $\mathcal{I}(X)$ by Theorem 2.10. In the proof of Theorem 2.10, we constructed $\tau \in \mathcal{I}(X)$ such that $\operatorname{dom}(\tau) = \operatorname{span}(\alpha)$, $\tau^{-1}\alpha\tau = \beta$, and $\tau\beta\tau^{-1} = \alpha$. Since $\operatorname{span}(\alpha) = \operatorname{dom}(\alpha) \cup \operatorname{im}(\alpha)$, we have $|\operatorname{span}(\alpha)| \le |\operatorname{dom}(\alpha)| + |\operatorname{im}(\alpha)| = \text{rank}(\alpha) + \text{rank}(\alpha) < r + r = r$ (since r is infinite). Thus $\text{rank}(\tau) = |\operatorname{dom}(\tau)| = |\operatorname{span}(\alpha)| < r$. Hence $\tau \in J_r$, and so $\alpha \sim_i \beta$ in J_r. □

2.4. Number of conjugacy classes in $\mathcal{I}(X)$

We will now count the i-conjugacy classes in $\mathcal{I}(X)$. Of course, we will have to distinguish between the finite and infinite X.

Let n be a positive integer. Recall that a *partition* of n is a sequence $\langle n_1, n_2, \dots, n_s\rangle$ of positive integers such that $n_1 \le n_2 \le \dots \le n_s$ and $n_1 + n_2 + \dots + n_s = n$. We denote by $p(n)$ the number of partitions of n and define $p(0)$ to be 1. For example, $n = 4$ has five partitions: $\langle 1,1,1,1\rangle$, $\langle 1,1,2\rangle$, $\langle 1,3\rangle$, $\langle 2,2\rangle$, and $\langle 4\rangle$; so $p(4) = 5$. Denote by $Q(n)$ the set of sequences $\langle (i_1,k_1),\dots,(i_u,k_u)\rangle$ of pairs of positive integers such that $k_1 < k_2 < \dots < k_u$ and $i_1k_1 + i_2k_2 + \dots + i_uk_u = n$. There is an obvious one-to-one correspondence between the set of partitions of n and the set $Q(n)$, so $|Q(n)| = p(n)$. For example, the partition $\langle 1,1,2,2,2,2,5\rangle$ of 15 corresponds to $\langle (2,1),(4,2),(1,5)\rangle \in Q(15)$. We define $Q(0)$ to be $\langle (0,0)\rangle$.

Notation 2.14. Let X be a finite set with $|X| = n$. Then every $\alpha \in \mathcal{I}(X)$ can be expressed uniquely as a join $\alpha = \sigma_\alpha \sqcup \eta_\alpha$, where σ_α is either 0 or a join of cycles, and η_α is either 0 or a join of chains. In other words, $\sigma_\alpha = \bigsqcup_{\delta\in\Delta_\alpha}\delta$ and $\eta_\alpha = \bigsqcup_{\theta\in\Theta_\alpha}\theta$. For example, if $\alpha = (2\,6\,8) \sqcup [1\,3] \sqcup [4\,5\,9]$, then $\sigma_\alpha = (2\,6\,8)$ and $\eta_\alpha = [1\,3] \sqcup [4\,5\,9]$. Note that $|\operatorname{span}(\sigma_\alpha)| = \sum_{k=1}^{n} k|\Delta_\alpha^k|$ and $|\operatorname{span}(\eta_\alpha)| = \sum_{k=1}^{n-1}(k+1)|\Theta_\alpha^k|$.

Let $C = \{[\alpha]_{\sim_i} : \alpha \in \mathcal{I}(X)\}$ be the set of i-conjugacy classes of $\mathcal{I}(X)$. For $r \in \{0,1,\dots,n\}$, denote by C_r the following subset of C:

$$C_r = \{[\alpha]_{\sim_i} \in C : |\operatorname{span}(\sigma_\alpha)| = r\}. \tag{2.4}$$

By Theorem 2.10, each C_r is well defined (if $\alpha \sim_i \beta$ then $|\operatorname{span}(\sigma_\alpha)| = |\operatorname{span}(\sigma_\beta)|$) and $C_0, C_1, \dots, C_n$ are pairwise disjoint.

Lemma 2.15. *Let X be a finite set with n elements, let $r \in \{0,1,\dots,n\}$, and let C_r be the set defined by (2.4). Then $|C_r| = p(r)p(n-r)$.*

Proof. Let $[\alpha]_{\sim_i} \in C_r$. Let $K = \{k \in \{1,\dots,n\} : \Delta_\alpha^k \ne \emptyset\}$. Write $K = \{k_1,k_2,\dots,k_u\}$ with $k_1 < k_2 < \dots < k_u$ ($u = 0$ if $K = \emptyset$). For $p \in \{1,\dots,u\}$, let $i_p = |\Delta_\alpha^{k_p}|$. By (2.3), the sequence $\langle (i_1,k_1),\dots,(i_u,k_u)\rangle$ (which we define to be $\langle (0,0)\rangle$ if $K = \emptyset$) does not depend on the choice of a representative in $[\alpha]_{\sim_i}$ and

$$i_1k_1 + \dots + i_uk_u = \sum_{k=1}^{n} k|\Delta_\alpha^k| = |\operatorname{span}(\sigma_\alpha)| = r. \tag{2.5}$$

Let $L = \{l \in \{1,\dots,n\} : l \ge 2 \text{ and} \Theta_\alpha^{l-1} \ne \emptyset \text{ or } l = 1 \text{ and } X \setminus \operatorname{span}(\alpha) \ne \emptyset\}$. (The reason we include l when $\Theta_\alpha^{l-1} \ne \emptyset$ is that there are l points in the span of each chain $[x_0\,x_1 \dots x_{l-1}]$ from Θ_α^{l-1}; and we include 1 when $X \setminus \operatorname{span}(\alpha) \ne \emptyset$ because $X \setminus \operatorname{span}(\alpha)$ consists of single points.) Write $L = \{l_1,l_2,\dots,l_v\}$ with $l_1 < l_2 < \dots < l_v$ ($v = 0$ if $L = \emptyset$). For $q \in \{1,\dots,v\}$, let $j_q = |\Theta_\alpha^{l_q-1}|$ (if $l_q \ge 2$) and $j_q = |X \setminus \operatorname{span}(\alpha)|$ (if $l_q = 1$). By (2.3), the sequence $\langle (j_1,l_1),\dots,(j_v,l_v)\rangle$ (which we define to be $\langle (0,0)\rangle$ if $L = \emptyset$) does not depend on the choice of a representative in $[\alpha]_{\sim_i}$ and

$$j_1l_1 + \cdots + j_vl_v = \sum_{l=1}^{n-1}(l+1)|\Theta_\alpha^l| + |X \setminus \operatorname{span}(\alpha)| = |\operatorname{span}(\eta_\alpha)| + |X \setminus \operatorname{span}(\alpha)| = n - r \tag{2.6}$$

(since $|\operatorname{span}(\sigma_\alpha)| + |\operatorname{span}(\eta_\alpha)| + |X \setminus \operatorname{span}(\alpha)| = n$).

Define a function $f : C_r \to Q(r) \times Q(n-r)$ (see the paragraph before Notation 2.14) by

$$([\alpha]_{\sim_i})f = (\langle (i_1,k_1),\ldots,(i_u,k_u)\rangle, \langle (j_1,l_1),\ldots,(j_v,l_v)\rangle).$$

Then f is well defined and one-to-one by (2.3), (2.5), and (2.6). Let

$$(\langle (i_1,k_1),\ldots,(i_u,k_u)\rangle, \langle (j_1,l_1),\ldots,(j_v,l_v)\rangle) \in Q(r) \times Q(n-r).$$

Then we can find $\alpha \in \mathcal{I}(X)$ that has i_p cycles of length k_p (for each $p \in \{1,\ldots,u\}$) and j_q chains of length $l_q - 1$ (for each $q \in \{1,\ldots,v\}$ such that $l_q \geq 2$). For such an α, $[\alpha]_{\sim_i} \in C_r$ and

$$([\alpha]_{\sim_i})f = (\langle (i_1,k_1),\ldots,(i_u,k_u)\rangle, \langle (j_1,l_1),\ldots,(j_v,l_v)\rangle),$$

so f is onto. Hence f is a bijection, and so $|C_r| = |Q(r) \times Q(n-r)| = |Q(r)||Q(n-r)| = p(r)p(n-r)$. □

If X is a finite set with n elements, then the symmetric group $\operatorname{Sym}(X)$ has $p(n)$ conjugacy classes [5, Prop. 11, p. 126]. The following theorem, which counts the i-conjugacy classes in the symmetric inverse semigroup $\mathcal{I}(X)$, generalizes this result.

Theorem 2.16. *Let X be a finite set with n elements. Then $\mathcal{I}(X)$ has $\sum_{r=0}^{n} p(r)p(n-r)$ i-conjugacy classes.*

Proof. Let C be the set of i-conjugacy classes of $\mathcal{I}(X)$. Then $C = C_0 \cup C_1 \cup \ldots \cup C_n$ and $C_0, C_1, \ldots, C_n$ are pairwise disjoint (see Notation 2.14.) The result follows by Lemma 2.15. □

For example, if $n = 5$, then the number of i-conjugacy classes of $\mathcal{I}(X)$ is

$$\sum_{r=0}^{5} p(r)p(5-r) = 1 \cdot 7 + 1 \cdot 5 + 2 \cdot 3 + 3 \cdot 2 + 5 \cdot 1 + 7 \cdot 1 = 36.$$

We will now count the i-conjugacy classes in $\mathcal{I}(X)$ for an infinite X. First, we need the following lemma. We denote by $\aleph_\varepsilon$ the infinite cardinal indexed by the ordinal ε [11, p. 131].

Lemma 2.17. *Let X be an infinite set with $|X| = \aleph_\varepsilon$ and let $\alpha \in \mathcal{I}(X)$. Then for all $k \geq 1$ and all $A \in \{\Delta^k_\alpha, \Theta^k_\alpha, \Omega_\alpha, \Upsilon_\alpha, \Lambda_\alpha\}$, $|A| \leq \aleph_\varepsilon$.*

Proof. Suppose $A = \Omega_\alpha$. Let $Z = \bigcup_{\omega\in\Omega_\alpha} \operatorname{span}(\omega) \subseteq X$. Since the elements of Ω_α are pairwise completely disjoint and $|\operatorname{span}(\omega)| = \aleph_0$ for every $\omega \in \Omega_\alpha$, we have

$$\aleph_\varepsilon = |X| \geq |Z| = |\bigcup_{\omega\in\Omega_\alpha} \operatorname{span}(\omega)| = |\Omega_\alpha| \cdot \aleph_0 \geq |\Omega_\alpha|.$$

Thus $|\Omega_\alpha| \leq \aleph_\varepsilon$. The proofs for the remaining values of A are similar. □

For sets A and B, we denote by A^B the set of all functions from B to A.

Theorem 2.18. *Let X be an infinite set with $|X| = \aleph_\varepsilon$. Let $\kappa = \aleph_0 + |\varepsilon|$. Then $\mathcal{I}(X)$ has $\kappa^{\aleph_0}$ i-conjugacy classes.*

Proof. Let M be the set of all cardinals μ such that $\mu \leq \aleph_\varepsilon$. Then M consists of $\aleph_0$ finite cardinals and $|\varepsilon| + 1$ infinite cardinals, hence $|M| = \aleph_0 + |\varepsilon| + 1 = \aleph_0 + |\varepsilon| = \kappa$. Let C be the set of i-conjugacy classes of $\mathcal{I}(X)$. Define a function $f : C \to M^{\mathbb{N}}$, where $\mathbb{N} = \{1, 2, 3, \ldots\}$, by

$$([\alpha)]_{\sim_i})f = \langle |\Omega_\alpha|, |\Upsilon_\alpha|, |\Lambda_\alpha|, |\Delta^1_\alpha|, |\Theta^1_\alpha|, |\Delta^2_\alpha|, |\Theta^2_\alpha|, |\Delta^3_\alpha|, |\Theta^3_\alpha|, \ldots\rangle.$$

By Theorem 2.10 and Lemma 2.17, f is well defined and one-to-one. Thus $|C| \leq |M^{\mathbb{N}}| = |M|^{|\mathbb{N}|} = \kappa^{\aleph_0}$.

We next define a one-to-one function $g : M^{\mathbb{N}} \to C$. Let

$$p = \langle \mu_1, \mu_2, \mu_3, \ldots\rangle \in M^{\mathbb{N}}.$$

Let $\mu = \sum_{k=1}^{\infty} k\mu_k$ (see [11, Ch. 9]). For every $k \geq 1$, $k\mu_k \leq \aleph_\varepsilon$ (since $\mu_k \leq \aleph_\varepsilon$ and $\aleph_\varepsilon$ is infinite). Thus

$$\mu = \sum_{k=1}^{\infty} k\mu_k \leq \aleph_0 \cdot \aleph_\varepsilon = \aleph_\varepsilon.$$

Hence, there is a collection $\{X_k\}_{k\geq1}$ of pairwise disjoint subsets of X such that $|X_k| = k\mu_k$ for every $k \geq 1$. Let $k \geq 1$. Since $|X_k| = k\mu_k$, there is a collection Δ^k of k-cycles in $\mathcal{I}(X)$ such that $|\Delta^k| = \mu_k$ and $\operatorname{span}(\bigsqcup_{\delta\in\Delta^k} \delta) = X_k$. Let $\alpha_k = \bigsqcup_{\delta\in\Delta^k} \delta$ and let $\alpha_p = \bigsqcup_{k\geq1} \alpha_k \in \mathcal{I}(X)$. We define $g : M^{\mathbb{N}} \to C$ by $pg = [\alpha_p]_{\sim_i}$.

Suppose $\alpha_p \sim_i \alpha_s$, where $p, s \in M^{\mathbb{N}}$. Then, by the definition of g, both α_p and α_s are joins of cycles and

$$\langle |\Delta^1_{\alpha_p}|, |\Delta^2_{\alpha_p}|, |\Delta^3_{\alpha_p}|, \ldots\rangle = \langle \mu_1, \mu_2, \mu_3, \ldots\rangle = \langle |\Delta^1_{\alpha_s}|, |\Delta^2_{\alpha_s}|, |\Delta^3_{\alpha_s}|, \ldots\rangle.$$

It follows from Theorem 2.10 that g is one-to-one. Hence $|C| \geq |M^{\mathbb{N}}| = |M|^{|\mathbb{N}|} = \kappa^{\aleph_0}$. The result follows. □

As an example, suppose $|X| = \aleph_{\omega_1}$, where ω_1 is the least uncountable ordinal. Then $\aleph_0 + |\omega_1| = \aleph_0 + \aleph_1 = \aleph_1$, and so the number of i-conjugacy classes in $\mathcal{I}(X)$ is $\aleph_1^{\aleph_0} = 2^{\aleph_0}$. (Clearly $2^{\aleph_0} \leq \aleph_1^{\aleph_0}$. On the other hand, $\aleph_1^{\aleph_0} \leq (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \aleph_0} = 2^{\aleph_0}$.) By a similar argument, if $|X| = \aleph_\varepsilon$, where ε is any countable ordinal or any ordinal of cardinality $\aleph_1$, then $\mathcal{I}(X)$ has $2^{\aleph_0}$ i-conjugacy classes. (The axioms of set theory cannot decide where in the aleph hierarchy the cardinal $2^{\aleph_0}$ occurs. If one assumes the Continuum Hypothesis, then $2^{\aleph_0} = \aleph_1$.)

3. Conjugacy in McAllister *P*-semigroups

Let S be an inverse semigroup with semilattice E of idempotents. We say that S is *E-unitary* if for all $a \in S$ and $e \in E$, if $ea \in E$ then $a \in E$ [10, §5.9].

Every E-unitary semigroup is isomorphic to a P-semigroup constructed by McAlister [22]. Consider a triple $(G, \mathcal{X}, \mathcal{Y})$, called a *McAlister triple* [10, p. 194], where G is a group, $\mathcal{X}$ is a set with a partial order relation $\leq$, and $\mathcal{Y}$ is a nonempty subset of $\mathcal{X}$ such that:

(1) $\mathcal{Y}$ is a lower semilattice under $\leq$, that is, if $A, B \in \mathcal{Y}$, then the greatest lower bound $A \wedge B$ exists and belongs to $\mathcal{Y}$;
(2) $\mathcal{Y}$ is an order ideal of $\mathcal{X}$, that is, if $A \in \mathcal{Y}$, $B \in \mathcal{X}$, and $B \leq A$, then $B \in \mathcal{Y}$;
(3) G acts on $\mathcal{X}$ by automorphisms, that is, there is a mapping $(g, X) \to gX$ from $G \times \mathcal{X}$ to $\mathcal{X}$ such that for all $g, h \in G$ and $A, B \in \mathcal{X}$, $g(hA) = (gh)A$ and $A \leq B \iff gA \leq gB$;
(4) $G\mathcal{Y} = \mathcal{X}$, and $g\mathcal{Y} \cap \mathcal{Y} \neq \emptyset$ for all $g \in G$.

Consider a set $P(G, \mathcal{X}, \mathcal{Y}) = \{(A, g) \in (\mathcal{Y}, G) : g^{-1}A \in \mathcal{Y}\}$ and define a multiplication on $P(G, \mathcal{X}, \mathcal{Y})$ by

$$(A, g)(B, h) = (A \wedge gB, gh). \tag{3.1}$$

The set $P(G, \mathcal{X}, \mathcal{Y})$ with multiplication (3.1) is a semigroup, called a *McAlister P-semigroup*. Every McAlister P-semigroup is an E-unitary inverse semigroup, and every E-unitary inverse semigroup is isomorphic to some McAlister P-semigroup [10, Thm. 5.9.2].

The following theorem describes i-conjugacy in any McAllister P-semigroup.

Theorem 3.1. *Let $S = P(G, \mathcal{X}, \mathcal{Y})$ be a McAlister P-semigroup. For $(A, g), (B, h) \in S$, the following are equivalent:*

(a) $(A, g) \sim_i (B, h)$*;*
(b) *there exists* $(C, k) \in S$ *such that* (i) $A = kB = C \wedge gC \wedge A$ *and* (ii) $g = khk^{-1}$*.*

Proof. Suppose $(A,g) \sim_i (B,h)$, that is, there is $(C,k) \in S$ such that $(C,k)^{-1}(A,g)(C,k) = (B,h)$ and $(C,k)(B,h)(C,k)^{-1} = (A,g)$. Since $(C,k)^{-1} = (k^{-1}C, k^{-1})$ [10, p. 194], by straightforward calculations we obtain

$$(B,h) = (k^{-1}C \wedge k^{-1}A \wedge (k^{-1}g)C, k^{-1}gk)\,,$$
$$(A,g) = (C \wedge kB \wedge (khk^{-1})C, khk^{-1})\,.$$

It follows that $g = khk^{-1}$ (so (ii) holds), $A = C \wedge kB \wedge gC$, and $kB = C \wedge A \wedge gC$. Thus $A \le kB$ and $kB \le A$, so $A = kB$. Further,

$$A = C \wedge kB \wedge gC = C \wedge (C \wedge A \wedge gC) \wedge gC = C \wedge A \wedge gC,$$

so (i) also holds. Conversely, suppose (i) and (ii) hold. Then

$$\begin{aligned}(C,k)^{-1}(A,g)(C,k) &= (k^{-1}C, k^{-1})(A \wedge gC, gk) = (k^{-1}C \wedge k^{-1}(A \wedge gC), k^{-1}gk)\\ &= (k^{-1}(C \wedge A \wedge gC), h) = (k^{-1}(kB), h) = (B,h)\,.\end{aligned}$$

Similarly, $(C,k)(B,h)(C,k)^{-1} = (A,g)$, and so $(A,g) \sim_i (B,h)$. □

4. Factorizable inverse monoids

We now describe i-conjugacy in factorizable inverse monoids, with the coset monoid of a group as a particular example.

First, recall that for a, b in an inverse semigroup S, the natural partial order is defined by $a \le b$ if there exists an idempotent e such that $a = eb$. Equivalently,

$$a \le b \iff b^{-1}a = a^{-1}a \iff a^{-1}b = a^{-1}a \iff ab^{-1} = aa^{-1} \iff ba^{-1} = aa^{-1}\,. \tag{4.1}$$

A subset $A \subseteq S$ is said to be *upward closed* if for all $a \in A$, $x \in S$, $a \le x$ implies $x \in A$.

For $a, b \in S$ with $a \sim_i b$, we set

$$\mathcal{C}_{a,b} = \{g \in S^1 \mid g^{-1}ag = b,\ gbg^{-1} = a\}\,.$$

Theorem 4.1. *Let S be an inverse semigroup. For all $a, b \in S$ with $a \sim_i b$, $\mathcal{C}_{a,b}$ is upward closed in S^1.*

Proof. Let $g \in \mathcal{C}_{a,b}$ and suppose $g \le h$ for $h \in S^1$. We use Proposition 1.3 and (4.1) to obtain: $h^{-1}ah = h^{-1} \cdot gbg^{-1} \cdot h = g^{-1}gbg^{-1}g = b$ and $hbh^{-1} = h \cdot g^{-1}ag \cdot h^{-1} = gg^{-1}agg^{-1} = a$. Thus $h \in \mathcal{C}_{a,b}$. □

An inverse monoid S is *factorizable* if $S = E(S)\,U(S)$. In other words, each element $a \in S$ can be written in the form $a = eg$ for some idempotent $e \in E(S)$ and some unit $g \in U(S)$.

For example, let G be a group and let $\mathcal{CM}(G) = \{Ha \mid H \leq G,\ a \in G\}$ be the *coset monoid* of G, where the multiplication on right cosets is defined by $Ha * Kb = (H \vee aKa^{-1})ab$, where $H \vee aKa^{-1}$ is the smallest subgroup of G that contains the subgroups H and aKa^{-1}. This is a factorizable inverse monoid [30], and every inverse semigroup embeds in the coset monoid of some group [23]. In this case, $E(S)$ is the set of all subgroups of G and $U(S)$ is the set of all singletons from G, that is, cosets of the trivial subgroup [6].

Corollary 4.2. *Let S be a factorizable inverse monoid. Then for all $a, b \in S$, $a \sim_i b$ if and only if $a \sim_u b$.*

Proof. We already noted in (1.10) that $\sim_u\ \subseteq\ \sim_i$ in any inverse monoid. For the other inclusion, suppose $a \sim_i b$ and let $h \in \mathcal{C}_{a,b}$ be given. Then there exist $e \in E(S)$, $g \in U(S)$ such that $h = eg$. But then $h \leq g$. By Theorem 4.1, $g \in \mathcal{C}_{a,b}$. □

Remark 4.3. If X is a finite set, then the symmetric inverse monoid $\mathcal{I}(X)$ is factorizable, so Corollary 4.2 gives another proof of the equivalence of parts (i) and (iii) of Proposition 2.11.

In particular, we have the following.

Corollary 4.4. *Let G be a group and let $\mathcal{CM}(G)$ be the coset monoid of G. For $Ha, Kb \in \mathcal{CM}(G)$, $Ha \sim_i Kb$ if and only if there exists $g \in G$ such that $g^{-1}Hag = Kb$ and $gKbg^{-1} = Ha$.*

5. Green's relations and Clifford semigroups

Let S be a semigroup. For $a, b \in S$, we say that $a\,\mathcal{L}\,b$ if $S^1a = S^1b$, $a\,\mathcal{R}\,b$ if $aS^1 = bS^1$, and $a\,\mathcal{J}\,b$ if $S^1aS^1 = S^1bS^1$. We set $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$. We also define $\mathcal{D}$ to be the join of $\mathcal{L}$ and $\mathcal{R}$, that is, the smallest equivalence relation on S containing both $\mathcal{L}$ and $\mathcal{R}$; it turns out that $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$ [10, p. 46]. These equivalences, called *Green's relations*, play an important role in semigroup theory [10, §2.1].

The conjugacy $\sim_i$ is always included in $\mathcal{D}$ (and so in $\mathcal{J}$).

Proposition 5.1. *Let S be an inverse semigroup. Then $\sim_i\ \subseteq \mathcal{D}$.*

Proof. Let $a, b \in S$ with $a \sim_i b$. By Proposition 1.3, there exists $g \in S^1$ such that $g^{-1} \cdot ag = b$, $ag = gb$, and $a \cdot gg^{-1} = a$. Since $ag \cdot g^{-1} = a$, we have $a\,\mathcal{R}\,ag$. Since $g^{-1} \cdot ag = b$ and $gb = ag$, we have $ag\,\mathcal{L}\,b$. Hence $a\,(\mathcal{R} \circ \mathcal{L})\,b$, that is, $a\,\mathcal{D}\,b$. □

For each element a in an inverse semigroup S, the unique idempotent in the $\mathcal{L}$-class of a is $a^{-1}a$, and the unique idempotent in the $\mathcal{R}$-class of a is aa^{-1}. These idempotents are i-conjugate.

Lemma 5.2. *Let S be an inverse semigroup. For all $x \in S$, $xx^{-1} \sim_i x^{-1}x$.*

Proof. This is immediate from $x^{-1} \cdot xx^{-1} \cdot x = x^{-1}x$ and $x \cdot x^{-1}x \cdot x^{-1} = xx^{-1}$. $\square$

In addition, i-conjugacy of elements implies i-conjugacy of their corresponding $\mathcal{L}$-related and $\mathcal{R}$-related idempotents.

Lemma 5.3. *Let S be an inverse semigroup and let $a, b \in S$ satisfy $a \sim_i b$. Then $a^{-1}a \sim_i b^{-1}b$ and $aa^{-1} \sim_i bb^{-1}$. More precisely, if $g \in \mathcal{C}_{a,b}$, then $g \in \mathcal{C}_{a^{-1}a,b^{-1}b}$ and $g \in \mathcal{C}_{aa^{-1},bb^{-1}}$.*

Proof. Let $g \in \mathcal{C}_{a,b}$. Then

$$b^{-1}b = (g^{-1}ag)^{-1}g^{-1}ag = g^{-1}a^{-1}\underbrace{gg^{-1}a}g = g^{-1}a^{-1}ag \qquad \text{and}$$

$$bb^{-1} = g^{-1}ag(g^{-1}ag)^{-1} = g^{-1}\underbrace{agg^{-1}}a^{-1}g = g^{-1}aa^{-1}g\,,$$

using Proposition 1.3 in both calculations. The equalities $gb^{-1}bg^{-1} = a^{-1}a$ and $gbb^{-1}g^{-1} = aa^{-1}$ follow similarly. $\square$

Let S be a semigroup. For $a \in S$, denote by H_a the $\mathcal{H}$-class containing a. Any $\mathcal{H}$-class of S containing an idempotent is a maximal subgroup of S. An element $a \in S$ is *completely regular* (or a *group element*) if its $\mathcal{H}$-class H_a is a group. If S is an inverse semigroup, the unique inverse a^{-1} of a completely regular element a is also the inverse of a in H_a, and so in particular, $aa^{-1} = a^{-1}a$. Conversely, if $aa^{-1} = a^{-1}a$, then $a \mathcal{H} aa^{-1}$, so H_a is a group.

Lemma 5.4. *Let S be an inverse semigroup and let $a, b \in S$ satisfy $a \sim_i b$. The following are equivalent:*

(a) *a is completely regular;*
(b) *b is completely regular;*
(c) *there exists $h \in \mathcal{C}_{a,b}$ such that $a \mathcal{R} h \mathcal{L} b$.*

Proof. (a) $\Longleftrightarrow$ (b) follows from Lemma 5.3.

Assume (a), (b), and fix $g \in \mathcal{C}_{a,b}$. Set $h = gb = ag$. Then

$$hh^{-1} = \underbrace{agg^{-1}}a^{-1} = aa^{-1} \quad \text{and} \quad h^{-1}h = b^{-1}\underbrace{g^{-1}gb} = b^{-1}b, \tag{5.1}$$

using Proposition 1.3 in both calculations. Thus $h \cdot h^{-1}a = aa^{-1}a = a$ and $bh^{-1} \cdot h = bb^{-1}b = b$, so $a \mathcal{R} h \mathcal{L} b$. Next, $h^{-1}ah = g^{-1}a^{-1}aag = g^{-1}aa^{-1}ag = g^{-1}ag = b$, using the complete regularity of a in the second equality. Similarly, $hbh^{-1} = gbbb^{-1}g^{-1} = gbb^{-1}bg^{-1} = gbg^{-1} = a$, using the complete regularity of b. Thus $h \in \mathcal{C}_{a,b}$. We have proven (a), (b) $\implies$ (c).

Now assume (c). We have $h^{-1}ah = b$ and $hbh^{-1} = a$. Moreover since each $\mathcal{R}$-class and each $\mathcal{L}$-class in an inverse semigroup contains exactly one idempotent [10, Thm. 5.1.1], we also have $hh^{-1} = aa^{-1}$ and $h^{-1}h = b^{-1}b$. We thus compute

$$b^{-1}b = h^{-1}h = h^{-1}\underbrace{hh^{-1}}h = h^{-1}\underbrace{a}a^{-1}h = \underbrace{h^{-1}hb}\underbrace{h^{-1}a^{-1}h} = b(h^{-1}ah)^{-1} = bb^{-1},$$

using Proposition 1.3 in the fifth equality. Therefore b is completely regular, that is, (b) holds. □

Proposition 5.5. *Let S be an inverse semigroup and let $a, b \in S$ satisfy $a \sim_i b$. If a, b lie in the same group $\mathcal{H}$-class H, then a and b are group conjugate in H.*

Proof. Since H is a group, both a and b are completely regular, and so by Lemma 5.4, there exists $h \in \mathcal{C}_{a,b}$ such that $a \mathcal{R} h \mathcal{L} b$. Since $a \mathcal{H} b$, we have $h \mathcal{H} a$, that is, $h \in H$. □

If every element of a semigroup S is completely regular, we say that S is a *completely regular semigroup*. A semigroup that is both inverse and completely regular is called a *Clifford semigroup*. One can characterize Clifford semigroups in several ways, some of which will be useful in what follows.

Proposition 5.6. [10, Thm. 4.2.1] *Let S be an inverse semigroup. The following are equivalent:*

(a) *S is a Clifford semigroup;*
(b) *for all $a \in S$, $aa^{-1} = a^{-1}a$;*
(c) *for all $a \in S$, $e \in E(S)$, $ea = ae$;*
(d) *$\mathcal{L} = \mathcal{R} = \mathcal{H}$.*

Theorem 5.7. *Let S be a Clifford semigroup. Then for all $a, b \in S$, $a \sim_i b$ if and only if a and b belong to the same $\mathcal{H}$-class H and they are group conjugate in H.*

Proof. The "if" direction is clear. For the converse, if $a \sim_i b$, then by Lemma 5.4 and Proposition 5.6(d), there exists $h \in \mathcal{C}_{a,b}$ such that $a \mathcal{H} h \mathcal{H} b$. The rest follows from Proposition 5.5. □

Using i-conjugacy, we can give new characterizations of Clifford semigroups in the class of inverse semigroups.

Theorem 5.8. *Let S be an inverse semigroup. The following are equivalent:*

(a) *S is a Clifford semigroup;*
(b) $\sim_i \subseteq \mathcal{H}$*;*
(c) $\sim_i \subseteq \mathcal{R}$*;*
(d) $\sim_i \subseteq \mathcal{L}$*;*
(e) *no two distinct idempotents in S are i-conjugate.*

Proof. We have (a) $\Longrightarrow$ (b) by Theorem 5.7. The implications (b) $\Longrightarrow$ (c) and (b) $\Longrightarrow$ (d) follow from $\mathcal{H} \subseteq \mathcal{R}$ and $\mathcal{H} \subseteq \mathcal{L}$. We have (c) $\Longrightarrow$ (e) and (d) $\Longrightarrow$ (e) by the fact that every $\mathcal{R}$-class and every $\mathcal{L}$-class of an inverse semigroup contains exactly one idempotent. Finally, suppose (e) holds. For $a \in S$, aa^{-1} and $a^{-1}a$ are idempotents and we have $aa^{-1} \sim_i a^{-1}a$ by Lemma 5.2. Thus $aa^{-1} = a^{-1}a$. Then (a) follows from Proposition 5.6. This completes the proof. □

Recall that a group G is abelian if and only if the conjugacy $\sim_G$ is the identity relation. This generalizes to inverse semigroups.

Theorem 5.9. *Let S be an inverse semigroup. Then S is commutative if and only if $\sim_i$ is the identity relation.*

Proof. Every commutative inverse semigroup is Clifford. On the other hand, if $\sim_i$ is the identity relation, then S is Clifford by Theorem 5.8. Thus we may assume from the outset that S is a Clifford semigroup. The desired result then follows from the following chain of equivalences: S is commutative if and only if each $\mathcal{H}$-class is an abelian group if and only if group conjugacy within each $\mathcal{H}$-class is the identity relation if and only if $\sim_i$ is the identity relation (by Theorem 5.7). □

Looking at conditions (b), (c) and (d) of Theorem 5.8, it is natural to ask what can be said if the opposite inclusions hold. We conclude this section with two results that answer this question.

Theorem 5.10. *Let S be an inverse semigroup. The following are equivalent:*

(a) *S is a semilattice;*
(b) $\mathcal{L} \subseteq \sim_i$*;*
(c) $\mathcal{R} \subseteq \sim_i$*.*

Proof. In a semilattice, $\mathcal{L}$ and $\mathcal{R}$ are trivial, so (a) $\Longrightarrow$ (b) and (a) $\Longrightarrow$ (c) follow. Assume (c). For each $a \in S$, $aa^{-1} \mathcal{R} a$, and so $g^{-1}aa^{-1}g = a$ for some $g \in S^1$. But every conjugate of an idempotent is an idempotent, so each $a \in S$ is idempotent. Thus (a) holds. The proof of (b) $\Longrightarrow$ (a) is similar. □

A semigroup is said to be $\mathcal{H}$-*trivial* if $\mathcal{H}$ is the identity relation.

Theorem 5.11. *Let S be an inverse semigroup. Then $\mathcal{H} \subseteq\ \sim_i$ if and only if S is $\mathcal{H}$-trivial.*

Proof. The "if" direction is obvious, so assume $\mathcal{H} \subseteq\ \sim_i$. If H is a group $\mathcal{H}$-class, then by Proposition 5.5, all elements of H are group conjugate, hence H is a trivial subgroup. Now suppose $a\,\mathcal{H}\,b$. We compute

$$\begin{aligned}(ba^{-1})^{-1}ba^{-1} &= ab^{-1}ba^{-1} = aa^{-1}aa^{-1} = aa^{-1} = bb^{-1} = bb^{-1}bb^{-1} \\ &= ba^{-1}ab^{-1} = ba^{-1}(ba^{-1})^{-1}\,.\end{aligned}$$

Thus ba^{-1} is completely regular, that is, it is in some group $\mathcal{H}$-class H. By the above computation, aa^{-1} is the identity in H. Since H is trivial, $ba^{-1} = aa^{-1}$. By (N), we have $a \le b$. Repeating the argument with the roles of a and b reversed, we also obtain $b \le a$. Thus $a = b$. Therefore $\mathcal{H}$ is the identity relation as claimed. □

6. The bicyclic monoid and stable inverse semigroups

The *bicyclic monoid* $\mathcal{B}$, which is an inverse semigroup, is usually defined in terms of a monoid presentation $\langle x, y \mid xy = 1\rangle$ [10, p. 32] [17, Sect. 3.4]. It has a more convenient isomorphic realization as the set $\mathcal{B}$ of ordered pairs of nonnegative integers with the following multiplication:

$$(a, b)(c, d) = (a - b + \max(b, c), d - c + \max(b, c))\,.$$

For any $(a, b) \in \mathcal{B}$, $(a, b)^{-1} = (b, a)$ and (a, b) is an idempotent if and only if $a = b$. The smallest group congruence σ in $\mathcal{B}$ is characterized as follows [17, p. 101]:

$$(a, b)\ \sigma\ (c, d) \iff a - b = c - d.$$

Theorem 6.1. *In $\mathcal{B}$, $\sim_i\ = \sigma$.*

Proof. Suppose $(a, b) \sim_i (c, d)$. Then for some $(e, f) \in \mathcal{B}$, we have $(a, b) = (f, e)(c, d)(e, f)$. Expanding this, we get

$$\begin{aligned}a &= f - e - d + c + m \\ b &= f - e + m\end{aligned}$$

where $m = \max(d - c + \max(e, c), e)$. Thus $a - b = c - d$, that is, $(a, b)\ \sigma\ (c, d)$.

Conversely, suppose that $(a, b)\ \sigma\ (c, d)$, so $a - b = c - d$. We claim that for $x = \min(c, d)$ and $y = \min(a, b)$, we have

$$(a, b) = (y, x)(c, d)(x, y) \quad \text{and} \quad (c, d) = (x, y)(a, b)(y, x)\,. \tag{6.1}$$

To prove this, we compute

$$\begin{aligned}(y,x)(c,d)(x,y) &= (y,x)(c-d+\underbrace{\max(d,x)}_{d}, y-x+\underbrace{\max(d,x)}_{d})\\ &= (y,x)(c, y-x+d)\\ &= (y-x+\underbrace{\max(x,c)}_{c}, y-x+d-c+\underbrace{\max(x,c)}_{c})\\ &= (y-x+c, y-x+d)\,,\end{aligned}$$

and similarly,

$$\begin{aligned}(x,y)(a,b)(y,x) &= (x,y)(a-b+\underbrace{\max(b,y)}_{b}, x-y+\underbrace{\max(b,y)}_{b})\\ &= (x,y)(a, x-y+b)\\ &= (x-y+\underbrace{\max(y,a)}_{a}, x-y_b-a+\underbrace{\max(y,a)}_{a})\\ &= (x-y+a, x-y+b)\,.\end{aligned}$$

Comparing the results of these calculations, and noting that $c-a=d-b$, we see that we will have established (6.1) once we have proven $x-y=c-a=d-b$.

Observe that $a-b=c-d$ implies that $a \leq b \iff c \leq d$, and also $a \geq b \iff c \geq d$. Thus $x=c \iff y=a$ and $x=d \iff y=b$. In the former case, $x-y=c-a$ and in the latter case, $x-y=d-b$. In both cases, we have $c-a=d-b$ because $a-b=c-d$. This completes the proof. □

A semigroup S is *left stable* if, for all $a,b \in S$, $S^1a \subseteq S^1ab$ implies $S^1a = S^1ab$, that is, $a\,\mathcal{L}\,ab$. This can be equivalently formulated as $a \in S^1ab$ implies $ab \in S^1a$ for all $a,b \in S$. *Right stability* is defined dually, and a semigroup is said to be *stable* if it is both left and right stable [3, p. 31]. Every periodic semigroup, and in particular every finite semigroup, is stable. In inverse semigroups, left and right stability are equivalent. We also have a useful characterization, which in fact holds more generally for regular semigroups.

Proposition 6.2 *([27], Ex. A.2.2(8), p. 595). Let S be an inverse semigroup. Then S is stable if and only if S does not contain an isomorphic copy of the bicyclic monoid as a subsemigroup.*

Here we give a new characterization of stability in terms of i-conjugacy and the natural partial order.

Theorem 6.3. *An inverse semigroup S is stable if and only if $\sim_i \cap \leq$ is the identity relation on S.*

Proof. Assume S is stable. Suppose $a \sim_i b$ and $a \leq b$. Then $g^{-1}ag = b$ and $gbg^{-1} = a$ for some $g \in S^1$. First, we compute

$$a = a\underbrace{a^{-1}a} = \underbrace{aa^{-1}}b = ab^{-1}b = ab^{-1}g^{-1}gb = a(\underbrace{gb})^{-1}\underbrace{gb} = a(ag)^{-1}ag\,, \tag{6.2}$$

where the second and third equalities follow from (N) and the fourth and sixth equalities follow from Proposition 1.3. Thus $a \in S^1ag$. Since S is stable, $ag \in S^1a$, that is, $ag = ca$ for some $c \in S^1$. Now

$$ag = ca = \underbrace{ca}a^{-1}a = \underbrace{ag}a^{-1}a = g\underbrace{ba^{-1}}a = gaa^{-1}a = ga\,,$$

where the fourth equality follows from Proposition 1.3 and the fifth equality follows from (N). Using this in (6.2), we have

$$\begin{aligned} a &= a(ga)^{-1}ga = \underbrace{aa^{-1}}g^{-1}ga = b\underbrace{a^{-1}g^{-1}}ga = b(\underbrace{ga})^{-1}\underbrace{ga} \\ &= \underbrace{b}(ag)^{-1}ag = g^{-1}\underbrace{ag(ag)^{-1}ag} = g^{-1}ag = b\,, \end{aligned}$$

where the third equality follows from (N).

Conversely, suppose that S is not stable, so by Proposition 6.2, S contains a copy of the bicyclic monoid $\mathcal{B}$. For nonnegative integers m, n with $m < n$, we have $(n, n) < (m, m)$ in $\mathcal{B}$. By Theorem 6.1, $(m, m) \sim_i (n, n)$. Thus $\sim_i \cap \leq$ strictly contains the identity relation. $\square$

Corollary 6.4. *Let S be a finite inverse semigroup. Then $\sim_i \cap \leq$ is the identity relation.*

7. Conjugacy in free inverse semigroups

For a nonempty set X (finite or infinite), denote by $\mathcal{FI}(X)$ the *free inverse semigroup* on X. In this section, we characterize $\sim_i$ in $\mathcal{FI}(X)$ (Theorem 7.9), show that every i-conjugacy class in $\mathcal{FI}(X)$ is finite (Corollary 7.7), and count the elements an arbitrary i-conjugacy class (Corollary 7.11). It follows from the characterization of $\sim_i$ that the i-conjugacy problem in $\mathcal{FI}(X)$ is decidable (Corollary 7.15).

Let X be a non-empty set. We say that an inverse semigroup F is a *free inverse semigroup* on X if it satisfies the following properties:

(1) X generates F;
(2) for every inverse semigroup S and every mapping $\phi : X \to S$, there is an extension of ϕ to a homomorphism $\overline{\phi} : F \to S$.

Since X generates F, an extension $\overline{\phi}$ is necessarily unique. It is well known that a free inverse semigroup on X exists and is unique [10, §5.10]. We will denote this unique object by $\mathcal{FI}(X)$. The semigroup $\mathcal{FI}(X)$ can be constructed as follows [10, Thm. 5.10.1].

Let $X^{-1} = \{x^{-1} : x \in X\}$ be a set that is disjoint from X, let $Y = X \cup X^{-1}$, and let Y^+ be the free semigroup on Y. We also consider the null word 1, and write $Y^* = Y^+ \cup \{1\}$.

For every $y \in Y$, we define y^{-1} to be x^{-1} if $y = x \in X$, and to be x if $y = x^{-1} \in X^{-1}$. Then $\mathcal{FI}(X)$ is isomorphic to the quotient semigroup Y^+/τ, where τ is the smallest congruence on Y^+ that contains the relation $\{(yy^{-1}y, y) : y \in Y\} \cup \{(yy^{-1}zz^{-1}, zz^{-1}yy^{-1}) : y, z \in Y\}$. For $w\tau \in \mathcal{FI}(X)$, where $w = y_1y_2 \dots y_n \in Y^+$, the unique inverse of $w\tau$ in $\mathcal{FI}(X)$ is $(w\tau)^{-1} = (y_n^{-1} \dots y_2^{-1}y_1^{-1})\tau$.

For $w \in Y^+$, we say that w is *reduced* if it does not contain any subword yy^{-1}, where $y \in Y$. We denote by $\overline{w}$ the unique reduced word in Y^* obtained from w by successively removing all subwords yy^{-1}. For example, $w = aba^{-1}ab^{-1}ab^{-1}$ is not reduced and $\overline{w} = aab^{-1}$. In the free group on X (which can be defined by Y^*/ρ, where ρ is the smallest congruence on Y^* that contains the relation $\{(yy^{-1}, 1) : y \in Y\}$), each congruence class modulo ρ contains exactly one reduced word [21, p. 3]. The situation is more complicated when one considers the congruence classes of $\mathcal{FI}(X)$. However, each congruence class $w\tau$ of $\mathcal{FI}(X)$ can be represented by a unique birooted word tree [8, Ch. 2]. Throughout this section, we will rely on this representation, which we will now describe following [8, Ch. 2].

Definition 7.1. A *word tree* T on X is a finite tree, with at least one edge, such that:

(i) each edge of T is oriented and labeled by some x in X;
(ii) T does not contain a word subtree of the form $\circ \xrightarrow{x} \circ \xleftarrow{x} \circ$ or $\circ \xleftarrow{x} \circ \xrightarrow{x} \circ$, where $x \in X$.

Let α, β be vertices of a word tree T. We will denote by $\Gamma(\alpha, \beta)$ any *walk* from α to β in T, that is, a sequence $(\alpha = \gamma_0, \gamma_1, \dots, \gamma_n = \beta)$, $n \geq 0$, of vertices of T such that for every $i \in \{0, \dots, n-1\}$, there is an edge between γ_{i-1} and γ_i (we ignore the orientation of the edges here). A walk $\Gamma(\alpha, \beta)$ is a *spanning walk* if it contains each vertex of T at least once; and it is a *path* if it contains no vertex of T more than once.

Let $\Gamma = (\gamma_0, \gamma_1, \dots, \gamma_n)$, be a walk in T. The *label* of Γ, denoted by $w(\Gamma)$, is the word $y_1y_2 \dots y_n$ in Y^* such that for every $i \in \{0, \dots, n-1\}$, $y_i = x$ if $\overset{\gamma_{i-1}}{\circ} \xrightarrow{x} \overset{\gamma_i}{\circ}$, and $y_i = x^{-1}$ if $\overset{\gamma_{i-1}}{\circ} \xleftarrow{x} \overset{\gamma_i}{\circ}$. We note that if Γ is a path, then $w(\Gamma)$ is reduced.

Definition 7.2. A *birooted word tree* is a triple (T, α, β), where T is a word tree on X and α and β are vertices of T.

Fix any transversal $\mathcal{S}_X$ of the isomorphism classes of word trees on X. There is a one-to-one correspondence between the elements of $\mathcal{FI}(X)$ and the birooted word trees (T, α, β) on X such that $T \in \mathcal{S}_X$ [8, Theorem 2.1.13].

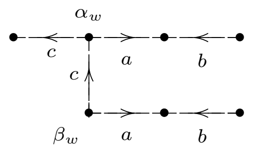

Fig. 7.1. (T_w, α_w, β_w) for w from Example 7.4.

Lemma 7.3. *Let $w \in Y^+$. Then:*

(a) *there exists a unique birooted tree (T_w, α_w, β_w) such that $T_w \in \mathcal{S}_X$ and there is a spanning walk $\Gamma(\alpha_w, \beta_w)$ in T_w such that $w(\Gamma(\alpha_w, \beta_w)) = w$;*
(b) *for every $v \in Y^+$, $w\tau = v\tau$ if and only if $(T_w, \alpha_w, \beta_w) = (T_v, \alpha_v, \beta_v)$;*
(c) *$w\tau \in \mathcal{FI}(X)$ is an idempotent if and only if $\alpha_w = \beta_w$.*

For any $w \in Y^+$, we will denote by (T_w, α_w, β_w) the unique birooted tree from Lemma 7.3. We note that, by [8, Cor. 2.1.5], if T is a word tree isomorphic to T_w, then there exists a unique isomorphism from T to T_w.

For $w = y_1 \dots y_n \in Y^+$, the birooted word tree (T_w, α_w, β_w) can be constructed as follows [8, Lemma 2.1.9]. Step 1 is to draw $\overset{\alpha_w}{\circ} \xrightarrow{x_1} \overset{\beta_w}{\circ}$ (if $y_1 = x_1$) or $\overset{\alpha_w}{\circ} \xleftarrow{x_1} \overset{\beta_w}{\circ}$ (if $y_1 = x_1^{-1}$), where $x_1 \in X$. Suppose that the birooted word tree has been constructed for the word $y_1 \dots y_i$, where $1 \leq i \leq n$. If $i = n$, we stop. If $i < n$, we perform step $(i + 1)$.

Suppose that the last edge constructed was $\circ \xrightarrow{x_i} \overset{\beta_w}{\circ}$. Then:

- if $y_{i+1} \neq x_i^{-1}$, we replace $\circ \xrightarrow{x_i} \overset{\beta_w}{\circ}$ with $\circ \xrightarrow{x_i} \circ \xrightarrow{x_{i+1}} \overset{\beta_w}{\circ}$ (if $y_{i+1} = x_{i+1}$) or with $\circ \xrightarrow{x_i} \circ \xleftarrow{x_{i+1}} \overset{\beta_w}{\circ}$ (if $y_{i+1} = x_{i+1}^{-1}$), where $x_{i+1} \in X$;
- if $y_{i+1} = x_i^{-1}$, we replace $\circ \xrightarrow{x_i} \overset{\beta_w}{\circ}$ with $\overset{\beta_w}{\circ} \xrightarrow{x_i} \circ$.

Suppose that the last edge constructed was $\circ \xleftarrow{x_i} \overset{\beta_w}{\circ}$. Then:

- if $y_{i+1} \neq x_i$, we replace $\circ \xleftarrow{x_i} \overset{\beta_w}{\circ}$ with $\circ \xleftarrow{x_i} \circ \xrightarrow{x_{i+1}} \overset{\beta_w}{\circ}$ (if $y_{i+1} = x_{i+1}$) or with $\circ \xleftarrow{x_i} \circ \xleftarrow{x_{i+1}} \overset{\beta_w}{\circ}$ (if $y_{i+1} = x_{i+1}^{-1}$), where $x_{i+1} \in X$;
- if $y_{i+1} = x_i$, we replace $\circ \xleftarrow{x_i} \overset{\beta_w}{\circ}$ with $\overset{\beta_w}{\circ} \xleftarrow{x_i} \circ$.

The birooted word tree (T_w, α_w, β_w) is constructed after n steps.

Example 7.4. Let $X = \{a, b\}$ and $w = ab^{-1}ba^{-1}cc^{-1}c^{-1}ab^{-1}ba^{-1} \in Y^+$. The birooted word tree (T_w, α_w, β_w) is presented in Fig. 7.1.

Lemma 7.5. *For all $w, v \in Y^+$, if $w\tau \sim_i v\tau$, then $T_w = T_v$.*

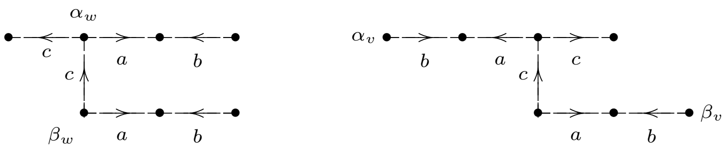

Fig. 7.2. (T_w, α_w, β_w) and (T_v, α_v, β_v) for w and v from Example 7.6.

Proof. Suppose $w, v \in Y^+$ with $w\tau \sim_i v\tau$. By Proposition 5.1, $\sim_i \subseteq \mathcal{D}$. By [8, Thm. 2.1.15], $w\tau \,\mathcal{D}\, v\tau$ if and only if $T_w = T_v$. The result follows. □

Example 7.6. Consider $w = ab^{-1}ba^{-1}cc^{-1}c^{-1}ab^{-1}ba^{-1}$ from Example 7.4. Let $v = ba^{-1}cc^{-1}c^{-1}ab^{-1}$. Then $w\tau{\sim_i}v\tau$ since for $u = ab^{-1}$, $(u^{-1}wu)\tau = v\tau$ and $(uvu^{-1})\tau = w\tau$. Both (T_w, α_w, β_w) and (T_v, α_v, β_v) are presented in Fig. 7.2.

Corollary 7.7. *All i-conjugacy classes in $\mathcal{FI}(X)$ are finite.*

Proof. Let $w\tau \in \mathcal{FI}(X)$. By [8, Thm. 2.1.15] and Lemma 7.3, the $\mathcal{D}$-class of $w\tau$ is finite. Thus, the i-conjugacy class of $w\tau$ is also finite since $\sim_i \subseteq \mathcal{D}$. □

Lemma 7.8. *Let $w\tau, v\tau \in \mathcal{FI}(X)$. Then $w\tau \sim_i v\tau$ if and only if there exists a reduced $g \in Y^*$ such that $(g^{-1}wg)\tau = v\tau$ and $(gvg^{-1})\tau = w\tau$.*

Proof. Suppose $w\tau \sim_i v\tau$. Then there exists $u \in Y^+$ such that $(u^{-1}wu)\tau = v\tau$ and $(uvu^{-1})\tau = w\tau$. By [8, p. 95], we may assume that $u = ge$, where $g \in Y^*$ is reduced and $e\tau$ is an idempotent in $\mathcal{FI}(X)$. Since $(ge)\tau \leq g\tau$, we have $(g^{-1}wg)\tau = v\tau$ and $(gvg^{-1})\tau = w\tau$ by Proposition 4.1. The converse is obvious. □

We can now characterize i-conjugacy in $\mathcal{FI}(X)$ in terms of properties of birooted word trees. For a word tree T and vertices α and β of T, we denote by $\Pi(\alpha, \beta)$ the (unique) path from α to β. If $\Gamma_1 = (\gamma_0, \dots, \gamma_k = \delta)$ and $\Gamma_2 = (\delta = \eta_0, \dots, \eta_m)$ are walks in T, then the *product* $\Gamma_1\Gamma_2$ is the walk $(\gamma_0, \dots, \gamma_{k-1}, \delta, \eta_1, \dots, \eta_m)$ [8, p. 83].

Theorem 7.9. *Let $w\tau, v\tau \in \mathcal{FI}(X)$, with the birooted word trees (T_w, α_w, β_w) and (T_v, α_v, β_v). Then $w\tau \sim_i v\tau$ if and only if $T_w = T_v$ and $w(\Pi(\alpha_w, \alpha_v)) = w(\Pi(\beta_w, \beta_v))$.*

Proof. Suppose $w\tau \sim_i v\tau$. Then, by Lemmas 7.5 and 7.8, $T_w = T_v$ $(= T)$, and there exists a reduced word $g \in Y^*$ such that $(g^{-1}wg)\tau = v\tau$ and $(gvg^{-1})\tau = w\tau$.

Suppose to the contrary that $w(\Pi(\alpha_w, \alpha_v)) \neq g$. By Lemma 7.3, there exists a spanning walk Γ in T such that $w(\Gamma) = g^{-1}wg$. Then $\Gamma = \Gamma_1\Gamma_2\Gamma_3$, where $w(\Gamma_1) = g^{-1}$, $w(\Gamma_2) = w$, and $w(\Gamma_3) = g$. Note that Γ_1 and Γ_3 are paths in T. Since $(g^{-1}wg)\tau = v\tau$, the path Γ_1 must begin at α_v (by Lemma 7.3). Moreover, Γ_1 must end at some $\delta \neq \alpha_w$ (since otherwise $w(\Pi(\alpha_w, \alpha_v))$ would be equal to g). Thus the walk Γ_2 begins at $\delta \neq \alpha_w$. Let T' be the word subtree of T spanned by Γ_2. Then, since $w(\Gamma_2) = w$, (T', δ, η) is

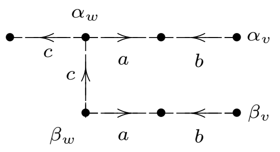

Fig. 7.3. (T_w, α_w, β_w) and (T_v, α_v, β_v) drawn in one diagram.

the birooted word tree for w, where η is a terminal vertex of Γ_2. This is a contradiction since $\delta \neq \alpha_w$. We have proved that $w(\Pi(\alpha_w, \alpha_v)) = g$. By a similar argument, $w(\Pi(\beta_w, \beta_v)) = g$, and so $w(\Pi(\alpha_w, \alpha_v)) = w(\Pi(\beta_w, \beta_v))$.

Conversely, suppose that $T_w = T_v$ $(= T)$ and $w(\Pi(\alpha_w, \alpha_v)) = w(\Pi(\beta_w, \beta_v))$ $(= g)$. Let $\Gamma(\alpha_w, \beta_w)$ be a spanning walk in T such that $w(\Gamma(\alpha_w, \beta_w)) = w$. Then $\Pi(\alpha_v, \alpha_w)\Gamma(\alpha_w, \beta_w)\Pi(\beta_w, \beta_v)$ is a spanning walk from α_v to β_v in T, and $w(\Pi(\alpha_v, \alpha_w)\Gamma(\alpha_w, \beta_w)\Pi(\beta_w, \beta_v)) = g^{-1}wg$. Hence $(g^{-1}wg)\tau = v\tau$ by Lemma 7.3. By a similar argument, $(gvg^{-1})\tau = w\tau$, and so $w\tau \sim_i v\tau$. $\square$

Example 7.10. Consider $w = ab^{-1}ba^{-1}cc^{-1}c^{-1}ab^{-1}ba^{-1}$ and $v = ba^{-1}cc^{-1}c^{-1}ab^{-1}$ from Example 7.6. Fig. 7.3 presents the birooted trees (T_w, α_w, β_w) and (T_v, α_v, β_v), where $T_w = T_v$ are drawn in one diagram. Note that $w(\Pi(\alpha_w, \alpha_v)) = w(\Pi(\beta_w, \beta_v)) = ab^{-1}$. Thus $w\tau \sim_i v\tau$ by Theorem 7.9.

We record three corollaries of Theorem 7.9. By Corollary 7.7, the i-conjugacy class of any $w\tau \in \mathcal{FI}(X)$ is finite. Using Theorem 7.9, we can calculate its cardinality.

Corollary 7.11. *Let $w\tau \in \mathcal{FI}(X)$ and let T be the word subtree of T_w spanned by the vertices of all paths $\Pi(\alpha_w, \gamma)$ in T_w such that $w(\Pi(\alpha_w, \gamma)) = w(\Pi(\beta_w, \eta))$ for some path $\Gamma(\beta_w, \eta)$ in T_w. Then the cardinality of the i-conjugacy class of $w\tau$ is equal to the number of vertices of T.*

Proof. Note that T is not empty since $w(\Pi(\alpha_w, \alpha_w)) = w(\Pi(\beta_w, \beta_w)) = 1$. By Theorem 7.9, there is a one-to-one correspondence between the paths $\Pi(\alpha_w, \gamma)$ in T and the elements of $[w\tau]_{\sim_i}$. The result follows since the number of paths in T that begin at α_w is equal to the number of vertices of T. $\square$

Example 7.12. Consider $w = ab^{-1}ba^{-1}cc^{-1}c^{-1}ab^{-1}ba^{-1}$ and see the birooted word tree (T_w, α_w, β_w) in Fig. 7.1. The word subtree T of T_w from Corollary 7.11 is presented in Fig. 7.4. Since T has four vertices, the i-conjugacy class of $w\tau$ has four elements. It follows from the proof of Theorem 7.9 that these elements are: $w\tau$, $(c^{-1}wc)\tau$, $(a^{-1}wa)\tau$, and $((ab^{-1})^{-1}w(ab^{-1}))\tau$.

Corollary 7.13. *Let $w\tau \in \mathcal{FI}(X)$ be an idempotent. Then the cardinality of the i-conjugacy class of $w\tau$ is equal to the number of vertices of T_w.*

Fig. 7.4. (T_w, α_w, β_w) and T from Corollary 7.11.

Proof. By Lemma 7.3, $\alpha_w = \beta_w$. Thus, for every vertex γ of T_w, $w(\Pi(\alpha_w, \gamma)) = w(\Pi(\beta_w, \gamma))$, which implies that the word subtree T from Corollary 7.11 is equal to T_w. Hence the cardinality of $[w\tau]_{\sim_i}$ is equal to the number of vertices of T_w. □

Note that if $w\tau$ is an idempotent, then $[w\tau]_{\sim_i}$ consists of all idempotents represented by the birooted word trees (T_w, γ, γ), where γ is a vertex of T_w. These are precisely the idempotents contained in the $\mathcal{D}$-class of $w\tau$.

Definition 7.14. We say that the i-conjugacy problem for $\mathcal{FI}(X)$ is *decidable* if there is an algorithm that given any pair (w, v) of words in Y^+, returns YES if $w\tau$ and $v\tau$ are i-conjugate in $\mathcal{FI}(X)$, and NO otherwise.

Corollary 7.15. *The i-conjugacy problem in $\mathcal{FI}(X)$ is decidable.*

Proof. Let $w, v \in Y^+$. Using the algorithm from [8, Lemma 2.1.9], we can construct a birooted word tree (T_1, α_1, β_1) on X such that $w = w(\Gamma(\alpha_1, \beta_1))$ for some spanning walk from α_1 to β_1 in T_1. We may assume that $T \in \mathcal{S}_X$, so $(T_1, \alpha_1, \beta_1) = (T_w, \alpha_w, \beta_w)$. Similarly, we can construct an analogous word tree (T_2, α_2, β_2) for v. It can be checked effectively if T_w and T_2 are isomorphic. If not, then $T_w \neq T_v$, and so the answer is NO by Lemma 7.5. Suppose T_w and T_2 are isomorphic. Then, by [8, Cor. 2.1.15], there exists a unique isomorphism, say φ, from T_2 to T_w. It follows that we can obtain T_v from T_2 by replacing each vertex γ in T_2 by the vertex $\gamma\varphi$, without changing any edges. Thus (T_v, α_v, β_v) can be constructed. We can now answer YES or NO by Theorem 7.9. □

8. Problems

Almost factorizable inverse semigroups naturally generalize factorizable inverse monoids in the sense that an inverse monoid is almost factorizable if and only if it is factorizable [17].

Problem 8.1. *Does i-conjugacy in almost factorizable inverse semigroups have a reasonable characterization suitably generalizing* Corollary 4.2?

The basic definition (1.1) of conjugacy in an inverse semigroup can, in principle, be extended to any class of semigroups in which there is some natural notion of unary (weak) inverse map. For example, let $(S, \cdot, ')$ be a unary E-inversive semigroup, that is, $(S, \cdot)$

is a semigroup and the identity $x'xx' = x'$ holds. Unary E-inversive semigroups include unary regular semigroups (in which the identity $xx'x = x$ also holds) and epigroups (in which $x \mapsto x'$ is the unique pseudoinverse [31]). Define a notion of conjugacy in such unary semigroups by $a \sim b$ if $g'ag = b$ and $gbg' = a$ for some $g \in S^1$. In general, these relations will not be transitive (except, for instance, when the identity $(xy)' = y'x'$ holds), so it is necessary to consider the transitive closure $\sim^*$.

Problem 8.2. *Study this notion of conjugacy in various interesting subclasses of unary E-inversive semigroups.*

Somewhat more promising is to consider the alternative formulations of i-conjugacy given in Proposition 1.3. For instance, part (c) of the proposition depends only on the idempotents gg^{-1} and $g^{-1}g$. This immediately suggests a generalization to *restriction semigroups* and their various specializations, such as *ample* semigroups (see [7] and the references therein). An algebra $(S, \cdot, {}^+, {}^*)$ is a restriction semigroup if $(S, \cdot)$ is a semigroup; $S \to S; x \mapsto x^+$ is a unary operation satisfying $x^+x = x$, $x^+y^+ = y^+x^+$, $(x^+y)^+ = x^+y^+$, $(xy)^+x = xy^+$; $S \to S; x \mapsto x^*$ is a unary operation satisfying dual identities; and $(x^+)^* = x^+$, $(x^*)^+ = x^*$. Here x^+ and x^* turn out to be idempotents. Any inverse semigroup is a restriction semigroup with $x^+ = xx^{-1}$ and $x^* = x^{-1}x$.

For a, b in a restriction semigroup S, define

$$a \sim_r b \quad \Longleftrightarrow \quad \exists_{g \in S^1} \left(ag = gb,\ ag^+ = a,\ g^*b = b \right).$$

Problem 8.3. *Study $\sim_r$ in restriction and ample semigroups. What is the relationship between $\sim_r$ and $\sim_{\mathfrak{n}}$?*

Acknowledgments

We are very grateful to the referee for rewriting and expanding (with proofs) Section 7. Originally, using the technique described in [26], we proved the results that are now Corollaries 7.7 and 7.15. The referee characterized i-conjugacy in $\mathcal{FI}(X)$ in terms of properties of birooted word trees (Theorem 7.9), from which our original results, and others, follow.

The first and second authors were partially supported by the Fundação para a Ciência e a Tecnologia (Portuguese Foundation for Science and Technology) through the project UID/MAT/00297/2019 (Centro de Matemática e Aplicações), the project PTDC/MHC-FIL/2583/2014, the FCT project PTDC/MAT-PUR/31174/2017, and CEMAT project UID/Multi/04621/2013.

The second author was also partially supported by a Simons Foundation Collaboration Grant 359872.

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